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Complementary 2-forms of Poisson structures

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Abstract. Let (M, P) be a Poisson manifold. A 2-form ω of M such that the Koszul bracket $\{\omega, \omega\}_P = 0$ is called a *complementary form* of P. Every complementary form yields a new Lie algebroid structure of TM, and, under some supplementary hypothesis, the form also defines a Poisson-Nijenhuis structure of M [12]. We give several examples of complementary forms, and new examples of Poisson-Nijenhuis manifolds. The general results are expressed in the framework of Hamiltonian structures [3] and Lie algebroids.

The motivation of the present paper comes from the theory of the *Poisson-Nijenhuis structures* which is being studied in connection with the integrability of Hamiltonian dynamical systems [3, 12, 7, 16] etc.

The main result will be that certain 2-forms of a Poisson manifold M, with the Poisson bivector P, provide the manifold with a Poisson-Nijenhuis structure, and, in particular, all the *symplectic-Nijenhuis structures* are generated in this way.

As a matter of fact, the interplay between Poisson-Nijenhuis structures and 2-forms was investigated in [12]. The difference is that the 2-forms ω of the present paper may not be closed. Instead, they are asked to satisfy the *dual Poisson condition* $\{\omega, \omega\}_P = 0$, where the bracket is that of Koszul [8]. Such *complementary forms* ω induce a new Lie algebroid structure on TM. Under a supplementary hypothesis (in particular, if closed), a complementary form yields a Poisson-Nijenhuis structure of M.

In the paper we give several examples of complementary 2-forms, and new examples of Poisson-Nijenhuis structures. These examples include the compact Hermitian symmetric spaces (where any harmonic 2-form is a complementary 2-form), foliated manifolds with a bundle-like symplectic form, the Kodaira-Thurston symplectic manifold, Riemannian manifolds with a parallel 2-form, etc.

To gain in generality, the main results will be given in the framework of Lie algebroids, and the existence of the bracket of 1-forms is proven in the case of a general Lie algebra endowed with a linear representation [3].

1. Preliminaries

To make this paper reasonably self-contained we shall review the most important notions and formulas needed. In particular, we shall start with a full review of the extension of a Poisson bracket to 1-forms because this bracket will be essential in what follows. Moreover, we consider this bracket in the general algebraic framework of [3], [4] since this framework is important in applications, and we believe that it is worth recording the proof of the fact that the bracket of 1-forms satisfies the Jacobi identity in its full generality. Essentially, the proof is the same as the one given in [7]. In the general case, the vector spaces involved are not reflexive, and one of the consequences of the fact that the proof also holds in this case is that there exists a Lie bracket of 1-forms on Poisson-Banach manifolds as well.

Let χ be a (real) Lie algebra with a given representation on a (real) vector space $\mathcal{F}$. The reader should think of χ as the Lie algebra of vector fields of a differentiable manifold M, and think of $\mathcal{F}$ as $C^\infty(M)$, but keep in mind that we have much less structure in the general case and, in particular, $\mathcal{F}$ is not a ring, and χ is not an $\mathcal{F}$-module. However, we may define various spaces of 'tensor fields' such as

$$\begin{aligned}\wedge^k(\chi) &= L_{\mathbf{R}\,\mathrm{alt}}(\underbrace{\chi\times\cdots\times\chi}_{k\text{ times}},\mathcal{F}),\\ \mathcal{V}^k(\chi) &= L_{\mathbf{R}\,\mathrm{alt}}(\underbrace{\wedge^1(\chi)\times\cdots\times\wedge^1(\chi)}_{k\text{ times}},\mathcal{F}),\end{aligned}$$

$$\mathcal{T}^k_h = L_{\mathbf{R}}(\underbrace{\chi\times\cdots\times\chi}_{h\text{ times}}\times\underbrace{\wedge^1(\chi)\times\cdots\times\wedge^1(\chi)}_{k\text{ times}},\mathcal{F}),$$

etc., where $L_{\mathbf{R}}$ denotes spaces of $\mathbf{R}$-linear mappings. But, of course, these spaces may not be the same as the tensor product spaces of the same type and, in particular, χ may be only a subspace of $\mathcal{V}^1(\chi)$.

Then, we shall denote by $\langle,\rangle$ the usual pairing on either $\chi\times\wedge^1(\chi)$ or $\wedge^1(\chi)\times\chi$, and, using the classical formulas of differentiable manifolds, we shall define the operators

$$d\colon\wedge^k(\chi)\to\wedge^{k+1}(\chi),\;\; i(X)\colon\wedge^k(\chi)\to\wedge^{k-1}(\chi),\;\; L_X:\mathcal{T}^k_h(\chi)\to\mathcal{T}^k_h(\chi),$$

where $X\in\chi$. All the algebraic properties of these operators which involve only the existing general structure hold in the general case as well. For instance, on $\wedge^k(\chi)$ we have

$$\begin{aligned}&L_X = di(X)+i(X)d,\\ &i([X,Y]) = L_X i(Y)-i(Y)L_X\quad (X,Y\in\chi),\end{aligned}\tag{1.1}$$

etc.

For our subject it is important to look at the space

$$\mathcal{A}(\chi)=\{H\in L_{\mathbf{R}}(\wedge^1(\chi),\chi)/\forall\lambda,\mu\in\wedge^1(\chi),\quad\langle\lambda,H\mu\rangle=-\langle H\lambda,\mu\rangle\}.\tag{1.2}$$

Clearly, $\mathcal{A}(\chi)\subseteq\mathcal{V}^2(\chi)$ (H is to be seen as $\langle H\lambda,\mu\rangle$), and the characteristic property of (1.2) is called the *skew-symmetry* of H. Furthermore, one has an important operation called here the *Gel'fand-Dorfman bracket* $[\,,\,]: \mathcal{A}(\chi) \times \mathcal{A}(\chi) \to \mathcal{V}^3(\chi)$, defined $\forall H, K \in \mathcal{A}(\chi)$ by [3]

$$[H, K](\alpha, \beta, \gamma) = \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{\langle K L_{H\alpha}\beta, \gamma\rangle + \langle H L_{K\alpha}\beta, \gamma\rangle\}. \tag{1.3}$$

Following [3], if $H \in \mathcal{A}(\chi)$ and $[H, H] = 0$, H is called a *Hamiltonian operator*. We shall also say that the same H is a *Poisson structure* on $\mathcal{F}$ since the *Poisson bracket* defined by

$$\{f, g\} = \langle H df, dg\rangle \quad (f, g \in \mathcal{F}) \tag{1.4}$$

is a Lie bracket:

$$\sum_{\mathrm{Cycl}(f,g,h)} \{\{f, g\}, h\} = \tfrac{1}{2}[H, H](df, dg, dh) = 0. \tag{1.5}$$

The following lemma is crucial [7]

LEMMA 1.1. Let $H \in \mathcal{A}(\chi)$ and let us define, $\forall \alpha, \beta \in \wedge^1(\chi)$,

$$\{\alpha, \beta\} = L_{H\alpha}\beta - L_{H\beta}\alpha - d\langle H\alpha, \beta\rangle. \tag{1.6}$$

Then, $\forall \gamma \in \wedge^1(\chi), \forall X \in \chi$, *one has*

$$\langle \gamma, H\{\alpha, \beta\}\rangle = \langle \gamma, [H\alpha, H\beta]\rangle + \tfrac{1}{2}[H, H](\alpha, \beta, \gamma), \tag{1.7}$$

$$\sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \langle\{\{\alpha, \beta\}, \gamma\}, X\rangle = [H, L_X H](\alpha, \beta, \gamma) + \tag{1.8}$$

$$+\frac{1}{2} \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} [H, H](\alpha, \beta, d\langle\gamma, X\rangle).$$

Proof. Let us make the following computation which uses the skew-symmetry of H:

$$\langle H\{\alpha, \beta\}, \gamma\rangle = \langle H L_{H\alpha}\beta, \gamma\rangle - \langle H L_{H\beta}\alpha, \gamma\rangle + (H\gamma)\langle H\alpha, \beta\rangle$$

$$= \langle H L_{H\alpha}\beta, \gamma\rangle - \langle H L_{H\beta}\alpha, \gamma\rangle + \langle L_{H\gamma}\beta, H\alpha\rangle + \langle \beta, [H\gamma, H\alpha]\rangle$$

$$= -\langle L_{H\alpha}\beta, H\gamma\rangle - \tfrac{1}{2}[H, H](\alpha, \beta, \gamma) - \langle L_{H\alpha}\gamma, H\beta\rangle$$

$$+\langle \beta, [H\gamma, H\alpha]\rangle.$$

If the Lie derivatives in the first and in the third term above are made explicit, and after reductions, we get exactly (1.7).

