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On the monodromy at infinity of a polynomial map

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Abstract. In this paper we study the semi-simple and unipotent part of the monodromy at infinity of a polynomial map which satisfies a natural restriction.

1. Introduction

(1.1) Let $f : \mathbf{C}^{n+1} \to \mathbf{C}$ be a map given by a polynomial with complex coefficients which will be also denoted by f. It is known (see, for example, ([17, Appendix A1]) that there is a finite set $\Gamma \subset \mathbf{C}$ such that the map

$$f\,|_{\mathbf{C}^{n+1}-f^{-1}(\Gamma)} \colon \mathbf{C}^{n+1} - f^{-1}(\Gamma) \to \mathbf{C} - \Gamma$$

is a locally trivial C^∞-fibration. The smallest set Γ veryfing this condition is called the bifurcation set of f and will be denoted by Γ_f. Γ_f contains the set Σ_f of critical values of f but it might be bigger, since f is not a proper map. For example, if $f = x(xy-1)$ then $\Sigma_f = \emptyset$ but $\Gamma_f = \{0\}$.

Fix $t_0 \in \mathbf{C}$ such that $|t_0| > \max\{|\gamma| : \gamma \in \Gamma_f\}$. The geometric monodromy associated with the path $s \mapsto t_0\,\mathrm{e}^{2\pi i s}$, $s \in [0,1]$ is a diffeomorphism of $f^{-1}(t_0)$ onto itself which induces an isomorphism:

$$T^\infty_{f,\mathbf{Z}} : H^n(f^{-1}(t_0), \mathbf{Z}) \to H^n(f^{-1}(t_0), \mathbf{Z})$$

that will be called the monodromy of f at infinity. It follows from the monodromy theorem (e.g. in the form it is stated in (4, III. 2.3)) that the eigenvalues of $T^\infty_f := T^\infty_{f,\mathbf{Z}} \otimes \mathrm{Id}_{\mathbf{C}}$ are roots of unity.

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$T^{\infty}_{f,\mathbf{Z}}$ is an invariant of the right equivalence class of f (we say that $f, g : \mathbf{C}^{n+1} \to \mathbf{C}$ are right equivalent if there is a diffeomorphism $\Phi : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ such that $f = g \circ \Phi$), in particular it is also an invariant of the embedded affine variety $\{f = 0\} \subseteq \mathbf{C}^{n+1}$. One expects that the study of T^{∞}_{f} can be useful in order to have a better understanding of the classification of polynomial maps up to right equivalence.

In this paper we study the monodromy at infinity of polynomials $f \in \mathbf{C}[X_1, \ldots, X_{n+1}]$ verifying the following condition:

$$\text{\textit{For } } t \in \mathbf{C} - \Sigma_f, \text{\textit{ the closure in } } \mathbf{P}^{n+1} \text{\textit{of the affine hypersurface} } \{f = t\} \subset \mathbf{C}^{n+1} \text{\textit{is non-singular.}} \qquad (*)$$

For the sake of brevity, a polynomial f verifying condition $(*)$ will be called a $(*)$-polynomial.

In Section 2 we list the main properties of $(*)$-polynomials. The most important ones are the following: the fibers of the map $f : \mathbf{C}^{n+1} \to \mathbf{C}$ have the homotopy type of a bouquet of n-spheres, and the fibration provided by f over a circe of large radius in $\mathbf{C}$ is equivalent to a fibration of type $S^{2n+1} - K \to S^1$ (i.e. to an open book decomposition). We also prove that T^{∞}_{f} is determined by the highest degree form f_d of f. If $X^{\infty}_{f} \subseteq \mathbf{P}^n$ denotes the hypersurface given by $f_d = 0$, it is easy to see that X^{∞}_{f} has only isolated singularities. The main goal of this paper is to obtain information about the complex monodromy of f at infinity in terms of topological invariants of the embedded hypersurface $X^{\infty}_{f} \subseteq \mathbf{P}^n$.

In Section 3 we study the semi-simple part of T^{∞}_{f} (equivalently, its characteristic polynomial $\mathrm{char}^{\infty}_{f}$), and we prove that it is completely determined by local data: it depends only on the number of variables, the degree of f, and the characteristic polynomials of the (local) monodromies of the singularities of X^{∞}_{f}.

In contrast to this, we show in Section 4 that the unipotent part of T^{∞}_{f} does not depend only on local data, but also on the position of the singularities of X^{∞}_{f} in $\mathbf{P}^n$. For example, it is proved in (4.6) that the number of Jordan blocks of T^{∞}_{f} associated with the eigenvalue 1 is the $(n-1)$th Betti number $b_{n-1}(X^{\infty}_{f})$ if n is even and $b_{n-1}(X^{\infty}_{f}) - 1$ if n is odd. On the other hand, it is well-known that the middle Betti number of a hypersurface with isolated singularities is not a purely local invariant but it depends on the position of the singular points (see for example [26], [5]). A concrete example showing how this reflects on the monodromy at infinity is given in (6.4).

In general, it is not possible to determine T^{∞}_{f} in terms of the Betti numbers of X^{∞}_{f} and local data, because even when the position of the singularities of X^{∞}_{f} does not have any influence on its Betti numbers (for example if X^{∞}_{f} is an irreducible curve), it can have an influence on the block structure of the monodromy at infinity. In Section 6 we exemplify the case when X^{∞}_{f} is a plane sextic with six cusps (Zariski's example). It is well known that the position of the cusps determines the Betti numbers of the 6-fold cyclic covering of $\mathbf{P}^2$ ramified along X_f^∞ (and then it determines whether the fundamental group of $\mathbf{P}^2 - X_f^\infty$ is abelian or not). In (6.5) we compute the unipotent part of T_f^∞ in terms of the Betti numbers of the 6-fold cyclic covering.

We study the nilpotent (or unipotent) part of T_f^∞ is given in two steps. In Section 4 we construct a compactification of the fibration of f at infinity, and we prove that its monodromy T determines completely T_f^∞. In Section 5 we consider a projective map $\pi': \mathcal{X}' \to D$ which is a fibration over the punctured disc $D - \{0\}$ with monodromy T^{-d}. Moreover, the singularities of $\mathcal{X}'$ and $(\pi')^{-1}(0)$ are isolated. This allows us to determine the monodromy T^d completely: while $(T^d)_{\neq 1}$ is determined in terms of the local monodromies of the singularities of X_f^∞, the block structure of $(T^d)_1$ is given by the weight filtration (of the mixed Hodge structure) of the d-fold cyclic covering of $\mathbf{P}^n$ branched along X_f^∞.

By the correspondance proved in Section 4, the results of Section 5 give much information about the unipotent part of T_f^∞.

In the computation of $(T^d)_1$ the results of the Appendix (written by the first author and J.H.M. Steenbrink) are crucial.

(1.2) Unless otherwise stated, all cohomology and homology groups have coefficients in the field $\mathbf{C}$ of complex numbers. The following notations will be used through the paper:

- $B_R = \{(x_1, \dots, x_{n+1}) \in \mathbf{C}^{n+1} \mid \sum |x_i^2| < R\}$, $\bar{B}_R$ its closure, $\partial \bar{B}_R$ its boundary.
- $D_r = \{t \in \mathbf{C} \mid |t| < r\}$, $\bar{D}_r$ its closure, $S_r = \partial \bar{D}_r$.
- Given $f \in \mathbf{C}[X_1, \dots, X_{n+1}]$ the gradient of f will be denoted $\partial f = (\partial f/\partial x_1, \dots, \partial f/\partial x_{n+1})$. If $\mathbf{C}^{n+1} \hookrightarrow \mathbf{P}^{n+1}$ is the embedding given by $(x_1, \dots, x_{n+1}) \mapsto [1 : x_1 : \dots : x_{n+1}]$ the hyperplane $\{x_0 = 0\} \subset \mathbf{P}^{n+1}$ will be denoted by H^∞. If f_d is the highest degree form of f we will denote the hypersurface in H^∞ given by $f_d = 0$ by X_f^∞ or by X^∞ if it is clear from the context which is the polynomial we are referring to. We always assume that $d > 1$.
- Let H be a finitely dimensional $\mathbf{C}$-vector space, $\varphi: H \to H$ a linear map, $\lambda \in \mathbf{C}$ a complex number. We denote by H_λ the space of generalized eigenvectors of eigenvalue λ, i.e.

  $$H_\lambda = \{x \in H \mid \exists n \in \mathbf{N}^* \quad \text{with} \quad (\varphi - \lambda \cdot \mathrm{Id})^n x = 0\}$$

  and $\varphi_\lambda := \varphi_{|H_\lambda}: H_\lambda \to H_\lambda$. We denote by $\#_k\varphi_\lambda$ the number of k-dimensional Jordan blocks of φ_λ, $\#\varphi_\lambda = \sum_{k\geqslant 1} \#_k\varphi_\lambda$ and $\#_k\varphi = \sum_{\lambda\in\mathbf{C}} \#_k\varphi_\lambda$.
- If $l > 0$ is an integer, we denote by $c_l(\varphi): H^{\oplus l} \to H^{\oplus l}$ the linear map defined by $c_l(\varphi)(x_1, \dots, x_l) = (\varphi(x_l), x_1, \dots, x_{l-1})$.

2. Polynomials with good behaviour at infinity

(2.1) In this section we will list the main properties of the $(*)$-polynomials and we will prove that the highest degree form of a $(*)$-polynomial determines its behaviour at infinity. For technical reasons, it will be more convenient to reformulate the $(*)$ condition as follows: Let $f \in \mathbf{C}[X_1, \ldots, X_{n+1}]$ be a polynomial of degree d and denote $f = f_d + f_{d-1} + \cdots$ its decomposition into homogeneous components. Then f is a $(*)$-polynomial if and only if

$$\{x \in \mathbf{C}^{n+1} \mid \partial f_d(x) = f_{d-1}(x) = 0\} = \{0\}.$$

The proof is easy and it is left to the reader. The $(*)$-polynomials have a number of good properties which are summarized in the following:

(2.2) THEOREM ([13], [15]). *Let f be a $(*)$-polynomial. Then:*

(i) *The bifurcation set Γ_f is exactly the set Σ_f of critical values.*

(ii) *The singular fibers $f^{-1}(s)$ $(s \in \Sigma_f)$ have only isolated singularities.*
For $s \in \Sigma_f$, denote by μ_s the sum of the Milnor numbers of the isolated singularities of $f^{-1}(s)$. Set $\mu^\infty = \Sigma_{s\in\Sigma_f}\mu_s$.

