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On the prime-to-p part of the groups of connected components of Néron models

Dedicated to Frans Oort on the occasion of his 60th birthday

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Abstract. Let A be the Néron model of an abelian variety over the field of fractions of a discrete valuation ring D with residue characteristic $p \geq 0$. Let $\Phi_{(p)}$ denote the prime-to-p part of the group of connected components of the geometric special fibre of A. Lorenzini has constructed a four step functorial filtration on $\Phi_{(p)}$ and proved certain upper bounds for its successive quotients. We improve these bounds, and show that they are sharp. As an application, we give a complete classification of the possible $\Phi_{(p)}$ for abelian vaieties whose reduction has toric part, abelian variety part and unipotent part of fixed dimensions.

1. Introduction

The aim of this article was originally to improve certain results of Dino Lorenzini concerning the groups of connected components of special fibres of Néron models of abelian varieties. Let D be a strictly henselian discrete valuation ring, K its field of fractions, k its residue field and A_K an abelian variety over K with Néron model A over D. Let $p \geqslant 0$ be the characteristic of k and let $\Phi_{(p)}$ denote the prime-to-p part of the group of connected components Φ of A_k. In [7] Lorenzini constructs a functorial four step filtration on $\Phi_{(p)}$ and shows that this filtration has certain properties. In particular, he gives upper bounds for the successive quotients. These bounds are of the following type. For a prime l and a finite abelian group G of l-power order, say $G \cong \oplus_{i \geqslant 1} \mathbf{Z}/l^{a_i}\mathbf{Z}$ with $a_1 \geqslant a_2 \geqslant \cdots$, he defines $\delta_l'(G) := l^{a_1} - 1 + (l-1)\sum_{i \geqslant 2} a_i$. Then he gives bounds for the δ_l' of certain successive quotients in terms of the dimensions of the toric and abelian variety parts of the special fibres of Néron models of A_K over various extensions of D. In [7, Remark 2.16] he remarks that the bounds might possibly be improved by replacing δ_l' by an other invariant δ_l defined as follows: for G as above one has $\delta_l(G) = \sum_{i \geqslant 1}(l^{a_i} - 1)$. This improvement is exactly what we do in this article. The results can be found in Section 3. Needless to say, we follow very much the approach of [7] in order to prove these sharper bounds. In fact, only Lemma 2.13

of [7] has to be changed, so the proof we give is rather short. We have taken this opportunity to weaken slightly the hypotheses of Lorenzini's results (he supposes D to be complete and k to be algebraically closed).

In Section 2 we recall Lorenzini's filtration. In Section 3 we state and prove the bounds on the δ_l of the l-parts of certain successive quotients and in Section 5 we show by some examples that the bounds of Section 3 are sharp; Section 4 is used to show some results on finite abelian groups that are needed in the other sections.

After all this work it turned out that a complete classification of the possible $\Phi_{(p)}$ for abelian varieties whose reduction has toric part, abelian variety part and unipotent part of fixed dimensions was in reach. The result, which is surprisingly simple to state, can be found in Theorem 6.1.

In this article we will frequently speak of the abelian variety part, the toric part and the unipotent part of the fibre over k of a Néron model over D. Since we are only interested in the characteristic polynomials of certain endomorphisms on the toric and abelian variety part, it suffices to define these parts after base change to an algebraic closure of k, and up to isogeny. Over the algebraic closure of k we can apply Chevalley's theorem; in [1, Theorems 9.2.1 and 9.2.2] one finds statements of the required results.

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2. Lorenzini's Filtration

Let D be a discrete valuation ring, let K be its field of fractions and k its residue field. Let A_K be an abelian variety over K, A its Néron model over D and $\Phi := A_k/A_k^0$ the finite étale group scheme over k of connected components of the special fibre A_k. Let $p \geqslant 0$ be the characteristic of k and let $\Phi_{(p)}$ be the prime-to-p part of Φ; if $p = 0$ we define $\Phi_{(p)}$ to be equal to Φ. In this section we will briefly recall the construction in [7] of a descending filtration

$$\Phi_{(p)} = \Phi_{(p)}^0 \supset \Phi_{(p)}^1 \supset \Phi_{(p)}^2 \supset \Phi_{(p)}^3 \supset \Phi_{(p)}^4 = 0 \tag{2.1}$$

which is functorial in A_K and invariant under base change by automorphisms of D. Since $\Phi_{(p)}$ is the direct sum of its l-parts Φ_l, with l ranging through the primes different from p, it suffices to describe the filtration on each Φ_l. We replace D by its strict henselization and view the group scheme Φ over the separably closed field k as just a group.

