COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 98, nº 1 (1995), p. 77-89

<http://www.numdam.org/item?id=CM_1995__98_1_77_0>

Compositio Mathematica **98:** 77–89, 1995.

Homogeneous varieties for semisimple groups of rank one

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Received 28 April 1994; accepted in final form 14 June 1994

1. Introduction

Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are an important part of classical mathematics. They are linked to many different topics like Dynkin diagrams, singularities of surfaces, Platonic solids, and equations of fifth degree (see Klein's book with comments by Slodowy [Kl]). Another way to look at a finite subgroup H is by its coset space $\mathrm{SL}_2(\mathbb{C})/H$. It is an homogeneous algebraic variety. All other homogeneous algebraic $\mathrm{SL}_2(\mathbb{C})$-varieties are of lower dimension and are easily determined.

In this paper we study homogeneous $\mathrm{SL}_2(k)$-varieties X which are defined over ground fields k different from $\mathbb{C}$ and we give a classification in two situations: (a) k is algebraically closed and X is arbitrary, and (b) k is arbitrary but $\dim X \leqslant 2$.

The main intricacy of this problem appears when k is of positive characteristic. First of all, there are many more finite subgroups than in characteristic zero, for example, the groups $\mathrm{SL}_2(\mathbb{F}_q)$. Essentially, these subgroups have been classified by Dickson [Di].

Therefore, the main focus of this paper is on the other problem in positive characteristic: A homogeneous variety is no longer determined by the conjugacy class of its isotropy groups. Instead, one has to consider more general objects, so-called subgroup schemes, i.e. algebraic groups whose ring of functions may have nilpotent elements. The best known example is the kernel of the Frobenius morphism, widely used in representation theory (see J. C. Jantzen's book [Ja]).

In this paper, we first define certain connected subgroup schemes of $\mathrm{SL}_2(k)$ and give a concrete realization of their coset spaces. Most of them are triangular (modulo Frobenius kernels), i.e. contained in a Borel subgroup. Only in the case of characteristic two we define two other series of subgroup schemes, named $\mathcal{B}(q)$ and $\mathcal{C}(q)$. Exactly one of them is of positive dimension, namely $\mathcal{B}(\infty)$. The corresponding homogeneous variety is realized as the open orbit in the projective space $\mathbf{P}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of $\mathrm{SL}_2(k)$.

We then show that our list contains all connected subgroup schemes up to conjugation. This is done by an induction. Combining this with the above-mentioned classification of Dickson, we obtain a complete list of subgroup schemes of $\mathrm{SL}_2(k)$.

Actually, my own interest for homogeneous varieties of $\mathrm{SL}_2(k)$ comes from the fact that many properties of actions of an arbitrary reductive group G can be reduced to the rank-one-case (see e.g. [Kn]). The results here will be used for that purpose in a subsequent paper.

NOTATION. All varieties and groups are defined over an algebraically closed field k of arbitrary characteristic (except in Section 5). Let p be its characteristic exponent, i.e. $p = \operatorname{char} k$ if the characteristic is positive and $p = 1$ otherwise. In the whole paper q, q_1, q_2, q' etc. denote either a power of p or ∞. If two subgroups are conjugated then we denote this by $H_1 \sim H_2$.

2. Construction of certain homogeneous varieties

For short, a *subgroup* of an algebraic group G is in this paper what is usually called a *subgroup scheme*. Then, homogeneous varieties for G are classified by conjugacy classes of subgroups of G. We start by defining some subgroups of

$$G = \mathrm{SL}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, ad - bc = 1 \right\}$$

by equations, where every equation involving an infinite power x^∞ has to be omitted.

In arbitrary characteristic we define

$$\mathcal{A}(n,\, q)\colon\ a^n = 1,\ b^q = 0,\ c = 0,\ d = a^{-1},\ \text{ where } 1 \leqslant n,\ q = p^l \leqslant \infty.$$

In particular, $1 = \mathcal{A}(1,\, 1)$, $T := \mathcal{A}(\infty,\, 1)$, $U := \mathcal{A}(1,\, \infty)$ and $B := \mathcal{A}(\infty,\, \infty)$ is the trivial subgroup, a maximal torus, a maximal unipotent subgroup and a Borel subgroup, respectively. The group $\mathcal{A}(n,\, q)$ is connected if and only if n equals ∞ or is a power of p. For $n < \infty$, we set $\mu_n := \mathcal{A}(n,\, 1)$.

Only in characteristic two we define

$$\mathcal{B}(q)\colon\ a^4 = 1,\ b^q = 0,\ c = a + a^{-1},\ d = a^{-1} + (1 + a^2)b,\ \text{ where } 1 \leqslant q \leqslant \infty,$$

and

$$\mathcal{C}(q)\colon\ a^2 = 1,\quad b^{2q} = 0,\quad c = ab^q,\quad d = a + b^{q+1},\quad \text{where } 1 \leqslant q < \infty.$$

It is easily verified that these equations actually define subgroups of G. Observe, $\mathcal{B}(1) \sim \mathcal{A}(4,\, 1)$ and $\mathcal{C}(1) \sim \mathcal{A}(2,\, 2)$.

For $q = p^l$ let $F_q\colon G \to G$ be the lth power of the *Frobenius morphism* defined by $(a,\, b,\, c,\, d) \mapsto (a^q,\, b^q,\, c^q,\, d^q)$. For a subgroup H of G and $q < \infty$ set ${}_qH := F_q^{-1}(H) \subseteq G$. For example, ${}_q\mathcal{A}(n,\, q')$ is defined by the equations

$$a^{qn} = 1,\quad b^{qq'} = 0,\quad c^q = 0\quad \text{and}\quad ad - bc = 1.$$

In particular, if the lth Frobenius kernel of an algebraic group H is denoted by $H^{(q)}$ then $G^{(q)} = {}_q1 = {}_q\mathcal{A}(1, 1)$.