(iii) *Any fiber $f^{-1}(s)$ has the homotopy type of a bouquet of n-dimensional spheres. The number of spheres in the generic fiber is μ^∞, the number of spheres in a singular fiber $f^{-1}(s)$ is $\mu^\infty - \mu_s$.*

(iv) *For any $r > 0$ with the property that $\Sigma_f \subset D_r$, there exists $R_0 \gg 0$ such that for any $t \in D_r$, $R \geqslant R_0$, $f^{-1}(t)$ intersects $\partial \bar{B}_R$ transversely and the restriction*

$$f: (f^{-1}(S_r) \cap \bar{B}_R, f^{-1}(S_r) \cap \partial \bar{B}_R) \to S_r$$

is a C^∞-locally trivial fibration of pairs of spaces.
The fibration $f: f^{-1}(S_r) \cap \bar{B}_R \to S_r$ is equivalent to the fibration $f: f^{-1}(S_r) \to S_r$ and it will be called the fibration of f at infinity. The fibration $f: f^{-1}(S_r) \cap \partial \bar{B}_R \to S_r$ extends to a C^∞-trivial fibration $f: f^{-1}(\bar{D}_r) \cap \partial \bar{B}_R \to \bar{D}_r$.

(v) *There exists $R_0' \gg 0$ such that for any $R' \geqslant R_0'$*

$$\varphi = \frac{f}{\|f\|} : \partial \bar{B}_{R'} - f^{-1}(0) \to S^1$$

is a C^∞-locally trivial fibration (called the Milnor fibration at infinity), which is equivalent to the fibration of f at infinity.

(vi) *Let X_f^∞ be the intersection of the hyperplane at infinity H^∞ with the projective closure $X_t = \overline{f^{-1}(t)} \subseteq \mathbf{P}^{n+1}$ of any fiber $f^{-1}(t)$. Then the hypersurface X_f^∞ has only isolated singularities.*

Proof. In [6] it is proved that a $(*)$-polynomial (in the sense of (2.1)) is quasi-tame, a condition introduced by the second author in [13]. Now (i)–(iv) follow from [13], (v) follows from [15] and (vi) from an easy verification. □

(2.3) Let $(F, \partial F)$ denote the fiber of the fibration (2.2.(iv)). Then the last part of (iv) implies that there is a smooth representative T_{geom} of the geometric monodromy which is the identity on ∂F. This allows us to define a variation map Var : $H^n(F) \to H^n_c(F)$ by $\mathrm{Var}[\omega] = [T^*_{\text{geom}}(\omega) - \omega]$. (For the definition at the level of integer homology, see for example [1]). The variation map fits in the following diagram:

$$\begin{tikzcd} H^n_c(F) \arrow[r, "T^\infty_{f,c} - \mathrm{Id}"] \arrow[d, "k"'] & H^n_c(F) \arrow[d, "k"] \\ H^n(F) \arrow[r, "T^\infty_f - \mathrm{Id}"'] \arrow[ur, "Var"] & H^n(F) \end{tikzcd}$$

where T^∞_f and $T^\infty_{f,c}$ are the corresponding monodromies and k is the natural map. The following result will be crucial in the study of the unipotent part of T^∞_f:

(2.4) PROPOSITION. Var *is an isomorphism.*

Proof. Since the fibration at infinity is equivalent to the Milnor fibration at infinity (2.2.(v)), Var is the variation map of a fibration of type $S^{2n+1} - K \to S^1$. Similarly as in the local case of isolated hypersurface singularities, the variation map is an isomorphism by Alexander duality (see [1] for more details). Actually, (2.4) is equivalent to the non-degeneracy of the Seifert form of the open book decomposition. □

(2.5) REMARK. Theorem (2.2) and proposition (2.4) hold not only for $(*)$-polynomials but for a larger class of polynomials which includes the 'tame' ([2]) and 'quasi-tame' polynomials ([13]), the 'M-tame' polynomials ([15]) and the convenient polynomials, non-degenerate with respect to their Newton boundary at infinity ([8]).

(2.6) THEOREM. *Let $f = f_d + f_{d-1} + \cdots$, $g = g_d + g_{d-1} + \cdots$ be two $(*)$-polynomials of degree d such that $f_d = g_d$. Then the fibrations at infinity of f and g are equivalent (in the sense of [25], p. 11). In particular all the invariants, introduced in (2.2) and (2.3) for f and g, are equivalent.*

Proof. Notice that the set of $(*)$-polynomials with fixed highest degree form f_d form a connected, smooth, quasiprojective variety. Thus, in order to prove the theorem, it is enough to prove the following claim:

CLAIM. Let $f = f_d + f_{d-1} + \cdots$ be a $(*)$-polynomial. Fix $0 \leqslant i \leqslant d-1$ and consider the family of polynomials $f_s = f + (s-1)f_i$, $s \in \mathbf{C}$. Fix $1 \gg \eta > 0$ such that each f_s is a $(*)$-polynomial for $|s-1| \leq \eta$. Then there exist $r \gg 0$ and $R_0 \gg 0$ such that:

(i) $\Sigma_{f_s} \subseteq D_r$ for $|s-1| \leqslant \eta$;
(ii) $f_s^{-1}(t)$ intersects $\partial \bar{B}_R$ transversely for $|s-1| \leqslant \eta$, $R \geqslant R_0$ and $t \in \bar{D}_r$.

Now, the proof of the theorem is the following: Set

$$\begin{aligned} \mathcal{E} &= \{(x,t,s) \in \bar{B}_{R_0} \times S_r \times B_1(\eta) \mid f_s(x) = t\}, \\ \partial\mathcal{E} &= \{(x,t,s) \in \mathcal{E} \mid x \in \partial \bar{B}_{R_0}\}, \end{aligned}$$

where $B_1(\eta) = \{s \mid |s-1| < \eta\}$. Then the projection $q : (\mathcal{E}, \partial\mathcal{E}) \to S_r \times B_1(\eta)$ is a locally trivial fibration and for any $s \in B_1(\eta)$, $q_{|q^{-1}(S_r \times \{s\})}$ is the fibration of f_s at infinity. Then the result follows from [25, p. 53].

Proof of the claim: Assume that (i) is not true. Then, by the curve selection lemma ([11], [15]), there exist real analytic curves $x(t) \in \mathbf{C}^{n+1}$, and $s(t) \in \mathbf{C}$ $(0 < t < \varepsilon)$ such that $|s(t)-1| \leq \eta, \partial f_{s(t)}(x(t)) \equiv 0$, and $\lim_{t\to 0} f_{s(t)}(x(t)) = \infty$. The last limit implies that $\lim_{t\to 0} \|x(t)\| = \infty$. Put $x(t) = t^{-n} y(t)$ with $n > 0$, $y(t) = y_0 + t y_1 + \cdots$, $y_0 \neq 0$. Then:

$$\partial f_d(y(t)) + t^n \partial f_{d-1}(y(t)) + \cdots + s(t) t^{(d-i)n} \partial f_i(y(t)) + \cdots \equiv 0. \tag{1}$$

In particular $\partial f_d(y_0) = 0$. Identity (1) gives

$$\partial f_d(y(t)) + t^n \cdot c \cdot \partial f_{d-1}(y_0) \equiv 0 \pmod{t^{n+1}}, \tag{2}$$

where $c = 1$ if $i < d-1$ and $c = s(0)$ if $i = d-1$. This identity, multiplied by $y(t)$, rewritten using the Euler–relations, differentiated with respect to t, and compared with its initial form (2), gives:

$$n \cdot t^{n-1} \cdot c \cdot (d-1) f_{d-1}(y_0) \equiv 0 \pmod{t^n}.$$

Therefore $f_{d-1}(y_0) = 0$, which contradicts condition $(*)$.

Part (ii) follows from a similar argument. Fix an $r \gg 0$ which satisfies (i). Assume that (ii) is not true for η, r and any $R_0 \gg 0$. Then there exist analytic curves $x(t) \in \mathbf{C}^{n+1}$, $\lambda(t) \in \mathbf{C}$ and $s(t) \in \mathbf{C}$, $(0 < t < \varepsilon)$, with $|s(t)-1| \leq \eta$, $|f_{s(t)}(x(t))| \leqslant r$, $\lim_{t\to 0} \|x(t)\| = \infty$, and

$$\partial f_{s(t)} \cdot (x(t)) = \lambda(t) \cdot \overline{x(t)}. \tag{3}$$

Let $x(t) = t^{-n} y(t)$ as above. Since $f_{s(t)}(x(t))$ has order 0,

$$\partial f_{s(t)}(x(t)) \cdot x'(t) + s'(t) \cdot f_i(x(t)) \equiv 0 \pmod{t^0}.$$

Hence $\lambda(t) \cdot \overline{x(t)} \cdot x'(t) + s'(t) \cdot t^{-ni} f_i(y(t)) \equiv 0 \pmod{t^0}$, which gives $\lambda(t) \equiv 0 \pmod{t^{(2-i)n+1}}$. Now (3) gives $(\bmod\ t^{n+1})$ the same equation as (2) which gives again the contradiction $\partial f_d(y_0) = f_{d-1}(y_0) = 0$. □

(2.7) Theorem (2.6) says that the behaviour of $f = f_d + f_{d-1} + \cdots$ at infinity depends only on f_d, in particular f can be replaced by any polynomial $f' = f_d + f'_{d-1}$ where f'_{d-1} is a polynomial such that its zero set in $\mathbf{P}^n$ does not intersect $\mathrm{Sing}(X^\infty)$. For example, we can take $f'_{d-1} = l^{d-1}$ where l is a generic linear form. This fact gives the hope that the topological invariants of f associated to its behaviour at infinity can be *explicitly* described in terms of invariants of the embedded hypersurface $X^\infty \subseteq \mathbf{P}^n$.

3. The semi-simple part of T_f^∞

(3.1) Let $f = f_d + f_{d-1}$ be a $(*)$-polynomial. We introduce some notations. Let $\mathrm{Sing}(X^\infty) = \{p_1, \ldots, p_k\}$. For $1 \leqslant i \leqslant k$, let $g_i\colon (H^\infty, p_i) \to (\mathbf{C}, 0)$ be a local equation defining the isolated hypersurface singularity germ (X^∞, p_i). We denote by μ_i its local Milnor number, by F_i its local Milnor fiber, and by T_i its algebraic monodromy acting on $H^{n-1}(F_i)$. In this section we determine the characteristic polynomial of T_f^∞ (or equivalently, its semi-simple part) in terms of the characteristic polynomials of the local algebraic monodromies $\{T_i\}_{i=1,\ldots,k}$.

(3.2) First assume that X^∞ is non-singular. Then $f_d\colon (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defines an isolated singularity and, by similar argument as in (2.6), the fibration of f at infinity is the same as the fibration of f_d at infinity, which is identical with the local Milnor fibration of the germ f_d defined at 0. In particular, its Milnor number is $\mu_{\mathrm{gen}}^\infty = (d-1)^{n+1}$, its monodromy has (finite) order d, and its characteristic polynomial is

$$\begin{aligned}\mathrm{char}_{\mathrm{gen}}(\lambda) &:= \det(\lambda \cdot \mathrm{Id} - T_{\mathrm{gen}}^\infty) \\ &= (\lambda - 1)^{(-1)^{n+1}} (\lambda^d - 1)^{((d-1)^{n+1} - (-1)^{n+1})/d}.\end{aligned}$$

Set $\mathrm{char}_f^\infty(\lambda) = \det(\lambda \cdot \mathrm{Id} - T_f^\infty)$, $\mathrm{char}_i(\lambda) = \det(\lambda \cdot \mathrm{Id} - T_i)$. Then we have:

(3.3) THEOREM. *Assume that f is a $(*)$-polynomial. Then, with the notations introduced above one has:*

$$\mathrm{char}_f^\infty(\lambda) = \mathrm{char}_{\mathrm{gen}}(\lambda) \cdot \prod_{i=1}^{k} \frac{\mathrm{char}_i(\lambda^{d-1})}{(\lambda^d - 1)^{\mu_i}}.$$

In particular, $\mu^\infty + \sum_{i=1}^k \mu_i = \mu_{\mathrm{gen}}^\infty$.