Now, in order to prove (1.8) we define an operator $d_*\colon \mathcal{V}^k(\chi) \to \mathcal{V}^{k+1}(\chi)$ by using the formula of an exterior differential, and the bracket $\{\ ,\ \}$ instead of $[\ ,\ ]$. Namely, $\forall X \in \chi$, we put

$$\begin{aligned}(d_*X)(\alpha,\beta) := {} & (H\alpha)\langle X,\beta\rangle - (H\beta)\langle X,\alpha\rangle - \\ & -\langle X,\{\alpha,\beta\}\rangle \overset{(1.6)}{=} \langle \alpha,(L_XH)\beta\rangle.\end{aligned} \tag{1.9}$$

Of course, the skew-symmetry of H was used, and it follows from it that

$$(L_XH)(\alpha) := [X,H\alpha] - H(L_X\alpha)$$

is also skew-symmetric. Then, we put

$$\begin{aligned}(d^{*2}X)(\alpha,\beta,\gamma) := {} & \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{(H\alpha)((d^*X)(\beta,\gamma)) \\ & -(d^*X)(\{\alpha,\beta\},\gamma)\},\end{aligned} \tag{1.10}$$

and, if we replace here d^*X by its definition (1.9) and use (1.7), we get

$$\begin{aligned}(d^{*2}X)(\alpha,\beta,\gamma) = {} & \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{\langle\{\{\alpha,\beta\},\gamma\},X\rangle - \\ & -\tfrac{1}{2}[H,H](\alpha,\beta,d\langle\gamma,X\rangle)\}.\end{aligned} \tag{1.11}$$

On the other hand, if we replace d^*X by the final result of (1.9) in (1.10), we obtain

$$\begin{aligned}(d^{*2}X)(\alpha,\beta,\gamma) = {} & \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{(H\alpha)\langle\beta,(L_XH)\gamma\rangle - \langle\{\alpha,\beta\},(L_XH)\gamma\rangle\} \\ = {} & \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{\langle L_{H\alpha}\beta,(L_XH)\gamma\rangle + \langle\beta,[H\alpha,(L_XH)\gamma]\rangle \\ & -\langle L_{H\alpha}\beta,(L_XH)\gamma\rangle + \langle L_{H\beta}\alpha,(L_XH)\gamma\rangle \\ & +((L_XH)\gamma)\langle H\alpha,\beta\rangle\} \\ = {} & \sum_{\mathrm{Cycl}(\alpha,\beta,\gamma)} \{\langle\beta,[H\alpha,(L_XH)\gamma]\rangle - \langle\gamma,(L_XH)L_{H\beta}\alpha\rangle \\ & +\langle\beta,[(L_XH)\gamma,H\alpha]\rangle + \langle L_{(L_XH)\gamma}\beta,H\alpha\rangle\} \\ & \overset{(1.3)}{=} [H,L_XH](\alpha,\beta,\gamma).\end{aligned} \tag{1.12}$$

The comparison of (1.12) and (1.11) yields the required formula (1.8). □

From Lemma 1.1 we deduce

THEOREM 1.2. (i) $[H, H] = 0$ *iff* $H\{\alpha, \beta\} = [H\alpha, H\beta]$. *ii) If* $[H, H] = 0$, *the bracket (1.6) is a Lie bracket on* $\wedge^1(\chi)$, *and* $(\alpha \in \wedge^1(\chi),\ f \in \mathcal{F}) \mapsto (H\alpha)f$ *is a representation of the Lie algebra* $(\wedge^1(\chi), \{\ ,\ \})$ *on* $\mathcal{F}$. *iii) Assume that* $(\chi, \mathcal{F})$ *satisfy the following restrictions: 1° $\mathcal{F}$ is a ring with unit and the representation of χ on $\mathcal{F}$ is by derivations; 2° if,* $\forall X \in \chi,\ Xf = 0\,(f \in \mathcal{F})$, *then* $f =$ const. *in* $\mathcal{F}$; *3° if* $\forall X \in \chi,\ L_X T = 0\,(T \in \mathcal{T}_h^k(\chi), (h,k) \neq (0,0))$, *then* $T = 0$; *4°* $\wedge^1(\chi)$ *is the span of the set* $\{df \mid f \in \mathcal{F}\}$ *over* $\mathcal{F}$. *Then, if the bracket* (1.6) *is a Lie bracket, one has* $[H, H] = 0$.

Proof. (i) [4], [7]. If $0 \neq X \in \chi$, obviously, there exists $\gamma \in \wedge^1(\chi)$ such that $\langle \gamma, X \rangle \neq 0$. Hence, if $[H, H] = 0$, (1.7) implies the required result.

(ii) [7]. Since $[H, L_X H] = (1/2)L_X[H, H]$, (1.8) shows that $[H, H] = 0$ implies the Jacobi identity for the bracket (1.6). The second assertion follows by using (1.7) again. (iii). (1.6) implies $\{df, dg\} = d\{f, g\}$, where $\{f, g\}$ is given by (1.4). Hence, the Jacobi identity for (1.6) implies $\sum_{\mathrm{Cycl}(f,g,h)} d(\{\{f, g\}, h\}) = 0$ i.e., in view of 2°, $\sum_{\mathrm{Cycl}(f,g,h)}(\{\{f, g\}, h\}) =$ const. Accordingly, (1.8) and (1.5) imply $L_X[H, H](df, dg, dh) = 0$, $\forall f, g, h \in \mathcal{F}$, and the restrictions 3°,4° allow us to conclude that $[H, H] = 0$. (1° is needed to make 2° and 4° meaningful.) □

Thus, if we agree to call a triple $(\chi, \mathcal{F}, H)$ as above, with $[H, H] = 0$, a *Hamiltonian algebra*, the main conclusion is that a Hamiltonian algebra also has a Lie bracket on $\wedge^1(\chi)$, which is given by (1.6), and is compatible with H. W e call it the *dual Lie bracket* of the given Hamiltonian algebra.

REMARK. Actually, in applications, an even more general algebraic structure is needed [4] namely, that of a Lie algebra χ with a *representation* on a given cochain complex $\mathcal{C} = (\sum_{k=0}^{\infty} \Omega^k,\ d\colon \Omega^k \to \Omega^{k+1},\ d^2 = 0)$. By such a representation we mean a mapping $X \mapsto i(X) \in L_{\mathbf{R}}(\Omega^k, \Omega^{k-1})$, defined $\forall X \in \chi$, $\forall k = 0, 1, 2, \ldots$, such that $i(X)i(Y) + i(Y)i(X) = 0$ and that (1.1) (with the formula of L_X seen as a definition) holds. Then, χ is represented on Ω^0 in the usual sense, and, if in the algebraic computations needed for the Lemma 1.1, we replace $\mathcal{F}$ by Ω^0, $\wedge^1(\chi)$ by Ω^1, Xf by $i(X)df$, $\langle X, \alpha \rangle$ by $i(X)\alpha$ $(X \in \chi,\ f \in \Omega^0,\ \alpha \in \Omega^1)$, the brackets (1.3), (1.4) will be defined, and Lemma 1.1 will be valid again. But, Theorem 1.2 will be true only if: (1) for (i), one also asks that, if for a given $X \in \chi$ and $\forall \alpha \in \Omega^1$, $\langle X, \alpha \rangle = 0$, then $X = 0$; (2) for (ii) one also asks that, if for a given $\alpha \in \Omega^1$ and $\forall X \in \chi$, $\langle X, \alpha \rangle = 0$, then $\alpha = 0$. A triple $(\chi, \mathcal{C}, H)$ as above, with $[H, H] = 0$, may be called a *Hamiltonian complex*, and such triples, precisely, are encountered in applications [3], [4].

