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Reduction numbers, Briançon–Skoda theorems and the depth of Rees rings

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1. Introduction

Let R be a commutative noetherian ring, and let $I \subseteq R$ be an ideal of R. Over the last fifteen years, there has been considerable effort in the study of the *Rees algebra* of I, $R[It] = \oplus_{n\geqslant 0} I^n$, and the *associated graded ring* of I, $G(I) = \oplus_{n\geqslant 0} I^n/I^{n+1}$. The associated Proj of these two graded rings are respectively the blowup of I and the normal cone of I. One of the main questions addressed has been when these algebras are Cohen–Macaulay (which we will abbreviate CM). If they are, there is a wealth of information about how I sits in R. See the introduction of [HH2] for a discussion of this topic, and our bibliography for many related papers. Recently, due to work of Sancho de Salas [SS] and Joseph Lipman [L], there is even more reason to study the Cohen–Macaulayness of these algebras. For instance, when R is CM with a canonical module, Sancho de Salas proves that $G(I^n)$ is CM for large n iff $X = \mathrm{Proj}(R[It])$ is CM and satisfies a Grauert–Riemenschneider vanishing property, namely that $R^i f_*(\omega_X) = 0$ for $i \geqslant 1$, where ω_X is the dualizing sheaf of X. Lipman proves that provided R is essentially of finite type over the complex numbers and X is nonsingular, then R has rational singularities iff $R[I^n t]$ is CM for large n.

A subsidiary question is the relationship between the CM property of $R[It]$ and the CM property of $G(I)$. In [Hu1], it was shown that if R is CM, then $R[It]$ CM implies $G(I)$ CM. The converse need not hold. (Indeed, putting together the results of Sancho de Salas and Lipman, one obtains, at least for rings essentially of finite type over the complex numbers, that if the converse holds for every ideal

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then the ring has a rational singularity. On the other hand, while this paper was being written, Lipman has shown that if R is pseudo-rational then for every ideal I of R, $G(I)$ CM implies that $R[It]$ is CM [L].) The first theorem giving necessary and sufficient conditions for the converse to hold in case I is m-primary was due to Goto and Shimoda [GS] in 1979. They use the concept of a reduction number. A reduction of I is an ideal $J \subseteq I$ for which there exists an integer n such that $JI^n = I^{n+1}$. The least integer n with this property is called the *reduction number* of I with respect to J and we will denote it by $r_J(I)$. The reduction number of I is defined by

$$r(I) := \min\{r_J(I) | J \text{ is a minimal reduction of } I\},$$

where J is said to be a *minimal reduction* of I if it is not properly contained in any other reduction of I. The theorem of Goto and Shimoda states:

THEOREM 1.1. *Let (R, m) be a d-dimensional CM local ring with infinite residue field and let I be an m-primary ideal of R. Then $R[It]$ is CM* iff *$G(I)$ is CM and $r(I) \leqslant d - 1$.*

This theorem was recently improved by Goto and Huckaba [GH], and related results have been given in [HM]. Trung and Ikeda gave a fairly definitive answer using the a-invariant of the graded ring: they proved in [TI] that if $G(I)$ is CM, then $R[It]$ is CM iff the a-invariant of $G(I)$ is negative (see also [V] and [GN]). Other authors have recently studied similar questions for the Gorenstein property, cf. [GNi], [HHR], [HRS], [I] for example. In this paper we will give a full generalization of the theorem of Goto and Shimoda. See Theorem 5.1. While we were writing this paper, we learned that independently Bernie Johnston and Dan Katz found this result (cf. [JK]) and Aron Simis, Bernd Ulrich, and Wolmer Vasconcelos found closely related results ([SUV]). A recent paper by Tang ([Ta]) contains some related results. We also will give necessary and sufficient conditions for the Gorenstein property to hold for the Rees and graded algebras in Theorem 5.8. Moreover our method of proof yields considerable information concerning the regularity of the Rees algebra and what we call the local reduction numbers of an ideal.

The exact statements of Theorems 5.1 and 5.8 are:

THEOREM 5.1. *Let (R, m) be a CM ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 1$. *Then $R[It]$ is CM if and only if the following conditions are satisfied*:

(i) *$G(I)$ is Cohen–Macaulay.*
(ii) *$r(I_\wp) \leqslant \mathrm{ht}(\wp) - 1$ for every prime $\wp \supseteq I$ with $\ell(I_\wp) = \mathrm{ht}(\wp)$.*

THEOREM 5.8. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 2$. *Then $R[It]$ is Gorenstein if and only if the following conditions are satisfied*:

(i) *$G(I)$ is Gorenstein.*
(ii) *$r(I_\wp) = \mathrm{ht}(\wp) - 2$ for every prime ideal $\wp \supseteq I$ with* $\mathrm{ht}(\wp/I) = 0$.

(iii) $r(I_\wp) \leqslant \mathrm{ht}(\wp) - 2$ *for every prime ideal* $\wp \supseteq I$ *with* $\ell(I_\wp) = \mathrm{ht}(\wp) \leqslant \ell(I)$.

Observe that applying (5.1) in both directions one obtains that if $G(I)$ is CM, then $R[It]$ is CM iff $R_p[I_pt]$ is CM for every prime p such that $\ell(I_p) = \mathrm{ht}(p)$. This is a very useful criterion.

If (R, m) is a local ring with infinite residue field R/m, every minimal reduction of I is minimally generated by

$$\ell(I) := \dim \oplus_{n\geqslant 0} I^n/mI^n$$

elements, and $\ell(I)$ is called the analytic spread of I. It is well-known that

$$\mathrm{ht}(I) \leqslant \ell(I) \leqslant \dim R.$$

The analytic deviation of I is defined by

$$\mathrm{ad}(I) := \ell(I) - \mathrm{ht}(I).$$

The relation type of I, $N(I)$, is defined to be the maximal degree of a minimal homogeneous generator of the ideal $\ker R[T_1, \dots, T_n] \to R[It]$, where I is n-generated and $T_j \mapsto i_jt$.

We introduce the notion of local reduction numbers of a graded algebra S over a local ring. By means of this notion we give upper bounds for the regularity of minimal reductions and the reduction number of the ideal S_+ generated by the elements of positive degree of S (cf. Theorem 3.2). These upper bounds can be used to estimate the a-invariant and the Castelnuovo–Mumford regularity of Cohen–Macaulay graded algebras and associated graded rings of ideals. As we mentioned above, this work allows us to derive general criteria for the Cohen–Macaulay and Gorenstein property of Rees algebras of ideals having arbitrary analytic deviation. We are also able to prove similar theorems relating the Serre property S_k of $R[It]$ to the Serre property S_{k-1} of $G(I)$ (see Theorem 6.8). We give numerous corollaries of our results when additional hypotheses on the ideal I are given. We also derive a bound on the relation type of an ideal I when both $R[It]$ and R are CM rings. The precise statement is as follows:

COROLLARY 6.7. *Let* (R, m) *be a CM ring with infinite residue field. If* $I \subseteq R$ *is an ideal of analytic spread* ℓ *and* $R[It]$ *is CM then*

$$N(I) \leqslant \max\{r_{\ell-1}(I), r(I) + 1\} \leqslant r_\ell(I) + 1.$$

The number $r_j(I)$ is the jth local reduction number (see the definition following Lemma 4.2).

As in the paper [AH], the bounds on reduction numbers of ideals have a great deal to do with theorems of Briançon–Skoda type (see [LT], [LS], and [AH]). The simplest version of the Briançon–Skoda theorem states:

THEOREM 1.2. *Let (R, m) be a d-dimensional regular local ring. For every ideal I and every reduction J of I, $I^d \subseteq J$.*

There are by now many versions giving more information. A basic problem in improving this result lies in understanding the coefficients in the theorem. By this we mean the following: let R be a regular local ring, and let I be an m-primary ideal. If R has an infinite residue field, then I has a reduction J generated by d elements $x_1, \dots, x_d$. Given an element $u \in I^d$, we may write

$$u = a_1 x_1 + \cdots + a_d x_d.$$

The "best" choice of the coefficients a_i is what we seek. Of course, the a_i are only defined up to trivial syzygies of the regular sequence $x_1, \dots, x_d$. The best one could hope for would be to say there is a choice of the a_i in I^{d-1}; this is exactly the case when the reduction number of I is at most $d - 1$, and this relates in turn to the Cohen–Macaulay property of the Rees algebra. In Theorem 7.6 we give a new Briançon–Skoda type theorem, valid for regular local rings containing a field, which gives very precise information about a choice of the coefficients a_i. The proof of Theorem 7.6 uses reduction to characteristic p, and in characteristic p uses ideas coming from the theory of tight closure. This material is of independent interest, and does not use the results in the rest of this paper. We refer the reader to Theorem 7.6 for the exact statement, which is somewhat technical. Although this section may be read independently from the rest of the paper, in Section 8 we apply our new Briançon–Skoda theorem to give bounds on the reduction number of an ideal I in an equicharacteristic regular local ring when $G(I)$ satisfies Serre's property S_ℓ, where $\ell = \ell(I)$. These results strengthen Theorem 5.1 when the graded ring $G(I)$ is not necessarily CM.

We now give a short summary of what is in each section. Section 2 contains basic definitions and material which we will need in the proofs of the main theorems in Section 5. In Section 2 we study standard graded algebras and develop graded versions of various local ideas such as analytic spread, reduction numbers, etc. Of particular note and importance in the proofs is the concept of a filter-regular sequence due to Trung. Section 3 contains the main theorems relating the local reduction numbers of a graded ring to its regularity and a-invariants. This section contains the heart of the proofs of the main results. In Section 4 we translate the local data coming from a local ring R and ideal I to the graded data of $G(I)$ and $R[It]$. Section 5 gives the proofs of the main results connecting the Rees and graded algebras, and gives many corollaries of these results. Section 6 gives information concerning the Serre properties of the Rees and graded rings, bounds on the Castelnuovo–Mumford regularity of the Rees algebra, and bounds on the relation type of I in terms of the local reduction numbers when R and $R[It]$ are CM. Section 7 gives our new Briançon–Skoda theorem, and Section 8 gives applications of it.

2. Filter-regular sequences and minimal reductions of a graded algebra

In this section we collect some basic results on filter-regular sequences and minimal reductions of graded algebras which we shall need later in our study of the Cohen–Macaulay property of the associated graded ring and the Rees algebra.

By a graded algebra we always mean a graded algebra $S = \oplus_{n\geqslant 0} S_n$ over a local ring S_0 with infinite residue field such that $S = S_0[S_1]$ (standard graded algebra). We denote by M the maximal graded ideal of S and by $S_+ = \oplus_{n>0} S_n$ the ideal generated by all homogeneous elements of positive degree of S. As usual, for any graded module L over S, and any integer n, L_n denotes the n-graded part of L.

A system of homogeneous elements $z_1, \ldots, z_r$ of S is called a *filter-regular sequence* on S (with respect to S_+) if $z_i \notin P$ for any prime $P \in \operatorname{Ass}(S/(z_1, \ldots, z_{i-1}))$, $P \not\supseteq S_+$, $i = 1, \ldots, r$. This notion originated from the theory of generalized Cohen–Macaulay rings [CST] where filter-regular sequences are defined for local rings with respect to the maximal ideal. We refer the reader to [CST] and [T2] for basic properties of filter-regular sequences.

Given any ideal Z of S we will put

$$\mathcal{U}(Z) := \cup_{n\geqslant 0} Z : (S_+)^n.$$

For any primary decomposition of $Z, \mathcal{U}(Z)$ is the intersection of the primary components of Z whose associated primes do not contain S_+. A sequence $z_1, \ldots, z_r$ of homogeneous elements in S is a filter-regular sequence on S if and only if

$$(z_1, \ldots, z_{i-1}) : z_i \subseteq \mathcal{U}((z_1, \ldots, z_{i-1})),$$

for $i = 1, \ldots, r$ (see e.g. [T2]).