Before we start the proof of theorem (3.3), we make some preparations:

(3.4) LEMMA. *Consider the family of polynomials* $f_s = f_d + sf_{d-1}$, *where* $s \in \mathbf{C}$. *There exists* $r > 0$ *such that the set of critical values of* f_s *is contained in* D_r *for any* s *with* $|s| \leq 1$.

The proof is similar to the proof of the Claim in (2.6), and it is left to the reader.

(3.5) We recall the definition of the zeta function of a locally trivial fibration $E \to S^1$ over the one-dimensional circle S^1. Let F be its fiber and $T^q\colon H^q(F) \to H^q(F)$ the algebraic monodromies induced by its characteristic map. Then define:

$$\zeta(E \to S^1) = \prod_q \det(\lambda \cdot \mathrm{Id} - T^q)^{(-1)^q}.$$

If f is an arbitrary polynomial and r is large enough so that $\Gamma_f \subset D_r$, then $\zeta^\infty(f)$ is, by definition, $\zeta(f^{-1}(S_r) \to S_r)$.

(3.6) *Proof of (3.3).* Fix r big enough so that the conclusion of Lemma 3.4 holds for r. For any $s \in \mathbf{C}$ with $|s| \leq 1$ and $t \in S_r$, set

$$X_{t,s} = \{[x] \in \mathbf{P}^{n+1} \mid f_d(x_1, \ldots, x_{n+1}) + sx_0 f_{d-1}(x_1, \ldots x_{n+1}) = tx_0^d\}.$$

$X_{t,s}$ is the projective closure of the affine variety $f_s^{-1}(t)$. Since for any $s \neq 0$ the polynomial f_s satisfies $(*)$, for any $s \neq 0$ the space $X_{t,s}$ is non-singular. The intersection $X_{t,s} \cap H^\infty$ is exactly X^∞, in particular it has only isolated singularities (cf. 2.2.(vi)). On the other hand, $X_{t,0}$ is singular with isolated singularities exactly at the singular points of X^∞.

Let B_i be a small open 'ball' in $\mathbf{P}^{n+1}$ with center at p_i, $(1 \leqslant i \leqslant k)$. More precisely, consider a real analytic function r_i defined in a neighborhood of p_i with non-negative values such that $r_i^{-1}(0) = \{p_i\}$, and take $B_i = r_i^{-1}([0, \eta_0))$. Here η_0 is small enough such that B_i's are disjoint and the following conditions hold:

(i) $r_i^{-1}(\eta)$ is smooth and intersects $X_{t,0}$, H^∞ and X^∞ transversely for any $\eta \leqslant \eta_0$, $t \in S_r$ and $i = 1, \ldots, k$,

(ii) The 'ball' $r_i^{-1}([0, \eta])$ and its intersections with $X_{t,0}$, H^∞ and X^∞ retract to p_i for any $\eta \leqslant \eta_0$, (in other words, B_i is a 'Milnor ball' at p_i for the analytic sets $X_{t,0}$, H^∞ and X^∞). Fix η_0 and the balls $\{B_i\}_{i=1,\ldots,k}$. Let $\bar{B}_i$ be the closure of B_i, and $\partial \bar{B}_i$ its boundary.
Now we are going to consider balls $\bar{B}_R$ in the affine space $\mathbf{C}^{n+1}$. Choose R_0 big enough so that one has:

(iii) $\partial \bar{B}_R$ intersects $f_0^{-1}(t)$ transversely for any $R \geqslant R_0$ and $t \in S_r$,

(iv) $\partial \bar{B}_R$ intersects $\partial \bar{B}_i$ transversely for any $R \geqslant R_0$ and $i = 1, \ldots, k$.
Fix a R_0 with these properties and set $C_{R_0} = \mathbf{C}^{n+1} - B_{R_0}$. Since $f_s^{-1}(t)$ is smooth for any s, t and $f_0^{-1}(t)$ intersects $\partial \bar{B}_{R_0}$ transversely, there exists $1 \gg \varepsilon > 0$ such that for any $s \in \mathbf{C}$ with $|s| \leqslant \varepsilon$ one has:

(v) $f_s^{-1}(t)$ intersects $\partial \bar{B}_{R_0}$ transversely for any t,

(vi) $X_{t,s}$ intersects $\partial\bar{B}_i$ transversely for any t and $i = 1, \ldots, k$.

Obviously, in general R_0 does not satisfy the condition (2.2.(iv)) for all s. With these choices, the map:

$$(f_s^{-1}(S_r), f_s^{-1}(S_r) \cap \bar{B}_{R_0}, f_s^{-1}(S_r) \cap C_{R_0}) \to S_r, \quad |s| \leqslant \varepsilon \tag{4}$$

is a locally trivial fibration of a triple of spaces. By the transversality conditions one has the equivalence of the fibrations:

$$f_0^{-1}(S_r) \cap B_{R_0} \to S_r \quad \text{and} \quad f_\varepsilon^{-1}(S_r) \cap B_{R_0} \to S_r$$

respectively of

$$(f_0^{-1}(S_r) \cap \partial\bar{B}_{R_0} \to S_r) \quad \text{and} \quad (f_\varepsilon^{-1}(S_r) \cap \partial\bar{B}_{R_0} \to S_r).$$

By a Mayer-Vietoris argument one has:

$$\frac{\zeta^\infty(f_\varepsilon)}{\zeta^\infty(f_0)} = \frac{\zeta(f_\varepsilon^{-1}(S_r) \cap C_{R_0} \to S_r)}{\zeta(f_0^{-1}(S_r) \cap C_{R_0} \to S_r)}. \tag{5}$$

Set:

$$\mathcal{F} = \{(x, t, s) \in C_{R_0} \times S_r \times \bar{D}_\varepsilon : f_s(x) = t\}.$$

If $A \subset C_{R_0}$ and $s \in \bar{D}_\varepsilon$ then set $\mathcal{F}(A)_s := \mathcal{F} \cap (A \times S_r \times \{s\})$. Denote:

$$\mathcal{Y}_s = \mathcal{F}((\cup_{i=1}^k \bar{B}_i) \cap C_{R_0})_s, \quad \partial\mathcal{Y}_s = \mathcal{F}((\cup_{i=1}^k \partial\bar{B}_i) \cap C_{R_0})_s,$$
$$\mathcal{Z}_s = \mathcal{F}(C_{R_0} - \cup_{i=1}^k B_i)_s.$$

Consider the map $\mathcal{F} \to S_r \times \bar{D}_\varepsilon$ induced by the projection. For any $s \in \bar{D}_\varepsilon$ the induced map $(\mathcal{X}_s, \mathcal{Y}_s, \mathcal{Z}_s) \to S_r$ is a locally trivial fibration of a triple of spaces, and the fibrations $\mathcal{Z}_s \to S_r$ and $\partial\mathcal{Y}_s \to S_r$ are independent on s. Then, by Mayer-Vietoris one has:

$$\zeta(\mathcal{F}(C_{R_0})_s \to S_r) = \frac{\zeta(\mathcal{Y}_s \to S_r) \cdot \zeta(\mathcal{Z}_s \to S_r)}{\zeta(\partial\mathcal{Y}_s \to S_r)}. \tag{6}$$

for any $|s| \leqslant \varepsilon$. Now (5) and (6) give

$$\frac{\zeta^\infty(f_\varepsilon)}{\zeta^\infty(f_0)} = \frac{\zeta(\mathcal{F}(C_{R_0})_\varepsilon \to S_r)}{\zeta(\mathcal{F}(C_{R_0})_0 \to S_r)} = \frac{\zeta(\mathcal{Y}_\varepsilon \to S_r)}{\zeta(\mathcal{Y}_0 \to S_r)}. \tag{7}$$

Notice that the right hand side of (7) is completely local, it is concentrated in the balls $\{B_i\}_i$. The rest of the proof is now devoted to compute $\zeta(\mathcal{Y}_0 \to S_r)$ and

$\zeta(\mathcal{Y}_\varepsilon \to S_r)$. Since the balls $\{B_i\}_i$ are disjoint, these zeta functions are products over the singular points p_i of X^∞: $\zeta(\mathcal{Y}_s \to S_r) = \prod_i \zeta(\mathcal{Y}_{si} \to S_r)$, where $\mathcal{Y}_{si} = \mathcal{Y}_s \cap B_i$ and $s = 0$ or $= \varepsilon$.

Fix a point $p_i \in \mathrm{Sing}(X^\infty)$ and choose coordinates so that $p_i = [0 : \ldots : 0 : 1]$ and $f_{d-1} = x_{n+1}^{d-1}$ (cf. 2.6–2.7). We recall that we fixed a circle of big radius S_r and after that we fixed the balls B_i and R_0 (thus R_0 and B_i depend on the choice of r). In local coordinates $(y_0, \ldots, y_n)$ (using the notation $y_i = x_i/x_{n+1}$ for $i = 0, \ldots, n$ and $y = (y_1, \ldots, y_n)$) one has:

$$C_{R_0} = \{(y_0, y) \mid 1 + \|y\|^2 \geqslant R_0|y_0|^2,\ y_0 \neq 0\},$$

and we can assume that $\bar{B}_i = \{|y_0|^2 + \|y\|^2 \leq \rho_i\}$ for some small $\rho_i > 0$. Then $\mathcal{Y}_{0i} = \{(y_0, y, t) \in (\bar{B}_i \cap C_{R_0}) \times S_r \mid g_i(y) = ty_0^d\}$ and the map $\mathcal{Y}_{0i} \to S_r$ is induced by the first projection. We will show that $\zeta(\mathcal{Y}_{0i} \to S_r) = 1$. For this consider the neighborhood $N_i = \{(y_0, y) \mid 1 \geqslant R_0|y_0|^2,\ |y_0|^2 + \|y\|^2 \leqslant \rho_i\}$. Then it is not difficult to see that

$$\mathcal{Y}'_{0i} := \{(y_0, y, t) \in N_i \times S_r \mid y_0 \neq 0,\ g_i(y) = ty_0^d\} \to S_r$$

is a subbundle of $\mathcal{Y}_{0i} \to S_r$ which is an equivariant strong deformation retract. In particular, $\zeta(\mathcal{Y}_{0i} \to S_r) = \zeta(\mathcal{Y}'_{0i} \to S_r)$.