Let $l \neq p$ be a prime number. Let $K \to K^{\mathrm{s}}$ be a separable closure, let D^{s} be the integral closure of D in K^{s} and let $\overline{k}$ be the residue field of D^{s}; note that $\overline{k}$ is an algebraic closure of k and that $k \to \overline{k}$ is purely inseparable. The first step in the construction of the filtration is the description of Φ_l in terms of the Tate module

$U_l := \mathrm{T}_l(A(K^{\mathrm{s}}))$ with its action by $I := \mathrm{Gal}(K^{\mathrm{s}}/K)$ given in Proposition 11.2 of [5]:

$$\Phi_l = (U_l \otimes \mathbf{Q}/\mathbf{Z})^I \,/\, \left(U_l^I \otimes \mathbf{Q}/\mathbf{Z}\right). \tag{2.2}$$

The long exact cohomology sequence of the short exact sequence

$$0 \to U_l \to U_l \otimes \mathbf{Q} \to U_l \otimes \mathbf{Q}/\mathbf{Z} \to 0$$

of continuous I-modules gives a canonical isomorphism (see [5, (11.3.8)])

$$\Phi_l = \mathrm{tors}(\mathrm{H}^1(I, U_l)) \tag{2.3}$$

where for M any abelian group, $\mathrm{tors}(M)$ denotes the subgroup of torsion elements. Let I_{t} be the quotient of I corresponding to the maximal tamely ramified extension of D, and let P be the kernel of $I \to I_{\mathrm{t}}$. Then I_{t} is canonically isomorphic to $\prod_{q\neq p} \mathrm{T}_q(\mathrm{G}_m(k)) = \prod_{q\neq p} \mathbf{Z}_q(1)$ and P is a pro-p group. The Hochschild–Serre spectral sequence shows that

$$\begin{aligned} \Phi_l &= \mathrm{tors}(\mathrm{H}^1(I, U_l)) = \mathrm{tors}(\mathrm{H}^1(I_{\mathrm{t}}, U_l^P)) \\ &= \mathrm{tors}((U_l^P)_{I_{\mathrm{t}}})(-1) = \mathrm{tors}((U_l)_I)(-1) \end{aligned} \tag{2.4}$$

with the lower indices I_{t} and I denoting coinvariants and "(-1)" a Tate twist. Let N_l be the submodule of U_l which is generated by the elements $\sigma(x) - x$ with σ in I and x in U_l. Then by definition we have $(U_l)_I = U_l/N_l$. As in [5, §2.5], we define $V_l := U_l^I$. Then V_l, which is called the fixed part of U_l, is canonically isomorphic to $\mathrm{T}_l(A_k(k))$. Let A'_K be the dual of A_K; i.e., $A'_K = \mathrm{Pic}^0_{A_K/K}$. We will denote by A' the Néron model over D of A'_K, by Φ' its group of connected components, etc. Let $\langle\cdot,\cdot\rangle\colon U_l \times U'_l \to \mathbf{Z}_l(1)$ be the Weil pairing. For any y in V'_l, σ in I and x in U_l we have $\langle\sigma(x) - x, y\rangle = \langle\sigma(x), y\rangle - \langle x, y\rangle = \sigma(\langle x, \sigma^{-1}(y)\rangle) - \langle x, y\rangle = 0$. It follows that N_l is contained in the orthogonal $V_l'^{\perp}$ of V'_l in U_l. Since $U_l/V_l'^{\perp}$ is torsion free, we conclude that

$$\Phi_l = \mathrm{tors}\left(V_l'^{\perp}/N_l\right)(-1). \tag{2.5}$$

Remark 2.6. In the proof of Theorem 3.3 we will see that $V_l'^{\perp}/N_l$ is in fact a finite group, hence we have $\Phi_l = (V_l'^{\perp}/N_l)(-1)$.

Now it is clear that any filtration on $V_l'^{\perp}$ induces a filtration on Φ_l. As in [5, §2.5], we define $W_l \subset V_l$ to be the submodule corresponding to the maximal torus in A_k. Let $\widetilde{W}_l \subset \widetilde{V}_l \subset V_l$ be the submodules called the essentially toric part and the essentially fixed part in [5, §4.1]; if G/k' is the connected component of the special fibre of a semi-stable Néron model of A_K over a suitable sub-extension of $K \to K^{\mathrm{s}}$ then $\widetilde{V}_l$ corresponds to $\mathrm{T}_l(G(\overline{k}))$ and $\widetilde{W}_l$ to the Tate module of the maximal torus in G. We denote by t, a and u the dimensions of the toric part, the abelian variety part and the unipotent part of $A^0_{\overline{k}}$; we denote by $\tilde{t}$ and $\tilde{a}$ the analogous dimensions of any semi-stable reduction of A_K. Note that $t + a + u = \tilde{t} + \tilde{a} = \dim(A_K)$. An easy application of the Igusa–Grothendieck orthogonality theorem (which states that $W_l = V_l \cap V_l'^{\perp}$, see [5, Theorem 2.4], or [12, Theorem 3.1]), gives us the following filtration of $V_l'^{\perp}$, in which the successive quotients are torsion free and of the indicated rank:

$$V_l'^{\perp} \overset{\tilde{t}-t}{\supset} \widetilde{V}_l \cap V_l'^{\perp} \overset{2(\tilde{a}-a)}{\supset} \widetilde{W}_l \overset{\tilde{t}-t}{\supset} W_l \overset{t}{\supset} 0. \tag{2.7}$$

Lorenzini's filtration (2.1) on Φ_l is the filtration induced by (2.5) and (2.7). Note that in fact any finite sub-extension of $K \to K^{\mathrm{s}}$ induces a filtration on Φ_l as above; see [7, Theorem 3.1] for results concerning those filtrations. The reason we only consider the filtration coming from extensions over which A_K has semi-stable reduction is that only that filtration matters for the bounds on $\Phi_{(p)}$ of the next section.

3. Bounds on $\Phi_{(p)}$

We keep the notation of the previous section. Recall that K is strictly henselian. First we define some invariants of finite abelian groups and fix some notation needed to state our results.