LEMMA 2.1. *Let $z_1, \ldots, z_r$ be a filter-regular sequence on S. Then*

(i) $\mathcal{U}((z_1, \ldots, z_{r-1})) : z_r = \mathcal{U}((z_1, \ldots, z_{r-1}))$.
(ii) $\operatorname{ht}(P) \geqslant r$ *for every associated prime P of $\mathcal{U}(z_1, \ldots, z_r)$.*

Proof. Since every associated prime P of $\mathcal{U}((z_1, \ldots, z_{r-1}))$ does not contain S_+, we have that $z_r \notin P$ by the definition of filter-regular sequences. From this (i) follows. For (ii) we first note that since $P \not\supseteq S_+$, $z_i \notin Q$ for any prime $Q \in \operatorname{Ass}(S/(z_1, \ldots, z_{i-1}))$, $Q \subseteq P$, $i = 1, \ldots, r$. Therefore, $z_1, \ldots, z_r$ is a regular sequence of S_P. It follows that $\dim(S_P) \geqslant r$. Since $\operatorname{ht}(P) = \dim(S_P)$, this gives the conclusion. □

COROLLARY 2.2. *Let $z_1, \ldots, z_d \in S_+$ be a filter-regular sequence on S with respect to S_+, $d = \dim S$. Then $\sqrt{S_+} = \sqrt{(z_1, \ldots, z_d)}$.*

Proof. Set $Z = (z_1, \ldots, z_d)$. By Lemma 2.1 (ii) we have $\mathcal{U}(Z) = S$. Hence $(S_+)^n \subset Z$ for n sufficiently large. Since $Z \subseteq S_+$, the conclusion is immediate. □

LEMMA 2.3. *Let S be a Cohen–Macaulay graded algebra. Then any filter-regular sequence $z_1, \ldots, z_s$ on S with length $s = \operatorname{ht}(S_+)$ is a regular sequence.*

Proof. We will prove by induction that $z_1, \dots, z_i$ is a regular sequence for $i = 0, \dots, s$. For $i = 0$ there is nothing to prove. For $i > 0$, we may assume that $z_1, \dots, z_{i-1}$ is a regular sequence of S. Let $P \in \mathrm{Ass}(S/(z_1, \dots, z_{i-1}))$. Since S is Cohen–Macaulay, $\mathrm{ht}(P) = i - 1$. Since $\mathrm{ht}(S_+) = s > i - 1$, $P \not\supseteq S_+$. By the definition of filter-regular sequences, $z_i \notin P$. Hence $z_1, \dots, z_i$ is also a regular sequence on S. □

A filter-regular sequence $z_1, \dots, z_r$ of S can also be characterized by its regularity. Following [T1] and [AH] we say that a sequence $z_1, \dots, z_r$ of homogeneous elements of S is $[t_1, \dots, t_r]$-*regular* if

$$[(z_1, \dots, z_{i-1}) : z_i]_n = (z_1, \dots, z_{i-1})_n,$$

for $n \geqslant t_i$, $i = 1, \dots, r$. The value of t_i is also allowed to be $-\infty$ and $+\infty$. According to [T1, Lemma 2.1] $z_1, \dots, z_r$ is filter-regular if and only if there exist integers $t_i < +\infty$ such that $z_1, \dots, z_r$ is $[t_1, \dots, t_r]$-regular.

Now we will establish some properties of minimal reductions of the ideal S_+.

LEMMA 2.4. *An ideal Z generated by 1-forms is a reduction of S_+ if and only if $\sqrt{S_+} = \sqrt{Z}$. In this case,*

$$r_Z(S_+) = \min\{n | S_{n+1} = Z_{n+1}\},$$

The proof is left to the reader. □

As in the local case [NR] one can show that a minimal reduction of S_+ is minimally generated by

$$\ell(S_+) := \dim \left(\bigoplus_{n \geqslant 0} S_n/\mathfrak{m}S_n \right)$$

elements, where $\mathfrak{m}$ denotes the maximal ideal of S_0. We call $\ell(S_+)$ the analytic spread of S_+. By [T1, Lemma 3.1] every minimal reduction of S_+ can be minimally generated by a filter-regular sequence of S. Hence we can introduce the following definitions.

DEFINITION 2.5. Let Z be a minimal reduction of S_+. We call the least number t such that there exists a minimal generating set $z_1, \dots, z_\ell$ of Z which is $[t, \dots, t]$-regular (resp. $[t+1, \dots, t+\ell]$-regular) the *regularity* (resp. the *sliding regularity*) of Z. This number will be denoted by $a(Z)$ (resp. $s(Z)$).

The numbers $s(Z)$, $a(Z)$ and $\ell(S_+)$ play a crucial role in the estimation of the a-invariant and the Castelnuovo–Mumford regularity of S. Recall that the a-invariant of S [GW] is defined by

$$a(S) := \max\{n \mid [H^d_M(S)]_n \neq 0\},$$

where $d := \dim S$ and $H^d_M(S)$ denotes the dth local cohomology module of S with respect to the maximal graded ideal M. The Castelnuovo–Mumford regularity of S [Mu] is defined by

$$\operatorname{reg}(S) := \max_{i \geqslant 0}\{\min\{t \mid [H^i_{S_+}(S)]_{n-i} = 0 \text{ for } n \geqslant t\} + i\} - 1,$$

where $H^i_{S_+}(S)$ denotes the ith local cohomology module of S with respect to S_+.

A special case of [T3, Theorem 2.2] and [T3, Corollary 2.3] is

THEOREM 2.6. *Let S be a Cohen–Macaulay graded algebra and let Z be a minimal reduction of S_+. Then $a(S) \leqslant s(Z)$ if and only if $r_Z(S_+) \leqslant \ell(S_+) + s(Z)$. In addition, for any $b > s(Z)$, $a(S) = b$ if and only if $r(S_+) = \ell(S_+) + b$.*

PROPOSITION 2.7. *Let S be a Cohen–Macaulay graded algebra. Let Z be a minimal reduction of S_+. Then*

$$\max\{s(Z),\, a(S)\} = \max\{s(Z),\, r_Z(S_+) - \ell(S_+)\}.$$

Proof. If $a(S) \leqslant s(Z)$ then $r_Z(S_+) - \ell(S_+) \leqslant s(Z)$ by Theorem 2.6 so both sides are equal to $s(Z)$. Suppose $a(S) > s(Z)$. Then by Theorem 2.6, $r_Z(S_+) - \ell(S_+) = a(S)$ and the equality again holds. □

REMARK 2.8. One can divide the above relation into two parts:
(1) $r_Z(S_+) \leqslant \max\{s(Z),\, a(S)\} + \ell(S_+)$.
(2) $a(S) \leqslant \max\{s(Z),\, r_Z(S_+) - \ell(S_+)\}$.

COROLLARY 2.9. *Let S be a Cohen–Macaulay graded algebra. Let Z be a minimal reduction of S_+. If Y is any minimal reduction of S_+ such that $r(S_+) = r_Y(S_+)$ then*

$$r_Z(S_+) \leqslant \max\{s(Z) + \ell(S_+),\, r(S_+),\, s(Y) + \ell(S_+)\}.$$

Proof. By applying Proposition 2.7 with both Z and Y we have $r_Z(S_+) \leqslant \max\{s(Z),\, a(S)\} + \ell(S_+) \leqslant \max\{s(Z),\, s(Y),\, r_Y(S_+) - \ell(S_+)\} + \ell(S_+)$ since $a(S) \leqslant \max\{s(Y),\, r_Y(S_+) - \ell(S_+)\}$. □

PROPOSITION 2.10. [T1, Corollary 3.3]. *Let S be a graded algebra and Z a minimal reduction of S_+. Then*

$$\operatorname{reg}(S) \leqslant \max\{a(Z) - 1,\, r_Z(S_+)\}.$$

REMARK 2.11. Mumford [Mu] showed that $\operatorname{reg}(S)$ gives upper bounds for the degrees of the generators and of all higher order syzygies of the defining ideal of S in a polynomial ring over S_0. In particular, the maximum degree of the elements of a homogeneous minimal basis of the defining ideal of S is bounded above by $\operatorname{reg}(S) + 1$ (see also [EG], [O]).

3. Regularity and local reduction numbers of a graded algebra

Let $S = \oplus_{n\geqslant 0} S_n$ be a (standard) graded algebra over a local ring S_0 with infinite residue field. As we have seen in Propositions 2.7 and 2.10 the regularities and the reduction numbers of minimal reductions of S_+ provide information on the a-invariant and the Castelnuovo–Mumford regularity of S. In this section we will estimate these invariants in terms of simpler invariants of S.

Given a prime ideal $\mathfrak{p}$ of S_0 we will denote by $S_\mathfrak{p}$ the localization of S at the multiplicatively closed set $S_0\backslash\mathfrak{p}$. $S_\mathfrak{p} = \oplus_{n\geqslant 0}(S_n)_\mathfrak{p}$ is a graded algebra over the local ring $(S_0)_\mathfrak{p}$. If $P = \mathfrak{p}S + S_+$ where $\mathfrak{p} \in \operatorname{Spec} S_0$ then $\dim S_\mathfrak{p} = \dim S_P = \operatorname{ht} P$. Set $s = \operatorname{ht}(S_+)$, $\ell = \ell(S_+)$. We introduce the following invariants:

$$r_i(S) := \begin{cases} i - s, & i < s, \\ \max\{r((S_\mathfrak{p})_+) - \operatorname{ht}(\mathfrak{p}) \mid \ell((S_\mathfrak{p})_+) = \operatorname{ht}(\mathfrak{p}) + s \leqslant i\} + i - s, & s \leqslant i \leqslant \ell \end{cases}$$

which we call the *ith local reduction number* of S. (Note that we allow i to take negative values.)

REMARK 3.1.

(1) For $i = s, \ldots, \ell$ we have the following relation between $r_{i-1}(S)$ and $r_i(S)$:

$$r_i(S) = \max\{r_{i-1}(S) + 1,\ r((S_\mathfrak{p})_+) \mid \ell((S_\mathfrak{p})_+) = \operatorname{ht}(\mathfrak{p}) + s = i\}.$$

(2) $r_0(S) < \cdots < r_\ell(S)$ is a strictly increasing sequence of numbers.

(3) If $\mathfrak{p} \in \operatorname{Spec} S_0$ then $r_i(S_\mathfrak{p}) \leqslant r_i(S)$ for $i \leqslant \ell((S_\mathfrak{p})_+)$.

THEOREM 3.2. *Let S be a Cohen–Macaulay graded algebra. Set $\ell = \ell(S_+)$ and let $Z = (z_1, \ldots, z_\ell)$ be a minimal reduction of S. Then*

(i) $z_1, \ldots, z_\ell$ *is* $[r_0(S) + 1, \ldots, r_{\ell-1}(S) + 1]$*-regular for any filter-regular sequence* $z_1, \ldots, z_\ell$ *of S which generates Z.*

(ii) $r_Z(S_+) \leqslant r_\ell(S)$.

We will prove Theorem 3.2 by proving:

$$(C_i)\colon [(z_1, \ldots, z_i)\colon z_{i+1}]_n = (z_i, \ldots, z_i)_n \ \text{ for } \ n \geqslant r_i(S) + 1 \ \ 0 \leqslant i < \ell.$$
$$(C_\ell)\colon r_Z(S_+) \leqslant r_\ell(S).$$

For each i such that $0 \leqslant i \leqslant \ell$, fix a primary decomposition of the ideal $(z_1, \ldots, z_i)$ and let U_i be the intersection of the primary components which contain S_+ and have height at most i. Also, let $V_i = \mathcal{U}((z_1, \ldots, z_i))$.

For the proof of Theorem 3.2 we need the following lemmas.

LEMMA 3.3. *Let $U \subseteq L$ be two graded ideals of S. For an integer $n \geqslant 0$, $U_n = L_n$ if and only if $[U_\mathfrak{p}]_n = [L_\mathfrak{p}]_n$ for every prime $\mathfrak{p}$ of S_0 which is the contraction of an associated prime of U.*

Proof. ($\Rightarrow$) is trivial. To see ($\Leftarrow$) we have to show that $L_n \subseteq U_n$. For this it suffices to show that $L_n \subseteq Q_n$ for every primary component Q of U. Let $\mathfrak{p}$ be the contraction in S_0 of the associated prime of Q. Then $[L_n]_\mathfrak{p} = [L_\mathfrak{p}]_n = [U_\mathfrak{p}]_n \subseteq [Q_n]_\mathfrak{p}$. From this is follows that $L_n \subseteq Q_n$. □

LEMMA 3.4. *Let S be a CM graded algebra with $s = \mathrm{ht}(S_+)$ and $\ell = \ell(S)$. Let $Z = (z_1, \dots, z_\ell)$ be a minimal reduction of S_+. Fix i such that $s \leqslant i \leqslant \ell$. Assume that the sequence $z_1, \dots, z_\ell$ satisfies (C_j) for $0 \leqslant j < i$. Let $P \supseteq S_+$ be a prime ideal of S with $\mathrm{ht}P > i$. Let $\mathfrak{p}$ be the contraction of P to S_0. Then*

$$[H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \dots, z_i)_\mathfrak{p})]_n = 0,$$

for $n \geqslant r_{i-1}(S) + 2$, $k < \mathrm{ht}(P) - i$.

Proof. We show the result by induction. Let $r_j = r_j(S)$ for $0 \leqslant j \leqslant \ell$. Notice that $P = \mathfrak{p}S + S_+$. The case $i = s$ follows from the fact that $S/(z_1, \dots, z_s)$ is a Cohen–Macaulay ring. For, since $S_\mathfrak{p}/(z_1, \dots, z_s)_\mathfrak{p}$ is also a Cohen–Macaulay ring with dimension $\mathrm{ht}(P) - s$

$$H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \dots, z_s)_\mathfrak{p}) = 0,$$

for $k < \mathrm{ht}(P) - s$.