Next, I want to give a concrete construction of the corresponding homogeneous varieties. It suffices to do this for subgroups not containing a Frobenius kernel $G^{(q)}$. In fact, let X be a G-variety. If we denote the same variety but with the twisted action $G \xrightarrow{F_q} G \to \operatorname{Aut} X$ by ${}_qX$ then $G/{}_qH = {}_q(G/H)$.

Consider $H = \mathcal{A}(n, q)$:

$n = q = \infty$: Then, of course $G/H = G/B = \mathbf{P}^1$ is the projective line.

$n < q = \infty$: Let $L_n \to \mathbf{P}^1$ be the line bundle of degree n and L'_n the complement of the zero-section. Then $H \subseteq B$ realizes G/H as bundle over $G/B = \mathbf{P}^1$ and in fact $G/H = L'_n$.

$q < \infty$: Let B^- be the group of lower triangular matrices. Then ${}_qB^-$ is defined by $b^q = 0$ and arises as an isotropy group of ${}_q\mathbf{P}^1$. Because

$$\mathcal{A}(n, q) = {}_qB^- \cap \mathcal{A}(n, \infty)$$

we can realize G/H as an orbit in $L'_n \times {}_q\mathbf{P}^1$ (where we set $L'_\infty := \mathbf{P}^1$). In fact, it is exactly the complement of the graph of the morphism

$$L'_n \twoheadrightarrow \mathbf{P}^1 \xrightarrow{F_q} \mathbf{P}^1.$$

Next, consider char $k = 2$ and $H = \mathcal{B}(\infty)$. Let $\mathfrak{g}$ be the adjoint representation of G, i.e. the space of 2×2-matrices of trace zero. Then G acts on the projective space $\mathbf{P}(\mathfrak{g})$ with three orbits: A curve defined by $\det A = 0$, a fixed point represented by the identity matrix, and the complement X_0 of both. Then $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right] \in X_0$ and its isotropy subgroup is $\mathcal{B}(\infty)$. Therefore, $G/\mathcal{B}(\infty) \cong X_0$.

For $1 < q < \infty$, we have

$$\mathcal{B}(q) = {}_qB^- \cap \mathcal{B}(\infty).$$

Hence, the space $G/\mathcal{B}(q)$ is realized as a dense open subset of $\mathbf{P}(\mathfrak{g}) \times {}_q\mathbf{P}^1$. To be precise, let

$$\left(\begin{bmatrix} x & y \\ z & x \end{bmatrix}, [u : v]\right)$$

be a point of $\mathbf{P}(\mathfrak{g}) \times \mathbf{P}^1$. Then $G/\mathcal{B}(q)$ is the complement of two divisors defined by

$$x^2 + yz \neq 0, \quad z^{q/2}u + y^{q/2}v \neq 0.$$

Finally, consider $H = \mathcal{C}(q)$ with $1 < q < \infty$. Consider the representation $V_0 = \mathfrak{g} \oplus {}_q\mathfrak{g}$ and choose coordinates as follows

$$\left(\begin{pmatrix} x_0 & y_0 \\ z_0 & x_0 \end{pmatrix}, \begin{pmatrix} x_1 & y_1 \\ z_1 & x_1 \end{pmatrix}\right).$$

Let W be the line spanned by $(1_2, 1_2)$ and $V := V_0/W$. Then $x := x_0 - x_1$, y_0, z_0, y_1, z_1 are coordinates on V. An easy calculation shows that $G/\mathcal{C}(q)$ is the generic orbit in V described by the equations

$$x^2 + y_0 z_0 + y_1 z_1 = \alpha, \quad y_0^q z_1 + z_0^q y_1 = \beta, \quad \alpha, \beta \in k, \beta \neq 0.$$

More precisely, $\mathcal{C}(q)$ is the isotropy group of any vector with $y_0 = z_1 \neq 0$ and $z_0 = 0$.

3. The normalizer

We want to show that this way be got all connected subgroups of G up to conjugacy. For this we need their normalizers.

3.1. LEMMA. *The normalizer subgroup N of a subgroup H together with the isomorphism type of N/H is given by the table below:*

H		N	N/H
$\mathcal{A}(q_1, 1)$	$1 \leqslant q_1 \leqslant 2$	G	$q_1 = 1 : \mathrm{SL}_2(k)$ $q_1 = 2 : \mathrm{PGL}_2(k)$
$\mathcal{A}(q_1, 1)$	$2 < q_1 \leqslant \infty$	$N_G(T)$	$q_1 < \infty : N_G(T)$ $q_1 = \infty : \mathbb{Z}/2\mathbb{Z}$
$\mathcal{A}(q_1, q_2)$	$2 < 2q_2 < q_1 \leqslant \infty$	$\mathcal{A}(\infty, q_2)$	$q_1 < \infty : \mathbf{G}_m$ $q_1 = \infty : 1$
$\mathcal{A}(q_1, q_2)$	$1 \leqslant q_1 \leqslant 2q_2 \leqslant \infty$ $q_1 \neq 2,\ 1 < q_2$	B	$q_2 < \infty : \mathbf{G}_m \ltimes \mathbf{G}_a[2q_2/q_1]$ $q_1 < q_2 = \infty : \mathbf{G}_m$ $q_1 = q_2 = \infty : 1$
$\mathcal{A}(2, q_2)$	$p = 2,\ 1 < q_2 \leqslant \infty$	${}_2B$	$q_2 < \infty : \mathbf{G}_m \ltimes (\mathbf{G}_a^{(2)}[-1] \times \mathbf{G}_a[q_2])$ $q_2 = \infty : \mathbf{G}_m \ltimes \mathbf{G}_a^{(2)}[-1]$
$\mathcal{B}(q)$	$p = 2,\ 1 < q \leqslant \infty$	$\mathcal{B}(\infty)$	$q < \infty : \mathbf{G}_a$ $q = \infty : 1$
$\mathcal{C}(q)$	$p = 2,\ 1 < q < \infty$	${}_2\mathcal{A}(q+1, \infty)$	$\mu_{q+1} \ltimes \mathbf{G}_a[q]$

The number in brackets refers to the character with which $\mathbf{G}_m$ acts by conjugation.