Now set $\rho = \sqrt{1/R_0}$ and consider the map $\mathcal{Y}'_{0i} \to S_r \times D^*_\rho$ given by $(y_0, y, t) \to (t, y_0)$. By construction, this is a fiber bundle (with fiber F_i), thus $\zeta(\mathcal{Y}'_{0i} \to S_r) = 1$ by [14, 3.3.9].

The next step is the computation of $\zeta(\mathcal{Y}_{\varepsilon i} \to S_r)$. Let r, R_0, ρ_i be as above. Then ε is fixed and is sufficiently small with respect to r, R_0, ρ_i (see conditions (v) and (vi) above). Then:

$$\mathcal{Y}_{\varepsilon i} = \{(y_0, y, t) \in (\bar{B}_i \times C_{R_0}) \times S_r \mid y_0 \neq 0 \quad \text{and} \quad ty_0^d - \varepsilon y_0 = g_i(y)\}.$$

Consider, similarly as above,

$$\mathcal{Y}'_{\varepsilon i} = \{(y_0, y, t) \in N_i \times S_r \mid y_0 \neq 0 \quad \text{and} \quad ty_0^d - \varepsilon y_0 = g_i(y)\}$$

with the projection onto S_r. Then $\mathcal{Y}_{\varepsilon i} \to S_r$ and $\mathcal{Y}'_{\varepsilon i} \to S_r$ are equivalent fiber bundles. Now take the projection $\tau_i\colon \mathcal{Y}'_{\varepsilon i} \to S_r \times D^*_\rho$ given by $(y_0, y, t) \to (t, y_0)$. This is a fiber bundle over $S_r \times D^*_\rho - \{(t, y_0) \mid ty_0^d = \varepsilon y_0\}$ with fiber F_i and the fiber of τ_i over any point of $\Delta := \{(t, y_0) \in S_r \times D^*_\rho \mid ty_0^d = \varepsilon y_0\}$ is contractible. For a fixed value $t = t_0 \in S_r$, the punctured disk $\{t_0\} \times D^*_\rho$ intersects Δ in $(d-1)$ points $q_1, \ldots, q_{d-1}$. If $t_0 \in \mathbf{R}_+$ (i.e. if $t_0 = r$), these points are:

$$q_k = (t_0, \mathrm{e}^{2\pi ik/d-1} \cdot \rho_\varepsilon) \ \ k = 1, \ldots, d-1.$$

They are situated on the circle

$$S(t_0) = \{(t, y_0) \in S_r \times D^*_\rho \mid t = t_0,\ |y_0| = \rho_\varepsilon\}.$$

Consider also the points $r_k \in S(t_0)$, $k = 1, \ldots, d-1$:

$$r_k = (t_0, \mathrm{e}^{2\pi i(k+\frac{1}{2})/d-1} \cdot \rho_\varepsilon).$$

Now the fiber $F_{\varepsilon i}$ of $\mathcal{Y}'_{\varepsilon i}$ over t_0 is $\tau^{-1}(\{t_0\} \times D^*_\rho)$. $S(t_0)$ is a strong deformation retract of $\{t_0\} \times D^*_\rho$ and the retraction can be lifted, therefore $F_{\varepsilon i}$ has the homotopy type of $\tau^{-1}(S(t_0))$. The points r_k lay on the arc $\widehat{q_k q_{k+1}}$, $\tau^{-1}(r_k) \simeq F_i$ and $\tau^{-1}(q_k)$ is contractible. Therefore $\tau^{-1}(\widehat{q_k q_{k+1}})$ has the homotopy type of the suspension $S(F_i)$ of F_i. In particular $\tau^{-1}(S(t_0)) \simeq S^1 \vee (\bigvee_{d-1} S(F_i))$.

If we now lift the path given by $\alpha \mapsto t_0\,\mathrm{e}^{2\pi i\alpha} \in S_r$, $\alpha \in [0,1]$, then the points q_k, respectively r_k, move on the path:

$$q_k(\alpha) = (t_0\,\mathrm{e}^{2\pi i\alpha},\ \mathrm{e}^{2\pi i(-\alpha+k)/(d-1)} \cdot \rho_\varepsilon),$$

respectively on

$$r_k(\alpha) = (t_0\,\mathrm{e}^{2\pi i\alpha},\ \mathrm{e}^{2\pi i(-\alpha+k+\frac{1}{2})/(d-1)} \cdot \rho_\varepsilon), \quad \alpha \in [0,1].$$

Obviously $r_k(0) = r_k$ and $r_k(1) = r_{k-1}$ (with the notation $r_0 = r_{d-1}$). Thus via the paths $r_2(\alpha), \ldots, r_{d-1}(\alpha)$, we can identify the fibers of the bundle τ over the points $\{r_k\}_{k=1}^{d-1}$. We determine now the geometric monodromy acting on $S^1 \vee (\bigvee_{d-1} S(F_i))$. It is clear that the action on S^1 is trivial. By the above identifications, the action on $\bigvee_{d-1} S(F_i)$ is given by $c_{d-1}(S(T_{\text{geom}}))$ (see (1.2)), where $S(T_{\text{geom}})$ is the suspension of the geometric monodromy T_{geom} induced by the loop $\gamma = r_1(\alpha) \circ \cdots \circ r_{d-1}(\alpha)$. If γ is given by $\alpha \mapsto (t(\alpha), y_0(\alpha))$ then the loops γ_u defined by $\alpha \mapsto (t(\alpha), uy_0(\alpha))$ for $u \in [c,1]$, $(1 > c > 0)$ are homotopic to γ. Moreover, the image of the loop γ_u is on the torus $\mathbf{T}_u = S_r \times \{y_0 \mid |y_0| = u\rho_\varepsilon\}$ and for $c \le u < 1$ the torus $\mathbf{T}_u$ does not intersect Δ. Now, the fibration induced by τ over $\mathbf{T}_c$ is a pullback of a representative G_i: $B_i \to D_\rho$ of g_i via the map $\xi : \mathbf{T}_c \to D^*_\rho$ given by $(t, y_0) \mapsto -\varepsilon y_0 + t y_0^d$.

Let L and M be oriented loops on $\mathbf{T}_c$ which generate its first homology group, i.e. $L = [s \mapsto (t_0, c\rho_\varepsilon\,\mathrm{e}^{2\pi i s})]$ and $M = [s \mapsto (t_0\,\mathrm{e}^{2\pi i s}, c\rho_\varepsilon)]$. Then we have $[\gamma_c] = -L + (d-1)M$ in $H_1(\mathbf{T}_c, \mathbf{Z})$. Now, first notice that the monodromy induced by M is trivial. To see this, notice that for c sufficiently small and $y_0 \neq 0$ fixed, the winding number of the loop $\alpha \mapsto y_0(-\varepsilon + t_0\,\mathrm{e}^{2\pi i\alpha} y_0^{d-1})$ (with respect to the origin) is zero. Therefore $\xi_*[M]$ is trivial in $H_1(D^*_\rho)$. On the other hand, $\xi_*[L] = 1 \in H_1(D^*_\rho)$. Therefore T_{geom} is the inverse the geometric monodromy of g_i. The corresponding algebraic monodromy acting on $H_n(\bigvee_{d-1} S(F_i))$ is $A_i = c_{d-1}(T_i^{-1})$. Thus $\zeta(\mathcal{Y}_{\varepsilon i} \to S_r) = (\det(\lambda \cdot \mathrm{Id} - A_i))^{(-1)^n}$ and since T_i is a real operator with eigenvalues on the unit circle $\det(\lambda - A_i) = \mathrm{char}_i(\lambda^{d-1})$. Then (7) reads:

$$\zeta^\infty(f_\varepsilon) = \zeta^\infty(f_0) \cdot \prod_{i=1}^{k} \mathrm{char}_i(\lambda^{d-1})^{(-1)^n}. \tag{8}$$

Now, for $s = 0$, f_s is exactly the homogeneous polynomial f_d, considered as a germ $f_d: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$. f_d defines a singularity with one-dimensional critical locus and its zeta function is computed in [21]:

$$\zeta(f_d) = \zeta(f_{\text{gen}}) \cdot \prod_{i=1}^{k} (\lambda^d - 1)^{\mu_i (-1)^{n+1}}, \tag{9}$$

where $\zeta(f_{\text{gen}})$ is the zeta function of a generic homogeneous singularity of degree d, i.e. $\zeta(f_{\text{gen}}) = (1-\lambda)(\text{char}_{\text{gen}}(\lambda))^{(-1)^n}$. Since $\zeta^\infty(f_\varepsilon) = (1-\lambda)\cdot(\text{char}_f^\infty)^{(-1)^n}$, the result follows from (8) and (9). □

4. The algebraic monodromy T_f^∞ via a compactification

(4.1) In the next two sections we will study the structure of the Jordan blocks of T_f^∞ for a $(*)$-polynomial f. For this purpose we will consider a fibration $\pi: \mathcal{X} \to S_r$ which compactifies the fibration of f at infinity. The main result of this section is that the algebraic monodromy T_f^∞ can be completely determined from the algebraic monodromy T of the projective fibration π (and conversely).

One interesting byproduct of this correspondence is that the number of Jordan blocks of T_f^∞ corresponding to eigenvalue one is the $(n-1)$-th Betti number $b_{n-1}(X^\infty)$ if n is even and $b_{n-1}(X^\infty) - 1$ if n is odd. In particular, the unipotent part of the monodromy at infinity depends not only on local data associated to the singularities of X^∞ but also on their position.

In the next section we determine the dth power of T. This gives much information about the unipotent part of T_f^∞ via the correspondence of this section.