DEFINITION 3.1. For l a prime number and $a = (a_1, a_2, \ldots)$ a sequence of integers $a_i \geqslant 0$ with $a_i = 0$ for i big enough, let $\delta_l(a) := \sum_i (l^{a_i} - 1)$. For l a prime number and $G \cong \oplus_i \mathbf{Z}/l^{a_i}\mathbf{Z}$ a finite abelian group of l-power order let $\delta_l(G) := \delta_l(a)$, where $a := (a_1, a_2, \ldots)$. For G a finite abelian group let $\delta(G) := \sum_l \delta_l(G_l)$, where $G = \oplus_l G_l$ is the decomposition of G into groups of prime power order.

Notation 3.2. Let $\widetilde{K}$ be the minimal sub-extension of K^{s} over which A_K has semi-stable reduction; it corresponds to the kernel of I acting on $\widetilde{V}$, see [5, §4.1]. We define K^{t} to be the maximal tame extension in $\widetilde{K}$, and for all $l \neq p$ we let K_l denote the maximal sub-extension of $\widetilde{K}$ whose degree over K is a power of l. We denote by $\tilde{t}$, $\tilde{a}$, t_{t}, a_{t}, u_{t}, t_l, a_l and u_l the dimensions of the toric parts, the abelian variety parts and the unipotent parts of the corresponding Néron models of A_K. For each prime $l \neq p$ we let $I_{(l)}$ be the subgroup of I such that $I/I_{(l)}$ is the quotient $\mathbf{Z}_l(1)$ of I_{t}.

Let A^{t} be the Néron model of A_K over the ring of integers D^{t} of K^{t}. Then $\mathrm{Gal}(K^{\mathrm{t}}/K)$ acts (from the right) on A^{t}, compatibly with its right-action on $\mathrm{Spec}(D^{\mathrm{t}})$. This action induces an action of $\mathrm{Gal}(K^{\mathrm{t}}/K)$ on the special fibre A^{t}_k. Let σ be a generator of the cyclic group $\mathrm{Gal}(K^{\mathrm{t}}/K)$. Let $l \neq p$ be a prime number and $i \geqslant 1$ an integer. Let $f_{l,i}$ denote the cyclotomic polynomial whose roots are the roots of unity of order l^i. We define $m_{\mathrm{a},l,i}$ and $m_{\mathrm{t},l,i}$ to be the multiplicities of $f_{l,i}$ in the characteristic polynomials of σ on the abelian variety part and on the toric part, respectively, of A_k^t (say one lets σ act on $T_l(A_k^t(\bar{k})) \otimes \mathbf{Q}$). Let $m_{l,i} := m_{a,l,i} + m_{t,l,i}$. Finally, for $j \geqslant 1$ we define $p_{a,l,j} := |\{i \geqslant 1 \mid m_{a,l,i} \geqslant j\}|$, $p_{t,l,j} := |\{i \geqslant 1 \mid m_{t,l,i} \geqslant j\}|$ and $p_{l,j} := |\{i \geqslant 1 \mid m_{l,i} \geqslant j\}|$. For an interpretation of $p_{a,l} = (p_{a,l,1}, p_{a,l,2}, \ldots)$ in terms of $m_{a,l} = (m_{a,l,1}, m_{a,l,2}, \ldots)$ etc. using partitions, see the beginning of the proof of Lemma 4.5.

THEOREM 3.3. *Let $l \neq p$ be a prime number and consider the filtration* (2.1) *on* Φ_l. *With the notations above, we have:*

1. *The group Φ_l^3 can be generated by t elements.*
2. $\delta_l(\Phi_l^2/\Phi_l^3) \leqslant \delta_l(p_{t,l}) \leqslant t_l - t$.
3. $\delta_l(\Phi_l^1/\Phi_l^2) \leqslant \delta_l(p_{a,l}) \leqslant 2(a_l - a)$.
4. $\delta_l(\Phi_l/\Phi_l^1) \leqslant \delta_l(p_{t,l}) \leqslant t_l - t$.
5. $\delta_l(\Phi_l/\Phi_l^2) \leqslant \delta_l(p_l) \leqslant (t_l - t) + 2(a_l - a)$.
6. $\delta_l(\Phi_l^1/\Phi_l^3) \leqslant \delta_l(p_l) \leqslant (t_l - t) + 2(a_l - a)$.

COROLLARY 3.4.

1. *The group $\Phi_{(p)}^3$ can be generated by t elements.*
2. $\delta(\Phi_{(p)}^2/\Phi_{(p)}^3) \leqslant t_t - t$.
3. $\delta(\Phi_{(p)}^1/\Phi_{(p)}^2) \leqslant 2(a_t - a)$.
4. $\delta(\Phi_{(p)}/\Phi_{(p)}^1) \leqslant t_t - t$.
5. $\delta(\Phi_{(p)}/\Phi_{(p)}^2) \leqslant (t_t - t) + 2(a_t - a)$.
6. $\delta(\Phi_{(p)}^1/\Phi_{(p)}^3) \leqslant (t_t - t) + 2(a_t - a)$.