Proof. With

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \quad \det X = 1,$$

the equality $XHX^{-1} = H$ leads for $H = \mathcal{A}(q_1, q_2)$ to following system of equalities:

(1) $x_{21}x_{22}(a - a^{-1}) - x_{21}^2 b = 0$,
(2) $(x_{11}x_{21})^{q_1} b^{q_1} = 0$,
(3) $(x_{11}x_{12})^{q_2}(a^{q_2} - a^{-q_2}) = 0$.

The evaluation of these equations is easy and is left mostly to the reader. Let me work out only the case $1 \leqslant q_1 \leqslant 2q_2 \leqslant \infty$, $q_1 \neq 2$, $q_2 > 1$. Assume first $q_1 = 1$. Then $a = 1$ and (3) holds. Because $q_2 > 1$ we have $b \neq 0$. Hence, (1) reduces to $x_{21}^2 = 0$ and (2) to $x_{11}x_{21} = 0$. Therefore, $x_{21} = x_{21} \det X = 0$ is the only relation, i.e. $X \in B$.

Consider now $q_1 > 2$. Then $a - a^{-1} \neq 0$ and (1) implies $x_{21}x_{22} = x_{21}^2 = 0$. As above, we get $x_{21} = 0$. Then (2) is satisfied. Finally, (3) follows from $q_1 \leqslant 2q_2$.

Next, let $H = \mathcal{C}(q)$ with $q > 1$ and char $k = 2$. Then

$$X \begin{pmatrix} a & b \\ ab^q & a + b^{q+1} \end{pmatrix} X^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = S, \quad \text{where } a^2 = 1,\ b^{2q} = 0$$

implies

$$A = a + x_{11}x_{21}b + x_{12}x_{22}ab^q + x_{12}x_{21}b^{q+1}$$

$$B = x_{11}^2 b + x_{12}^2 ab^q + x_{11}x_{12}b^{q+1}$$

$$C = x_{21}^2 b + x_{22}^2 ab^q + x_{21}x_{22}b^{q+1}$$

Then $S \in \mathcal{C}(q)$ is equivalent to $A^2 = 1$, $B^{2q} = 0$, and $C = AB^q$. Therefore, we get the following equations

(1) $x_{21}^2 = 0$,
(2) $x_{22}^2 = x_{11}^{2q}$,
(3) $x_{21}x_{22} = x_{21}x_{11}^{2q+1}$,
(4) $x_{11}x_{22} - x_{12}x_{21} = 1$.

This implies

$$x_{11}^{2q+2} \overset{(2)}{=} x_{11}^2 x_{22}^2 \overset{(4)}{=} (1 + x_{12}x_{21})^2 \overset{(1)}{=} 1.$$

Conversely, the three equations

$$x_{11}x_{22} - x_{12}x_{21} = 1, \quad x_{21}^2 = 0, \quad (x_{11}^2)^{q+1} = 1,$$

are equivalent to (1)–(4) above, i.e. N is the preimage of $\mathcal{A}(q + 1, \infty)$ under F_2 as claimed.

Finally, consider $H := \mathcal{B}(q)$. Then a direct calculation as above becomes quite involved. Therefore, we proceed differently. First observe, that $\mathcal{B}(1)$ is conjugated to $\mathcal{A}(4, 1) \cong \mathbf{G}_m^{(4)}$ by $s = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Moreover, multiplication gives a morphism

$$\mathcal{B}(1) \times \mathcal{A}(1, q) \to \mathcal{B}(q)$$

which is an isomorphism of schemes but not a homomorphism. It is easy to check, that $U = \mathcal{A}(1, \infty)$ normalizes $\mathcal{B}(q)$. Hence, $\mathcal{B}(\infty) = \mathcal{B}(1)U \subseteq N$. To see the opposite inclusion, consider the action of G on $G/\mathcal{B}(\infty) \cong X_0 \subset \mathbf{P}(\mathfrak{g})$. We calculate the fixed point scheme X_0^H. First $X_0^{B(1)}$ is reduced and consists of all points of the form

$$\begin{bmatrix} 1 & 1 \\ y & 1 \end{bmatrix}, \quad y \neq 1.$$

Then intersection with the fixed point scheme of $\mathcal{A}(1, q)$ implies $y = 0$, i.e. $X_0^{\mathcal{B}(q)}$ is scheme-theoretically the point x_0 defined by $y = 0$. Hence, $Nx_0 \subseteq X_0^{\mathcal{B}(q)} = \{x_0\}$ which implies $N \subseteq G_{x_0} = \mathcal{B}(\infty)$.