If $j \geqslant s + 1$, we consider the exact sequence

$$\begin{aligned} 0 \to [(z_1, \dots, z_{j-1}) : z_j]/(z_1, \dots, z_{j-1}) &\to S_+/(z_1, \dots, z_{j-1}) \\ &\to S/[(z_1, \dots, z_{j-1}) : z_j] \to 0. \end{aligned}$$

Localized at $\mathfrak{p}$ this induces the following exact sequence of local cohomology modules

$$\begin{aligned} &[H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \dots, z_{j-1})_\mathfrak{p})]_n \\ &\quad \to [H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/((z_1, \dots, z_{j-1}) : z_j)_\mathfrak{p})]_n \\ &\quad \to [H^{k+1}_{P_\mathfrak{p}}(([(z_1, \dots, z_{j-1}) : z_j]/(z_1, \dots, z_{j-1}))_\mathfrak{p})]_n. \end{aligned}$$

By the inductive hypothesis we have

$$[H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \dots, z_{j-1})_\mathfrak{p})]_n = 0,$$

for $n \geqslant r_{j-2} + 2$, $k < \mathrm{ht}(P) - j + 1$ (this holds even if $j = 1$ and $s = 0$). Further, (C_{j-1}) is satisfied. Hence

$$[H^m_{P_\mathfrak{p}}([(z_1, \dots, z_{j-1})_\mathfrak{p} : z_j]/(z_1, \dots, z_{j-1})_\mathfrak{p})]_n = 0,$$

for $n \geqslant r_{j-1} + 1$ and m arbitrary by Lemma 3.7 below. By Remark 3.1, $r_{j-2} + 2 \leqslant r_{j-1} + 1$, so we can deduce that

$$[H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/((z_1, \dots, z_{j-1}) : z_j)_\mathfrak{p})]_n = 0,$$

for $n \geqslant r_{j-1}+1$, $k < \mathrm{ht}(P) - j + 1$. On the other hand, the exact sequence

$$\begin{aligned} 0 \to S/[(z_1, \ldots, z_{j-1}) : z_j](-1) &\xrightarrow{z_j} S/(z_1, \ldots, z_{j-1}) \\ &\longrightarrow S/(z_1, \ldots, z_j) \to 0 \end{aligned}$$

yields the following exact sequence of local cohomology modules

$$\begin{aligned} [H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \ldots, z_{j-1})_\mathfrak{p})]_n &\to [H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \ldots, z_j)_\mathfrak{p})]_n \\ &\to [H^{k+1}_{P_\mathfrak{p}}(S_\mathfrak{p}/((z_1, \ldots, z_{j-1}) : z_i)_\mathfrak{p})]_{n-1}. \end{aligned}$$

Hence we can conclude that $[H^k_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \ldots, z_j)_\mathfrak{p})]_n = 0$, for $n \geqslant r_{j-1}+2$, $k < \mathrm{ht}(P) - j$. □

LEMMA 3.5. *Let S, Z and i be as in Lemma* 3.4. *Let $d = \dim S$. Assume that (C_j) holds for $0 \leqslant j < i$. Then $(z_1, \ldots, z_i)_n = [U_i \cap V_i]_n$ for $n \geqslant r_{i-1}(S)+2$.*

Proof. Let $r_j = r_j(S)$ for $0 \leqslant j \leqslant \ell$. For every prime ideal $P \supseteq S_+$ of S let $U(P)$, resp. $U_0(P)$, denote the intersection of the primary components of $(z_1, \ldots, z_i)$ (from our fixed primary decomposition) whose associated primes are contained, resp. properly contained, in P ($U(P) = S$, resp. $U_0(P) = S$, if there are no such primes). Let $\mathfrak{p}$ be the contraction of P in S_0. Then

$$U_0(P)_\mathfrak{p}/U(P)_\mathfrak{p} = H^0_{P_\mathfrak{p}}(S_\mathfrak{p}/(z_1, \ldots, z_i)_\mathfrak{p}).$$

This can be seen easily after localizing at $\mathfrak{p}$, so that $\mathfrak{p} = m$ and $P = m \oplus S_+$. Note that then $U(P) = (z_1, \ldots, z_i)$ so

$$U_0(P)/U(P) = \left[\bigcup_n (z_1, \ldots, z_i) : P^n\right] / (z_1, \ldots, z_i) = H^0_P(S/(z_1, \ldots, z_i)).$$

If $\mathrm{ht}(P) > i$, Lemma 3.4 applied with $k = 0$ implies that $[U(P)_\mathfrak{p}]_n = [U_0(P)_\mathfrak{p}]_n$ for $n \geqslant r_{i-1}+2$. Therefore $[U(P)_\mathfrak{q}]_n = [U_0(P)_\mathfrak{q}]_n$ for any prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$ and $n \geqslant r_{i-1}+2$. By Lemma 3.3, this means that

$$U(P)_n = U_0(P)_n, \tag{$*$}$$

for $n \geqslant r_{i-1}+2$.

For every integer $j \geqslant i$, let W_j be the intersection of the (fixed) primary components of $(z_1, \ldots, z_i)$ whose associated primes contain S_+ and have height $\leqslant j$ ($W_i = U_i$). It is easily seen that $(z_1, \ldots, z_i) = W_d \cap V_i$ and

$$W_j \cap V_i = \bigcap_{P \supseteq S_+,\, \mathrm{ht}(P)=j} U(P) \cap V_i,$$

$$W_{j-1} \cap V_i = \bigcap_{P \supseteq S_+,\, \mathrm{ht}(P)=j} U_0(P) \cap V_i.$$

According to $(*)$, $[W_j \cap V_i]_n = [W_{j-1} \cap V_i]_n$ for $n \geqslant r_{i-1} + 2$, $j = i+1, \dots, d$. Therefore we can conclude that $(z_1, \dots, z_i)_n = [U_i \cap V_i]_n$ for $n \geqslant r_{i-1} + 2$. □

Proof of Theorem 3.2. We will use induction on $d = \dim(S)$. If $d = 0$ then $Z = 0$ and $s = \ell = 0$ so we need only to show (C_0). This holds since $r_0(S) = r(S_+)$ in this case.

Assume now that $d > 0$. We will prove by induction that (C_i) holds whenever $i < d$. If $i < s$ then this is clear. Assume that $i \geqslant s$ and assume by induction that (C_j) holds for $0 \leqslant j < i$.

By Lemma 3.5 we know that $(z_1, \dots, z_i)_n = [U_i \cap V_i]_n$ for $n \geqslant r_i(S) + 1$ since $r_i(S) + 1 \geqslant r_{i-1}(S) + 2$. Hence we must show that $[U_i \cap V_i]_n : z_{i+1} = [U_i \cap V_i]_n$ for $n \geqslant r_i(S) + 1$. By Lemma 2.1(i), $V_i : z_{i+1} = V_i$. We claim that $[U_i]_n = S_n$ for $n \geqslant r_i(S) + 1$. Let $P \in \mathrm{Ass}(S/U_i)$ and let $\mathfrak{p}$ be the contraction of P to S_0. Since $P \supseteq S_+$, $\mathfrak{p}$ is not the maximal ideal of S_0. Let $k = \min\{i, \ell((S_\mathfrak{p})_+)\}$. Applying our induction hypothesis to $S_\mathfrak{p}$ we know that $r_{(z_1,\dots,z_i)_\mathfrak{p}}((S_\mathfrak{p})_+) \leqslant r_k(S_\mathfrak{p}) \leqslant r_i(S)$. The last inequality follows from Remark 3.1 (3). This shows that $[(U_i)_\mathfrak{p}]_n = [S_\mathfrak{p}]_n$ for $n \geqslant r_i(S) + 1$, so $[U_i]_n = S_n$ for $n \geqslant r_i(S) + 1$ by Lemma 3.3. Hence for $n \geqslant r_i(S) + 1$,

$$\begin{aligned}[U_i \cap V_i]_n : z_{i+1} &= ([U_i]_n : z_{i+1}) \cap ([V_i]_n : z_{i+1}) \\ &= (S_n : z_{i+1}) \cap [V_i]_n = S_n \cap [V_i]_n = [U_i \cap V_i]_n,\end{aligned}$$

which shows (C_i) if $i < \ell$. Note that knowing (C_j) holds for $j < \ell$ implies that $s(Z) \leqslant r_{l-1}(S) - \ell + 1$, and then Corollary 2.9 gives that

$$r_Z(S_+) \leqslant \max\{r(S_+), r_{\ell-1}(S) + 1\} \leqslant \max\{r(S_+), r_\ell(S)\},$$

hence (C_ℓ) holds. □

We obtain from Theorem 3.2 the following estimates for the regularities and the reduction number of minimal reductions of S_+ in terms of the local reduction numbers.

COROLLARY 3.6. *Let S be a Cohen–Macaulay graded algebra. Let Z be an arbitrary minimal reduction of S_+. Set $\ell = \ell(S_+)$. Then*

(i) $s(Z) \leqslant r_{\ell-1}(S) - \ell + 1$.
(ii) $a(Z) \leqslant r_{\ell-1}(S) + 1$.
(iii) $\max\{r_{\ell-1}(S) + 1, r_Z(S_+)\} = r_\ell(S)$.

Proof. (i) and (ii) follow from Theorem 3.2(i) and Remark 3.1 (2). For (iii) we have $r((S_\mathfrak{p})_+) \leqslant r(S_+)$ for every prime ideal $\mathfrak{p}$ of S_0 with $\ell((S_\mathfrak{p})_+) = \mathrm{ht}(\mathfrak{p}) + s = \ell$

because the image of a minimal reduction of S_+ in $S_\mathfrak{p}$ is also a minimal reduction of $(S_\mathfrak{p})_+$. By the definition of $r_i(S)$ this implies

$$r_\ell(S) \leqslant \max\{r_{\ell-1}(S)+1,\ r(S_+)\} \leqslant \max\{r_{\ell-1}(S)+1,\ r_Z(S_+)\}.$$

On the other hand, we have $r_Z(S_+) \leqslant r_\ell(S)$ by Theorem 3.2 (ii). Since $r_{\ell-1}(S) < r_\ell(S)$, we get $\max\{r_{\ell-1}(S)+1,\ r_Z(S_+)\} \leqslant r_\ell(S)$. □

LEMMA 3.7. *Let S be an $\mathbb{N}$-graded Noetherian ring and $I \subseteq R$ a homogeneous ideal of S. If M is a graded S-module such that $M_n = 0$ for all $n \geqslant j$ then $[H_I^k(M)]_n = 0$ for all $n \geqslant j$ and for all k.*

Proof. Let $f_1, \ldots, f_t$ be homogeneous generators of I. Then $H_I^k(M)$ is a quotient of a submodule of the direct sum of the $\mathbb{Z}$-graded modules $M_{f_{i_1}\cdots f_{i_k}}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant t$. Since $\deg f_i \geqslant 0$, these modules vanish in degree j and higher. □

4. Local reduction numbers of an ideal

Throughout this section let (R, m) be a local Noetherian ring with infinite residue field. We will apply the results of Section 3 to the associated graded ring $G(I) := \oplus_{n\geqslant 0} I^n/I^{n+1}$ of an ideal I in R. For this we need to make some observations on the relationship between certain invariants of $G(I)_+$ and I.

LEMMA 4.1. *The following relations hold for any ideal I in R:*

(i) $\ell(G(I)_+) = \ell(I)$.

(ii) $r_Z(G(I)_+) = r_J(I)$ *whenever Z is generated by the initial forms of the elements of J in $G(I)$.*

(iii) $r(G(I)_+) = r(I)$.

Proof. Set $S = G(I)$. For (i) let $\mathfrak{m}$ denote the maximal ideal m/I of the local ring $R/I = S_0$. It is clear that $S_n/\mathfrak{m}S_n = I^n/mI^n$ for all $n \geqslant 0$. Therefore

$$\ell(S_+) = \dim\left(\bigoplus_{n\geqslant 0} S_n/\mathfrak{m}S_n\right) = \dim\left(\bigoplus_{n\geqslant 0} I^n/mI^n\right) = \ell(I).$$

For (ii) we note that $r_Z(S_+)$ is the least integer n for which $Z_{n+1} = S_{n+1}$. This is equivalent to $JI^n + I^{n+2} = I^{n+1}$ or, by Nakayama's Lemma, $JI^n = I^{n+1}$ so that $r_Z(S_+) = r_J(I)$. Finally (iii) follows from (ii) because every minimal reduction Z of S_+ is generated by the initial forms of the elements of a minimal reduction J of I and conversely. □

Let $L(I)$ denote the set of all prime ideals $\wp \supseteq I$ of R with $\ell(I_\wp) = \mathrm{ht}(\wp)$. This set is relatively small by the following lemma.

LEMMA 4.2. *$L(I)$ is a finite set and* $\mathrm{ht}(\wp) \leqslant \ell(I)$ *for every* $\wp \in L(I)$.

Proof. $L(I)$ is finite because $L(I)$ is contained in the set $\bar{A}^*(I) := \cup_{n\geqslant 1}\mathrm{Ass}(R/(\overline{I^n}))$, where $(\overline{I^n})$ denotes the integral closure of I^n [Mc, Proposition 4.1] and $\bar{A}^*(I)$ is a finite set. For the second statement we note that every minimal reduction of an ideal is minimally generated by a number of elements which is equal to the analytic spread. Therefore, since the image of every minimal reduction of I in $R_\wp$ is a reduction of $I_\wp$, $\ell(I) \geqslant \ell(I_\wp) = \mathrm{ht}(\wp)$. □

Put $s = \mathrm{ht}(I)$ and $\ell = \ell(I)$. We introduce the following invariants:

$$r_i(I) := \begin{cases} i - s, & i < s, \\ \max\{r(I_\wp) - \mathrm{ht}(\wp) | \wp \in L(I) \text{ and } \mathrm{ht}(\wp) \leqslant i\} + i, & i = s, \ldots, \ell, \end{cases}$$

which we call the *ith local reduction number of I*. Note that

$$r_s(I) = \max\{r(I_\wp) | \wp \supseteq I \text{ is a prime ideal of } R \text{ with } \mathrm{ht}(\wp) = s\}.$$

LEMMA 4.3. *Assume that $G(I)$ is equidimensional and catenary (for instance, this happens if R is universally catenary and equidimensional). Then*

$$r_i(G(I)) = r_i(I),$$

for $i = 0, \ldots, \ell(I)$.