From the geometric point of view, it is natural to look at Hamiltonian algebras in the framework of Lie algebroids. We agree that all the differentiable manifolds bellow are finite dimensional, unless specifically mentioned otherwise, and the frame-category is the C^∞ category.

First we remember that a *Lie algebroid* (e.g., [11]) is a vector bundle $\pi : E \to M$ endowed with a Lie bracket $[\ ,\ ]_E$ on the space ΓE of the global cross sections of E, and with an *anchor* bundle morphism $A : E \to TM$ which is Lie bracket preserving and satisfies the condition

$$[s_1, fs_2]_E = ((As_1)f)s_2 + f[s_1, s_2]_E, \quad \forall s_1, s_2 \in \Gamma E. \tag{1.13}$$

For a Lie algebroid E it is possible to extend the usual calculus on differentiable manifolds if cross sections of E, $\wedge^k E$ and $\wedge^k E^\star$ are used instead of vector fields, multivector fields and differential forms, respectively, and if derivatives $(As)f$ $(s \in \Gamma E)$ are used instead of those given by vector fields. In other words, the classical case is just the case where $E = TM$, $[\ ,\ ]_E$ is the Lie bracket of vector fields, and $A = Id$. If the classical formulas are changed as indicated above, we obtain an 'exterior differential' d_E, a 'Lie derivative' L^E_s etc. which have the usual properties [11, 7], etc.

In particular, we also obtain a *Schouten-Nijenhuis bracket of 'E-multivector fields'* defined, for instance, as the **R**-linear extension of the formula

$$\begin{aligned}
&[s_1 \wedge \cdots \wedge s_k, s'_1 \wedge \cdots \wedge s'_h]_E = \\
&= (-1)^{k+1} \sum_{i=1}^{k} \sum_{j=1}^{h} (-1)^{i+j} [s_i, s'_j]_E \wedge s_1 \wedge \cdots \wedge \hat{s}_i \\
&\quad \wedge \cdots \wedge s_k \wedge s'_1 \wedge \cdots \wedge \hat{s}'_j \wedge \cdots \wedge s'_h,
\end{aligned} \tag{1.14}$$

where $s_i, s'_j \in \Gamma E$ and a hat means the absence of a factor. Our sign conventions are those of [10] and [15], and we may see as in the classical case (e.g., [15]) that, if (1.14) is extended to a bracket $[\lambda, \mu]_E$ $(\lambda \in \Gamma \wedge^k E, \mu \in \Gamma \wedge^h E)$, the result is independent of the decomposition of λ and μ into sums of products as in (1.14).

The reader will find in Section 6 of [7] a complete study of the operations described above. In particular, if one defines grade(λ) $= k - 1$ for $\lambda \in \Gamma \wedge^k E$, and one multiplies (1.14) by $(-1)^{k+1}$, $[\ ,\ ]_E$ is the operation of a *graded Lie algebra* where d_E is an antiderivation.

Now, from the previous definitions, it follows that, if we put $\chi = \Gamma E$ with the bracket $[,]_E$, $\mathcal{F} = C^\infty(M)$, and with the representation $sf = (As)f (s \in \Gamma E, f \in C^\infty(M))$, we can use the algebraic results of the first part of this section.

In particular, the space $\mathcal{A}(\chi)$ defined by (1.2) is just $\Gamma \wedge^2 E$, and the operators d^E, $i(s)$, L^E_s are those which were denoted previously by d, $i(X)$, L_X. Concerning the Gel'fand-Dorfman bracket (1.3), one can check that it is just the Schouten-Nijenhuis bracket of the corresponding E-bivector fields, and, hereafter, we shall speak only of Schouten-Nijenhuis brackets. (There are no general Schouten-Nijenhuis brackets on the spaces $\mathcal{V}^k(\chi)$ of a general pair $(\chi, \mathcal{F})$.)

Accordingly, we define a *Poisson structure* or a *Poisson bivector* on E as an element $\Theta \in \Gamma \wedge^2 E$ which satisfies the condition

$$[\Theta, \Theta]_E = 0. \tag{1.15}$$

Moreover, $\forall\Theta \in \Gamma\wedge^2 E$, there is an associated *dual bracket* of ΓE^* defined by (1.6). We shall denote the Hamiltonian operator H associated with Θ by $\sharp_\Theta (= \sharp)$: $E^* \to E$. More precisely, $\sharp$ is defined by

$$\langle \sharp\alpha, \lambda\rangle = \Theta(\alpha, \lambda), \forall\lambda \in \Gamma E^*. \tag{1.16}$$

The corresponding dual bracket (1.6) will be written as

$$[\alpha, \beta]_{E^*} := L^E_{\sharp\alpha}\beta - L^E_{\sharp\beta}\alpha - d_E(\Theta(\alpha, \beta)), \tag{1.17}$$

where $\alpha, \beta \in \Gamma E^*$.

Now, Theorem 1.2 yields

THEOREM 1.3. *$\Theta \in \Gamma\wedge^2 E$ is a Poisson structure of E iff*

$$\sharp[\alpha, \beta]_{E^*} = [\sharp\alpha, \sharp\beta]_E \tag{1.18}$$

holds. A Poisson structure Θ of a Lie algebroid E induces a Lie algebroid structure on the dual vector bundle E^* with the bracket (1.17) and with anchor $A^* = A \circ \sharp_\Theta$.

Proof. The only fact that we still must check is (1.13), and this follows by a simple computation. □

We shall say that the obtained structure of E^* is the *dual Lie structure with respect to* Θ. In view of (1.18), $\sharp$ is a homomorphism of Lie algebroids. It is also important to notice that, in view of (1.14), (1.18) extends to the general Schouten-Nijenhuis brackets of $E^*, [\ ,\ ]_{E^*}$ and $E, [\ ,\ ]_E$, if we extend $\sharp$ to $\lambda \in \Gamma\wedge^k E^*$ by

$$(\sharp\lambda)(\alpha_1, \ldots \alpha_k) = (-1)^k\lambda(\sharp\alpha_1, \ldots \sharp\alpha_k) \quad (\alpha_i \in \Gamma E^*). \tag{1.19}$$

Using the terminology defined above, it is well known that a *Poisson manifold* is just a differentiable manifold M with a Poisson structure P on the Lie algebroid $(TM, \mathrm{Id.})$. We refer the reader, for instance, to [18] or [15] for the theory of Poisson manifolds. Accordingly, (1.17), where L and d are the classical operators on a manifold, is a Lie bracket of 1-forms, and, following the usual custom, we shall denote it by $\{\alpha, \beta\}_P$. This bracket and $\sharp$ make T^*M into a Lie algebroid *dual* to TM with respect to P.

The dual Lie structure of T^*M, where M is a Poisson manifold, was discovered and studied by many authors (e.g., see the references given in [7]). In particular, the dual Schouten-Nijenhuis bracket $\{\ ,\ \}_P$ of differential forms on a Poisson manifold, which is essential for the present paper, was first studied by Koszul [8]. One also has

PROPOSITION 1.4. *If $\varphi : (M_1, P_1) \to (M_2, P_2)$ is a Poisson mapping, $\forall\lambda \in \wedge^k M_2$, $\mu \in \wedge^h M_2$, we have*

$$\varphi^*\{\lambda, \mu\}_2 = \{\varphi^*\lambda, \varphi^*\mu\}_1. \tag{1.20}$$

Proof. In this proposition we used the notation $\Gamma \wedge^k T^*M = \wedge^k M$ (and we shall continue to do so hereafter), and the indices 1, 2 signify brackets associated to the Poisson structures P_1, P_2. That φ is a Poisson morphism means

$$\{f, g\}_2 \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_1 \tag{1.21}$$

where $f, g \in C^\infty(M_2)$ and $\{f, g\} = P(df, dg)$.

It follows straightforwardly from (1.21), (1.17) and (1.13) that $\forall f, g, h \in C^\infty(M_2)$ one has

$$\varphi^*\{df, dg\}_2 = \{\varphi^* df, \varphi^* dg\}_1,$$

$$\varphi^*\{df, hdg\}_2 = \{\varphi^* df, \varphi^*(hdg)\}_1,$$

and, then, it is clear that (1.20) holds for any 1-forms λ, μ of M_2. This result extends to arbitrary differential forms using (1.14) □

The previous definitions and results were described in the finite dimensional case, but many of the known applications of Poisson-Nijenhuis structures to integrability questions are for infinite dimensional situations. These applications can be studied either by using infinite dimensional manifolds [12] or by an algebraic machinery [3], [4], [7].