The calculations for N/H are easy and left to the reader. For future reference let me just mention the computation for $H = \mathcal{A}(2, q)$, $1 < q < \infty$. In this case $\bar{N} := \mathbf{G}_m \ltimes (\mathbf{G}_a^{(2)}[-1] \times \mathbf{G}_a[q])$ is isomorphic to the matrix group

$$\begin{pmatrix} t^{-1} & 0 & x \\ & t^q & y \\ & & 1 \end{pmatrix} \quad \text{with } x^2 = 0.$$

Then $t = a^2$, $x = a^{-1}c$ and $y = (ab)^q$ defines a homomorphism of ${}_2B$ onto $\bar{N}$ with kernel H. □

4. The connected subgroups

Now we are in the position to prove our main classification result:

4.1. THEOREM. *Every connected subgroup of $G = \mathrm{SL}_2(k)$ is conjugated to exactly one of these:*

(a) *In arbitrary characteristic, G and ${}_q\mathcal{A}(q_1, q_2)$ with $1 \leqslant q < \infty$ and $1 \leqslant q_1, q_2 \leqslant \infty$.*
(b) *In characteristic two, additionally ${}_q\mathcal{B}(q')$ with $1 \leqslant q < \infty$, $1 < q' \leqslant \infty$ and ${}_q\mathcal{C}(q')$ with $1 \leqslant q < \infty$, $1 < q' < \infty$.*

Proof. If $H \neq G$ then there is a maximal Frobenius kernel $G^{(q)}$ which is contained in H. Because the list above is closed under taking preimages of the Frobenius morphism, we may replace H by $F_q(H)$. Hence we may assume that $G^{(q)} \subseteq H$ implies $q = 1$ or what is the same $\mathrm{Lie}\, H \neq \mathrm{Lie}\, G$.

We consider first zero-dimensional connected subgroups H. Then there is a smallest power $q = q(H)$ of p such that $H \subseteq G^{(q)}$. We prove, that our list above is complete by induction on $q(H)$. Obviously, $q(H) = 1$ implies $H = 1 = {}_1\mathcal{A}(1, 1)$.

Assume now $q(H) > 1$. Then $H' := H \cap G^{(q/p)}$ is proper subgroup of H with $q(H') < q = q(H)$, hence, by induction, it is on our list. Because the Frobenius kernels are normal subgroups, H is contained in the normalizer N of H'. Furthermore, $Q := H/H' \hookrightarrow G^{(q)}/G^{(q/p)} \cong G^{(p)}$ is a group scheme of height one. Hence, it suffices to check, that our list is closed under normal extensions of height one. This is not very difficult because these correspond bijectively to p-closed subalgebras of Lie N/H' (see [DG] II, 7.4).

Assume H' is the second or third $\mathcal{A}$-case in Lemma 3.1, or $\mathcal{B}(q)$ or $\mathcal{C}(q)$: In this case $\dim \operatorname{Lie} N/H' = 1$, which implies that H' has a unique normal extension of height one which is $\mathcal{A}(pq_1, 1)$, $\mathcal{A}(pq_1, q_2)$, $\mathcal{B}(pq)$ or ${}_2\mathcal{A}(1, q)$ respectively. All of these are on the list.

Assume $H' = \mathcal{A}(1, 1) = 1$: Then H is determined by a p-closed Lie subalgebra of G. Hence, up to conjugacy, H equals $\mathcal{A}(p, 1)$, $\mathcal{A}(1, p)$, $\mathcal{A}(p, p)$ or $G^{(p)} = {}_p\mathcal{A}(1, 1)$.

Assume $H' = \mathcal{A}(2, 1)$, with $p = 2$. Then we have to determine the subalgebras of $\operatorname{Lie} \mathrm{PGL}_2(k)$ which are the 2×2-matrices modulo scalars. We get, up to G-conjugacy,

$$\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \quad \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & * \\ * & * \end{bmatrix}.$$

To these subalgebras correspond $H = \mathcal{A}(2, 2)$, $\mathcal{A}(4, 1)$, $G^{(2)} = {}_2\mathcal{A}(1, 1)$, $\mathcal{A}(4, 2)$, and ${}_2\mathcal{A}(2, 1)$, respectively.

Assume $H' = \mathcal{A}(q_1, q_2)$ with $1 \leqslant q_1 \leqslant 2q_2 < \infty$ and $q_1 \neq 2$, $1 < q_2$: Up to B-conjugacy there are only three non-trivial subalgebras of Lie N/H' leading to $\mathcal{A}(pq_1, q_2)$, $\mathcal{A}(q_1, pq_2)$ and $\mathcal{A}(pq_1, pq_2)$.

Assume $H' = \mathcal{A}(2, q)$ with $p = 2$ and $1 < q < \infty$: This is the most interesting case because here the groups $\mathcal{B}(q)$ and $\mathcal{C}(q)$ appear. By the calculation at the end of the proof of Lemma 3.1, Lie N/H' has a presentation as

$$\left\{ \begin{pmatrix} t & 0 & x \\ & 0 & y \\ & & 0 \end{pmatrix} \right\}.$$

A short calculation shows that there are, up to $(N/H')^{\mathrm{red}}$-conjugacy, only the following subalgebras $\mathfrak{s}$ which are closed under $A \mapsto A^2$:

$\mathfrak{s}$:	$y = t = 0$	$x = t = 0$	$x = y,\ t = 0$	$x = y = 0$	$x = t,\ y = 0$
H:	${}_2\mathcal{A}(1, q/2)$	$\mathcal{A}(2, 2q)$	$\mathcal{C}(q)$	$\mathcal{A}(4, q)$	$\mathcal{B}(q)$

$\mathfrak{s}$:	$t = 0$	$x = 0$	$y = 0$	$x = t$	–
H:	${}_2\mathcal{A}(1, q)$	$\mathcal{A}(4, 2q)$	${}_2\mathcal{A}(2, q/2)$	$\mathcal{B}(2q)$	${}_2\mathcal{A}(2, q)$

This finishes the proof for finite subgroups.