Proof. Set $S = G(I)$. For every prime ideal $\wp \supseteq I$ of R we denote by $\mathfrak{p}$ its image in $R/I = S_0$. This gives a one-to-one correspondence between prime ideals of R which contain I and prime ideals of S_0. Since $S_\mathfrak{p} = G(I_\wp)$, we have $\ell(I_\wp) = \ell((S_\mathfrak{p})_+)$ and $r(I_\wp) = r((S_\mathfrak{p})_+)$ by Lemma 4.1. If S is equidimensional and catenary, then $\mathrm{ht}(\wp) = \mathrm{ht}(\mathfrak{p}) + s$, hence

$$r((S_\mathfrak{p})_+) - \mathrm{ht}(\mathfrak{p}) + i - s = r(I_\wp) - \mathrm{ht}(\wp) + i.$$

By the definitions of the local reduction numbers of S and I, this implies $r_i(S) = r_i(I)$ for $i = 0, \ldots, \ell(I)$. □

THEOREM 4.4. *Assume that $G(I)$ is a Cohen–Macaulay ring. Set $\ell = \ell(I)$ and let J be an arbitrary minimal reduction of I. Then*

(i) $\max\{r_{\ell-1}(I) + 1, r_J(I)\} = r_\ell(I)$.

(ii) $\max\{r_{\ell-1}(I) + 1, a(G(I)) + \ell\} = r_\ell(I)$.

Proof. Set $S = G(I)$ and let Z be the minimal reduction of S_+ generated by the initial forms of the elements of J in S. By Corollary 3.6(iii) and using Lemma 4.1 and Lemma 4.3 we have

$$\begin{aligned} \max\{r_{\ell-1}(I) + 1, r_J(I)\} &= \max\{r_{\ell-1}(S) + 1, r_Z(S_+)\} \\ &= r_\ell(S) = r_\ell(I) \end{aligned}$$

which proves (i). For (ii) we have

$$\max\{s(Z)+\ell,\ a(S)+\ell\} = \max\{s(Z)+\ell,\ r_Z(S_+)\}$$

by Proposition 2.7. By Corollary 3.6 (i), $s(Z)+\ell \leqslant r_{\ell-1}(S)+1$. Hence

$$\max\{r_{\ell-1}(S)+1,\ a(S)+\ell\} = \max\{r_{\ell-1}(Z)+1,\ r_Z(S_+)\}.$$

Using Lemma 4.1 and Lemma 4.3 we can interpret this relation as

$$\max\{r_{\ell-1}(I)+1,\ a(S)+\ell\} = \max\{r_{\ell-1}(I)+1,\ r_J(I)\}$$

which together with (i) proves (ii). □

COROLLARY 4.5. *Assume that $G(I)$ is a Cohen–Macaulay ring. Set $\ell = \ell(I)$. If there exists a minimal reduction J of I such that $r(I_\wp) - \mathrm{ht}(\wp) < r_J(I) - \ell$ for every prime ideal $\wp \supseteq I$ with $\ell(I_\wp) = \mathrm{ht}(\wp) < \ell$, then*

$$a(G(I)) = r_J(I) - \ell.$$

Proof. The assumption on J means that $r_J(I) > r_{\ell-1}(I)+1$. By Theorem 4.4 we get

$$r_J(I) = r_\ell(I) = \max\{r_{\ell-1}(I)+1,\ a(G(I))+\ell\}$$

which implies $r_J(I) = a(G(I))+\ell$. □

DEFINITION 4.6. Let c be an integer. We say that I has *sliding reduction number* c if $r_i(I) = c+i-\mathrm{ht}(I)$, for $i = \mathrm{ht}(I), \ldots, \ell(I)-1$.

REMARK 4.7. Set $\ell = \ell(I)$ and $s = \mathrm{ht}(I)$. If $\ell = s$, i.e. I is an equimultiple ideal, then I can have any sliding reduction number. If $\ell > s$, i.e. $\mathrm{ad}(I) > 0$, then I has sliding reduction number c if and only if

$$\begin{aligned} &r(I_\wp) = c \ \text{ for some prime } \wp \supseteq I \ \text{ with } \mathrm{ht}(\wp) = s \ \text{ and} \\ &r(I_\wp) \leqslant c + \mathrm{ht}(\wp) - s \ \text{ for every prime } \wp \supseteq I \ \text{ with } \ell(I_\wp) = \mathrm{ht}(\wp) \leqslant \ell - 1. \end{aligned}$$

Recall that the $r_i(I)$'s are strictly increasing. Having sliding reduction number means that the first $\ell - 1$ increase by exactly one, the least possible. For example, I has sliding reduction number 0 if $I_\wp$ is a complete intersection for every prime ideal $\wp \supseteq I$ with $\mathrm{ht}(\wp) = \ell(I) - 1$.

The following result shows that if I has sliding reduction number, then the a-invariant of $G(I)$ can be computed by means of the (global) reduction numbers.

THEOREM 4.8. *Assume that $G(I)$ is a Cohen–Macaulay ring and I has sliding reduction number c. Then for any minimal reduction J of I*

$$a(G(I)) = \max\{c - \mathrm{ht}(I),\ r_J(I) - \ell(I)\}.$$

Proof. Set $S = G(I)$, $s = \mathrm{ht}(I)$, and $\ell = \ell(I)$. We have $r_i(I) = c + i - s$, $i = s, \ldots, \ell - 1$. If $r_J(I) - \ell > c - s$, then $r_J(I) > r_{\ell-1}(I) + 1$, which is equivalent to having $r(I_\wp) - \mathrm{ht}(\wp) < r_J(I) - \ell$ for every prime $\wp \supseteq I$ with $\ell(I_\wp) = \mathrm{ht}(\wp) < \ell$. Hence $a(S) = r_J(I) - \ell$ by Corollary 4.5. If $r_J(I) - \ell \leqslant c - s$, then $r_J(I) \leqslant r_{\ell-1}(I) + 1$. By Theorem 4.4 we have $r_{\ell-1}(I) + 1 = r_\ell(I) \geqslant a(S) + \ell$, hence

$$c - s = r_{\ell-1}(I) - \ell + 1 \geqslant a(S).$$

On the other hand, by [T3, Lemma 2.4] we know that

$$a(S) \geqslant \max\{r((S_+)_P) | P \in \mathrm{Ass}(S/S_+),\ \mathrm{ht}(P) = s\} - s.$$

Let $\mathfrak{p}$ be the contraction of such a prime P in S_0. Then $P = \mathfrak{p} \oplus S_+$ because $P \supseteq S_+$. Since $\mathrm{ht}(P) = s = \mathrm{ht}(S_+)$, we have $\mathrm{ht}(\mathfrak{p}) = 0$. It is easily seen that $\ell((S_+)_P) = \ell((S_\mathfrak{p})_+) = s$ and $r((S_+)_P) = r((S_\mathfrak{p})_+)$. Therefore

$$\begin{aligned} &\max\{r((S_+)_P) | P \in \mathrm{Ass}(S/S_+),\ \mathrm{ht}(P) = s\} \\ &\quad = \max\{r((S_\mathfrak{p})_+) | \mathfrak{p} \in \mathrm{Spec}(S_0) \text{ with } \ell((S_\mathfrak{p})_+) = s,\ \mathrm{ht}(\mathfrak{p}) = 0\} = r_s(S). \end{aligned}$$

Therefore $a(S) \geqslant r_s(S) - s$. By Lemma 4.3, $r_s(S) = r_s(I)$ so that $a(S) \geqslant r_s(I) - s = c - s$. Hence $a(S) = c - s$. □

REMARK 4.9. Suppose that I has sliding reduction number c. The formula for $a(G(I))$ in Theorem 4.7 can be reformulated as follows:

$$a(G(I)) = \min\{n \geqslant c + \mathrm{ad}(I) | JI^n = I^{n+1}\} - \ell(I).$$

This formula has been proved for $\mathrm{ad}(I) = 1$ [T3, Corollary 3.2] (cf. [GH, Proposition 2.4] for the case I is generically a complete intersection) and for $\mathrm{ad}(I) = 2$ and $I_\wp$ is a complete intersection for every prime $\wp \supseteq I$ with $\mathrm{ht}(\wp) = \mathrm{ht}(I) + 1$ [GN1, Theorem 1.3], [T3, Corollary 4.2].

Theorem 4.7 has the following interesting consequence on the invariance of the reduction numbers of minimal reductions.

COROLLARY 4.10. *Assume that $G(I)$ is a Cohen–Macaulay ring. If I has sliding reduction number less than or equal to $r(I) - \mathrm{ad}(I)$, then*

$$r_J(I) = a(G(I)) + \ell(I)$$

for any minimal reduction J of I.

Proof. If $c \leqslant r(I) - \mathrm{ad}(I) = r(I) - \ell(I) + \mathrm{ht}(I)$ then $\max\{c - \mathrm{ht}(I),\ r_J(I) - \ell(I)\} = r_J(I) - \ell(I)$, giving the desired equality. □

When $G(I)$ is CM we can bound the Castelnuovo–Mumford regularity index of $G(I)$ in terms of the local reduction numbers.

PROPOSITION 4.11. *Assume that* $G(I)$ *is a Cohen–Macaulay ring. Set* $\ell = \ell(I)$ *and let* J *be an arbitrary minimal reduction of* I. *Then*

$$\operatorname{reg}(G(I)) \leqslant \max\{r_{\ell-1}(I),\, r_J(I)\} \leqslant r_\ell(I).$$

Proof. Set $S = G(I)$ and let Z be the minimal reduction of S_+ generated by the initial forms of the elements of J in S. By Proposition 2.10 and Corollary 3.6(ii) we have

$$\operatorname{reg}(S) \leqslant \max\{a(Z) - 1,\, r_Z(S_+)\} \leqslant \max\{r_{\ell-1}(S),\, r_Z(S_+)\}.$$

Using Lemma 4.1 and Lemma 4.3 this can be interpreted as $\operatorname{reg}(S) \leqslant \max\{r_{\ell-1}(I),\, r_J(I)\}$, which together with Theorem 4.4(i) implies the conclusion. □

5. Cohen–Macaulay and Gorenstein properties of Rees algebras

Throughout this section let (R, m) be a local Noetherian ring with infinite residue field. We will derive criteria for the Cohen–Macaulay and Gorenstein properties of the Rees algebra $R[It] := \oplus_{n\geqslant 0} I^n t^n$ of an ideal I in R by means of the Cohen–Macaulay and Gorenstein properties of the associated graded ring $G(I)$ and the reduction numbers of I.

THEOREM 5.1. *Let* (R, m) *be a CM ring and* I *an ideal of* R *with* $\operatorname{ht}(I) \geqslant 1$. *Then* $R[It]$ *is CM if and only if the following conditions are satisfied*:
(i) $G(I)$ *is Cohen–Macaulay*.
(ii) $r(I_\wp) \leqslant \operatorname{ht}(\wp) - 1$ *for every prime* $\wp \supseteq I$ *with* $\ell(I_\wp) = \operatorname{ht}(\wp)$.

Proof. ($\Rightarrow$) Set $S = G(I)$. By [Hu1, Proposition 1.1], [TI, Theorem 7.1] we have (i) and $a(S) < 0$. Note that $R_\wp[I_\wp t]$ is CM for every prime ideal $\wp \supseteq I$. Using induction on $d = \dim(R)$ we may assume that

$$r(I_\wp) \leqslant \operatorname{ht}(\wp) - 1 \text{ if } \ell(I_\wp) = \operatorname{ht}(\wp) \leqslant d - 1.$$

To prove (ii) we need only to consider the case $\ell(I_\wp) = \operatorname{ht}(\wp) = d$. Then $\wp = m$ and $I_\wp = I$. By the inductive assumption, $r_{d-1}(I) \leqslant d - 2$ (including the base case $d = s$). Choose $J \subseteq I$ a minimal reduction such that $r_J(I) = r(I)$. Using Theorem 4.4 we get $\max\{r_{\ell-1}(I)+1,\, r(I)\} = \max\{r_{\ell-1}(I)+1,\, a(S)+\ell\}$. Since $\ell = d$ we have

$$r(I) \leqslant \max\{r_{d-1}(I) + 1,\, a(S) + d\}.$$

This is at most $d - 1$ since $a(S) < 0$ and $r_{d-1}(I) \leqslant d - 2$.

($\Leftarrow$) Set $\ell = \ell(I)$. By (ii) we have $r_\ell(I) \leqslant \ell - 1$ and by Theorem 4.4, $a(S) + \ell \leqslant r_\ell(I)$. Thus, $a(S) \leqslant r_\ell(I) - \ell < 0$ which together with (i) implies that $R[It]$ is Cohen–Macaulay [TI, Theorem 7.1]. □

COROLLARY 5.2. *Let (R, m) be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 1$. *Assume that $I_\wp$ is a complete intersection for every prime $\wp \supset I$ with* $\mathrm{ht}(\wp) = \ell(I)$. *Then $R[It]$ is Cohen–Macaulay if and only if $G(I)$ is Cohen–Macaulay.*

Proof. Let $\wp \supseteq I$ be a prime ideal of R with $\mathrm{ht}(\wp) = \ell(I_\wp) \leqslant \ell(I)$. By the assumption, $I_\wp$ is a complete intersection, hence $\ell(I_\wp) = \mathrm{ht}(I)$ so that $\mathrm{ht}(\wp) = \mathrm{ht}(I)$. Since $r(I_\wp) = 0$, condition (ii) of Theorem 5.1 is satisfied, and the conclusion follows. □

Recall that $P^{(n)} = \{r \mid$ there exists $s \notin P$ such that $sr \in P^n\}$.