Proof (of Theorem 3.3). We begin with some generalities. We always have $M_I = (M_{I_{(l)}})_{\mathbf{Z}_l(1)}$. The functors $M \mapsto M_{I_{(l)}}$ and $M \mapsto M^{I_{(l)}}$ are exact on the category of finitely generated $\mathbf{Z}_l$-modules with continuous $I_{(l)}$-action, and, for such modules, the canonical map $M^{I_{(l)}} \to M_{I_{(l)}}$ is an isomorphism, hence $M_{I_{(l)}}$ is torsion free if M is torsion free. For M a finitely generated $\mathbf{Z}_l$-module with continuous $\mathbf{Z}_l(1)$-action we have $M_{\mathbf{Z}_l(1)} = M/(\sigma-1)M$ and $M^{\mathbf{Z}_l(1)} = M[\sigma-1]$, where σ is any topological generator of $\mathbf{Z}_l(1)$.

Next we recall some general facts on the action of I on U_l. Let $\tilde{I}$ denote the subgroup $\mathrm{Gal}(K^s/\widetilde{K})$ of I. Then $\tilde{I}$ acts trivially on $\widetilde{V}_l = U_l^{\tilde{I}}$ and on $U_l/\widetilde{V}_l$; the action of $\tilde{I}$ on U_l factors through the biggest pro-l quotient $\mathbf{Z}_l(1)$ of $\tilde{I}$ and is given by an isogeny $U_l/\widetilde{V}_l \to \widetilde{W}_l(-1)$ (see [5, §9.2, Theorem 10.4]). It follows that $N_l \cap \widetilde{W}_l$ is open in $\widetilde{W}_l$. The group I acts on $\widetilde{V}_l$ via its finite quotient $\mathrm{Gal}(\widetilde{K}/K) = I/\tilde{I}$; this action can be described in terms of an action of $I/\tilde{I}$ on the special fibre of the Néron model of $A_{\widetilde{K}}$ (see [5, §4.2]). Dually, I acts with finite image on $U_l/\widetilde{W}_l$.

As promised in Remark 2.6 we will show that $\Phi_l = V_l'^{\perp}/N_l$. It suffices to show that $V_l'^{\perp}$ and N_l have the same rank. We have $\mathrm{rank}(U_l/V_l'^{\perp}) = t + 2a$. From the generalities at the beginning of the proof it follows that $U_l/N_l = (U_l)_I =$

$((U_l)_{I_{(l)}})_{\mathbf{Z}_l(1)} = (U_l^{I_{(l)}})_{\mathbf{Z}_l(1)}$. Let σ be a topological generator of $\mathbf{Z}_l(1)$. The exact sequence

$$0 \longrightarrow U_l^I \longrightarrow U_l^{I_{(l)}} \xrightarrow{\sigma-1} U_l^{I_{(l)}} \longrightarrow \left(U_l^{I_{(l)}}\right)_{\mathbf{Z}_l(1)} \longrightarrow 0 \tag{3.5}$$

shows that $\mathrm{rank}((U_l)_I) = \mathrm{rank}(U_l^I) = \mathrm{rank}(V_l) = t + 2a$. In order to prove Theorem 3.3 we may neglect the Tate twist in (2.5).

By definition, we have $\Phi_l^3 = W_l/N_l \cap W_l$. Since W_l is a free $\mathbf{Z}_l$ module of rank t, Φ_l^3 can be generated by t elements.

Let us now consider Φ_l^2/Φ_l^3. Since $\Phi_l^2 = \widetilde{W}_l/\widetilde{W}_l \cap N_l$, the group Φ_l^2/Φ_l^3 is a quotient of $(\widetilde{W}_l/W_l)_I = ((\widetilde{W}_l/W_l)_{I_{(l)}})_{\mathbf{Z}_l(1)}$. Lemma 4.4 implies that $\delta_l(\Phi_l^2/\Phi_l^3) \leqslant \delta_l(((\widetilde{W}_l/W_l)_{I_{(l)}})_{\mathbf{Z}_l(1)})$. By the generalities above, $(\widetilde{W}_l/W_l)_{I_{(l)}}$ is isomorphic as $\mathbf{Z}_l(1)$-module to $\widetilde{W}_l^{\mathrm{Gal}(\widetilde{K}/K_l)}/W_l$. Note that $\widetilde{W}_l^{\mathrm{Gal}(\widetilde{K}/K_l)}$ is the Tate module of the toric part of the special fibre of the Néron model of A_K over the ring of integers of K_l, and that $W_l = \widetilde{W}_l^{\mathrm{Gal}(\widetilde{K}/K)}$. It follows that for all $i \geqslant 1$ the multiplicity of $f_{l,i}$ in the characteristic polynomial of a generator σ of $\mathrm{Gal}(K^{\mathrm{t}}/K)$ on $(\widetilde{W}_l/W_l)_{I_{(l)}}$ is $m_{\mathrm{t},l,i}$ and that 1 is not a root of this characteristic polynomial. Applying Lemma 4.5 and Corollary 4.7 gives the second part of the theorem.