Let H be a positive dimensional subgroup of G with $\operatorname{Lie} H \neq \operatorname{Lie} G$. Assume first that H is smooth. Then $\dim H \leqslant 2$ implies that H is solvable, hence conjugate to a subgroup of B. Hence, either $H \sim B = \mathcal{A}(\infty, \infty)$ or $\dim H = 1$. In the latter case we get either $H \cong \mathbf{G}_a$ or $H \cong \mathbf{G}_m$. Therefore, $H \sim U = \mathcal{A}(1, \infty)$ or $H \sim T = \mathcal{A}(\infty, 1)$. This settles in particular the case of $\operatorname{char} k = 0$. From now on we assume $p \geqslant 2$. Then $H_q := H \cap G^{(q)}$ is a finite normal subgroup of H with $q(H_q) = q$. Let $q > 2$.

Assume that $H_q = \mathcal{B}(q)$. Then $H^{\mathrm{red}} \subseteq N_G(\mathcal{B}(q))^{\mathrm{red}} = U$ and $\mathcal{B}(\infty) = \mathcal{B}(q)U = N_G(\mathcal{B}(q))$ we conclude $H = \mathcal{B}(\infty)$.

Next assume $H_q = \mathcal{C}(q/2)$. Then again $H^{\mathrm{red}} = U$. But $G^{(2)} \subseteq \mathcal{C}(q/2)U \subseteq H$ in contradiction to the choice of H.

We are left with the case that all H_q are conjugated to $\mathcal{A}(q_1, q_2)$ where q_1, q_2 depend on q. More precisely, $\max(q_1, q_2) = q(H_q) = q$. Then $H_q \subseteq H_{pq}$ and the fact that $\mathcal{A}(q_1, q_2)$ is conjugated to a subgroup of $\mathcal{A}(q_1', q_2')$ if and only if $q_1 \leqslant q_1'$ and $q_2 \leqslant q_2'$ imply that three cases may happen:

q_1 is constant and $q_2 = q$: Then for $2q > q_1$ there is a *unique* subgroup of type $\mathcal{A}(q_1, pq_2)$ containing $\mathcal{A}(q_1, q_2)$. Hence we may assume that H_q not only is conjugated but equals $\mathcal{A}(q_1, q)$ for every q. Therefore, $H \cap G^{(q)} = \mathcal{A}(q_1, \infty) \cap G^{(q)}$ for all q which implies $H = \mathcal{A}(q_1, \infty)$ (see [MO] 2.1).

q_2 is constant and $q_1 = q$, or $q_1 = q_2 = q$. As above, one concludes $H \sim \mathcal{A}(\infty, q_2)$ or $H \sim \mathcal{A}(\infty, \infty)$. □

5. The infinite subgroups

From Theorem 4.1 it is not difficult to derive a complete classification of all subgroups H of G: It is determined by its connected component H^0, known from Theorem 4.1, and a finite reduced subgroup of $N_G(H^0)/H^0$, known from Lemma 3.1. Let us start with the case $\dim H > 0$. Finite subgroups are handled in the next section.

5.1. THEOREM. *Let H be a positive dimensional subgroup of $G = \mathrm{SL}_2(k)$. Then H is conjugated to ${}_{q_0}H'$ where H' is one of G, B, $\mu_n \ltimes U$ where $1 \leqslant n < \infty$, $T \ltimes U^{(q)}$ where $1 \leqslant q < \infty$, $N_G(T)$, or $B(\infty)$ when $p = 2$.*

Now it is easy to generalize this to an arbitrary ground field k_0. Let $\mathrm{SL}_2(k_0)$ act on the Lie algebras $\mathfrak{sl}_2(k_0)$, $\mathfrak{gl}_2(k_0)$ and $\mathfrak{pgl}_2(k_0)$ by conjugation. For $\alpha, \beta \in k_0$ we define the quadrics $Q(\alpha, \beta) := \{A \in \mathfrak{gl}_2 | \operatorname{tr} A = \alpha, \det A = B\}$.

5.2. THEOREM. *Let k_0 be any field and X a homogeneous $\mathrm{SL}_2(k_0)$-variety defined over k_0. Assume $1 \leqslant \dim X \leqslant 2$. Then X is a Frobenius twist of one of the varieties in the right-hand column. The first column describes the isotropy group over an algebraic closure of k_0.*

B:	*The projective line* $\mathbf{P}^1$.
$\mu_n \ltimes U$:	L'_n, *the line bundle over* $\mathbf{P}^1$ *of degree* $n \geqslant 1$ *with the zero-section removed.*
$\mu_2 \ltimes U,\ p = 2$:	$Q(0,\ \beta)\backslash[\mathbf{1}_2]$ *where* $\beta \in k_0$ *is not a square.*
$\mathcal{B}(\infty),\ p = 2$:	$\{[A] \in \mathbf{P}(\mathfrak{sl}_2) \mid [A] \neq [\mathbf{1}_2],\ \det A \neq 0\}$.
$T \ltimes U^{(q)},\ q > 1$:	$\mathbf{P}^1 \times_q \mathbf{P}^1 \backslash \Gamma_q$ *where* Γ_q *is the graph of the Frobenius morphism* F_q.
T:	$Q(1,\ \beta)$ *where* $\beta \in k_0,\ 4\beta \neq 1$.
$N_G(T)$:	*The open orbit in* $\mathbf{P}(\mathfrak{pgl}_2)$, *which is described by* $4 \det A - (\operatorname{tr} A)^2 \neq 0$.

REMARKS: 1. We denote the line through a matrix A by $[A]$.