COROLLARY 5.3. *Let R be a Cohen–Macaulay ring and P a prime ideal of R with* $\mathrm{ht}(P) \geqslant 1$ *such that R_P is regular. Assume that $P^{(n)} = P^n$ for $n \geqslant 1$. Then $R[Pt]$ is Cohen–Macaulay if and only if $G(P)$ is Cohen–Macaulay.*

Proof. It is known [CN] that the assumption $P^{(n)} = P^n$ for $n \geqslant 1$ implies $\ell(P_\wp) < \mathrm{ht}(\wp)$ for every prime ideal $\wp \supsetneq P$ of R. Hence we need to check condition (ii) of Theorem 5.1 only for $\wp = P$. Since R_P is regular, PR_P is a complete intersection, hence the conclusion follows. □

Corollary 5.3 is proved for R Gorenstein in [Va3, Theorem 5.7.7].

Recently Lipman [L] has proved the following result (We refer to [LT] for a discussion of and definition of pseudo-rational rings. However note that by [LT], regular rings are pseudo-rational):

THEOREM 5.4. *Let (R, m) be a pseudo-rational local ring. Let I be any ideal of R. Then $G(I)$ is CM if and only if $R[It]$ is CM.*

Combining this result with Theorem 5.1 gives us

PROPOSITION 5.5. *Let (R, m) be a CM local ring and I an ideal of R. Assume that R_P is pseudo-rational whenever $\ell(I_P) = \dim R_P$. Then $G(I)$ is CM if and only if $R[It]$ is CM.*

Proof. [Hu, Proposition 1.1] shows that $G(I)$ is CM if $R[It]$ is CM. Assume that $G(I)$ is CM. By Theorem 5.1 we need only show that $r(I_P) \leqslant \mathrm{ht}(P) - 1$ whenever $\ell(I_P) = \mathrm{ht}(P)$. By our assumption, R_P is regular at such primes P, so by Theorem 5.4, $R_P[I_Pt]$ is CM, therefore Theorem 5.1 applied to $R_P[I_Pt]$ gives the desired conclusion. □

In fact, Lipman uses his methods to prove (5.5) also, but we noticed it independently and the proof points out the efficacy of (5.1).

A slightly different version of Theorem 5.1 is the following criterion for the Cohen–Macaulay property of $R[It]$ which involves the reduction number of I. This criterion generalizes results in the cases $\mathrm{ad}(I) = 0$ [GS], [GHO]; $\mathrm{ad}(I) = 1$ [HuHu1], [GH], [Vi], [T3]; $\mathrm{ad}(I) = 2$ [GN1], [T3].

THEOREM 5.6. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 1$. *Then $R[It]$ is Cohen–Macaulay if and only if the following conditions are satisfied*:

(i) *$G(I)$ is Cohen–Macaulay.*
(ii) $r(I_\wp) \leqslant \mathrm{ht}(\wp) - 1$ *for every prime* $\wp \supseteq I$ *with* $\ell(I_\wp) = \mathrm{ht}(\wp) < \ell(I)$.
(iii) $r_J(I) \leqslant \ell(I) - 1$ *for some (equivalently, every) minimal reduction J of I.*

Proof. ($\Rightarrow$) Set $\ell = \ell(I)$. By Theorem 5.1 we already have (i), (ii), and $r_\ell(I) \leqslant \ell - 1$. By Theorem 4.4 (i) the latter relation implies $r_J(I) \leqslant \ell - 1$, hence (iii) holds.
($\Leftarrow$) By Theorem 5.1 we need only to show that $r(I_\wp) \leqslant \ell - 1$ for every prime $\wp \supseteq I$ with $\ell(I_\wp) = \mathrm{ht}(\wp) = \ell$. But this follows from (iii) because $J_\wp$ is a minimal reduction of $I_\wp$ so that $r(I_\wp) \leqslant r_J(I)$. □

COROLLARY 5.7. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 1$. *Assume that $I_\wp$ is a complete intersection for every prime $\wp \supseteq I$ with* $\mathrm{ht}(\wp) < \ell(I)$. *Then $R[It]$ is Cohen–Macaulay if and only if the following conditions are satisfied*:

(i) *$G(I)$ is Cohen–Macaulay.*
(ii) $r_J(I) \leqslant \ell(I) - 1$ *for some (equivalently, every) minimal reduction J of I.*

Proof. Condition (ii) of Theorem 5.6 is satisfied by the assumption on I, since $r(I_\wp) = 0$ for all $\wp \supseteq I$ such that $\mathrm{ht}(\wp) < \ell$. □

THEOREM 5.8. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 2$. *Then $R[It]$ is Gorenstein if and only if the following conditions are satisfied*:

(i) *$G(I)$ is Gorenstein.*
(ii) $r(I_\wp) = \mathrm{ht}(\wp) - 2$ *for every prime ideal* $\wp \supseteq I$ *with* $\mathrm{ht}(\wp/I) = 0$.
(iii) $r(I_\wp) \leqslant \mathrm{ht}(\wp) - 2$ *for every prime ideal* $\wp \supseteq I$ *with* $\ell(I_\wp) = \mathrm{ht}(\wp) \leqslant \ell(I)$.

Proof. ($\Rightarrow$) Set $S = G(I)$. By [I, Theorem 3.1] and [TI, Theorem 7.1] we have (i) and $a(S) = -2$. Note that $R[I_\wp t]$ is Gorenstein for every prime ideal $\wp \supseteq I$. To see (ii), let $\wp$ be a minimal prime of I. After localizing we may assume that I is m-primary, so $\ell = d = \dim S$. Since $r_{d-1}(I) = -1$ we can apply Theorem 4.4(i) to get $r_d(I) = \max\{-1 + 1, r(I)\}$ so $r(I) = r_d(I)$. Then by Theorem 4.4(ii)

$$r(I) = \max\{r_{d-1}(I) + 1, a(S) + d\} = d - 2.$$

This gives (ii).

We now prove (iii) by induction on $d = \dim S$. If $d = s$ then (ii) gives us the result. Assume now that the inductive hypothesis holds for dimension smaller than

d. If $\ell(I) < d$ then (iii) holds by induction. Thus we may assume that $\ell(I) = d$. For $\wp \subsetneq m$, the induction hypothesis shows that (iii) holds and also that $r_i(I) = i - 2$ for $s \leqslant i < \ell$. Hence I has sliding reduction number $s - 2$. By Theorem 4.8

$$-2 = a(S) = \max\{-2, r(I) - d\}.$$

From this it follows that $r(I) \leqslant d - 2$, which proves (iii).
($\Leftarrow$) By (ii) and (iii) we have $r_i(I) = i - 2$ for $i = s, \dots, \ell$. Hence I has sliding reduction number $s - 2$ and $r(I) \leqslant r_\ell(I) = \ell - 2$ by Theorem 4.4(i). Using Theorem 4.8 we get

$$a(S) = \max\{-2, r(I) - \ell\} = -2$$

which together with (i) implies by [I, Theorem 3.1] and [TI, Theorem 7.1] that $R[It]$ is Gorenstein. □

REMARK 5.9. Suppose $R[It]$ is Gorenstein. If I is generically a complete intersection, that is, $I_\wp$ is a complete intersection for every prime ideal $\wp \supseteq I$ with $\dim R/\wp = \dim R/I$, then $r(I_\wp) = 0$ and $\mathrm{ht}(\wp/I) = 0$ for such a prime $\wp$. Hence condition (ii) of Theorem 5.8 implies that $\mathrm{ht}(\wp) = 2$ so that $\mathrm{ht}(I) = 2$ (cf. [I, Corollary 4.5]).

COROLLARY 5.10. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 2$. *Assume that $I_\wp$ is a complete intersection for every prime $\wp \supseteq I$ with* $\mathrm{ht}(\wp) = \ell(I)$. *Then $R[It]$ is Gorenstein if and only if* $\mathrm{ht}(I) = 2$, *R/I is equidimensional, and $G(I)$ is Gorenstein.*

Proof. ($\Rightarrow$) If P is minimal over I then $\mathrm{ht}(P) \leqslant \ell(I)$. Hence I_P is a complete intersection and Theorem 5.8(ii) shows that $\mathrm{ht}(P) = 2$. Thus R/I is equidimensional and $\mathrm{ht}(I) \doteq 2$.
($\Leftarrow$) The only primes for which $\ell(I_P) = \mathrm{ht}(P) \leqslant \ell(I)$ are the minimal primes of I. Since then I_P is a complete intersection, conditions (ii) and (iii) are satisfied. □

COROLLARY 5.11. *Let R be a Cohen–Macaulay ring and P a prime ideal of R with* $\mathrm{ht}(P) \geqslant 2$ *such that R_P is regular. Assume that $P^{(n)} = P^n$ for $n \geqslant 1$. Then $R[Pt]$ is Gorenstein if and only if* $\mathrm{ht}(P) = 2$ *and $G(P)$ is Gorenstein.*

Proof. The hypothesis that $P^n = P^{(n)}$ for $n \geqslant 1$ implies that $\ell(I_\wp) < \mathrm{ht}(\wp)$ for every $\wp \supsetneq P$. Since PR_P is a complete intersection of dimension $\mathrm{ht}\, P$, we have $R[Pt]$ Gorenstein if and only if $\mathrm{ht}(P) = 2$. □

The Gorenstein property of $R[It]$ can be also expressed by means of the reduction number of I. The following criterion generalizes results in the cases $\mathrm{ad}(I) = 0$ [I]; $\mathrm{ad}(I) = 1$ [GH], [T3]; and $\mathrm{ad}(I) = 2$ [GN2], [T3]. See also [HHR].

THEOREM 5.12. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 2$. *Then $R[It]$ is Gorenstein if and only if the following conditions are satisfied*:

(i) $G(I)$ *is Gorenstein.*
(ii) $r(I_\wp) = \mathrm{ht}(\wp) - 2$ *for every prime ideal* $\wp \supseteq I$ *with* $\mathrm{ht}(\wp/I) = 0$.
(iii) $r(I_\wp) \leqslant \mathrm{ht}(\wp) - 2$ *for every prime ideal* $\wp \supseteq I$ *with* $\ell(I_\wp) = \mathrm{ht}(\wp) < \ell(I)$.
(iv) $r_J(I) \leqslant \ell(I) - 2$ *for some (equivalently, every) minimal reduction* J *of* I.

Proof. ($\Rightarrow$) Set $\ell = \ell(I)$. By Theorem 5.8 we already have (i), (ii), (iii) and $r_\ell(I) = \ell - 2$. By Theorem 4.4(i), $r_J(I) \leqslant r_\ell(I) \leqslant \ell - 2$ for any reduction $J \subseteq I$, giving (iv).
($\Leftarrow$) By Theorem 5.8 we need only to show that $r(I_\wp) \leqslant \ell - 2$ for every prime $\wp \supseteq I$ with $\ell(I_\wp) = \mathrm{ht}(\wp) = \ell$. But this follows from (iv) because if $J \subseteq I$ is a minimal reduction then $J_\wp$ is a minimal reduction of $I_\wp$ so that $r(I_\wp) \leqslant r_J(I)$. □

REMARK 5.13. We may replace condition (iii) of Theorem 5.12 by the condition that I has sliding reduction number $\mathrm{ht}(I) - 2$.

COROLLARY 5.14. *Let R be a Cohen–Macaulay ring and I an ideal of R with* $\mathrm{ht}(I) \geqslant 2$. *Assume that $I_\wp$ is a complete intersection for every prime $\wp \supseteq I$ with* $\mathrm{ht}(\wp/I) = 0$ *and for all $\wp \supseteq I$ such that* $\mathrm{ht}(\wp) < \ell(I)$. *Then $R[It]$ is Gorenstein if and only if* $\mathrm{ht}(I) = 2$, *R/I is equidimensional and the following conditions are satisfied*:
(i) $G(I)$ *is Gorenstein.*
(ii) $r_J(I) \leqslant \ell - 2$ *for some (equivalently, every) minimal reduction* J *of* I.

6. Castelnuovo–Mumford regularity and Serre properties of Rees algebras

Throughout this section let (R, m) be a Noetherian local ring with infinite residue field. We will study the Castelnuovo–Mumford regularity and the Serre properties of the Rees algebra $R[It]$ of an ideal I in R.

LEMMA 6.1. *Let I be a proper ideal of R with* $\mathrm{ht}(I) \geqslant 1$. *Then* $\ell(R[It]_+) = \ell(I)$ *and* $r_i(R[It]) = i - 1$ *for* $i = 1, \ldots, \ell(I)$.