The proof of parts 3, 4, 5 and 6 of the theorem follows the same lines. For example, Φ_l^1/Φ_l^2 is a quotient of $((\widetilde{V}_l \cap V_l'^{\perp})/\widetilde{W}_l)_I$. The group I acts with finite image on $(\widetilde{V}_l \cap V_l'^{\perp})/\widetilde{W}_l$. The Grothendieck–Igusa orthogonality theorem [5, Theorem 2.4] shows that $(\widetilde{V}_l \cap V_l'^{\perp}/\widetilde{W}_l)_{I_{(l)}}$ has rank $2(a_l - a)$. We have

$$\begin{aligned}\left(\frac{\widetilde{V}_l \cap V_l'^{\perp}}{\widetilde{W}_l}\right)_I \otimes \mathbf{Q} &= \frac{(\widetilde{V}_l \cap V_l'^{\perp})^I}{\widetilde{W}_l^I} \otimes \mathbf{Q} = \frac{V_l \cap V_l'^{\perp}}{W_l} \otimes \mathbf{Q} \\ &= \frac{W_l}{W_l} \otimes \mathbf{Q} = 0 \end{aligned} \tag{3.6}$$

which shows that the hypotheses of Lemma 4.5 are satisfied. Since $\widetilde{V}_l \cap V_l'^{\perp}/\widetilde{W}_l$ is isogenous to $\widetilde{V}_l/V_l$, the multiplicities of the $f_{l,i}$ in the characteristic polynomial of a generator σ of $\mathrm{Gal}(K^{\mathrm{t}}/K)$ on $(\widetilde{V}_l \cap V_l'^{\perp}/\widetilde{W}_l)_{I_{(l)}}$ are precisely the $m_{\mathrm{a},l,i}$.

The proof of part 6 is entirely similar to the proofs of parts 2 and 3. For parts 4 and 5 one notes that $V_l'^{\perp}/\widetilde{V}_l \cap V_l'^{\perp}$ is dual to $\widetilde{W}_l'/W_l'$, that $V_l'^{\perp}/\widetilde{W}_l$ is dual to $\widetilde{V}_l'/V_l'$ and one uses that A_K and A_K' are isogenous. □

Proof (of Corollary 3.4). One just considers the factorization into irreducible factors of the characteristic polynomial of a generator σ of $\mathrm{Gal}(K^{\mathrm{t}}/K)$ acting on the semi-abelian variety part of A_k^{t}. □

4. Some Abelian Group Theory

In this section we prove some results needed in the proof of Theorem 3.3. We fix a prime number l and consider finite $\mathbf{Z}_l$-modules, i.e., finite abelian groups of l-power order. Recall that there is a bijection between the set of isomorphism classes of finite $\mathbf{Z}_l$-modules and the set of partitions (i.e., sequences $m = (m_1, m_2, \ldots)$ of non-negative integers such that $m_1 \geqslant m_2 \geqslant \cdots$ and $m_i = 0$ for i big enough): a finite $\mathbf{Z}_l$-module M corresponds to the partition $m = (m_1, m_2, \ldots)$ which satisfies $M \cong \oplus_{i\geqslant 1}\mathbf{Z}/l^{m_i}\mathbf{Z}$. To any partition m we attach the number $\delta_l(m) := \sum_{i\geqslant 1}(l^{m_i} - 1)$. Note that with these definitions, we have $\delta_l(M) = \delta_l(m)$, with $\delta_l(M)$ as in Definition 3.1.

LEMMA 4.1. *Let $0 \to B \to E \to A \to 0$ be an extension of finite $\mathbf{Z}_l$-modules. Let $b = (b_1, b_2, \ldots)$, e and a denote their invariants. Define $n_i := a_i + b_i$ and $n = (n_1, n_2, \ldots)$. Let $m = (m_1, m_2, \ldots)$ be the invariant of $A \oplus B$; i.e., m is the sequence obtained by reordering $(a_1, b_1, a_2, b_2, \ldots)$. Then we have $m \leqslant e \leqslant n$, with "$\leqslant$" the lexicographical ordering.*

Proof. Let us first prove that $e \leqslant n$. We use induction on $|E|$. We have $e_1 \leqslant a_1 + b_1 = n_1$ since $l^{a_1+b_1}$ kills E. If $e_1 < n_1$ there is nothing to prove, so we suppose that $e_1 = n_1$. Choose any element x in E of order l^{e_1} and consider the subgroup it generates. We get a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z}/l^{b_1}\mathbf{Z} & \to & \mathbf{Z}/l^{e_1}\mathbf{Z} & \to & \mathbf{Z}/l^{a_1}\mathbf{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & E' & \to & A' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{4.1.1}$$

in which the rows and columns are exact. Now the columns are split, since l^{b_1} is the exponent of B, etc. Hence $b' := (b_2, b_3, \ldots)$, $e' := (e_2, e_3, \ldots)$ and $a' := (a_2, a_3, \ldots)$ are the invariants of B', E' and A', respectively. The proof is finished by induction.

Let us now prove that $e \geqslant m$. By passing to Pontrjagin duals, if necessary, we may assume that $b_1 \geqslant a_1$. Then $m_1 = b_1$. If $e_1 > m_1$ there is nothing to prove, hence we suppose that $e_1 = m_1 = b_1$. We choose any element x in B of order l^{b_1}. Just as above we find a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbf{Z}/l^{b_1}\mathbf{Z} & \xrightarrow{\text{id}} & \mathbf{Z}/l^{b_1}\mathbf{Z} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & B & \to & E & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & E' & \to & A & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{4.1.2}$$

in which the columns are split. Induction finishes the proof. □

Remark 4.2. It would be nice to have a complete description of the possible invariants of extensions E of finite $\mathbf{Z}_l$-modules A by B in terms of the invariants of A and B. As Hendrik Lenstra pointed out to me, the problem can be phrased in terms of Hall polynomials, see for example [8]. An interesting question is whether there exist partitions a, b and e such that the corresponding Hall polynomial is not zero but has a prime number as root (note that this Hall polynomial is non-zero if and only if there exists a so-called Littlewood–Richardson sequence of type $(a, b; e)$).