We end this section by the indication of the way towards a theory of Poisson-Banach manifolds [12]. Let M be a Hausdorff C^∞ Banach manifold [1]. Then, we define a Poisson structure of M to be a Lie bracket $\{\ ,\ \}$ on $C^\infty(M)$ such that, $\forall f \in C^\infty(M)$, $\{f, .\} := X_f$ is a vector field called the *Hamiltonian vector field* of f. Thus

$$\{f, g\} = X_f g = dg(X_f) = -X_g f = -df(X_g), \tag{1.22}$$

and we see that there exists a cross-section H of the vector bundle Hom(T^*M, TM) such that

$$X_f = H(df), \qquad \{f, g\} = \langle H(df), dg\rangle. \tag{1.23}$$

It is not very useful to look at the 'Poisson bivector' $P(df, dg) = \{f, g\}$ for the following two reasons: (i) the model of M may not be reflexive and P may not define H, (ii) the Schouten-Nijenhuis bracket can be defined in the usual way for sections of $\wedge^k TM$ but not for sections of $L_{alt}(T^*M \times \cdots \times T^*M, \mathbf{R})$.

Thus, the characteristic feature of the theory is that we must look rather at $\sharp_P := H$ than at P itself, and a *Poisson-Banach manifold* is a Banach manifold M endowed with an operator H such that (1.23) is a Lie bracket of functions. This means that H is a *Hamiltonian operator* as defined in the first part of this section i.e., $[H, H] = 0$, for the bracket (1.3) [3]. The definition of a Poisson-Banach structure H may be extended to *Lie-Banach algebroids* and the results of Theorem 1.3 remain true.

2. Symplectic-Nijenhuis manifolds

We begin with the definition of the *Poisson-Nijenhuis structures* on Lie algebroids.

Let $\pi: E \to M$ be a Lie algebroid of anchor A, and let $B: E \to E$ be a vector bundle morphism. Then the *Nijenhuis tensor* of B is

$$\mathcal{N}_B(s_1, s_2) := [Bs_1, Bs_2]_E - B[s_1, Bs_2]_E - B[Bs_1, s_2]_E + B^2[s_1, s_2]_E, \quad (2.1)$$

$\forall s_1, s_2 \in \Gamma E$. If $\mathcal{N}_B = 0$, B is called a *Nijenhuis endomorphism* of E, and (2.1) shows that

$$[s_1, s_2]_B := [Bs_1, s_2]_E + [s_1, Bs_2]_E - B[s_1, s_2]_E \quad (2.2)$$

defines a new Lie algebroid structure of anchor $A \circ B$ on E. Now, a *Poisson-Nijenhuis structure* of E is a pair (Θ, B), where Θ is a Poisson structure and B is a Nijenhuis endomorphism of E, such that

$$\Theta_1(\sigma_1, \sigma_2) = \langle B\sharp_\Theta \sigma_1, \sigma_2 \rangle \quad (\sigma_1, \sigma_2 \in \Gamma E^*) \quad (2.3)$$

is skew-symmetric, and the dual brackets (1.17) defined by the pairs $([\ ,\ ]_E, \Theta_1)$ and $([\ ,\ ]_B, \Theta)$ coincide.

Moreover, if $[\ ,\ ]'_E$ is any Lie algebroid structure of E with an anchor of the form $A' = A \circ C$ for some Lie algebroid endomorphism $C : E \to E$, it follows easily that, $\forall s_1, s_2 \in \Gamma E$,

$$[s_1, s_2]'_E = [s_1, s_2]_C + S(s_1, s_2), \quad (2.4)$$

where S is an E-valued 2-'E-form'. (Check that $[s_1, s_2]'_E - [s_1, s_2]_C$ is $C^\infty(M)$-bilinear.) The triple $([,]'_E, C, \Theta)$ where Θ is a Poisson structure of E and $A' = A \circ C$ is called a *generalized Poisson-Nijenhuis structure* of E [16] if the conditions of the definition of a Poisson-Nijenhuis structure given above are satisfied, except for the fact that B is replaced by C, and that we may have $\mathcal{N}_C \neq 0$, but we must have $\forall \sigma \in \Gamma E^*$

$$i(\sharp_\Theta \sigma) S = 0. \quad (2.5)$$

A Poisson-Nijenhuis structure is characterized by the skew-symmetry of Θ_1 of (2.3) and by [7] (see also [16])

$$\sigma \circ L^E_{\sharp_\Theta \tau} B - \tau \circ L^E_{\sharp_\Theta \sigma} B + d_E(\Theta(\tau, \sigma)) \circ B - d_E(\Theta(\tau \circ B, \sigma)) = 0, \quad (2.6)$$

$\forall\sigma,\tau\in\Gamma E^*$. The same relation with C instead of B characterizes the generalized Poisson-Nijenhuis structures.

As mentioned in Section 1, the Poisson-Nijenhuis structures are interesting in the integrability theory of Hamiltonian systems. This is a consequence of the following main property [7]: if (Θ, B) is a Poisson-Nijenhuis structure of the Lie algebroid E, the pairs $(\Theta_{(k)}, B^p)$, where $p,k = 0,1,2,\cdots$, $\Theta_{(0)} = \Theta$, and $\Theta_{(k)}$, $k \geqslant 1$, are defined inductively by formula (2.3), are again Poisson-Nijenhuis structures, and $[\Theta_{(k)},\Theta_{(h)}]_E = 0$. Similarly [16], if $([\ ,\ ]'_E, C, \Theta)$ is a generalized Poisson-Nijenhuis structure so are $([\ ,\ ]'_E, C, \Theta_{(k)})$ and, again, $[\Theta_{(k)},\Theta_{(h)}]_E = 0$. The last equality is known as the *compatibility* of the corresponding Poisson structures. The above mentioned structures are called the *Poisson-Nijenhuis hierarchy* of the original structure. Under certain regularity conditions, the existence of the hierarchy implies that the eigenvalues of B are Poisson commuting first integrals of Hamiltonian systems [12].

As we already said, most of the applications of the theory of Poisson-Nijenhuis structures to integrability problems regard infinite dimensional Hamiltonian systems, and they appear either in the context of Hamiltonian complexes (see Section 1) [3], [4] or in the context of Poisson-Banach manifolds [12]. Therefore, if not geometry but integrability is the main interest, the definition of the (generalized) Poisson-Nijenhuis structures should be formulated in corresponding terms. It is easy to understand that this definition can be given in exactly the same way for a Hamiltonian complex $(\chi, \mathcal{C}, H)$, and a 'tensor' $B\in L_{\mathbf{R}}(\chi,\chi)$ and, in particular, for the case of the Hamiltonian algebra with $\chi = \chi(M)$, $\mathcal{F} = C^\infty(M)$, where M is a Banach manifold endowed with a Poisson structure H as defined in Section 1. In the last case, we get the notion of a *Poisson-Nijenhuis-Banach manifold.* The existence of the Poisson hierarchy of a Poisson-Nijenhuis-Banach manifold was proven in [12], along with the finite dimensional case, since the computations of [12] are also valid on Banach manifolds. Moreover, in fact, these computations require only algebraic properties of Lie derivatives, exterior differentials and Gel'fand-Dorfman brackets. It follows from this argument that they remain also valid for Hamiltonian complexes which satisfy part i) of Theorem 1.2, and for Hamiltonian algebras. Furthermore, the existence of the Poisson hierarchy is ensured in these cases as well and, in particular, in the case of Lie- Banach algebroids. (On the contrary, we couldn't use the proof of [16] for the same purpose since this proof uses the fact that the Jacobi identity for the bracket of 1-forms implies the Poisson character of a bivector, and this may not be true in the general cases mentioned above.)