(4.2) We introduce some notations. Let $f \in \mathbf{C}[X_1, \dots, X_{n+1}]$ be a $(*)$-polynomial. By (2.6) we can assume that f has the form $f = f_d + x_{n+1}^{d-1}$, where f_d is homogeneous of degree d. Fix r such that $\Sigma_f \subset D_r$. Set:

$$\mathcal{X} = \{([x_0, \dots, x_{n+1}], t) \in \mathbf{P}^{n+1} \times S_r \mid f_d(x) + x_0 x_{n+1}^{d-1} = t x_0^d\}$$

and let $\pi: \mathcal{X} \to S_r$ be the second projection map. Put:

$$\begin{aligned} X^\infty \times S_r &= \{([x], t) \in \mathcal{X} \mid x_0 = 0\}, & \mathcal{X}^0 &= \mathcal{X} - X^\infty \times S_r, \\ X_t &= p^{-1}(t) \ \ (t \in S_r), & X_t^0 &= X_t \cap \mathcal{X}^0 \ \ (t \in S_r). \end{aligned}$$

Fix $t_0 \in S_r$. Then $\pi: (\mathcal{X}, \mathcal{X}^0) \to S_r$ is a locally trivial fibration of pairs of spaces and $\pi_{|\mathcal{X}^0}$ is exactly the fibration of f at infinity. Let $T : H^n(X_{t_0}) \to H^n(X_{t_0})$ be the algebraic monodromy of π. Property (2.2.(v)) implies that there exists a sufficiently large ball B_{R_0} and a representative $T_{\text{geom}} : X_{t_0} \to X_{t_0}$ of the geometric monodromy of π such that $T_{\text{geom}\,|X_{t_0} - B_{R_0}}$ is the identity. Therefore, the diagram in (2.3) can be extended to the following diagram:

$$\begin{tikzcd} H^n(X_{t_0}) \arrow[r, "i^*"] \arrow[d, "T - \mathrm{Id}"'] & H^n(X^0_{t_0}) \arrow[d, "Var"] \\ H^n(X_{t_0}) & H^n_c(X^0_{t_0}) \arrow[l, "i_*"'] \end{tikzcd}$$

where i_* and i^* are the natural maps. Recall that $i^* i_* \mathrm{Var} = k \circ Var = T_f^\infty - \mathrm{Id}$.
(4.3) Since f is a $(*)$-polynomial, X_{t_0} is a smooth hypersurface in $\mathbf{P}^{n+1}$. Let S be the polarization form of $H^n(X_{t_0})$, i.e. $S(\alpha, \beta) = \int_{X_{t_0}} \alpha \wedge \beta$ for $\alpha, \beta \in H^n(X_{t_0})$. Then it is well-known that $H^q(X_{t_0}) = H^q(\mathbf{P}^{n+1})$ if $q \neq n$ and $H^n(X_{t_0})$ decomposes in a direct sum $H^n(\mathbf{P}^{n+1}) \oplus \mathrm{P}^n(X_{t_0})$, orthogonal with respect to S.

Since the hyperplane section at infinity X^∞ has only isolated singularities, the primitive decomposition $H_q(X^\infty) = H_q(\mathbf{P}^n) \oplus \mathrm{P}_q(X^\infty)$ satisfies $\mathrm{P}_q(X^\infty) = 0$ if $q \neq n-1, n$. The numbers $p_q(X^\infty) = \dim \mathrm{P}_q(X^\infty)$ $(q = n-1, n)$, are in general non-zero, and in general, *they depend on the position of the singularities of* X^∞.

On the other hand, the Euler characteristic of X^∞ does not depend on it. One has:

$$p_{n-1}(X^\infty) - p_n(X^\infty) = \frac{d-1}{d}[(d-1)^n - (-1)^n] - \sum_{i=1}^{k} \mu_i.$$

(4.4) Since $X^0_{t_0}$ has the homotopy type of a bouquet of n-spheres (cf. 2.2.(iii)), the exact sequence of cohomology with supports has the following form:

$$0 \to H^n_{X^\infty}(X_{t_0}) \xrightarrow{j_*} H^n(X_{t_0}) \xrightarrow{i^*} H^n(X^0_{t_0}) \to H^{n+1}_{X^\infty}(X_{t_0}) \to H^{n+1}(X_{t_0}) \to 0.$$

The sequence is equivariant with respect to the monodromy action. This action on $H^n_{X^\infty}(X_{t_0})$, $H^{n+1}_{X^\infty}(X_{t_0})$ and $H^{n+1}(X_{t_0})$ is the identity. Recall also the duality isomorphism $H^*_{X^\infty}(X_{t_0}) \simeq H_{2n-*}(X^\infty)$. In the sequel we will identify $H^n_{X^\infty}(X_{t_0})$ with its image in $H^n(X_{t_0})$.

(4.5) LEMMA. *There is an equivariant direct sum decomposition*

$$H^n(X_{t_0}) \simeq H^n_{X^\infty}(X_{t_0}) \oplus (H^n_{X^\infty}(X_{t_0}))^\perp,$$

which is orthogonal with respect to the form S. *In particular, the monodromy* T *decomposes as* $T = Id \oplus T'$.

Proof. Consider the orthogonal decomposition $H^n_{X^\infty}(X_{t_0}) = H^n(\mathbf{P}^{n+1}) \oplus \mathrm{P}^n_{X^\infty}(X_{t_0})$ and the corresponding decomposition of j_*:

$$j_* = \mathrm{Id} \oplus j' \colon H^n(\mathbf{P}^{n+1}) \oplus \mathrm{P}^n_{X^\infty}(X_{t_0}) \to H^n(\mathbf{P}^{n+1}) \oplus \mathrm{P}^n(X_{t_0}).$$

Then $\mathrm{P}^n_{X^\infty}(X_{t_0})$ has a natural polarized Hodge structure and the inclusion j' is a morphism of polarized Hodge structures (where on $\mathrm{P}^n(X_{t_0})$ the polarization is induced by S). It follows that the restriction of S to $\mathrm{P}^n_{X^\infty}(X_{t_0})$ is non-degenerate hence the result follows. □

We have a perfect pairing $Q : H^n_c(X^0_{t_0}) \otimes H^n(X^0_{t_0}) \to \mathbf{C}$ given by $\alpha \otimes \beta \mapsto \int_{X^0_{t_0}} \alpha \wedge \beta$ and the morphisms i_* and i^* are adjoint with respect to Q. Thus $\mathrm{im}(i_*) \simeq (\ker(i^*))^\perp = (H^n_{X^\infty}(X_{t_0}))^\perp$. Since Q is compatible with the monodromy action this isomorphism is equivariant.

The following theorem describes the monodromy T^∞_f in terms of T and the Betti numbers of X^∞.

(4.6) THEOREM.

(a) *For any* $\lambda \neq 1$, $(T^\infty_f)_\lambda = T_\lambda$.
(b) *Assume* $\lambda = 1$. *Then:*
   (i) $\#_1(T^\infty_f)_1 = b_n(X^\infty) + p_{n-1}(X^\infty) - \#T_1$,
   (ii) $\#_2(T^\infty_f)_1 = \#_1 T_1 - b_n(X^\infty)$,
   (iii) $\#_{l+1}(T^\infty_f)_1 = \#_l T_1$ *for* $l \geqslant 2$.

In particular $\#(T^\infty_f)_1 = \dim(\ker T^\infty_f - Id) = p_{n-1}(X^\infty)$.

Proof. Part (a) follows from (4.4). Let V_1 denote the composed map

$$H^n(X^0_{t_0})_1 \xrightarrow{\mathrm{Var}_1} H^n_c(X^0_{t_0})_1 \xrightarrow{(i_*)_1} \mathrm{im}(i_*)_1.$$

By (2.4) we have that V_1 is onto. By (4.2), (4.4) and (4.5) one has the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{im}(i_*)_1 \arrow[r] \arrow[d, "T'_1 - \mathrm{Id}"'] & H^n(X^0_{t_0})_1 \arrow[r] \arrow[d, "(T^\infty_f)_1 - \mathrm{Id}"] \arrow[dl, "V_1"'] & \mathrm{P}_{n-1}(X^\infty) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{im}(i_*)_1 \arrow[r] & H^n(X^0_{t_0})_1 \arrow[r] & \mathrm{P}_{n-1}(X^\infty) \arrow[r] & 0
\end{tikzcd}$$

Now part (b) follows from this diagram and (4.5). □

(4.7) COROLLARY. *Assume* 1 *is not an eigenvalue of any of the local monodromies* T_i $(i = 1, \ldots, k)$, *(cf. 3.1). Then* $\#_l(T^\infty_f)_1 = 0$ *for* $l > 1$ *(i.e.* $(T^\infty_f)_1 = \mathrm{Id}$*).*

Proof. From (3.3) and from the Euler–characteristic formula (4.3) we deduce that $p_{n-1}(X^\infty) - p_n(X^\infty) = \dim H^n(X^0_{t_0})_1$. Then the exact sequence (4.4) gives that $\mathrm{P}^n(X_{t_0})_1 = 0$, or $(i^*)_1 = (i_*)_1 = 0$. Hence $(T_f^\infty)_1 = \mathrm{Id}$ from the last diagram. □

(4.8) COROLLARY. *Let $\langle , \rangle$ be the intersection form on $H_n(X_{t_0}^0)$ (or, by Poincare duality, the form $(\alpha, \beta) \mapsto \int \alpha \wedge \beta$ on $H_c^n(X_{t_0}^0)$). Then one has:*

$$\begin{aligned}\dim H_n(\partial X_{t_0}^0) &= \operatorname{rank}\langle , \rangle = \operatorname{rank}(T_{f,c}^\infty - Id)\\ &= \operatorname{rank}(T_f^\infty - Id) = \mu^\infty - p_{n-1}(X^\infty).\end{aligned}$$

In particular, the intersection form depends on the position of the singular points of X^∞.

(4.9) REMARK. Notice that, similarly as in the local case of isolated hypersurface singularities, the intersection form $\langle , \rangle$ and the monodromy T_f^∞ can be determined from the variation map Var. To see this, set $H := H_n(X_{t_0}^0, \mathbf{R})$, denote by $b : H \to H^*$ the map given by $b(x) = \langle x, \cdot \rangle$ and by $T_{f,c}^\infty$ the monodromy action on H. Then the variation map is a map Var: $H^* \to H$ and (after a canonical identification of H^{**} and H) one has:

$$\begin{aligned}T_{f,c}^\infty &= (-1)^{n+1}\mathrm{Var} \circ (\mathrm{Var}^*)^{-1}\\ T_f^\infty &= ((T_{f,c}^\infty)^*)^{-1}; \text{ and}\\ b &= -(\mathrm{Var})^{-1} - (-1)^n \circ (\mathrm{Var}^*)^{-1}.\end{aligned}$$

Actually, all these invariants are defined over the integers, Var is unimodular and it is equivalent to the Seifert form of the Milnor fibration at infinity (cf. 2.2.(v)).

5. The dth power of the monodromy

(5.1) In this section we determine the dth power of T (T being the transformation introduced in Section 4). Since the Jordan block structure of the unipotent parts of T and T^d are the same, this will provide much information about the unipotent part of T_f^∞ via (4.6).

In the computation of T^d there are two (rather different) cases. If $\lambda \neq 1$ then $(T^d)_\lambda$ is completely local: in (5.3) we describe it in terms of the local transformations T_i. The result (and also its proof) is topological. On the other hand, if $\lambda = 1$, then $(T^d)_1$ is described Hodge theoretically (cf. 5.5) in terms of local data and the weight filtration (of the mixed Hodge structure) of the d-fold cyclic covering of H^∞ branched along X_f^∞.

For this purpose we will introduce a map π': $\mathcal{X}' \to D'$, where D' is a disk in the complex plane, which induces a fibration over the punctured disk with algebraic monodromy T^{-d}, and such that the central fiber has only isolated singularities. The map π' provides smoothings of these singularities and we will determine the relation between the monodromies of these smoothings and the transformations T_i.

(5.2) We start with some preliminary constructions.