Proof. Set $T = R[It]$. The first statement follows from the fact that $T_n/mT_n \simeq I^n/mI^n$, $n \geqslant 0$. Recall that if $\mathrm{ht}(I) \geqslant 1$ then $\dim T = \dim R + 1$ and $\mathrm{ht}(T_+) = 1$ (see [Ma]). For the second statement we note that $\mathrm{ht}(T_+) = 1$. If $\wp \supseteq I$ is a prime ideal of R, then $\ell((T_\wp)_+) = \ell(I_\wp) \leqslant \mathrm{ht}(\wp)$. If $\wp \not\supseteq I$, then $T_\wp = R_\wp[t]$, hence $\ell((T_\wp)_+) = 1$ and $r((T_\wp)_+) = 0$. Therefore, by the definition of $r_i(T)$ we get

$$\begin{aligned} r_i(T) &= \max\{r((T_\wp)_+) - \mathrm{ht}(\wp) | \ell((T_\wp)_+) = \dim T_\wp = \mathrm{ht}(\wp) + 1 \leqslant i\} + i - 1 \\ &= \max\{r((T_\wp)_+) | \mathrm{ht}(\wp) = 0\} + i - 1 = i - 1. \end{aligned}$$ □

PROPOSITION 6.2. *Let I be a proper ideal of R with* $\mathrm{ht}(I) \geqslant 1$. *Assume that $R[It]$ is a Cohen–Macaulay ring. Then*

$$\mathrm{reg}(R[It]) \leqslant \ell(I) - 1.$$

Proof. We have by Proposition 2.10 and Corollary 3.6, that $\operatorname{reg}(R[It]) \leqslant r_\ell(R[It])$ and by Lemma 6.1, $r_\ell(R[It]) = \ell(I) - 1$. □

Recall that the relation type $N(I)$ of $R[It]$ is the maximal degree of the elements of a homogeneous minimal basis of the defining ideal of $R[It]$ represented as a quotient ring of a polynomial ring over R.

COROLLARY 6.3. *Let I be an ideal of R with* $\operatorname{ht}(I) \geqslant 1$. *Assume that $R[It]$ is a Cohen–Macaulay ring. Then*

$$N(I) \leqslant \ell(I).$$

Proof. See Remark 2.11. □

We show now that when R is assumed to be CM as well (in which case $G(I)$ will be CM) then $N(I)$ can be bounded in terms of the local reduction numbers of I.

LEMMA 6.4. *Let (R, m) be a CM ring and let I be an ideal of height s and analytic spread ℓ. Let $J = (a_1, \ldots, a_\ell)$ be a minimal reduction of I. If $G(I)$ is CM and $a_1^*, \ldots, a_\ell^*$ is an $[0, \ldots, 0, n_{s+1}, \ldots, n_\ell]$-regular sequence where $n_{s+1} \leqslant \cdots \leqslant n_\ell$ then for all $1 \leqslant k \leqslant \ell$ and all $n \geqslant n_k + 2$ we have*

$$(a_1, \ldots, a_k)I^{n-2} \cap I^n \subseteq (a_1, \ldots, a_k)I^{n-1}.$$

Proof. Let $J_k = (a_1, \ldots, a_k)$. If $k \leqslant s$ then $a_1, \ldots, a_k$ and $a_1^*, \ldots, a_k^*$ are regular sequences so $J_k \cap I^n = J_k I^{n-1}$ for all n. So assume that $k > s$ and that the claim is true for $k - 1$.

Let $n \geqslant n_k + 2$ and let $x \in J_k I^{n-2} \cap I^n$. Say $x = a_1 i_1 + \cdots + a_k i_k$ where $i_j \in I^{n-2}$. If $i_k \in I^{n-1}$ then we are done by induction. If not then

$$i_k^* \in [(a_1^*, \ldots, a_{k-1}^*) : a_k^*]_{n-2} = [(a_1^*, \ldots, a_{k-1}^*)]_{n-2}$$

since $n - 2 \geqslant n_k$. Thus $i_k \in J_{k-1} I^{n-3} + I^{n-1}$. Hence $x = a_1 i_1' + \cdots a_{k-1} i_{k-1}' + a_k y \in J_{k-1} I^{n-2} + a_k I^{n-1}$. Then $a_1 i_1' + \cdots a_{k-1} i_{k-1}' \in J_k I^{n-2} \cap I^n \subseteq J_{k-1} I^{n-1}$ since $n \geqslant n_k \geqslant n_{k-1}$. Hence $x \in J_k I^{n-1}$. □

PROPOSITION 6.5. *Let (R, m) be a CM ring and I an ideal of height s and analytic spread ℓ. Let $J = (a_1, \ldots, a_\ell)$ be a minimal reduction of I. Suppose that $R[It]$ is CM. If $a_1^*, \ldots, a_\ell^*$ is $[0, \ldots, 0, n_{s+1}, \ldots, n_\ell]$-regular in $G(I)$, where $n_{s+1} \leqslant \cdots \leqslant n_\ell$, then $a_1 t, \ldots, a_\ell t$ is $[0, 1, \ldots, 1, n_{s+1}, \ldots, n_\ell]$-regular in $R[It]$ (there are $s - 1$ 1's).*

Proof. Let $J_k = (a_1, \ldots, a_k)$. Since a_1 is R-regular we have that $a_1 t$ is $R[It]$-regular. Now suppose that $1 < k \leqslant s$, $n \geqslant 1$ and $x t^n \in [(a_1 t, \ldots, a_{k-1} t) :$

$a_k t]_n$. Then $x \in I^n \cap J_k : a_k = I^n \cap J_k = J_k I^{n-1}$ so $xt^n \in J_k I^{n-1} t^n = (J_k t) I^{n-1} t^{n-1} = (J_k t)_n$.

Assume now that $k > s$. Let $n \geqslant n_k$ and suppose $(xt^n)(a_k t) \in (a_1 t, \ldots, a_{k-1} t)_{n+1}$. Then $x a_k \in J_{k-1} I^n$. Interpreting this in $G(I)$ gives $x^* \in [(a_1^*, \ldots, a_{k-1}^*) : a_k^*]_n = (a_1^*, \ldots, a_{k-1}^*)_n$. Hence $x \in J_{k-1} I^{n-1} + I^{n+1}$, say $x = a_1 i_1 + \cdots + a_{k-1} i_{k-1} + y$ where $i_j \in I^{n-1}$ and $y \in I^{n+1}$. Then $a_k y \in J_{k-1} I^n \cap I^{n+2} \subseteq J_{k-1} I^{n+1}$ by Lemma 6.4 since $n \geqslant n_{k-1}$. By the same argument as for x we obtain $y \in J_{k-1} I^n + I^{n+2}$. Continuing this way we get $x \in \bigcap_{m \geqslant n} (J_{k-1} I^{n-1} + I^m) = J_{k-1} I^{n-1}$, so $xt^n \in (a_1 t, \ldots, a_{k-1} t)_n$. □

COROLLARY 6.6. *With R, I, J and* $1 \leqslant n_{s+1} \leqslant \cdots \leqslant n_\ell$ *as in Proposition* 6.5 *we have* $\operatorname{reg}(R[It]) \leqslant \max\{n_{s+1} - 1, \ldots, n_\ell - 1, r_J(I)\}$. *In particular* $\operatorname{reg}(R[It]) \leqslant \max\{r_{\ell-1}(I), r(I)\}$.

Proof. By Proposition 2.10 we have $\operatorname{reg}(R[It]) \leqslant \max\{a(Jt) - 1, r_{Jt}(It)\}$. But $r_{Jt}(It) = r_J(I)$ and $a(Jt) \leqslant \max\{1, n_{s+1}, \ldots, n_\ell\}$ by Proposition 6.5. This gives the desired bound.

Now, since $G(I)$ is CM we have $n_j \leqslant r_{j-1}(I) + 1$ for $1 \leqslant j \leqslant \ell$ by Theorem 3.2. Hence $\operatorname{reg}(R[It]) \leqslant \max\{r_{\ell-1}(I), r(I)\}$ by applying Proposition 6.5 with J a minimal reduction such that $r(I) = r_J(I)$. □

COROLLARY 6.7. *Let R and* $R[It]$ *be CM rings. Then* $N(I) \leqslant \max\{r_{\ell-1}(I) + 1, r(I) + 1\} \leqslant r_\ell(I) + 1$.

Proof. This follows from Remark 2.11 and Corollary 3.6 (iii). □

The bound for $N(I)$ given in Corollary 6.7 is proved in the case $\operatorname{ad}(I) = 0$ or $\operatorname{ad}(I) = 1$ and I is generically a complete intersection [HuHu1], [GH].

Now we will use Theorem 5.1 to compare the Serre condition (S_k) in the associated graded ring and the Rees algebra. Recall that the Serre condition (S_k) on a Noetherian ring T is defined by the condition that

$$\operatorname{depth}(T_P) \geqslant \min\{\operatorname{ht}(P), k\}$$

holds for every prime ideal P of T. We are here inspired by a result of Noh and Vasconcelos [NV, Theorem 2.2].

THEOREM 6.8. *Let R be a Noetherian ring satisfying* (S_{k+1}) *and let I be an ideal of R with* $\operatorname{ht}(I) \geqslant 1$. *Then* $R[It]$ *satisfies* (S_{k+1}) *if and only if the following conditions are satisfied*:

(i) $G(I)$ *satisfies* (S_k).

(ii) $r(I_\wp) \leqslant \operatorname{ht}(\wp) - 1$ *for every prime* $\wp \supseteq I$ *with* $\ell(I_\wp) = \operatorname{ht}(\wp) \leqslant k$.

Proof. $(\Rightarrow)$ Set $S = G(I)$. As in the proof of [NV, Theorem 2.2], to show (i) we only need to prove that S_P is Cohen–Macaulay for every prime $P \supseteq S_+$ of S with $\operatorname{ht}(P) \leqslant k$. Note that $S = R[It]/IR[It]$ and let Q be the inverse image of P in $R[It]$. Then $Q \supseteq R[It]_+$ and $\operatorname{ht}(Q) \leqslant k + 1$. Hence $R[It]_Q$ is a

Cohen–Macaulay ring by the condition (S_{k+1}) of $R[It]$. Let $\wp = Q \cap R$. Then $R[It]_Q$ is the localization of $R_\wp[I_\wp t]$ at its (unique) maximal graded ideal. Hence $R_\wp[I_\wp t]$ is Cohen–Macaulay [HR, MR]. Note that $\mathrm{ht}(\wp) \leqslant \mathrm{ht}(Q) \leqslant k+1$. Then $R_\wp$ is Cohen–Macaulay since R satisfies (S_k). Hence $G(I_\wp)$ is Cohen–Macaulay by [Hu, Proposition 1.1]. Since S_P is a localization of $G(I_\wp)$, S_P is Cohen–Macaulay too.

To prove (ii) let $\wp \supseteq I$ be a prime ideal of R with $\ell(I_\wp) = \mathrm{ht}(\wp) \leqslant k$ and $Q = (\wp, It)$. Then $\mathrm{ht}(Q) \leqslant \mathrm{ht}(\wp) + 1 \leqslant k + 1$. Similarly as above we can show that $R_\wp[I_\wp t]$ is Cohen–Macaulay. Since $R_\wp$ is Cohen–Macaulay, we can apply Theorem 5.1 to deduce that $r(I_\wp) \leqslant \mathrm{ht}(\wp) - 1$.

($\Leftarrow$) Let Q be an arbitrary prime ideal of $R[It]$ and $\wp = Q \cap R$. Then $\mathrm{ht}(\wp) \leqslant \mathrm{ht}(Q)$. Note that $R[It]_Q$ is a localization of $R_\wp[I_\wp t]$. If $\wp \not\supseteq I$, then $R_\wp[I_\wp t] = R_\wp[t]$. Since $R_\wp$ has (S_{k+1}), so has $R_\wp[t]$. Hence

$$\operatorname{depth} R[It]_Q \geqslant \min\{\mathrm{ht}(Q),\ k+1\}.$$

Now let $\wp \supseteq I$. If $\mathrm{ht}(\wp) \leqslant k$, then $R_\wp$ is a Cohen–Macaulay ring. Since $\dim G(I_\wp) = \mathrm{ht}(\wp) \leqslant k$, by (i) we have that $G(I_\wp)$ is a Cohen–Macaulay ring. So we can apply Theorem 5.1 to deduce from (ii) that $R_\wp[I_\wp t]$ and hence $R[It]_Q$ is a Cohen–Macaulay ring. If $\mathrm{ht}(\wp) \geqslant k+1$, we proceed as in the proof for ($\Leftarrow$) of [NV, Theorem 2.2] to show that $\operatorname{depth} R[It]_Q \geqslant \min\{\mathrm{ht}(Q),\ k+1\}$. □

REMARK 6.9.

(1) To prove ($\Rightarrow$) we only need condition (S_k) on R.

(2) It is assumed in [NV, Theorem 2.2] that $\mathrm{ht}(I) \geqslant k+1$. In this case condition (ii) of Theorem 6.8 is automatically satisfied.

THEOREM 6.10. *Let (R, m) be a regular local ring. Let I be an ideal of R of analytic spread ℓ. If $G(I)$ satisfies (S_ℓ) then $R[It]$ satisfies $(S_{\ell+1})$.*

Proof. By Theorem 6.8 we need to show that $r(I_P) \leqslant \mathrm{ht}(P) - 1$ for every prime $P \supseteq I$ such that $\ell(I_P) = \mathrm{ht}(P) \leqslant \ell$. This follows from Theorems 5.1 and 5.4. □

7. Briançon–Skoda theorem with coefficients for regular rings containing a field

The purpose of this section is to give an improved version of the Briançon–Skoda theorem in [AH, Sect. 3]. The point of the improvement is to obtain more accurate information about coefficients for ideals of higher analytic deviation. Since the proof uses tight closure methods we will obtain the results only for regular local rings containing a field. The case of mixed characteristic is still open. The results from this section will be applied in Section 8 in order to bound reduction numbers for ideals of small analytic deviation when $G(I)$ satisfies Serre's condition (S_ℓ), where $\ell = \ell(I)$.