Now, we come back to the finite dimensional Lie algebroids, and we restrict our attention to the most important case, that of a *Poisson-Nijenhuis manifold*, where $E = TM$ with the usual Lie bracket, and $A =$ Id., and, particularly, to the *symplectic-Nijenhuis manifolds*. These are symplectic manifolds M, with the symplectic 2-form σ $(d\sigma = 0)$, and with a *Nijenhuis* $(1,1)$*-tensor field* B such that (P, B) is a Poisson-Nijenhuis structure of M (i.e., of TM), where P is the Poisson bivector which has the same Poisson brackets of functions as the symplectic form σ. If $\sharp_\sigma^{-1} : TM \to T^*M$ is defined by $\sharp_\sigma^{-1}(X) = i(X)\sigma\ (X \in TM)$, P is characterized by $\sharp_P \circ \sharp_\sigma^{-1} = -\mathrm{Id.}$, or by the fact that its matrices of local coordinates are $P = {}^t\sigma^{-1}$.

The case of the symplectic-Nijenhuis manifolds is the case where the basic applications of the theory to integrability problems actually appear [12], and, for this reason, they provide us with the motivation of our study.

On a symplectic-Nijenhuis manifold (M, σ, B) there exists a unique 2-form ω such that

$$\sharp_\omega^{-1} = \sharp_\sigma^{-1} \circ B, \tag{2.7}$$

where $\sharp_\omega^{-1}$ is defined in the same way as $\sharp_\sigma^{-1}$ but, while $\sharp_\sigma^{-1}$ is indeed the inverse of a mapping, $\sharp_\omega^{-1}$ is just a notation which, for uniformity, we prefer instead of the more common $\flat_\omega$; $\sharp_\omega$ itself does not exist if ω is not nondegenerate. Indeed, (2.7) and (2.3) imply $\forall X, Y \in TM$

$$\omega(X,Y) = \sigma(BX,Y) = P_1(\sharp_\sigma^{-1}X, \sharp_\sigma^{-1}Y), \tag{2.8}$$

where P_1 is the Θ_1 of (2.3) in the present situation, and (2.8) accounts for the skew-symmetry of ω. We say that ω is the *associated* 2-*form* of (M, σ, B), and we prove

THEOREM 2.1. The associated 2-form ω of a symplectic-Nijenhuis manifold (M, σ, B) satisfies the conditions

$$\{\omega, \omega\}_P = 0, \quad d\omega = 0. \tag{2.9}$$

Conversely, if a 2-form ω of a symplectic manifold (M, σ) satisfies (2.9), ω is associated with a symplectic-Nijenhuis structure (M, σ, B), where B is defined by (2.7).

Proof. Since $\sharp_P$ is an isomorphism, the first relation (2.9) is equivalent to $[\sharp_P\omega, \sharp_P\omega] = 0$ i.e., to the fact that $\sharp_P\omega$ defined by (1.19) is a Poisson structure of M. But, it follows easily from (2.8) that $\sharp_P\omega = P_1$, and, since P_1 belongs to the Poisson hierarchy of the structure, the required condition holds.

Now, let us make the interesting remark that, on every symplectic manifold (M, σ), and for every differential form λ one has [7]

$$\mathrm{d}\,\lambda = \{\sigma, \lambda\}_P. \tag{2.10}$$

Indeed, if the coordinate expressions of σ and λ, and the formula (1.14) are used to compute $\{\sigma, \lambda\}_P$, the result is the coordinate expression of $\mathrm{d}\,\lambda$. (See [15] for the general formula of the coordinate expression of the bracket of two forms.)

Accordingly, the second condition (2.9) is equivalent to $\{\sigma, \omega\}_P = 0$ and, therefore, to $[\sharp_P\sigma, \sharp_P\omega] = [P, P_1] = 0$. But, the last equality holds since the structures of the Poisson hierarchy are compatible.

Conversely, if we have a 2-form ω satisfying (2.9) on a symplectic manifold (M, σ), $P_1 = \sharp_P \omega$ will be a Poisson structure of M, compatible to P (as above), and, by a known result [12], [16], $(P, -\sharp_{P_1} \circ \sharp_\sigma^{-1})$ is a Poisson-Nijenhuis structure of M. But, $\forall \alpha \in T^*M$, and $\forall X \in TM$ at the same point of M, we have

$$\langle \alpha, \sharp_{P_1} \sharp_\sigma^{-1} X \rangle = P_1(\sharp_\sigma^{-1} X, \alpha) = \omega(\sharp_P \sharp_\sigma^{-1} X, \sharp_P \alpha) = \langle \alpha, \sharp_P \sharp_\omega^{-1} X \rangle, \quad (2.11)$$

i.e., $-\sharp_{P_1} \circ \sharp_\sigma^{-1} = B$, for B defined by (2.7). □

REMARK. A comparison of Theorem 2.1 with Proposition 2.1 of [12] shows that a closed 2-form ω on a symplectic manifold satisfies $\{\omega, \omega\}_P = 0$ iff it satisfies the hypothesis $d\tilde{\omega} = 0$ of [12] for the 2-form $\tilde{\omega}(X, Y) = \omega(X, \sharp_P \sharp_\omega^{-1} Y)$ $(X, Y \in \Gamma TM)$.

Theorem 2.1 is advantageous since it is easier to work with differential forms than with Nijenhuis tensors. This fact is illustrated by the following examples.

PROPOSITION 2.2. Let M be a compact Hermitian symmetric space with metric g and Kähler form σ. Then, any harmonic 2-form ω of M is associated with a symplectic-Nijenhuis structure (M, σ, B), where B is defined by (2.7).

Proof. The result is proven if we check that ω satisfies the first condition (2.9) since, by harmonicity, $d\omega = 0$.

By the results of Koszul [8] (see also [15]) the bracket $\{\lambda, \mu\}_P$ of differential forms $\lambda \in \wedge^k M$, μ on a Poisson manifold is expressible by

$$\{\lambda, \mu\}_P = (\delta\lambda) \wedge \mu + (-1)^k \lambda \wedge (\delta\mu) - \delta(\lambda \wedge \mu), \quad (2.12)$$

where $\delta = i(P)d - di(P)$. Hence, the first condition (2.9) means

$$2(\delta\omega) \wedge \omega - \delta(\omega \wedge \omega) = 0. \quad (2.13)$$

Furthermore, Brylinski [2] (see also [15]) proved that, in the case of a symplectic manifold, δ is just the *symplectic codifferential*, which is known from [9] where it is also proven that a Kähler manifold has the property $\delta = C\delta_g C$. In this formula, δ_g is the *Riemannian codifferential* of the Kähler metric g, and C is defined on forms by applying the complex structure tensor J to the arguments of a form (e.g., [17]).

Now, if ω is harmonic, $C\omega$ is also harmonic (e.g., [17]) and, in particular, $\delta_g C\omega = 0$. Therefore, $\delta\omega = 0$. Finally, on a compact Riemannian symmetric space, the exterior product of harmonic forms is harmonic (e.g., [5]). Thus, in our case $\omega \wedge \omega$ is harmonic, and $\delta(\omega \wedge \omega) = 0$. Hence (2.13) holds. □

In order to give a second example we define first an auxiliary notion. Let $\mathcal{F}$ be a foliation of a differentiable manifold M. A symplectic form σ of M (if it exists) is said to be *$\mathcal{F}$-bundle-like* if: (i) the leaves of $\mathcal{F}$ are symplectic submanifolds of (M, σ); (ii) for any pair of $\mathcal{F}$-projectable vector fields (i.e., vector fields which have projections onto the space of leaves of $\mathcal{F}$) X, Y which are σ-orthogonal to $\mathcal{F}$, $\sigma(X, Y)$ is constant along the leaves of $\mathcal{F}$. The name *bundle-like* is used because the situation is similar to that of *bundle-like Riemannian metrics* (e.g., [14]).

PROPOSITION 2.3. Let (M, σ) be a symplectic manifold which has a foliation $\mathcal{F}$ such that σ is an $\mathcal{F}$-bundle-like symplectic form. Let B be the projection on $T\mathcal{F}$ according to the decomposition $TM = N\mathcal{F} \oplus T\mathcal{F}$ where $N\mathcal{F}$ is the σ-orthogonal bundle of $T\mathcal{F}$. Then (M, σ, B) is a symplectic-Nijenhuis structure.

Proof. First, we notice that, in this Proposition, what we have is a Dirac bracket (i.e., a Poisson bracket computed along the leaves of a foliation with symplectic leaves e.g., [10], [15]) which satisfies the supplementary condition (ii).