LEMMA 4.3. *Suppose that $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ are partitions of N (i.e., $\sum_{i\geqslant 1} a_i = N = \sum_{i\geqslant 1} b_i$) and that $a \geqslant b$ in the lexicographical ordering. Then $\delta_l(a) \geqslant \delta_l(b)$, with equality if and only if $a = b$.*

Proof. Consider the set X of all partitions of N with its lexicographical ordering. From the inequality

$$(l^{n+1} - 1) + (l^{m-1} - 1) > (l^n - 1) + (l^m - 1)$$

satisfied for any integers $n \geqslant m$ it follows that $\delta_l \colon X \to \mathbf{Z}$ is strictly increasing. □

LEMMA 4.4.

1. *For M a finite $\mathbf{Z}_l$-module we have $\delta_l(M) \geqslant 0$, with equality if and only if $M = 0$.*
2. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite $\mathbf{Z}_l$-modules. Then $\delta_l(M) \geqslant \delta_l(M') + \delta_l(M'')$, with equality if and only if the sequence is split.*
3. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite $\mathbf{Z}_l$-modules. Suppose that M is killed by l^a and that $|M'| = l^b$. Then $\delta_l(M) \leqslant \delta_l(M'') + b(l^a - l^{a-1})$.*

Proof. This follows directly from Lemmas 4.1 and 4.3.

LEMMA 4.5. *Let M be a finitely generated free $\mathbf{Z}_l$-module with an automorphism σ of finite order. Suppose that $M/(\sigma - 1)M$ is finite, or, equivalently, that the automorphism $\sigma \otimes 1$ of the $\mathbf{Q}_l$-vector space $M \otimes \mathbf{Q}$ does not have 1 as eigenvalue. For $i \geqslant 1$ let m_i be the multiplicity, in the characteristic polynomial of σ, of the cyclotomic polynomial f_i whose roots are the roots of unity of order l^i. For each $j \geqslant 1$, let $p_j := |\{i \geqslant 1 \mid m_i \geqslant j\}|$. Then $\delta_l(M/(\sigma - 1)M) \leqslant \sum_{i\geqslant 1}(l^{p_i} - 1)$.*

Proof. Let $q = (q_1, q_2, \ldots)$ be the partition obtained by reordering $(m_1, m_2, \ldots)$. Then $p := (p_1, p_2, \ldots)$ is what is usually called the conjugate of q: when viewing partitions as Young diagrams, p and q are obtained from each other by interchanging rows and columns. In particular, we have $\sum_{i\geqslant 1} p_i = \sum_{i\geqslant 1} m_i$.

Let n be the order of σ. Then M is a module over the ring $\mathbf{Z}_l[x]/(x^n - 1)$. Let us write $n = l^r n'$ with n' not divisible by l. Then $\mathbf{Z}_l[x]/(x^n - 1)$ is the product of the ring $\mathbf{Z}_l[x]/(x^{l^r} - 1)$ by another ring R and $x - 1$ is invertible in R. This implies that M is the direct sum of two modules, one over $\mathbf{Z}_l[x]/(x^{l^r} - 1)$ and the other over R, and that the module over R does not contribute to $M/(\sigma - 1)M$. Hence we have reduced the problem to the case where the order of σ is l^r.

Let $i_1 < i_2 < \cdots < i_{p_1}$ denote the integers $i \geqslant 1$ such that $m_i > 0$. For $1 \leqslant j \leqslant p_1$, let $F_j := f_{i_j}$ be the corresponding cyclotomic polynomials, and let $F := F_1 \cdot F_2 \cdots F_{p_1}$. Since M is torsion free as $\mathbf{Z}_l$-module, M is a module over the ring $A := \mathbf{Z}_l[x]/(F)$. For any A-module N, we define $\overline{N} := N/(x - 1)N$. Let us first note that for all j we have $F_j(1) = l$. It follows that $\overline{A} = \mathbf{Z}/l^{p_1}\mathbf{Z}$. For N an A-module, $\overline{N}$ is an $\overline{A}$-module, hence l^{p_1} annihilates $\overline{N}$.

We claim that $|\overline{M}| = l^{\sum_{i\geqslant 1} p_i}$. To prove this, note that $|\overline{M}| = |\det(\sigma - 1)|_l^{-1}$, with $|\cdot|_l$ the l-adic absolute value on $\mathbf{Q}_l$, normalized by $|l|_l = 1/l$. So in order to compute $|\overline{M}|$ we may replace M by any σ-stable lattice M' in $M \otimes \mathbf{Q} \cong \oplus_{i\geqslant 1}(\mathbf{Q}_l[x]/(f_i))^{m_i}$. Taking $M' := \oplus_{i\geqslant 1}(\mathbf{Z}_l[x]/(f_i))^{m_i}$ and noting that $f_i(1) = l$ gives the result.