As in the previous section, we can assume that $f = f_d + x_{n+1}^{d-1}$. Set:

$$\mathcal{X}_\infty = \{([x], t) \in \mathbf{P}^{n+1} \times D \mid t \cdot (f_d(x_1, \ldots, x_{n+1}) + x_0 x_{n+1}^{d-1}) = x_0^d\},$$

where D denotes a disk of small radius in the complex plane and let $\pi_\infty\colon \mathcal{X}_\infty \to D$ denote the map induced by the projection onto D. The map π_∞ induces a locally trivial fibration over $D - \{0\}$ with fiber $\pi_\infty^{-1}(t) = X_{1/t}$, and the algebraic monodromy acting on $H^n(X_{1/t})$ is T^{-1}. Notice that the singular locus of $\mathcal{X}_\infty$ is $X^\infty \times \{0\}$.

Consider now the pullback of the map π_∞ over the map $\delta\colon D' \to D$ defined by $\delta(t) = t^d$, D' being again a disk of small radius. Then $\mathcal{X}_\infty \times_\delta D'$ can be identified with

$$\{([x], t) \in \mathbf{P}^{n+1} \times D' \mid t^d \cdot (f_d(x_1, \ldots, x_{n+1}) + x_0 x_{n+1}^{d-1}) = x_0^d\}.$$

Moreover, the pullback π'_∞ of π_∞ is induced by the second projection. Obviously, $\mathcal{X}_\infty \times_\delta D'$ over $D'^* := D' - \{0\}$ is a fiber bundle with the same fiber as π_∞, and with algebraic monodromy T^{-d}.

Consider now the space:

$$\mathcal{X}' = \{([x], t) \in \mathbf{P}^{n+1} \times D' \mid f_d(x_1, \ldots, x_{n+1}) + t \cdot x_0 x_{n+1}^{d-1} = x_0^d\}$$

and the map $\pi'\colon \mathcal{X}' \to D'$ induced by the second projection. Then $\theta\colon \mathcal{X}' \to \mathcal{X}_\infty \times_\delta D'$, given by $\theta([x], t) \mapsto ([tx_0 : x_1 : \ldots : x_{n+1}], t)$ is the normalization map, and identifies the fiber bundles over D'^* induced by π'_∞ and π'. Denote $X'_s = (\pi')^{-1}(t)$ for $t \in D'$.

Therefore, we have constructed a map $\pi'\colon \mathcal{X}' \to D'$ which is a fiber bundle over D'^*, with the same fiber as π_∞ and with algebraic mondromy T^{-d}. Moreover, $\mathrm{Sing}(\mathcal{X}') = \mathrm{Sing}(X'_0)$ is the *finite set* $\mathrm{Sing}(X^\infty) \times \{0\}$. The singularities of X'_0 are the d-th suspensions of the singularities of X^∞ and the map π' provides their smoothings.

Set $\mathrm{Sing}(X'_0) = \{p'_1, \ldots, p'_k\}, p'_i = [0 : p_i]$ and let F'_i, T'_i denote the Milnor fiber and the algebraic monodromy (acting on $H^n(F'_i)$) of the smoothing of (X'_0, p'_i) given by π'.

Now we will formulate the first part of the main result of this section.

(5.3) THEOREM. *Let $H^n(F)_\xi$ be the generalized eigenspace with respect to the eigenvalue ξ of T_f^∞. Then, for any root of unity $\lambda \neq 1$, we can identify $\oplus_{\xi^d=\lambda} H^n(F)_\xi$ with the generalized λ-eigenspace of $\oplus_{i=1}^k (H^{n-1}(F_i)^{\oplus(d-1)})$ provided by the operator $\oplus_{i=1}^k c_{d-1}(T^{-d})$. By this identification:*

$$[(T_f^\infty)^d]_\lambda = \oplus_{i=1}^k [c_{d-1}(T_i^{-d})]_\lambda.$$

REMARK. Recall that for $\lambda \neq 1$ one has: $[(T_f^{\infty})^d]_\lambda = [T^d]_\lambda$ (cf. 4.6.a).

Proof of (5.3): From the Leray spectral sequence associated to the specialization map restricted to X'_t (for $t \neq 0$ and fixed) we get a sequence of vanishing cycles, equivariant with respect to the monodromy:

$$0 \to H^n(X'_0) \to H^n(X'_t) \to \bigoplus_{i=1}^{k} H^n(F'_i) \to \mathrm{P}^{n+1}(X'_0) \to 0 \tag{10}$$

where $\mathrm{P}^{n+1}(X'_0) \simeq \ker[H^{n+1}(X'_0) \to H^{n+1}(X'_t)]$ denotes the primitive cohomology. In particular, for $\lambda \neq 1$ we have

$$H^n(X'_t)_\lambda \simeq \bigoplus_{i=1}^{k} H^n(F'_i)_\lambda.$$

We want to relate now the action of T'_i on $H^n(F'_i)$ with that of T_i on $H^{n-1}(F_i)$. This is now a local problem. Fix $p_i \in \mathrm{Sing}(X^\infty)$ and assume that $p_i = [0 : \ldots : 0 : 1]$. We have to study the (local) smoothing:

$$\mathcal{Y} = \{(y_0, y, t) \mid g_i(y) + ty_0 = y_0^d\} \to D$$

given by $(y_0, y, t) \mapsto t$. In the sequel D denotes a sufficiently small disc. Consider the map $\varphi: \mathcal{Y} \to D \times D$ given by $\varphi(y_0, y, t) = (t, y_0)$. Then it is not difficult to verify that φ defines an isolated complete intersection singularity. In the sequel $\varphi: \mathcal{Y} \to D \times D$ will denote a 'good representative' of this icis in the sense of [10]. The discriminant of φ is given by $\Delta = \{y_0^d = ty_0\} \subset D \times D$. Moreover, the following properties hold:

(i) Over the complement of Δ, φ is a fiber bundle with fiber F_i,
(ii) $\varphi^{-1}((t, y_0))$ is contractible for $(t, y_0) \in \Delta$,
(iii) the monodromy of the fiber bundle (over $D \times D - \Delta$) is abelian, the monodromy induced by a small oriented circle around Δ (constructed in any transversal slice at a smooth point of Δ) is T_i,
(iv) $pr_1 \circ \varphi$ is the local smoothing of X'_0 given by π', in particular F'_i can be identified with $\varphi^{-1}(\{t = t_0\})$ for $t_0 \neq 0$ sufficiently small.

Fix $t_0 > 0$ sufficiently small. Then $\{(t, y_0) \in D \times D \mid t = t_0\}$ intersects Δ at the points $q_0 = (t_0, 0)$ and $q_k = (e^{2\pi ik/d-1} \cdot \sqrt[d-1]{t_0}, t_0)$, where $1 \leqslant k \leqslant d-1$. Let I_k be the real segment $\overline{q_0 q_k}$ (k=1,...,d-1) and $I = \bigcup_{k=1}^{d-1} I_k \subset \{t = t_0\} \subset D \times D$. Then I is a strong deformation retract of $\{t = t_0\}$ and this retraction can be lifted via φ. In particular, F'_i has the homotopy type of $\varphi^{-1}(I)$. Let r_k be the middle point of the segment I_k. Since $\varphi^{-1}(q_k)$ is contractible for any $k = 0, \ldots, d-1$ and $\varphi^{-1}(r_k) \simeq F_i$, the space $\varphi^{-1}(I_k)$ has the homotopy type of the suspension $S(F_i)$ of F_i. It follows then that $F'_i \simeq \bigvee_{d-1} S(F_i)$.

In order to compute the monodromy action on $\bigvee_{d-1} S(F_i)$, notice first that the reduced homology of $\bigvee_{d-1} S(F_i)$ is generated by the suspension of the cycles of the spaces $\varphi^{-1}(r_k)$. Let $t = t_0 \, e^{2\pi i\alpha}$, $\alpha \in [0,1]$. Then the points $r_1, \ldots, r_{d-1}$ move along the paths

$$r_k(\alpha) = (t_0 \, e^{2\pi i\alpha}, e^{2\pi i(\alpha+k)/(d-1)} \cdot \sqrt[d-1]{t_0}) \, , \quad \alpha \in [0,1] \, , \quad 1 \leq k \leq d-1 .$$

We identify the fibers $\varphi^{-1}(r_k)$ $(1 \leqslant k \leqslant d-1)$ via the paths $r_k(\alpha)$ $(1 \leqslant k \leqslant d-2)$. Then, with the notations introduced in (1.2) the monodromy T_i' is of the form $c_{d-1}(M)$, where M is induced by the geometric monodromy $M_{\text{geom}}: \varphi^{-1}(r_1) \to \varphi^{-1}(r_1)$ induced by the loop $\gamma = r_{d-1}(\alpha) \circ \cdots \circ r_1(\alpha)$. Now the linking number of γ with $\{y_0 = 0\}$ is one, and with $\{y_0^{d-1} = t\}$ is $d-1$. Thus $M = T_i^d$ and one has:

$$(T^{-d})_\lambda = \bigoplus_{i=1}^{k} [c_{d-1}(T_i^d)]_\lambda .$$

By (4.6), $(T^{-d})_\lambda = (T_f^{-d})_\lambda$. Also, $c_{d-1}(T_i)^{-1}$ is conjugate to $c_{d-1}(T_i^{-1})$. Thus taking inverses in the above equality the theorem follows. □

(5.4) EXAMPLE. The theorem above determines, in terms of the local monodromies, the Jordan blocks of T_f^∞ corresponding to eigenvalues ξ, with $\xi^d \neq 1$. In some cases all the eigenvalues ξ satisfy $\xi^d \neq 1$, this fact can be verified by the computation of the characteristic polynomial of T_f^∞ (cf. 3.3). In these cases, the above theorem describes completely T_f^∞.

For example, if $f_d \in \mathbf{C}[X, Y, Z]$ is a product of d linear forms defining d lines in $\mathbf{P}^2$ intersecting at one point, then the monodromy at infinity of any $(*)$-polynomial with highest degree form f_d satisfies the above condition. It turns out that T_f^∞ is of finite order, in particular it can be completely determined from (3.3).

As we will see (in the next theorem and in some of the examples of Section 6), it is not possible to obtain a similar description of the stucture of the Jordan blocks associated to the dth roots of unity, because this information is not purely local anymore.

(5.5) THEOREM. *Consider the morphism of mixed Hodge structures:*

$$N := \log((T^{-d})_1): H^n(X_t')_1 \to H^n(X_t')_1(-1).$$

Recall that X_0' is the d-fold cyclic covering of H^∞ branched along X^∞. Then there is an exact sequence of mixed Hodge structures:

$$0 \to H^n(X_0') \to \ker N \to \bigoplus_{i=1}^{k} H^{n+1}_{\{p_i'\}}(\mathcal{X}') \to 0.$$

In particular, for any l one has:

$$\#_l(T^{-d})_1 = \dim \mathrm{Gr}^W_{n-l+1} H^n(X_0') + \sum_{i=1}^{k} \#_l(T_i)_1,$$

and

$$\#(T^{-d})_1 = \dim H^n(X_0') + \sum_{i=1}^{k} \dim\ker(T_i - \mathrm{Id}).$$

Proof. The exact sequence is given by Theorem 2 in the Appendix, since sp^* is injective (cf. the exact sequence (10) in the proof of (5.3)). Since the weight filtration on $H^n(X_t')_1$ is the monodromy weight filtration of the nilpotent endomorphism N one has: $\#_l(T^{-d})_1 = \dim \mathrm{Gr}^W_{n-l+1}\ker N$.