In the following computation we shall use the *bigrade* or *type* of forms induced by the decomposition $TM = N\mathcal{F} \oplus T\mathcal{F}$, and the corresponding decomposition

$$d = d'_{(1,0)} + d''_{(0,1)} + \partial_{(2,-1)} \tag{2.14}$$

of the exterior differential (e.g., [14]). In particular, we have a decomposition

$$\sigma = \sigma'_{(2,0)} + \sigma''_{(0,2)}, \tag{2.15}$$

and $\mathrm{d}\,\sigma = 0$ is equivalent to

$$d'\sigma' = 0,\ d''\sigma'' = 0,\ d'\sigma'' = 0,\ d''\sigma' + \partial\sigma'' = 0. \tag{2.16}$$

Since the leaves of $\mathcal{F}$ are symplectic, the leafwise Poisson brackets of functions in $C^\infty(M)$ exist and yield a Poisson structure, say D, of M, known as a *Dirac bracket*. More exactly, $\forall f \in C^\infty(M)$, we have a leafwise Hamiltonian vector field X_f^D defined by $i(X_f^D)\sigma'' = i(X_f^D)\sigma = -d''f$. Therefore, $X_f^D = \sharp_P d''f$ ($\sharp_P \circ \sharp_\sigma^{-1} = -\mathrm{Id.}$), and

$$D(df, dg) = \{f, g\}_D = \sigma''(X_f^D, X_g^D) = \sigma''(\sharp_P d''f, \sharp_P d''g) = \sigma''(\sharp_P df, \sharp_P dg),$$

since the added terms $d'f$, $d'g$ have no contribution to the result. Hence, $D = \sharp_P \sigma''$, and, because D is a Poisson bivector, we have

$$\{\sigma'', \sigma''\}_P = 0. \tag{2.17}$$

Until now, we referred to any Dirac bracket. If we add the condition (ii) of the definition of an $\mathcal{F}$-bundle-like symplectic form, we also get that $d''\sigma' = 0$ (the equivalence of the latter condition with property (ii) is rather obvious). Then, it follows from (2.16) that we also have $d\sigma'' = 0$, and, by Theorem 2.1, σ'' is associated with a symplectic-Nijenhuis structure. The Nijenhuis tensor of this structure is $B = -\sharp_P \circ \sharp_{\sigma''}^{-1}$, and it is easy to see that, $\forall X \in TM$, if $X = X_{(1,0)} + X_{(0,1)}$, then $BX = X_{(0,1)}$. □

We end this section by a nice concrete example of a compact symplectic-Nijenhuis manifold. Namely, we take M to be the Kodaira-Thurston example of a non-Kähler symplectic manifold: the quotient of $\mathbf{R}^4$ by the group of transformations

$$(x^1, x^2, x^3, x^4) \mapsto (x^1 + p, x^2 + q, x^3 + qx^4 + m, x^4 + n), \tag{2.18}$$

where p, q, m, n are integers. The symplectic form of M is

$$\sigma = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4, \tag{2.19}$$

which is invariant by (2.18). Assume that $\varphi = \varphi(x^1, x^2)$ and $\psi = \psi(x^4)$ are well defined on M (i.e.,invariant by (2.18)), and take

$$\omega = \varphi(x^1, x^2)\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \psi(x^4)\mathrm{d}x^3 \wedge \mathrm{d}x^4 \tag{2.20}$$

(invariant by (2.18), again). Clearly, $\mathrm{d}\omega = 0$, and a straightforward computation based on (1.14) yields $\{\omega, \omega\}_P = 0$, where $\sharp_P \circ \sharp_\sigma^{-1} = -Id$. Hence, by Theorem 2.1, ω is associated with a Poisson-Nijenhuis structure whose Nijenhuis tensor B will be given by

$$B\frac{\partial}{\partial x^1} = \varphi\frac{\partial}{\partial x^1}, \quad B\frac{\partial}{\partial x^2} = \varphi\frac{\partial}{\partial x^2}, \quad B\frac{\partial}{\partial x^3} = \psi\frac{\partial}{\partial x^3}, \quad B\frac{\partial}{\partial x^4} = \psi\frac{\partial}{\partial x^4}. \tag{2.21}$$

Notice that B has exactly two eigenvalues, hence, it leads to the integrability of the Hamiltonian dynamical systems which leave ω invariant [12].

3. General complementary 2-forms

Theorem 2.1 suggests the following general definition: let $\pi : E \to M$ be a Lie algebroid of anchor A, and Θ a Poisson structure of E; then a 2-E-form $\omega \in \Gamma\wedge^2 E^*$ will be called a *complementary 2-form* of Θ if

$$[\omega, \omega]_{E^*} = 0. \tag{3.1}$$

Then, (Θ, ω) is a *complemented Poisson structure* of E, and, in particular, if $E = TM$, $A = \mathrm{Id}.$, (M, Θ, ω) is a *complemented Poisson manifold.*

Generally, the complementarity condition (3.1) is not equivalent with the condition of [12] quoted in the Remark which follows Theorem 2.1. The main result on complementary 2-forms is

THEOREM 3.1. Let (Θ, ω) be a complemented Poisson structure of the Lie algebroid E, and let $B\colon E \to E$ be defined by

$$B = \sharp_\Theta \circ \sharp_\omega^{-1}. \tag{3.2}$$

Then, the bracket defined $\forall s_1, s_2 \in \Gamma E$ *by*

$$[s_1, s_2]'_E = [s_1, s_2]_B + \sharp_\Theta i(s_1)i(s_2)\,\mathrm{d}_E\omega \tag{3.3}$$

provides E *with a new Lie algebroid structure of anchor map* $A \circ B$.

Proof. By Theorem 1.1, we may see the dual bundle E^* of E as a Lie algebroid with the dual Lie structure induced by Θ and the anchor $A^* = A \circ \sharp_\Theta$. Then, by (3.1), the complementary 2-form ω is a Poisson structure of E^*, and it induces a dual Lie algebroid structure on E with the anchor map $A^* \circ \sharp_\omega^{-1} = A \circ B$. By (2.4), the bracket of this structure is

$$[s_1, s_2]'_E = [s_1, s_2]_B + S(s_1, s_2), \tag{3.4}$$

where $s_1, s_2 \in \Gamma E$ and $S \in \Gamma(E \otimes \wedge^2 E^*)$. We shall prove that S is precisely the last term of (3.3).

By (1.17) we have

$$[s_1, s_2]'_E = L^{E^*}_{\sharp_\omega^{-1} s_1} s_2 - L^{E^*}_{\sharp_\omega^{-1} s_2} s_1 - d_{E^*}(\omega(s_1, s_2)). \tag{3.5}$$

Here, we may replace $\forall \alpha \in \Gamma E^*, \forall s \in \Gamma E$

$$L^{E^*}_\alpha s = i(\alpha) d_{E^*} s + d_{E^*} i(\alpha) s, \tag{3.6}$$

and, furthermore, by known results of [10] and [7],

$$d_{E^*} s = -[\Theta, s]_E = -L^E_s \Theta. \tag{3.7}$$

We get

$$[s_1, s_2]'_E = i(\sharp_\omega^{-1} s_2) L^E_{s_1}\Theta - i(\sharp_\omega^{-1} s_1) L^E_{s_2}\Theta + d_{E^*}(\omega(s_1, s_2)). \tag{3.8}$$

Now, let us take $\alpha \in \Gamma E^*$, and evaluate:

$$\langle i(\sharp_\omega^{-1} s_2) L^E_{s_1}\Theta, \alpha\rangle = (L^E_{s_1}\Theta)(\sharp_\omega^{-1} s_2, \alpha)$$

$$= (A s_1)(\Theta(\sharp_\omega^{-1} s_2, \alpha)) - \Theta(L^E_{s_1}\sharp_\omega^{-1} s_2, \alpha) - \Theta(\sharp_\omega^{-1} s_2, L^E_{s_1}\alpha)$$

$$= (A s_1)\langle B s_2, \alpha\rangle - \Theta(L^E_{s_1} i(s_2)\omega, \alpha) - \langle B s_2, L^E_{s_1}\alpha\rangle$$

$$= \langle \alpha, [s_1, B s_2]_E\rangle - \Theta(L^E_{s_1} i(s_2)\omega, \alpha).$$