Let $a = (a_1, a_2, \ldots)$ be the invariant of $\overline{M}$; i.e., $\overline{M} \cong \oplus_{i\geqslant 1}\mathbf{Z}/l^{a_i}\mathbf{Z}$ and $a_1 \geqslant a_2 \geqslant \cdots$. Note that a and p are partitions of the same number, hence in view of Lemma 4.3, it suffices to show that $a \leqslant p$ in the lexicographical ordering. Since l^{p_1} annihilates $\overline{M}$, we have $a_1 \leqslant p_1$. If $a_1 < p_1$ there is nothing to prove, so we assume that $a_1 = p_1$. Let y be in M such that its image $\overline{y}$ in $\overline{M}$ corresponds to $(1, 0, 0, \ldots)$. Let A' denote the submodule Ay of M. Since M is free as a $\mathbf{Z}_l$-module, A' is free as a $\mathbf{Z}_l$-module, and we have $A' = \mathbf{Z}_l[x]/(G)$, with G dividing F. Let $0 \leqslant s \leqslant p_1$ be the number of irreducible factors of G. Then we have $\overline{A'} = \mathbf{Z}/l^s\mathbf{Z}$. We have a short exact sequence

$$0 \to A' \to M \to M' \to 0 \tag{4.5.1}$$

of A-modules, with M' not necessarily free as $\mathbf{Z}_l$-module. Multiplication by $x - 1$ on this sequence induces an exact sequence

$$0 \to M'[x - 1] \to \overline{A'} \to \overline{M} \to \overline{M'} \to 0. \tag{4.5.2}$$

The element $\overline{1}$ of $\overline{A'}$, which is annihilated by l^s, is mapped to $\overline{y}$ which has annihilator l^{p_1}. It follows that $s = p_1$, that $G = F$ and that $M'[x-1] = 0$. Let us now consider the finite A-module tors(M'). Multiplication by $x - 1$ acts injectively, hence bijectively. Since $x - 1$ is in the maximal ideal of A, it follows that tors$(M') = 0$, hence that M' is free as $\mathbf{Z}_l$-module. The proof is now finished by induction on rank(M), since $\overline{M'} \cong \oplus_{i\geqslant 2}\mathbf{Z}/l^{a_i}\mathbf{Z}$ and the partition p' obtained from M' is $(p_2, p_3, \ldots)$. □

Remark 4.6. Lemma 4.5 can be seen as a bound on the cohomology group $\mathrm{H}^1(\mathbf{Z}/n\mathbf{Z}, M)$, where 1 in $\mathbf{Z}/n\mathbf{Z}$ acts on M via σ. It is an interesting question, raised by Xavier Xarles, to obtain similar bounds for non-cyclic groups.

COROLLARY 4.7. *Let M be a finitely generated free $\mathbf{Z}_l$-module with an automorphism σ of finite order. Suppose that $M/(\sigma-1)M$ is finite. Then*

$$\delta_l(M/(\sigma-1)M) \leqslant \mathrm{rank}(M).$$

Proof. We use the notation of the beginning of the proof of Lemma 4.5. Then one has:

$$\begin{aligned}\mathrm{rank}(M) &\geqslant \sum_{i\geqslant 1} m_i\phi(l^i) \geqslant \sum_{i\geqslant 1} q_i\phi(l^i)\\ &= \sum_{i\geqslant 1}\sum_{j=1}^{p_i}\phi(l^j) = \sum_{i\geqslant 1}(l^{p_i}-1).\end{aligned} \tag{4.7.1}$$

The proof is finished by applying Lemma 4.5. □

LEMMA 4.8. *Let M be a finitely generated free $\mathbf{Z}_l$-module with an automorphism σ of finite order. Suppose that $M/(\sigma-1)M$ is finite and that $\delta_l(M/(\sigma-1)M) = \mathrm{rank}(M)$. Then M is a direct sum of $\mathbf{Z}_l$-modules of the type $\mathbf{Z}_l[x]/(f_1{\cdot}f_2\cdots f_r)$ with σ acting as multiplication by x and where f_i denotes the cyclotomic polynomial whose roots are the roots of unity of order l^i.*

Proof. The proof is by induction on rank(M) and consists of an inspection of the proofs of Lemma 4.5 and Corollary 4.7. First of all we must have that $n' = 1$. Secondly, we note that $m_i = q_i$ for all $i \geqslant 1$ since the inequalities in (4.7.1) are equalities (here we use that $\sum_{i\geqslant 1} m_i = \sum_{i\geqslant 1} q_i$ and that $\phi(l^i) < \phi(l^j)$ if $1 \leqslant i < j$). So m is the conjugate partition of p, hence $A = \mathbf{Z}_l[x]/(F)$ with $F = f_1{\cdot}f_2\cdots f_{p_1}$. The formula for the number of elements of $|\overline{M}|$ in the proof of Lemma 4.5 shows that a and p are partitions of the same number. By the hypotheses of the lemma we are proving, we have $\delta_l(a) = \delta_l(p)$. Lemma 4.3 implies that $a = p$. The end of the proof of Lemma 4.5 shows that $A' = A$ and that M' is free as $\mathbf{Z}_l$-module. By induction on rank(M), we know that M' is of the indicated type. It remains to show that the short exact sequence (4.5.1) splits. To do that, it is sufficient to show that $\mathrm{Ext}^1_A(A_i, A) = 0$, where $A_i = \mathbf{Z}_l[x]/(f_1 \cdots f_i)$ with $i \leqslant p_1$. This Ext^1 is easily computed using the projective resolution

$$\cdots \longrightarrow A \xrightarrow{f} A \xrightarrow{g} A \xrightarrow{f} A \longrightarrow A_i \longrightarrow 0 \tag{4.8.1}$$

with $f = f_1 \cdots f_i$ and $g = f_{i+1} \cdots f_{p_1}$. □

The following lemmas will be used in Sections 5 and 6.