The local equation defining $\mathcal{X}'$ in a neighborhood of p_i' (in $\mathbf{C}^{n+2}$) is $\hat{g}_i := g_i + x_0 t - x_0^d = 0$. Let $\hat{F}_i, \hat{T}_i$ denote the corresponding Milnor fiber and monodromy acting on $H^{n+1}(\hat{F}_i)$. It follows from the Sebastiani-Thom formula that there is an isomorphism $H^{n+1}(\hat{F}_i) \simeq H^{n-1}(F_i)$ compatible with the actions of $\hat{T}_i$, respectively of T_i.

From the following exact sequence (cf. [22]):

$$0 \to H^{n+1}_{\{p_i'\}}(\mathcal{X}') \to H^{n+1}_c(\hat{F}_i)_1 \xrightarrow{j} H^{n+1}(\hat{F}_i)_1 \to H^{n+2}_{\{p_i'\}}(\mathcal{X}') \to 0$$

one has the identifications: $\dim \mathrm{Gr}^W_{n-l+1} H^{n+1}_{\{p_i'\}}(\mathcal{X}') = \dim \mathrm{Gr}^W_{n-l+1}\ker j = \dim \mathrm{Gr}^W_{n-l+1}\ker N_{\hat{g}_i} = \#_l(\hat{T}_i)_1 = \#_l(T_i)_1$. □

Notice that Theorems (5.3), (5.5) and (4.6) give an almost complete description of the nilpotent (or unipotent) part of T_f^∞.

The next criterion shows that even in the case $\lambda = 1$ the transformation $(T^{-d})_1$ can be local if the local transformations T_i satisfy some restrictions.

(5.6) PROPOSITION. *With the notations previously introduced, assume that one has:*

(a) $\#_s(T_i)_1 = 0$ *for* $s > 1$ *and* $i = 1, \dots, k$.
(b) T_i *has no dth root of unity different from 1 as eigenvalue for* $i = 1, \dots, k$.

Then $(T^d)_1$ *(acting on* $H^n(X_t')_1$*) is the identity. In particular:*

(a′) $\#_k(T_f^\infty)_1 = 0$ *for* $k > 2$.
(b′) $\#_k(T_f^\infty)_\lambda = 0$ *for* $\lambda^d = 1$, $\lambda \neq 1$ *and* $k > 1$.

Actually, in this case the monodromy at infinity T_f^∞ *can be completely determined from the local monodromies* T_i *and the Betti numbers of* X^∞. *(See (5.3) and the last relation* $\#_2(T_f^\infty)_1 = p_{n-1}(X^\infty)$ *of (4.6).)*

Proof. From the exact sequence (10) one has that

$$\dim \frac{H^n(X_t')_1}{H^n(X_0')} = \sum_{i=1}^{k} \dim H^n(F_i')_1 - \dim \mathrm{P}^{n+1}(X_0'). \tag{11}$$

The relation $T_i' = c_{d-1}(T_i^d)$ (see the proof of (5.3)), and (b) give that $\dim H^n(F_i')_1 = \dim H^{n-1}(F_i)_1$.

On the other hand, from (5.5) or from Theorem 2 in the Appendix one has:

$$\dim \frac{\ker[T^{-d} - \mathrm{Id}: H^n(X_t')_1 \to H^n(X_t')_1]}{H^n(X_0')} = \sum_{i=1}^{k} \dim H^{n+1}_{\{p_i'\}}(\mathcal{X}'). \tag{12}$$

Then from (a) above and from the Sebastiani–Thom formula one has

$$\begin{aligned} \dim H^{n+1}_{\{p_i'\}}(\mathcal{X}') &= \dim(\ker[\hat{T}_i - \mathrm{Id}: H^{n+1}(\hat{F}_i) \to H^{n+1}(\hat{F}_i)]) \\ &= \dim H^{n-1}(F_i)_1. \end{aligned}$$

Now from (11) and (12) we get that $(T^d)_1 = \mathrm{Id}$ on $H^n(X_t')_1$, therefore (a′) and (b′) follow from (4.6).

Notice also that $\mathrm{P}^{n+1}(X_0') = 0$. □

(5.7) REMARK. The vanishing of $\mathrm{P}^{n+1}(X_0') = 0$ under assumptions (a) and (b) above means that the only root of the Alexander polynomial $\Delta^1_{X^\infty}$ of the hypersurface X^∞ is 1 (cf. [9], [5, Chapter 6, 3.24]), and then it is $\Delta^1_{X^\infty}(t) = (t-1)^\delta$, $\delta = \dim \mathrm{P}^n(X^\infty)$. We recall that $\Delta^1_{X^\infty}$ agrees with the characteristic polynomial of the monodromy acting on $H^{n-1}(F_d)$, where F_d is the Milnor fiber of the map germ $f_d: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$, cf. [5].

The above proposition has the following consequence, which is significant on its own, and can be formulated independently of the results of the paper:

(5.8) COROLLARY. Let $X^\infty \subset \mathbf{P}^n$ be a hypersurface with isolated singularities, and of degree d. Assume that the local monodromies T_i of these singular points satisfy:

(a) $\#_s(T_i)_1 = 0$ for $s > 1$ and $i = 1, \ldots, k$; and
(b) T_i has no d-th root of unity different from 1 as eigenvalue for $i = 1, \ldots, k$.

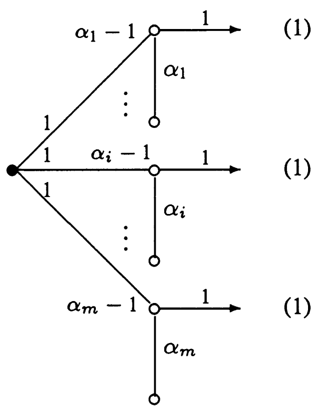
Then the d-fold covering Y of $\mathbf{P}^n$ branched along X^∞ has the following properties:

(a′) the primitive cohomology $P^{n+1}(Y) = 0$, and
(b′) the mixed Hodge strucure on $H^n(Y)$ is pure of weight n.

6. Examples

(6.1) *The case* $n = 1$.

Write f_d in the form $f_d = \prod_{i=1}^{m} l^{\alpha_i}$, where l_i are different linear forms. If f is a $(*)$-polynomial then by (2.6) its monodromy at infinity is completely determined by the integers $\alpha_i \in \mathbf{N}^*$. Moreover, f is "good" in the sense of Neumann (cf. [16]) and it has a RPI splice diagram which describes completely the link at infinity of an arbitrary fiber $f^{-1}(t)$ (defined as $f^{-1}(t) \cap S_R$, $R \gg 0$) and the Waldhausen (splice) decomposition of the link complement $S_R - f^{-1}(t)$. This diagram provides the whole set of invariants of the Milnor fibration at infinity (cf. [7], [16]). By the algorithm described in ([16, Section 4]) we deduce easily that the splice diagram of f at infinity is:



(the root vertex is marked '•').

It follows (from (3.3) or [7, 11.3]) that the characteristic polynomial of T_f^∞ is:

$$\operatorname{char}_f^\infty(\lambda) = (\lambda - 1) \cdot (\lambda^d - 1)^{m-2} \cdot \prod_{i=1}^{m} \frac{\lambda^{(d-1)\alpha_i} - 1}{\lambda^{d-1} - 1}.$$

Set $q = d(d-1)\prod_{i=1}^{m} \alpha_i$. Then $\lambda^q = 1$ for any root λ of the characteristic polynomial. Then by [7, Theorem 14.1] one has

$$\det((\lambda \cdot \mathrm{Id} - T_f^\infty) \mid_{\mathrm{Im}((T_f^\infty)^q - Id)}) = \frac{\prod_{i=1}^{m} (\lambda^{d_i} - 1)}{(\lambda - 1)^{m-1} (\lambda^\alpha - 1)}$$

where $d_i = \gcd(d, \alpha_i)$ and $\alpha = \gcd(\alpha_1, \dots, \alpha_m)$. It follows that:

- $\mu^\infty = d^2 - 3d + 1 + m$,
- $\#_2 T_f^\infty = \sum_{i=1}^{m} d_i - \alpha - m + 1$,
- $(T_f^\infty)_1 = \mathrm{Id}$,
- T_f^∞ is of finite order if and only if $d_i = 1$ for $i = 1, \dots, m$.

(6.2) REMARK. In the above case, the link of f at infinity can be realized as the link of an isolated curve singularity g: $(\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ if and only if either

$m = 1$ and $\alpha_1 = d$ or $m = d$ and $\alpha_1 = \cdots = \alpha_d = 1$. In these cases f can be assumed to be $f = x_1^d + x_2^{d-1}$ or $f = x_1^d + x_2^d$ respectively. Exactly in these cases the minimal RPI splice diagrams do not have edges. In all other cases the minimal splice diagram satisfies 'reverse Puiseaux inequalities' (cf. [16]), i.e. all 'edge determinants' are negative (recall that the 'edge determinants' associated with the germ of a plane curve singularity are positive).

(6.3) *Lines in general position.*

Let $l_1, \ldots, l_d \in \mathbf{C}[X_1, X_2, X_3]$ be distinct linear forms defining a set of lines in $\mathbf{P}^2$ such that no more than two lines meet at a point. Let f be a $(*)$-polynomial with highest degree form $l_1 \cdot \cdots \cdot l_d$. It follows from Theorem (5.3) and Proposition (5.6) that T_f^∞ can have Jordan blocks of size bigger than one only for eigenvalue 1. The number of the blocks associated with eigenvalue one can be computed from Theorem (4.6), it is $(d-1)(d-2)/2$. The generalized eigenspace of $(T_f^\infty)_1$ has dimension $(d-1)(d-2)$ (from 3.3). Therefore, there are no Jordan blocks of size one associated to the eigenvalue 1. The monodromy at infinity can now be completely determined using Theorem (3.3).

(6.4) *Quintic hypersurfaces in* $\mathbf{P}^4$.

As a first example of the influence of the position of the singularities of X^∞ on the block structure of the monodromy at infinity, consider the hypersurfaces Y, Z described in ([26, pp. 50 and 55]). Both are quintic hypersurfaces in $\mathbf{P}^4$ with 108 nodes but $b_4(Y) = 19$ while $b_4(Z) = 20$. Let f_Y (resp. f_Z) be $(*)$- polynomials of degree 5 which have as highest degree form a polynomial defining Y (resp. Z) in $\mathbf{P}^4$. Set $T_Y = T_{f_Y}^\infty$, $T_Z = T_{f_Z}^\infty$ As in the previous example, T_Y and T_Z can have Jordan blocks of size bigger than one only for eigenvalue 1. The number of the blocks of size two can be computed using Theorem (4.6) and it turns out to be 100 for Y and 99 for Z ($\dim H^4(X_{t_0}^0)_1 = 204$ in both cases). Combining this with Theorem (3.3) we can completely describe T_Y and T_Z.