Therefore, we have

$$i(\sharp_\omega^{-1} s_2) L^E_{s_1}\Theta = [s_1, B s_2]_E - \sharp_\Theta L^E_{s_1} i(s_2)\omega. \tag{3.9}$$

Of course, we have a similar expression for the second term of (3.8) and, with that, (3.8) takes the form (3.4) with

$$S(s_1, s_2) = d_{E^*}(\omega(s_1, s_2)) + \sharp_\Theta(L^E_{s_2} i(s_1)\omega - (L^E_{s_1} i(s_2)\omega) + B[s_1, s_2]_E. \quad (3.10)$$

In (3.10) we may replace B by (3.2) and, also, use the fact that $\forall f \in C^\infty(M)$, we have

$$d_{E^*} f = -\sharp_\Theta d_E f, \quad (3.11)$$

which follows from the definitions of d_E, d_{E^*} and from $A^* = A \circ \sharp_\Theta$. Then, we obtain

$$\begin{aligned} S(s_1, s_2) = {} & \sharp_\Theta(-d_E(\omega(s_1, s_2)) + L^E_{s_2} i(s_1)\omega \\ & -L^E_{s_1} i(s_2)\omega) + i([s_1, s_2]_E)\omega), \end{aligned} \quad (3.12)$$

and if we use (3.6) for ω instead of α and the classical formula

$$i([s_1, s_2]_E)\omega = L^E_{s_1} i(s_2)\omega - i(s_2) L^E_{s_1}\omega, \quad (3.13)$$

we arrive at the required expression

$$S(s_1, s_2) = \sharp_\Theta i(s_1) i(s_2) d_E \omega. \quad (3.14)$$

□

The result which connects the complementary forms and the Poisson-Nijenhuis structures is

THEOREM 3.2. *With the same notation as in Theorem 3.1, if*

$$i(\sharp_\Theta \alpha) i(\sharp_\Theta \beta) d_E \omega = 0, \quad \forall \alpha, \beta \in \Gamma E^*, \quad (3.15)$$

then the Lie algebroid structure (3.3), together with the morphism (3.2) and the Poisson structure Θ*, is a generalized Poisson-Nijenhuis structure of* E*. If the stronger condition*

$$i(\sharp_\Theta \alpha) d_E \omega = 0, \quad \forall \alpha \in \Gamma E^*, \quad (3.16)$$

holds (in particular, if $d_E \omega = 0$*), and if the anchor* A *of* E *is injective, then* (Θ, B) *is a Poisson-Nijenhuis structure of* E*.*

Proof. That (Θ, B) give rise to a E-tensor (2.3) which is skew-symmetric is trivial. To check (2.5) for S of (3.14) we take $\alpha, \beta \in \Gamma E^*$ and $t \in \Gamma E$, and compute

$$\begin{aligned}&\langle (i(\sharp_\Theta\beta)S)(t), \alpha\rangle = \langle S(\sharp_\Theta\beta, t), \alpha\rangle \\ &= \langle \sharp_\Theta i(\sharp_\Theta\beta)i(t)d_E\omega), \alpha\rangle = -d_E\omega(t, \sharp_\Theta\beta, \sharp_\Theta\alpha) = 0,\end{aligned}$$

if (3.15) holds.

Furthermore, if (3.16) holds, we have $\forall\alpha \in \Gamma E^*, \forall s, t \in \Gamma E$,

$$\langle \sharp_\Theta i(s)i(t)d_E\omega, \alpha\rangle = -d_E\omega(t, s, \sharp_\Theta\alpha) = 0,$$

and S of (3.14) vanishes. If this happens, (3.3) reduces to $[s_1, s_2]'_E = [s_1, s_2]_B$. Applying the anchor $A \circ B$ to this equality, and using (2.2), (2.1), we get

$$\begin{aligned}[ABs_1, ABs_2] &= AB[s_1, s_2]'_E = A([Bs_1, Bs_2]_E - \mathcal{N}_B(s_1, s_2)) = \\ &= [ABs_1, ABs_2] - A\mathcal{N}_B(s_1, s_2),\end{aligned}$$

and, if A is injective, this implies $\mathcal{N}_B = 0$.

To finish the proof of Theorem 3.2, we have to check, for instance, that (2.6) holds.

First, we notice the following auxiliary formula whose validity is checked $\forall\alpha \in \Gamma E^*$ by evaluating on $s \in \Gamma E$:

$$\alpha \circ (\sharp_\Theta \sharp_\omega^{-1}) = \sharp_\omega^{-1}\sharp_\Theta\alpha = i(\sharp_\Theta\alpha)\omega. \tag{3.17}$$

Then $\forall s \in \Gamma E$, and with the notation of (2.6), we get:

$$\begin{aligned}&\langle \sigma \circ L^E_{\sharp_\Theta\tau}(\sharp_\Theta \circ \sharp_\omega^{-1}) - \tau \circ L^E_{\sharp_\Theta\sigma}(\sharp_\Theta \circ \sharp_\omega^{-1}), s\rangle \\
=\ & \langle \sigma, [\sharp_\Theta\tau, (\sharp_\Theta \circ \sharp_\omega^{-1}s]_E\rangle - \langle \sigma, \sharp_\Theta \circ \sharp_\omega^{-1}[\sharp_\Theta\tau, s]_E\rangle \\
& -\langle \tau, [\sharp_\Theta\sigma, (\sharp_\Theta \circ \sharp_\omega^{-1}s]_E\rangle + \langle \tau, \sharp_\Theta \circ \sharp_\omega^{-1}[\sharp_\Theta\sigma, s]_E\rangle \\
=\ & -\langle \sharp_\Theta\sigma, [\tau, \sharp_\omega^{-1}s]_{E^*}\rangle + \langle \sharp_\Theta\tau, [\sigma, \sharp_\omega^{-1}s]_{E^*}\rangle \\
& +\langle \sharp_\Theta\sigma, \sharp_\omega^{-1}[\sharp_\Theta\tau, s]_E - \langle \sharp_\Theta\tau, \sharp_\omega^{-1}[\sharp_\Theta\sigma, s]_E\rangle \\
=\ & -\Theta(\sigma, [\tau, i(s)\omega]_{E^*}) + \Theta(\tau, [\sigma, i(s)\omega]_{E^*}) + \omega([\sharp_\Theta\tau, s]_E, \sharp_\Theta\sigma) \\
& -\omega([\sharp_\Theta\sigma, s]_E, \sharp_\Theta\tau) = (d_{E^*}\Theta)(\tau, \sigma, i(s)\omega) - (A\sharp_\Theta\tau)(\Theta(\sigma, i(s)\omega)) \\
& +(A\sharp_\Theta\sigma)(\Theta(\tau, i(s)\omega)) - (A\sharp_\Theta\sharp_\omega^{-1}s)(\Theta(\tau, \sigma)) + \Theta([\tau, \sigma]_{E^*}, i(s)\omega) \\
& +d_E\omega(\sharp_\Theta\tau, \sharp_\Theta\sigma, s) - (A\sharp_\Theta\tau)(\omega(\sharp_\Theta\sigma, s)) + (A\sharp_\Theta\sigma)(\omega(\sharp_\Theta\tau, s)) \\
& -(As)(\omega(\sharp_\Theta\tau, \sharp_\Theta\sigma)) + \omega(\sharp_\Theta[\tau, \sigma]_{E^*}, s) = d_{E^*}\Theta)(\tau, \sigma, i(s)\omega) \\
& +d_E\omega(\sharp_\Theta\tau, \sharp_\Theta\sigma, s) - (A\sharp_\Theta\sharp_\omega^{-1}s)(\Theta(\tau, \sigma)) + (As)(\Theta(\tau \circ \sharp_\Theta \circ \sharp_\omega^{-1}, \sigma)).\end{aligned}$$

In this computation, we used the expression of an exterior differential, and the auxiliary formula (3.17), and we made all the possible reductions. Now, if we look at the obtained result, and take into account that

$$d_{E^*}\Theta = -[\Theta, \Theta]_E = 0$$

(see (3.7)) and that we have (3.15), we see that (2.6) is true, and the proof is finished. □

Theorem 3.2 shows the existence of an interesting class of (generalized) Poisson-Nijenhuis structures. A complementary 2-form ω will be called a *(generalized) Nijenhuis complementary form* if it satisfies (3.16) ((3.15) in the generalized case).