First we shall establish that given any ideal I in any local ring (R, m) with infinite residue field, we may pick generators of I such that the first i are a reduction of I_P for any prime $P \supseteq I$ of height i. The idea behind this comes from Lemma 2.3 of [AN] where good generating sets of I are chosen subject to the condition that $\mu(I_P) \leqslant \operatorname{ht} P$ up to a certain height. Since we are choosing reductions we are using the "hypothesis" that $\ell(I_P) \leqslant \operatorname{ht} P$ for primes P up to a certain height. But by the nature of analytic spread this is always satisfied.

DEFINITION 7.1. Let (R, m) be any local ring and let $I \subseteq R$ be an ideal and let J be a minimal reduction of I. Then the generating set $a_1, \ldots, a_n$ of J is a *basic* generating set for J if for all primes $P \supseteq I$ such that $i = \operatorname{ht} P < \ell(I)$ we have that $(a_1, \ldots, a_i)_P$ is a reduction of I_P.

We use the term "basic" to suggest the parallels with basic element theory.

The following lemma allows us to make the Briançon–Skoda Theorem statements.

LEMMA 7.2. *Let (R, m) be a local ring with infinite residue field. Let $I \subseteq R$ be any ideal and let $\ell = \ell(I)$. For every minimal reduction J of I, there exists a minimal generating set $a_1, \ldots, a_\ell$ such that*

(1) *if P is a prime ideal containing I and* $\operatorname{ht} P = i < \ell$ *then $(a_1, \ldots, a_i)_P$ is a reduction of I_P, and*
(2) $\operatorname{ht}((a_1, \ldots, a_i)I^n : I^{n+1} + I) \geqslant i + 1$ *for all $n \gg 0$.*

Proof. Let $G = G(I)$. Note that $(a_1, \ldots, a_\ell)$ is a reduction of I if and only if $(a_1^*, \ldots, a_\ell^*)$ is a reduction of G_+. Fix a minimal reduction J of I. Choose $a_1, \ldots, a_\ell$ a minimal generating set of J such that $a_1^*, \ldots, a_\ell^*$ is a filter regular sequence with respect to G_+ (see [T2]). Let $P \supseteq I$ be a prime of height $i < \ell$. Then $a_1^*, \ldots, a_\ell^*$ is still filter regular in G_P with respect to $(G_P)_+$. Since $\dim G_P = \operatorname{ht} P = i$, we may apply Corollary 2.2 and Lemma 2.4 to see that $(a_1^*, \ldots, a_i^*)_P$ is a reduction of $(G_P)_+$. Hence $(a_1, \ldots, a_i)_P$ is a reduction of I_P.

To see (2) we note that if $\operatorname{ht} P < i + 1$ then $(a_1, \ldots, a_i)_P$ is a reduction of I_P so P cannot contain $\cup_n((a_1, \ldots, a_i)I^n) : I^{n+1} + I$. □

REMARK 7.3. There is an alternative approach to the proof of Lemma 7.2 which follows along the lines of the proof of [AN, Lemma 2.3]. Given an ideal I and $K \subseteq I$ we can define

$$L_i(K, I) = \sum_{x_1, \ldots, x_{i-1} \in I} \bigcup_{n \geqslant 0} (K + (x_1, \ldots, x_{i-1}))I^n : I^{n+1},$$

for all i. So if $J \subseteq I$ is a minimal reduction we pick a minimal generating set $a_1, \ldots, a_\ell$ inductively such that letting $J_r = (a_1, \ldots, a_r)$ for $r < \ell$ and letting $L_i^r = L_i(J_r, I)$ we have

(1) $\operatorname{ht} L_i^r \geqslant i + r$ for $i > 1$ and $r + i < \ell$,
(2) $\operatorname{ht}(L_1^r + I) \geqslant r + 1$, and
(3) $\operatorname{ht} L_1^r \geqslant r$.

In order to state our Briançon–Skoda theorem we will need some terminology indicating the removal of primary components whose height is too large. In other words, we need a refinement of the notion of I^{un} given in [AH, Definition 3.1].

DEFINITION 7.4. Fix an ideal K of a ring R. Let I be an ideal and let $t > \operatorname{ht} K$ be an integer. Let $I^{\langle t,K\rangle} = S^{-1}IR \cap R$ where $S = R - \cup\{P \in \operatorname{Ass}(R/K) \mid \operatorname{ht}(P) < t\}$. Often, if the ideal K is understood we will write simply $I^{\langle t\rangle}$. The ideal $I^{\langle t,K\rangle}$ is the intersection of the primary components of I whose radicals are contained in an associated prime of K of height less than t.

REMARK 7.5. If R is a regular ring of characteristic p and A is an ideal of R then $(A^{\langle j,K\rangle})^{[q]} = (A^{[q]})^{\langle j,K\rangle}$.

Proof. We note that for any ideals I and J, $(I \cap J)^{[q]} = I^{[q]} \cap J^{[q]}$ and if I is P-primary then $I^{[q]}$ is also P-primary, since the Frobenius endomorphism is flat. Let $A = W_1 \cap \cdots \cap W_n \cap V_1 \cap \cdots \cap V_m$ be a primary decomposition where $\sqrt{W_i}$ is contained in an associated prime of K of height less than j, and the V_i are the other components. Then $A^{\langle j\rangle} = W_1 \cap \cdots \cap W_n$ and $A^{[q]} = W_1^{[q]} \cap \cdots \cap W_n^{[q]} \cap V_1^{[q]} \cap \cdots \cap V_m^{[q]}$, so $(A^{[q]})^{\langle j\rangle} = W_1^{[q]} \cap \cdots \cap W_n^{[q]} = (A^{\langle j\rangle})^{[q]}$. □

THEOREM 7.6. *Let (R, m) be a regular local ring containing a field. Let I be any ideal of R of height g and analytic spread ℓ. Let $J = (a_1, \ldots, a_\ell)$ be a minimal reduction generated by a basic generating set. For each $i \geqslant g$ let $J_i = (a_1, \ldots, a_i)$. Then for every $w \geqslant 0$*

$$\overline{I^{\ell+w}} \subseteq J^{w+1}(J_{\ell-1}(\cdots(J_g)^{\langle g+1,I\rangle}\cdots)^{\langle \ell-1,I\rangle})^{\langle \ell,I\rangle}.$$

Proof. We give the proof in characteristic $p > 0$ and then the result follows in equicharacteristic 0 by standard methods of reduction to characteristic p. See [AH] for a thorough treatment of a similar reduction.

For $g \leqslant i < \ell$ let $A_{i+1} = \cup_n(J_iJ_{i+1}^n : J_{i+1}^{n+1})$. Note that by Lemma 7.2 (1) we have $\operatorname{ht} A_{i+1} + I \geqslant i + 1$. Let n_i be the value where the union stabilizes. Then for any prime P of height $i + 1$ containing I we have $r_{(J_i)_P}((J_{i+1})_P) \leqslant n_i$. For each $q = p^e$ let $A_{i+1,q} = (J_i^{iq}J_{i+1}^{n_i} : J_{i+1}^{iq+n_i})$. Note that for all q, $\sqrt{A_i} = \sqrt{A_{i,q}}$. Now choose $f_g, \ldots, f_{\ell-1}$ such that $f_i \in J_{i+1}^{n_i}$ for each $g \leqslant i < \ell$.

Let $d \neq 0$ be an element such that $dI^t \subseteq J^t$ for all t (e.g. $d \in I^{r_J(I)}$ will work since J is a reduction of I). Now let x be a nonzero element of $\overline{I^{\ell+w}}$. Then letting $c = x^k$ for k large enough we have $cdx^q \in dI^{(\ell+w)q} \subseteq J^{(\ell+w)q}$. Then by Lemma 7.8 below

$$f_g \cdots f_{\ell-1}cdx^q \in f_g \cdots f_{\ell-1}J^{(\ell+w)q} \subseteq (J^{w+1})^{[q]}f_g \cdots f_{\ell-1}J^{(\ell-1)q}.$$

We wish to show that

$$f_g \cdots f_{\ell-1} J^{(\ell-1)q} \subseteq (J_{\ell-1}(J_{\ell-2} \cdots (J_g)^{\langle g+1\rangle} \cdots)^{\langle \ell-1\rangle})^{\langle \ell\rangle [q]}.$$

It then follows that x is in the tight closure of the desired ideal and since all ideals in a regular ring are tightly closed we are done.

We first note that $A_{i+1,q} f_i J_{i+1}^{iq} \subseteq J_i^{[q]} J_i^{(i-1)q}$. In particular, $A_{g+1,q} f_g J_{g+1}^{gq} \subseteq J_g^{[q]}$. Now, since $\operatorname{ht} A_{g+1,q} + I \geqslant g+1$, the ideal $A_{g+1,q}$ is not in any associated prime of $(J_g^{[q]})^{\langle g+1\rangle}$. Hence $f_g J_{g+1}^{gq} \subset (J_g^{[q]})^{\langle g+1\rangle}$. Suppose inductively that

$$f_{i-1} \cdots f_g J_i^{(i-1)} \subseteq (J_{i-1}^{[q]} \cdots (J_g^{[q]})^{\langle g+1\rangle} \cdots)^{\langle i\rangle}, \tag{$*$}$$

for some i with $g \leqslant i-1 < \ell - 1$. Then

$$\begin{aligned} A_{i+1,q} f_i \cdots f_g J_{i+1}^{iq} &\subseteq f_{i-1} \cdots f_g J_i^{[q]} J_i^{(i-1)q} \\ &\subseteq J_i^{[q]} (J_{i-1}^{[q]} \cdots (J_g^{[q]})^{\langle g+1\rangle} \cdots)^{\langle i\rangle} \end{aligned}$$

(the second inclusion comes from multiplying $(*)$ by $J_i^{[q]}$). Now, since $\operatorname{ht}(A_{i+1,q} + I) \geqslant i+1$ we get that

$$f_i \cdots f_g J_{i+1}^{iq} \subseteq (J_i^{[q]}(\cdots)^{\langle i\rangle})^{\langle i+1\rangle}.$$

In particular $f_{\ell-1} \cdots f_g J^{(\ell-1)g} \subseteq (J_\ell^{[q]} \cdots)^{\langle \ell\rangle}$ and Remark 7.5 gives the desired result. □

REMARK 7.7. This result is stronger than [AH, Theorem 3.3], especially in the case where the ideal I has minimal components of different heights.

LEMMA 7.8. [AH, Lemma 3.4]. *Let R be any commutative ring and $A = (u_1, \ldots, u_n) \subseteq R$. Then for all w, $k \geqslant 0$ we have*

$$A^{(n+w)k} \subseteq (u_1^k, \ldots, u_n^k) A^{(n-1)k}.$$

8. Reduction numbers of ideals in regular rings

We can now apply the results of Sections 6 and 7 to ideals of small analytic deviation in regular rings containing a field to get bounds on reduction numbers. The main results of this section are Theorems 8.4 and 8.5, in which we prove that, under certain conditions on I, $r(I) < \ell(I)$ if $G(I)$ satisfies Serre's condition $(S_{\ell(I)})$. We will use repeatedly the result of [VV] that if $g = \operatorname{ht} I$ and $a_1, \ldots, a_g \in I - I^2$ are a regular sequence then $a_1^*, \ldots, a_g^*$ is $G(I)$-regular if and only if $(a_1, \ldots, a_g) \cap I^n = (a_1, \ldots, a_g) I^{n-1}$ for all n.

For the rest of this section, if I is an ideal and $J = (a_1, \ldots, a_\ell)$ is a minimal reduction generated by a basic generating set then we will let $J_i = (a_1, \ldots, a_i)$ and $K_i = (J_i \cdots (J_g)^{\langle g+1, I\rangle} \cdots)^{\langle i+1, I\rangle}$ for $g \leqslant i \leqslant \ell - 1$. For any ideal $K \subseteq R$ we use $\mathrm{Min}^0(K)$ to denote the set of minimal primes of K which have maximal dimension.

We will need notation to handle the image of an element $u \in R$ in $G(I)$ when we do not know precisely the degree of u.

DEFINITION 8.1. Let $I \subseteq R$ and $u \in R$. If $u \in I^j$ then we denote by $u^{*(j)}$ the element $u + I^{j+1}$ considered to be in the jth graded piece of $G(I)$. Note that when $u \in I^j - I^{j+1}$ we have $u^{*(j)} = u^*$ and when $u \in I^{j+1}$ then $u^{*(j)} = 0$.

PROPOSITION 8.2. *Let (R, m) be a CM ring with infinite residue field and let $I \subseteq R$ with $\ell = \ell(I)$ and $g = \mathrm{ht}\, I > 0$. If I has sliding reduction number zero, $G(I)$ is CM, and $I^\ell \subseteq JI^{\ell-g}$ for some minimal reduction J then $R[It]$ is CM.*

Proof. Since I has sliding reduction number zero we need only show that $r(I) < \ell$, in order to apply Theorem 5.6. Let $a_1, \ldots, a_\ell$ be a basic generating set of the reduction J and let $J_i = (a_1, \ldots, a_i)$ for $1 \leqslant i \leqslant \ell$. Because I has sliding reduction number zero, we know by Corollary 3.6(i) that $a_1^*, \ldots, a_\ell^*$ is $[-g, \ldots, 0, 1, 2, \ldots, \ell - g]$-regular.