(6.5) *Zariski's sextics.*

Let $f_6 \in \mathbf{C}[X_1, X_2, X_3]$ be a form defining a plane sextic in $\mathbf{P}^2$ with six cusps and no other singularities. Let f be any $(*)$-polynomial with highest degree form f_6. Then from (4.7) and (5.3) it follows that T_f^∞ has no Jordan blocks of size bigger than one associated neither to the eigenvalue 1 nor to eigenvalues λ with $\lambda^6 \neq 1$. Also, from the sequence (10), the computation of the action of monodromy on $\oplus_{i=1}^k H^n(F_i')$ in the proof of (5.3) and from (4.6.a), it follows that T_f^∞ can have Jordan blocks of size at most two. The number of them depends on the position of the cusps, more precisely:

- $\#_2 T_f^\infty = 10$ if the six cusps are on a conic,

- $\#_2 T_f^\infty = 12$ if they are not.

Because from the computation of the characteristic polynomial of T_f^∞ in Section 2 and the sequence in (4.4) one gets that

$$\dim H^2(X_t')_1 = \sum_{\lambda^6=1} \dim H^2(X_t)_\lambda = 58,$$

(where X_t' is defined in the proof of (5.3)). On the other hand, from Theorem (5.5) one gets that:

$$\#(T^{-6})_1 = \dim H^2(X_0') = 46 + \delta,$$

where $\delta = \dim H^3(X_0')$, X_0' being the hypersurface in $\mathbf{P}^3$ defined by $x_0^6 = f_6$, i.e., the 6-fold cyclic covering of $\mathbf{P}^2$ branched along the curve X^∞. The possible values of δ are known to be 2 if the six cusps are on a conic or 0 if they are not (cf. [27, VIII, Sect. 3]), and then the result follows.

APPENDIX: On the local invariant cycle theorem
by R. García López and J.H.M. Steenbrink

In this note all cohomology groups will be assumed to have coefficients in the field $\mathbf{Q}$ of rational numbers. We prove the following two theorems:

THEOREM 1. *Let X be a complex analytic space which can be embedded in a projective variety as an open analytic subset. Let $\pi: X \to D$ be a flat projective holomorphic map onto the unit disk D in the complex plane. Let Z be the singular locus of X, set $Y = \pi^{-1}(0)$ and assume that $Z \subset Y$. Let X_t be the generic fiber of π. Let $k \in \mathbf{N}$ and let $T \in \mathrm{Aut}(H^k(X_t))$ be the monodromy transformation of π around the critical value 0. Then the sequence*

$$H^k(X - Z) \to H^k(X_t) \xrightarrow{T-\mathrm{Id}} H^k(X_t)$$

is exact.

REMARKS. 1. The first map in the sequence above is the restriction map.

2. If $Z = \emptyset$, the theorem is due to Katz in the setting of l-adic cohomology and to Clemens and Schmid in the Kähler case ([3]).

3. The hypothesis $Z \subset \pi^{-1}(0)$ is equivalent to the generic fiber of π being smooth.

Proof. After possibly shrinking D, we may assume that the restriction of π over the punctured disk $D - \{0\}$ is a C^∞- fiber bundle and that the inclusion $Y \hookrightarrow X$ is a homotopy equivalence. Let then $\tilde{X}$ be the limit fiber of π, defined as $\tilde{X} = X \times_D \mathbf{H}$, where $\mathbf{H}$ is the universal covering space of $D - \{0\}$. We recall that X_t and $\tilde{X}$ are of the same homotopy type. In the sequence

$$H^k(X - Z) \xrightarrow{\alpha} H^k(X - Y) \xrightarrow{\beta} H^k(\tilde{X})$$

one has $\mathrm{Im}(\beta) = \mathrm{Ker}(T - \mathrm{Id})$ by the Wang sequence. The terms in this sequence carry mixed Hodge structures (MHS) such that α and β become morphisms of MHS. We use Saito's formalism of mixed Hodge modules ([18]).

- For $H^k(\tilde{X})$ one has the limit MHS ([20], [23]) given by $H^k(\tilde{X}) \simeq \mathbf{H}^k(Y, \Psi_f \mathbf{Q}_X^H)$.
- Let $C \subset Y$ be any closed analytic subset, let $i: Y \hookrightarrow X$ and $j: X - C \hookrightarrow X$ be the inclusion maps. Then

  $$H^k(X - C) \simeq \mathbf{H}^k(Y, i^* Rj_* j^* \mathbf{Q}_X^H)$$

  gives $H^k(X - C)$ a MHS.

By [20], $\mathrm{Ker}(T - \mathrm{Id})$ has weight $\leq k$. Hence it suffices to show that $W_k H^k(X - Y) = \alpha(W_k H^k(X - Z))$, where $W_\bullet$ denotes the corresponding weight filtration. One has the exact sequence of MHS

$$H^k(X - Z) \to H^k(X - Y) \to H^{k+1}(X - Z, X - Y).$$

Fix a projective variety W containing X as an open analytic subset. Without loss of generality we can assume that $W - Z$ is smooth. By excision we have an isomorphism of MHS $H^{k+1}(W - Z, W - Y) \simeq H^{k+1}(X - Z, X - Y)$. We also have the exact sequence of MHS

$$H^k(W - Z) \to H^k(W - Y) \to H^{k+1}(W - Z, W - Y) \to H^{k+1}(W - Z).$$

Now $W_k H^{k+1}(W - Z) = 0$ as $W - Z$ is smooth, moreover $W_k H^k(W - Z) = \mathrm{Im}(H^k(W) \to H^k(W - Z))$ and similarly for $W_k H^k(W - Y)$, so $W_k H^k(W - Z) \to W_k H^k(W - Y)$ is surjective. We conclude that $W_k H^{k+1}(W - Z, W - Y) = 0$. Hence $\alpha : W_k H^k(X - Z) \to W_k H^k(X - Y)$ is surjective. □

REMARK. M. Saito has informed us that the theorem above follows also from the results in [19]. Actually, if $IH^*(X)$ denotes the intersection cohomology of X then, with the notations above one has a factorization

$$IH^k(X) \to H^k(X - Z) \to H^k(X_t)$$

and Theorem 1 follows then from [19, (3.8)].

If the central fiber has only isolated complete intersection singularities (icis) then we have:

THEOREM 2. *In addition to the hypothesis of Theorem 1 and with the same notations, assume that* $Y = \pi^{-1}(0)$ *has only icis and set* $\dim(X) = n + 1$. *Then there is an isomorphism:*

$$\frac{\ker[T - Id : H^n(X_t) \to H^n(X_t)]}{\mathrm{im}[sp^* : H^n(Y) \to H^n(X_t)]} \simeq H_Z^{n+1}(X),$$

where sp^* *denotes the morphism induced in cohomology by the specialization map.*

REMARKS. (1) The isomorphism above is also an isomorphism of mixed Hodge structures.

(2) In the applications in Section 5–6 of the paper above, X is a hypersurface with isolated singularities. Given $p \in Z$, let $g_p \colon (\mathbf{C}^{n+2}, 0) \to (\mathbf{C}, 0)$ be a map germ defining the germ (X, p) and let F_p, T_p be the corresponding Milnor fiber and local monodromy acting on $H^{n+1}(F_p)$. Then we recall that there is an isomorphism:

$$H^{n+1}_{\{p\}}(X) \simeq \operatorname{coker}[T_p - \mathrm{Id} \colon H^{n+1}(F_p) \to H^{n+1}(F_p)].$$

Proof. We claim first that there is an isomorphism $W_nH^n(X-Z) \simeq W_nH^n(X-Y)$. One can prove as in the proof of Theorem 1 that $W_nH^{n+1}(X-Z, X-Y) = 0$, so from the exact sequence of the pair $(X - Z, X - Y)$ it follows that in order to prove the claim it is enough to show that the map $H^{n-1}(X - Y) \to H^n(X - Z, X - Y)$ is surjective. Since the singularities of Y are icis, it follows from the long exact sequence of vanishing cycles that the monodromy acts as the identity on $H^k(\tilde{X})$ for $k \neq n$. Assume that $n \geqslant 2$. Then the map above fits in a commutative diagram with exact row:

$$\begin{tikzcd}
 & H^{n-2}(\tilde{X})(-1) \arrow[d] \arrow[dr, "\gamma"] & \\
H^{n-1}(X - Z) \arrow[r] & H^{n-1}(X - Y) \arrow[r] & H^n(X - Z, X - Y)
\end{tikzcd}$$

and the MHS of $H^{n-2}(\tilde{X})(-1)$ is pure of weight n. Since the singularities of the total space X are also icis, we have that $H^{n-1}(X - Z) \simeq H^{n-1}(X) \simeq H^{n-1}(Y)$ and since Y is complete the weights of $H^{n-1}(Y)$ are $\leq n - 1$. It follows then that the map γ above is injective. On the other hand, one has isomorphisms:

$$\begin{aligned} H^n(X - Z, X - Y) &\simeq H^{n-2}(Y - Z)(-1) \\ &\simeq H^{n-2}(Y)(-1) \simeq H^{n-2}(\tilde{X})(-1). \end{aligned}$$

The first is a Thom isomorphism, the second follows from the fact that the singularities of Y are icis (so $H^{n-2}_Z(Y) = H^{n-1}_Z(Y) = 0$) and the third is induced by the specialization map. So $\dim H^n(X - Z, X - Y) = \dim H^{n-2}(\tilde{X})$, thus γ is an isomorphism and the claim follows. The case $n = 1$ is similar and left to the reader.

Since $Y \hookrightarrow X$ is a homotopy equivalence, from the exact sequence of the couple $(X, X - Z)$ we get the exact sequence:

$$H^n(Y) \xrightarrow{\delta} W_nH^n(X - Z) \to W_nH^{n+1}_Z(X) \to W_nH^{n+1}(Y).$$

Since the singularities of Y and X are isolated, it follows from [24], [12] that $W_nH^{n+1}(Y) = 0$ and $W_nH^{n+1}_Z(X) \simeq H^{n+1}_Z(X)$. So we have:

$$\begin{tikzcd} & H^n(Y) \arrow[d, "\delta"'] \arrow[dr, "sp^*"] & & \\ 0 \arrow[r] & W_nH^n(X-Z) \arrow[r] & H^n(\tilde{X}) \arrow[r, "T-\mathrm{Id}"] & H^n(\tilde{X}) \end{tikzcd}$$

with $\mathrm{coker}(\delta) \simeq H_Z^{n+1}(X)$. The horizontal sequence comes from the Wang sequence and is exact by the claim above and the fact that the weights of $\ker(T-\mathrm{Id})$ are $\leq n$. The theorem follows then from an easy diagram-chase. □

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