In what follows, we use these notions in the case of Poisson manifolds. We have

PROPOSITION 3.3. *Let $\varphi : (M_1, P_1) \to (M_2, P_2)$ be a Poisson mapping, and let ω_2 be a complementary 2-form of P_2. Then $\omega_1 := \varphi^*\omega_2$ is a complementary 2-form of P_1. Furthermore, if ω_2 is (generalized) Nijenhuis, and if*

$$\varphi_*(\mathrm{im}\,\sharp_{P_1}) = \mathrm{im}\,\sharp_{P_2} \tag{3.18}$$

(e.g., if (M_2, P_2) is symplectic) then ω_1 is also (generalized) Nijenhuis.

Proof. The result is an immediate consequence of Proposition 1.2 and Theorem 3.2. □

COROLLARY 3.4. *The symplectic realizations of a complemented Poisson manifold are complemented symplectic manifolds.*

COROLLARY 3.5. *If (M, P, ω) is a complemented Poisson manifold, the pullback of ω to either a symplectic leaf S of P or the local transversal germ N at $x_0 \in S$ is a complementary 2-form of S or N, respectively. In particular, if ω is a Nijenhuis complementary form, the leaves S are symplectic-Nijenhuis manifolds.*

Proof. The inclusions of S and N in M are Poisson mappings and, obviously, the inclusion of S in M satisfies (3.18). □

It is less difficult to find examples of complemented Poisson manifolds than it is to find Poisson-Nijenhuis structures.

First, notice that if (M, σ) is a symplectic manifold with $P = {}^t\sigma^{-1}$ then ω is a complementary 2-form of σ iff $\sharp_P\omega$ is a Poisson structure P_1 of M. Thus, on symplectic manifolds, complementary 2-forms are equivalent to Poisson structures, and every new Poisson structure P_1 of (M, σ) yields a new Lie algebroid structure of the form (3.3) on TM with anchor $B = \sharp_P \circ \sharp_\omega^{-1}$. It is easy to see that $\sharp_{P_1} = -B \circ \sharp_P$, whence the symplectic foliation of P_1, im $\sharp_{P1}$ coincides with im B. Of course, the symplectic 2-form σ is complementary to its own Poisson structure P.

Next, let (M^m, P) be a regular Poisson manifold with rank $P = 2k < m$. Then, the symplectic foliation $\mathcal{S}$ of P is regular, we may choose a transversal distribution $N\mathcal{S}$, and we may use the bigrading technique induced by $TM = N\mathcal{S} \oplus T\mathcal{S}$ as we did in Section 2. In particular, P has type $(0, 2)$, the space $\wedge^{1,0}M$ of the forms of type $(1, 0)$ is equal to ker $\sharp_P$, and the latter is an abelian subalgebra with respect to the bracket (1.17) of the present case (e.g., see [15]). Accordingly, by (1.14), every form ω of type $(2, 0)$ will be a complementary 2-form of P, and $B = \sharp_P \circ \sharp_\omega^{-1} = 0$. We also have $d\omega = d'\omega + d''\omega$, where the terms have the type $(3, 0)$ and $(2, 1)$, respectively. Therefore, (3.3) reduces to

$$[X, Y]' = \sharp_P i(X) i(Y) d\omega \quad (X, Y \in \Gamma TM), \tag{3.19}$$

and (3.15) holds. But this remark is not interesting since $B = 0$.

Furthermore, if a regular Poisson manifold (M, P) has a 2-form ω which is parallel with respect to a Poisson connection (i.e., a torsionless connection ∇ such that $\nabla P = 0$) then ω is a closed complementary 2-form of (M, P), and it yields a Poisson-Nijenhuis structure on (M, P). Indeed, using (1.14) and a technical computation, one gets the local components [15]

$$\{\omega, \omega\}_{P\, i_1 i_2 i_3} = -\delta^{j_1 j_2 k}_{i_1 i_2 i_3} (P^{su} \omega_{uk} \nabla_s \omega_{j_1 j_2} + \omega_{u j_1} \omega_{v j_2} \nabla_k P^{uv}), \tag{3.20}$$

where ∇ is an arbitrary torsionless connection of M. If $\nabla P = \nabla\omega = 0$, $\{\omega, \omega\}_P = 0$ as claimed.

For instance, if a Riemannian manifold (M, g) has a parallel 2-form ω, and if we put $P = \sharp_g \omega$, then it is clear that P is a Poisson structure of M, and the Riemannian connection ∇ of (M, g) is a Poisson connection. Accordingly, the form ω is complementary to P and $(P, \sharp_P \circ \sharp_\omega^{-1})$ is a Poisson-Nijenhuis structure.

Another example is provided by the results of Section 12 of [12]. Namely, if G is a Lie group, and P is a right(left)-invariant Poisson structure on G, one has $\{\alpha, \beta\}_P = 0$ for any left(right)-invariant 1-forms α, β of G. Indeed, for any left(right)-invariant vector field X we have $L_X P = 0$, and we get

$$\begin{aligned}\langle \{\alpha, \beta\}_P, X\rangle &= \langle L_{\sharp\alpha}\beta - L_{\sharp\beta}\alpha - d(P(\alpha, \beta)), X\rangle \\ &= (\sharp\alpha)(\beta(X)) + \beta(L_X(\sharp\alpha)) - (\sharp\beta)(\alpha(X)) - \alpha(L_X(\sharp\beta)) \\ &\quad - X(P(\alpha, \beta)) = P(L_X\alpha, \beta) + P(\alpha, L_X\beta) - L_X(P(\alpha, \beta)) = 0.\end{aligned}$$

(In the computation above, we wrote $\sharp$ instead of $\sharp_P$.)

Now, it is clear from (1.14) that if ω is a left(right)-invariant 2-form we shall get $\{\omega, \omega\}_P = 0$. Hence, any left(right)-invariant 2-form is complementary to any right(left)-invariant Poisson structure of G. If we also require $d\omega = 0$, we get the Poisson-Nijenhuis structure of Section 12 of [12].

For a final example, let P be the *Lie-Poisson structure* of the dual space $\mathcal{G}^*$ of a finite dimensional Lie algebra $\mathcal{G}$ (e.g., [18], [15]), and let $\mathbf{r} \in \wedge^2\mathcal{G}$ be a solution of the *classical Yang-Baxter equation* $[\mathbf{r}, \mathbf{r}] = 0$, where the bracket is the *algebraic Schouten-Nijenhuis bracket* of $\mathcal{G}$ (the extension of the bracket of $\mathcal{G}$ by a formula of the type (1.14), e.g., [15]). Then $\mathbf{r}$ can be seen as a 'constant' (i.e., with constant components) 2-form on $\mathcal{G}^*$, and, since it follows easily from the definition of the Lie-Poisson structures that the brackets $\{\ ,\ \}_P$ and $[\ ,\ ]$ are the same for 'constant' forms, the 2-form $\mathbf{r}$ is a closed complementary 2-form of $(\mathcal{G}^*, P)$ and it provides the latter with a 'constant' Poisson-Nijenhuis structure. (See also Section 14 of [12] and [6].)

Finally, let us notice that it is not possible to extend Theorems 3.1 and 3.2, as they stand, to either Poisson-Banach manifolds or Hamiltonian algebras or complexes. Indeed, if $(\chi, \mathcal{F}, H)$ is a Hamiltonian algebra as in Section 1, we might try to define a complementary form as an element $\sharp_\omega^{-1} \in L_{\mathbf{R}}(\chi, \wedge^1(\chi))$, which can be extended to the bidual of χ with respect to $\mathcal{F}$, and such that the Gel'fand-Dorfman bracket $\{\sharp_\omega^{-1}, \sharp_\omega^{-1}\}$, defined for the dual bracket of the given Hamiltonian algebra, vanishes. But, the dual bracket of such a complementary form is on the bidual of χ and not on χ itself.

Added in Proofs. The new book: Dorfman I., Dirac structures and integrability of nonlinear evolution equations, J. Wiley & Sons, New York, 1993, is also available now for the general algebraic Hamiltonian theory of Gel'fand and Dorfman to which we referred in Section 1.

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