If $I^\ell \not\subseteq JI^{\ell-1}$ then choose $k < \ell - 1$ maximal such that $I^\ell \subseteq JI^k$ and t minimal such that $I^\ell \subseteq J_t I^k + I^{k+1}$. Note that $k \geqslant \ell - g$. Let $f = a_1u_1 + \cdots + a_tu_t + w \in I^\ell$ where $u_i \in I^k$ and $w \in I^{k+1}$. Then $u_t^{*(k)} \in [(a_1^*, \ldots, a_{t-1}^*) : a_t^*]_k = (a_1^*, \ldots, a_{t-1}^*)_k$. Hence $u_t \in J_{t-1}I^{k-1} + I^{k+1}$. Thus $f \in J_{t-1}I^k + I^{k+1}$, contradicting our choice of k and t. Thus $I^\ell \subseteq JI^{\ell-1}$ so $r_J(I) \leqslant \ell - 1$. □

LEMMA 8.3. *Let (R, m) be a regular local ring containing a field and having infinite residue field. Let $I \subseteq R$ be an ideal of analytic spread ℓ such that $G(I)$ satisfies (S_ℓ). Let J be a minimal reduction. Then $I^\ell \subseteq J(I \cap K_{\ell-1})$. More generally, let $a_1, \ldots, a_\ell$ be a basic generating set for J. If $f = a_1u_1 + \cdots a_nu_n \in I^2$ with $u_i \in K_m$ with $m \geqslant n - 1$ then $u_i \in I \cap K_m$.*

Proof. Since J is generated by a basic generating set we have n_i such that $\mathrm{ht}(J_iI^{n_i} : I^{n_i+1}) + I > i$ for $g \leqslant i < \ell$. Choose

$$x_{g+1} \in (I : K_m) \cap (J_gI^{n_g} : I^{n_g+1}) - \cup\{P \mid P \in \mathrm{Min}^0(I)\}.$$

Now choose x_{i+1} inductively for $i < n$ so that

$$x_{i+1} \in (I : K_m) \cap (J_iI^{n_i} : I^{n_i+1}) - \cup\{P \mid P \in \mathrm{Min}^0(I + (x_{g+1}, \ldots, x_i)\}.$$

Under these conditions the ideal $(a_1^*, \ldots, a_g^*, a_{g+1}^* + x_{g+1}^*, \ldots, a_j^* + x_j^*)$ has depth j on $G(I)$ for $j \leqslant \ell$ since $G(I)$ satisfies (S_ℓ). Thus the Koszul complex on $a_1^*, \ldots, a_j^* + x_j^*$ is acyclic.

Let f be as above. Then $a_1^* u_1^{*(0)} + \cdots + a_n^* u_n^{*(0)} = 0$ in $G(I)$. Note that $x_i^* u_i^{*(0)} = 0$ for $g+1 \leqslant i \leqslant n$. If the coefficients are not all in I then let $i = \max\{t \mid u_t^{*(0)} \neq 0\}$. Then

$$a_1^* u_1^{*(0)} + \cdots + a_g^* u_g^{*(0)} + (a_{g+1}^* + x_{g+1}^*) u_{g+1}^{*(0)} + \cdots + (a_i^* + x_i^*) u_i^{*(0)} = 0$$

and since $G(I)$ satisfies (S_ℓ) we have $u_i^{*(0)} \in (a_1^*, \ldots, a_g^*, a_{g+1}^* + x_{g+1}^*, \ldots, a_{i-1}^* + x_{i-1}^*)$. Evaluating this sum in degree zero gives $u_i \in (x_{g+1}, \ldots, x_i) + I$. But the same analysis with $x_{g+1}^n, \ldots, x_i^n$ in place of $x_{g+1}, \ldots, x_i$ shows that $u_i \in \cap_n ((x_{g+1}^n, \ldots, x_i^n) + I) = I$. □

THEOREM 8.4. *Let (R, m) be a regular local ring containing a field and having infinite residue field. Let $I \subseteq R$ be an ideal with* ad$(I) = 1$. *If $G(I)$ satisfies (S_{g+1}) then $r_J(I) \leqslant g$ for all minimal reductions $J \subseteq I$.*

Proof. Let $J = (a_1, \ldots, a_{g+1})$ be a minimal reduction of I generated by a basic generating set. If $P \supseteq I$ and ht $P < g+1$ then P is minimal over I. Hence $r(I_P) < g$ since $G(I_P)$ is CM.

Choose $k \gg 0$ such that ht$(J_g I^k : I^{k+1}) + I > g$ and let

$$x \in J_g : (J_g)^{\langle g+1 \rangle} \cap J_g I^k : I^{k+1} - \cup\{P \mid P \in \mathrm{Min}^0(I)\}.$$

Let $f \in I^{g+1}$. Suppose inductively that for some $n < g$, $f = a_1 v_1 + \cdots + a_{g+1} v_{g+1} \in J I^n$ where $v_i \in I^n$ and $x v_{g+1} \in J_g \cap I^n = J_g I^{n-1}$. For $n = 1$ this holds by Lemma 8.3. Then $(a_{g+1}^* + x^*) v_{g+1}^{*(n)} \in (a_1^*, \ldots, a_g^*)$, so $v_{g+1}^{*(n)} \in (a_1^*, \ldots, a_g^*)_n$. Say $v_{g+1} = a_1 r_1 + \cdots + a_g r_g + w$ with $w \in I^{n+1}$ and $r_i \in I^{n-1}$. Then $xw \in J_g \cap I^{n+1} = J_g I^n$. Also

$$f - a_{g+1} w \in J_g \cap I^{n+2} = J_g I^{n+1},$$

so the inductive hypotheses are preserved. Hence $I^{g+1} = J I^g$ and $r_J(I) \leqslant g$. □

THEOREM 8.5. *Let (R, m) be a regular local ring containing a field and having infinite residue field. Let $I \subseteq R$ be an ideal of height g and* ad$(I) = 2$. *Suppose that $r(I_P) \leqslant 1$ for every $P \in$* Min$^0(I)$. *If $G(I)$ satisfies (S_{g+2}) then $r_J(I) \leqslant g+1$ for every minimal reduction $J \subseteq I$.*

Proof. By Theorem 8.4 we have $r(I_P) < \ell(I_P)$ for $P \supseteq I$ and ht$(P/I) < 2$.

Let $J = (a_1, \ldots, a_{g+2})$ be a minimal reduction of I generated by a basic generating set and choose $k \gg 0$ such that ht$(J_{g+1} I^k : I^{k+1}) + I > g+1$. Let

$$x_1 \in J_g J_{g+1} : K_{g+1} \cap J_g : K_g \cap J_g I : I^2 - \cup\{P \mid P \in \mathrm{Min}^0(I)\},$$
$$x_2 \in J_{g+1} J_g^{\langle g+1 \rangle} : K_{g+1} \cap J_{g+1} I^k : I^{k+1} - \cup\{P \mid P \in \mathrm{Min}^0(I, x_1)\}.$$

Let $f \in I^{g+2}$ and say $f = a_1u_1 + \cdots + a_{g+2}u_{g+2}$ where $u_i \in K_{g+1} \cap I$. Then $x_2^*u_{g+2}^{*(1)} = a_1^*t_1^{*(0)} + \cdots + a_{g+1}^*t_{g+1}^{*(0)}$ where $t_i \in J_g^{\langle g+1\rangle}$. Then since $x_1^*u_i^{*(1)} = x_1^*t_i^{*(0)} = 0$ we have that $(u_1^{*(1)} - t_1^{*(0)}, \dots, u_{g+1}^{*(1)} - t_{g+1}^{*(0)}, u_{g+2}^{*(1)})$ is a sum of Koszul relations on the regular sequence $a_1^*, \dots, a_g^*, a_{g+1}^* + x_1^*, a_{g+2}^* + x_2^*$. Write this vector as a sum of Koszul relations and let $(y_0^{*(0)} + y_1^{*(1)} + \cdots)$ be the coefficient of $(0, \dots, 0, -(a_{g+2}^* + x_2^*), a_{g+1}^* + x_1^*)$. We obtain the following information:

(1) $u_{g+2}^{*(1)} - y_1^{*(1)}x_1^* - y_0^{*(0)}a_{g+1}^* \in (a_1^*, \dots, a_g^*)$,

(2) $y_0^{*(0)}x_1^* = 0$, and

(3) $y_1^{*(1)}a_{g+1}^* \in (a_1^*, \dots, a_g^*)$.

From (1) and (2) we get $(x_1^*)^2y_1^{*(1)} \in (a_1^*, \dots, a_g^*)$, which combined with (3) gives $y_1^{*(1)}x_1^* \in (a_1^*, \dots, a_g^*) : (a_{g+1}^* + x_1^*) = (a_1^*, \dots, a_g^*)$. Hence $u_{g+2}^{*(1)} - y_0^{*(0)}a_{g+1}^* \in (a_1^*, \dots, a_g^*)$. Write $u_{g+2} = a_1r_1 + \cdots + a_gr_g + y_0a_{g+1} + w$ with $w \in I^2$. Note that $x_2w = a_1r_1' + a_gr_g' + a_{g+1}(t - x_2y_0)$ where $t \in J_g^{\langle g+1\rangle}$. Then, since $x_1(t - x_2y_0) \in I$ we get $(t - x_2y_0)^{*(0)} \in (a_1^*, \dots, a_g^*)$. Thus $t - x_2y_0 \in I$. We may conclude that $x_2w \in J_{g+1}I$.

Now, $f - a_{g+2}w = a_1s_1 + \cdots + a_gs_g + a_{g+1}(u_{g+1} + a_{g+2}y_0) \in I^3$, where $s_i \in I$. Since $x_1^*(u_{g+1}^{*(1)} + a_{g+2}^*y_0^{*(0)}) \in (a_1^*, \dots, a_g^*)$ we get $u_{g+1} + a_{g+2}y_0 \in J_g + I^2$, say $u_{g+1} + a_{g+2}y_0 = a_1v_1 + \cdots + a_gv_g + z$. Thus $f - a_{g+2}w - a_{g+1}z \in J_g \cap I^3 = J_gI^2$, so $f \in JI^2$.

Suppose now that $f \in I^{g+2} \cap JI^n$ for some n with $2 \leqslant n < g+1$ and $f = a_1r_1 + \cdots + a_gr_g + a_{g+1}z + a_{g+2}w$ where $z, w \in I^n$ and $x_2w \in J_{g+1}I^{n-1}$. Note that $x_1z \in J_gI^{n-1}$ by hypothesis. If $x_2w = a_1t_1 + \cdots + a_{g+1}t_{g+1}$ with $t_i \in I^{n-1}$ then we get that $(\dots, z^{*(n)} - t_{g+1}^{*(n-1)}, w^{*(n)})$ is a sum of Koszul relations on $a_1^*, \dots, a_g^*, a_{g+1}^* + x_1^*, a_{g+2}^* + x_2^*$. As before write this as a sum of Koszul relations and let $(\cdots + y_{n-1}^{*(n-1)} + y_n^{*(n)} + \cdots)$ be the coefficients of $(0, \dots, 0, -(a_{g+2}^* + x_2^*), a_{g+1}^* + x_1^*)$. Then

(1) $w^{*(n)} - x_1^*y_n^{*(n)} - a_{g+1}^*y_{n-1}^{*(n-1)} \in (a_1^*, \dots, a_g^*)$,

(2) $x_1^*y_{n-1}^{*(n-1)} \in (a_1^*, \dots, a_g^*)$, and

(3) $y_n^{*(n)}a_{g+1}^* \in (a_1^*, \dots, a_g^*)$.

From (1) and (2) we have $(x_1^*)^2y_n^{*(n)} \in (a_1^*, \dots, a_g^*)$, which combined with (3) gives $y_n^{*(n)}x_1^* \in (a_1^*, \dots, a_g^*)_n$. Hence $w^{*(n)} - a_{g+1}^*y_{n-1}^{*(n-1)} \in (a_1^*, \dots, a_g^*)$. Write $w = a_1r_1 + \cdots + a_gr_g + y_{n-1}a_{g+1} + w'$ with $w' \in I^{n+1}$. Then $x_2w' \in J_{g+1}I^{n-1} \cap I^{n+1}$, say $x_2w' = a_1r_1' + \cdots + a_{g+1}r_{g+1}'$. Then since $x_1r_{g+1}' \in J_gI^{n-2}$ we get $r_{g+1}' \in J_gI^{n-2} + I^n$, so $x_2w' \in J_{g+1}I^n$. Hence $f - a_{g+2}w' =$

$a_1s_1 + \cdots + a_gs_g + a_{g+1}(z + a_{g+2}y_{n-1}) \in I^{n+2}$ with $s_i \in I^n$. Finally, since $x_1(z + a_{g+2}y_{n-1}) \in J_gI^{n-1}$ we get $z + a_{g+2}y_{n-1} \in J_gI^{n-1} + I^{n+1}$. Hence $f - a_{g+2}w' - a_{g+1}z' \in J_g \cap I^{n+2} = J_gI^{n+1}$, so that $f \in JI^{n+1}$. Inductively, we now get $I^{g+2} = JI^{g+1}$. □

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