2. The intersection $Q(0,\ \beta) \cap [\mathbf{1}_2]$ consists only of one point, namely $\sqrt{\beta}\mathbf{1}_2$. Furthermore, $Q(0,\ \beta_1) \cong Q(0,\ \beta_2)$ if and only if $\beta_1 - \beta_2 \in k_0^2$.

3. If $p \neq 2$ then $Q(1,\ \beta) \cong Q(0,\ \beta')$ with $\beta' = \beta - \frac{1}{4} \neq 0$. Furthermore, $Q(0,\ \beta_1) \cong Q(0,\ \beta_2)$ if and only if $\beta_1\beta_2^{-1}$ is a square in k_0.

4. If $p = 2$ then consider $f\colon\ k_0 \to k_0\colon\ t \mapsto t^2 - t$. Then $Q(1,\ \beta_1) \cong Q(1,\ \beta_2)$ if and only if $\beta_1 - \beta_2 \in f(k_0)$.

Proof. Let k be an algebraic closure of k_0. Then $X_k \cong G/H$ where H is one of the groups in Theorem 5.1. Let N be the normalizer of H in G. Because $\operatorname{Aut}^G X_k \cong N/H$, the different forms of X_k over k_0 are classified by the cohomology set $H^1(k/k_0,\ N/H)$. More precisely, an element of this cohomology set is a N/H-torsor over k_0. In our case, it is the fixed point scheme X^H.

Because $N_G({}_qH) = {}_qN_G(H)$ we get $\operatorname{Aut}^G {}_qX_k = \operatorname{Aut}^G X_k$. Therefore, we may assume that X_k is not a Frobenius twist.

It is known that the cohomology set for any smooth connected solvable group is trivial ([DG] IV, Section 4, 3.7a). By the computations in Lemma 3.1 this means that there is only one form, namely the one given above, unless H is either T or $\mu_2 \ltimes U$ with $p = 2$.

For $H = T$ we get $N/H = \mathbb{Z}/2\mathbb{Z}$ which means that an N/H-tensor is just a Galois extension of k_0 of degree two. Let x_{ij} be coordinates of $\mathfrak{gl}_2$. Then $Q(1,\ \beta)^T$ is described by the equations $x_{12} = x_{21} = 0,\ x_{22} = 1 - x_{11}$, and $x_{11}(1 - x_{11}) = \beta$. Hence, $Q(1,\ \beta)^T \cong \operatorname{Spec} k_0[x]/(x^2 - x + \beta)$. Every Galois extension of k_0 of degree two is of this form. Moreover, two of them are isomorphic if and only if the conditions in Remark 3 and 4 are met.

Finally, consider the case $H = \mu_2 \ltimes U$ in characteristic two. Then $N/H = \mathbf{G}_m \ltimes \mathbf{G}_a^{(2)}$, hence $H^1(k/k_0,\ N/H) = H^1(k/k_0,\ \mathbf{G}_a^{(2)})$. The last set classifies inseparable field extensions of degree two. Now $Q(0,\ \beta)^H$ is given by $x_{21} = 0,\ x_{22} = x_{11}$, and $x_{11}^2 = \beta$. This shows again that we got all N/H-torsors. □

REMARK: If $H = N_G(T)$ and the characteristic is two then G/H is the complement of the hyperplane $\operatorname{tr} A = 0$ in $\mathbf{P}(\mathfrak{pgl}_2)$. This means $G/H \cong \mathbf{A}^2$. In particular, $SL_2(k)$ can act transitively on an affine space. This provides a counterexample to Theorem 1.1(ii) in [Bo]. (The erroneous statement in [Bo] occurs on p. 77, l. 10. The argument is valid if H is connected. In particular, the main result of the paper, Theorem 1.1(i), is correct.)

6. The finite subgroups

We start with a simple observation which follows immediately from Lemma 3.1.

6.1. LEMMA. *Let $H \subseteq G$ be a subgroup with $H^0 = \mathcal{A}(q_1, q_2)$ and $q_2 > 1$. Then $H \subseteq B$.*

For this reason we start by looking for subgroups in B or more generally in $\mathbf{G}_m \ltimes \mathbf{G}_a[d]$, i.e. the group defined by

$$(x, y)(\bar{x}, \bar{y}) = (x\bar{x}, y + x^d\bar{y}).$$

Note, that for $d = 2$ we get B via the isomorphism

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto (a, ab).$$

Recall, that a *p-polynomial* is a polynomial $f \in k[t]$ which is of the form $\Sigma_i a_i t^{p^i}$. It is additive, i.e. satisfies $f(x + y) = f(x) + f(y)$. Let n, l be integers. Then $f = \Sigma_i a_i t^i \in k[t]$ is called *n-homogeneous of degree l* if $a_i \neq 0$ implies $i \equiv l \bmod n$. In this case, $f(at) = a^l f(t)$ when $a^n = 1$.

6.2. LEMMA. *Every subgroup of $\mathbf{G}_m \ltimes \mathbf{G}_a[d]$ is conjugated to one defined by the equations $x^n = 1$, $f(y) = 0$ where $n \geqslant 0$ and f is a $\frac{n}{(n,d)}$-homogeneous p-polynomial.*

Proof. By [Wa] Thm. 4, H is a semidirect product of a subgroup S of $\mathbf{G}_m$ and a subgroup N of $\mathbf{G}_a$. Then S is defined by $x^n = 1$ and N by $f(y) = 0$ for some integer n, and p-polynomial f. The fact that S normalizes N means $f(x^d y) = x^l f(y)$ for some l, i.e., $dq \equiv l \bmod n$ for every t^q occurring in f. This means that f is $\frac{n}{(n,d)}$-homogeneous. □

The case $d = 2$ implies that every subgroup of B is conjugated to one of the following form:

DEFINITION. Let $n \geqslant 0$ be an integer and f an $\frac{n}{(n,2)}$-homogeneous p-polynomial. Then define a subgroup of $SL_2(k)$ by the equations

$$\mathcal{A}(n, f)\colon\ a^n = 1, \quad f(ab) = 0, \quad c = 0, \quad d = a^{-1}.$$

In particular, we get $\mathcal{A}(n, q) = \mathcal{A}(n, t^q)$, $\mathcal{A}(\infty, q) = \mathcal{A}(0, t^q)$, $\mathcal{A}(n, \infty) = \mathcal{A}(n, 0)$, $\mathcal{A}(\infty, \infty) = \mathcal{A}(0, 0)$ for $1 \leqslant n < \infty$, $q < \infty$.

Next, we consider subgroups H with $H^0 = \mathcal{B}(q)$, $1 < q < \infty$, $p = 2$. Then $y = (ab)^q$ identifies $\mathcal{B}(\infty)/\mathcal{B}(q)$ with $\mathbf{G}_a$. Hence, H is defined by $\bar{f}((ab)^q) = 0$ where $\bar{f}$ is a 2-polynomial with $\bar{f}'(0) \neq 0$. As $\bar{f}$ and q vary, $f(t) = \bar{f}(t^q)$ runs through the set of all 2-polynomials with $f' = 0$. Therefore, H is one of these:

DEFINITION. For $p = 2$ let f be a 2-polynomial with $f' \equiv 0$. Then define a subgroup of $\mathrm{SL}_2(k)$ by the equations

$$\mathcal{B}(f)\colon\ a^4 = 1, \quad f(ab) = 0, \quad c = a + a^{-1}, \quad ad - bc = 1.$$

In particular, we get $\mathcal{B}(q) = \mathcal{B}(t^q)$ for $1 < q < \infty$ and $\mathcal{B}(\infty) = \mathcal{B}(0)$.

Now consider the case $H^0 = \mathcal{C}(q)$, $1 < q < \infty$, $p = 2$. Then

$$N_G(\mathcal{C}(q)) =$$

$$= {}_2\mathcal{A}(q+1, \infty) \to \mu_{q+1} \ltimes \mathbf{G}_a[-1]\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a^2, a^{-1}c + (ab)^q)$$

is a surjective homomorphism with kernel $\mathcal{C}(q)$. Hence, H is described by equations $a^{2n} = 1$, $\bar{f}(a^{-1}c + (ab)^q) = 0$ where $n|q+1$ and $\bar{f}$ is an n-homogeneous 2-polynomial with $\bar{f}'(0) = \lambda \neq 0$. Then $c^2 = 0$ implies $\bar{f}(a^{-1}c) = \lambda a^{-1}c$, hence $c = a\lambda^{-1}\bar{f}((ab)^q)$. Because the linear term of $\bar{f}$ does not vanish, the degree of $\bar{f}$ is actually one. Hence, $f(t) = \bar{f}(t^q)$ is an n-homogeneous 2-polynomial of degree $-1 \equiv q \bmod n$. Therefore, H is one of these:

DEFINITION. For $p = 2$ let $n \geqslant 1$ be odd and f an n-homogeneous 2-polynomial of degree -1 with $f' \equiv 0$. Then define a subgroup of $\mathrm{SL}_2(k)$ by the equations

$$\mathcal{C}(n, f)\colon\ a^{2n} = 1, \quad c = af(ab), \quad c^2 = 0, \quad ad - bc = 1.$$

Note that only those integers n occur which divide a number of the form $2^m + 1$. As special cases, we get $\mathcal{C}(q) = \mathcal{C}(1, t^q)$, $1 < q < \infty$ and $\mathcal{A}(2n, \infty) = \mathcal{C}(n, 0)$.

It remains the classification of ordinary, i.e. reduced, finite subgroups of $\mathrm{SL}_2(k)$ which has been done by Dickson. Then we get:

6.3. THEOREM. *Let H be a finite subgroup of $G = \mathrm{SL}_2(k)$ with* $\mathrm{Lie}\, H \neq \mathrm{Lie}\, G$. *Then H is conjugated to one of the following*:

1. $\mathcal{A}(n, f)$, *where $n \geqslant 1$ and $f \neq 0$.*
2. $\tilde{D}_n := \mu_n \cup s \cdot \mu_n$ *where* $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $n \geqslant 3$, *and n is even if $p \neq 2$;*
3. *When $p \neq 2$: The binary tetrahedral group $\tilde{A}_4$, the binary octahedral group $\tilde{S}_4$, and the binary isosahedral group $\tilde{A}_5$;*
4. $\mathrm{SL}_2(\mathbb{F}_q)$ *where $q > 1$;*
5. $\widetilde{\mathrm{PGL}}_2(\mathbb{F}_q)$, $q > 1$: *The (schematic) preimage of* $\mathrm{PGL}_2(\mathbb{F}_q) \subseteq \mathrm{PGL}_2(k)$ *in G;*
6. *When $p = 2$: $\mathcal{B}(f)$ with $f \neq 0$;*

7. *When* $p = 2$: $\mathcal{C}(n, f)$ *with* $f \neq 0$.

All the redundancies of this list are the following:

(a) $\mathcal{A}(n, f(t)) \sim \mathcal{A}(n, \alpha f(\beta t))$ *where* $\alpha, \beta \in k^*$.
(b) *When* $p = 2$: $\mathcal{B}(n, f) = \mathcal{B}(n, \alpha f)$, $\mathcal{C}(n, f(t)) \sim \mathcal{C}(n, \alpha f(\alpha t))$ *where* $\alpha \in k^*$, *and* $\mathrm{SL}_2(\mathbb{F}_2) \sim \tilde{D}_3$, $\widetilde{\mathrm{PGL}}_2(\mathbb{F}_2) \sim \tilde{D}_6$.
(c) *When* $p = 3$: $\tilde{A}_4 = \mathrm{SL}_2(\mathbb{F}_3)$, $\tilde{S}_4 = \widetilde{\mathrm{PGL}}_2(\mathbb{F}_3)$.
(d) *When* $p = 5$: $\tilde{A}_5 = \mathrm{SL}_2(\mathbb{F}_5)$.

REMARKS: 1. For $p > 2$, the normalizer of $\mathrm{SL}_2(\mathbb{F}_q)$ is $\widetilde{\mathrm{PGL}}_2(\mathbb{F}_q)$ in which it has index two. An element of the non-trivial coset is $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ where $\alpha \in \mathbb{F}_{q^2} - \mathbb{F}_q$, $\alpha^2 \in \mathbb{F}_q$. For $p = 2$ we get simply $\widetilde{\mathrm{PGL}}_2(\mathbb{F}_q) = \mu_2 \times \mathrm{SL}_2(\mathbb{F}_q)$.

2. In characteristic zero, one gets the well-known classification: The cyclic groups μ_n (case 1), and the binary polyhedral groups $\tilde{D}_{2n} (n \geqslant 2)$, $\tilde{A}_4$, $\tilde{S}_4$, and $\tilde{A}_5$ (case 2 and 3).

3. In characteristic two, G contains a subgroup isomorphic to A_5, namely $\mathrm{SL}_2(\mathbb{F}_4)$.

4. For $p > 1$ we have the equality $B(\mathbb{F}_q) = \mathcal{A}(q - 1, t - t^q)$.

Proof of the Theorem: After the discussion above, the only remaining cases to check are $H^0 = \mathcal{A}(q_1, 1)$, $1 \leqslant q_1 < \infty$. If $q_1 > 2$ then $H \subseteq N_G(T) = T \cup s \cdot T$. Then $H \cap T = \mu_n$ for some $n \geqslant 1$. Therefore, either $H = \mu_n = \mathcal{A}(n, 1)$ or $H \sim \tilde{D}_n$. If $p \neq 2$ then $s^2 = -\mathbf{1}_2 \in \mu_n$ which implies $n \equiv 0 \bmod 2$.

Now assume $q_1 = 1$, i.e., H is reduced, or $p = q_1 = 2$. In the latter case $H = \mu_2 \times H^{\mathrm{red}}$. But the list of the theorem is closed for multiplication with μ_2 : For $n \geqslant 1$ odd, we get $\mu_2 \times \mathcal{A}(n, f) = \mathcal{A}(2n, f)$, $\mu_2 \times \tilde{D}_n = \tilde{D}_{2n}$ and $\mu_2 \times \mathrm{SL}_2(\mathbb{F}_q) = \widetilde{\mathrm{PGL}}_2(\mathbb{F}_q)$. Therefore, it suffices to consider reduced subgroups H of G.

We may assume that the representation of H on $V = k^2$ is irreducible, because otherwise H is contained in a Borel subgroup and therefore conjugated to some $\mathcal{A}(n, f)$. Because a finite group has only a finite number of irreducible representations some Frobenius twist ${}_qV$ is isomorphic to V, i.e., V is defined over the field $\mathbb{F}_q$. Therefore, we may assume that H is contained in $\mathrm{SL}_2(\mathbb{F}_q)$. Now we use Dickson's classification of subgroups in $\mathrm{PSL}_2(\mathbb{F}_q)$ (see [Di] 260 or [Hu] II. 8.27). It follows that every such subgroup which has no fixed point in $\mathbf{P}^1(k)$ has even order. Hence, the order of H is even, too. If $p \neq 2$ then $-\mathbf{1}_2$ is the only element of order two in G. This implies $-\mathbf{1}_2 \in H$, i.e., H is the full preimage of a subgroup of $\mathrm{PSL}_2(\mathbb{F}_q)$. This holds also for $p = 2$, because the projection $\mathrm{SL}_2(\mathbb{F}_q) \to \mathrm{PSL}_2(\mathbb{F}_q)$ is bijective. Now the theorem follows easily from the table in [Di], [Hu]. The discussion in [Di] also shows that for each type there is only one conjugacy class in G. This finishes the classification.

The redundancies are easy except maybe those for $\mathcal{C}$. First, conjugation with a diagonal matrix shows $\mathcal{C}(n, f(t)) \sim \mathcal{C}(n, \alpha f(\alpha t))$. Conversely, assume $H_1 = \mathcal{C}(n_1, f_1) \sim H_2 = \mathcal{C}(n_2, f_2)$. Then $H_i^0 = \mathcal{C}(1, \lambda_i t^{q_i})$ where $\lambda_i t^{q_i}$ is the lowest degree term of f_i. After conjugation with elements of T we may assume $\lambda_1 = \lambda_2 = 1$, hence $H^0 := H_1^0 = H_2^0 = C(1, q)$ where $q = q_1 = q_2$. In particular, if $H_1 = gH_2g^{-1}$ then $g \in N_G(H^0)^{\mathrm{red}} = \mu_{q+1} \ltimes U$. Therefore, $H_1 \cap U = g(H_2 \cap U)g^{-1}$. This shows $f_1(t) = \alpha f_2(\alpha t)$ with $\alpha^{q+1} = 1$. Finally, $H_i \cap B/H_i \cap U = \mu_{2n_i}$ which shows $n_1 = n_2$. □

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