LEMMA 4.9. *Let M be a finite $\mathbf{Z}_l$-module and let*

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^r = 0$$

be a strictly descending filtration. Suppose that for all i with $0 \leqslant i \leqslant r-2$ the group M^i/M^{i+2} is cyclic. Then M is cyclic.

Proof. For $r \leqslant 2$ there is nothing to prove. If we know the result for $r = 3$, the general case follows by induction since then M^0/M^3 is cyclic and the filtration $M^0 \supset M^2 \supset M^3 \supset \cdots \supset M^r$ has length $r-1$. So assume now that $r = 3$. Let x be an element of M^0 such that its image in M^0/M^2 is a generator. Then a certain multiple ax of x gives a generator of M^1/M^2. Since M^1 is cyclic, and M^1/M^2 a non-trivial quotient, ax is a generator of M^1. The subgroup of M^0 generated by x contains M^1 and its quotient by M^1 is M^0/M^1. We conclude that x generates M^0. □

Remark 4.10. The proof of Lemma 4.9 generalizes immediately to a proof of the following assertion. Let A be a local ring and M an A-module with a finite strictly descending filtration M^i such that the M^i/M^{i+2} are cyclic. Then M is cyclic.

LEMMA 4.11. *Let $0 \to B \to E \to A \to 0$ be a short exact sequence of finite $\mathbf{Z}_l$-modules with invariants b, e and a. Let $t \geqslant 0$ be an integer and suppose that B is generated by t elements. Then for all $i \geqslant 1$ we have $a_i \geqslant e_{i+t}$.*

Proof. For a partition p, let p' denote its conjugate. Then for all $i \geqslant 1$ we have $l^{a'_i} = |A[l^i]/A[l^{i-1}]|$. Let d be the endomorphism of the set of partitions defined by: $d(p)_i = p_{i+1}$ for all $i \geqslant 1$. Let d' be the conjugate of d: $d'(p) = d(p')'$. Then $d'(p)_i = \max(0, p_i - 1)$. When viewing a partition p as a Young diagram in which the p_i are the lengths of the columns, d and d' remove the longest column and row, respectively. For a finite $\mathbf{Z}_l$-module M with invariant m, the submodule lM has invariant $d'(m)$. The maps d and d' commute. In the rest of this proof we will consider the partial ordering on the set of partitions in which $p \leqslant q$ if and only if for all $i \geqslant 1$: $p_i \leqslant q_i$. Note that $p \leqslant q$ is equivalent to $p' \leqslant q'$. Below we will use that $p \geqslant q$ if and only if: $p'_1 \geqslant q'_1$ and $d'(p) \geqslant d'(q)$. We will also use that if N and M are finite $\mathbf{Z}_l$-modules with invariants n and m such that N is a subquotient of M, then $n \leqslant m$.

The proof of the lemma is by induction on $|E|$. What we have to prove is that $a \geqslant d^t(e)$. The exact sequence $0 \to B[l] \to E[l] \to A[l]$ shows that $a'_1 \geqslant e'_1 - t$. Note that $d^t(e)'_1 = \max(0, e'_1 - t)$, hence we have $a'_1 \geqslant d^t(e)'_1$. The exact sequence $0 \to B \cap lE \to lE \to lA$ shows (induction hypothesis) that $d'(a) \geqslant d^t(d'(e)) = d'(d^t(e))$. The two inequalities just proved imply that $a \geqslant d^t(e)$. □

LEMMA 4.12. *Let l be a prime. Let $0 \to B \to E \to A \to 0$ be a short exact sequence of finite $\mathbf{Z}_l$-modules with invariants b, e and a, respectively. Then*

$$\delta_l(b) + \delta_l(a) \geqslant 2 \sum_{i \geqslant 1} \left(\frac{l^{\lfloor e_i/2 \rfloor} + l^{\lceil e_i/2 \rceil}}{2} - 1 \right) \tag{4.12.1}$$

where for any real number x, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest (resp. smallest) integer $\leqslant x$ (resp. $\geqslant x$).

Proof. We have $\sum_i (a_i + b_i) = \sum_i e_i$. Lemma 4.1 asserts that $a + b \geqslant e$ in the lexicographical ordering. Consider the set S of all pairs (r, s) of partitions, such that $r + s \geqslant e$ and $\sum_i (r_i + s_i) = \sum_i e_i$. Let $f \colon S \to \mathbf{Z}$ be the map which sends (r, s) to $\delta_l(r) + \delta_l(s)$. We will show that f achieves its minimum at all (r, s) in S with the property that, for all $i \geqslant 1$, one has $\{r_i, s_i\} = \{\lfloor e_i/2 \rfloor, \lceil e_i/2 \rceil\}$.

Suppose now that (r, s) is an element of S where f has a minimum. We have to show that $|r_i - s_i| \leqslant 1$ for all $i \geqslant 1$. Suppose that this is not the case. Let $j \geqslant 1$ be minimal for the property that $|r_j - s_j| > 1$ and $|r_i - s_i| \leqslant 1$ for all $i < j$. We may and do suppose that $r_j - s_j > 1$. Note that if $j > 1$ we have $s_{j-1} > s_j$. We define r' and s' as follows: $(r'_i, s'_i) = (r_i - 1, s_i + 1)$ if $i \geqslant j$ and $r_i = r_j$; in all other cases $(r'_i, s'_i) = (r_i, s_i)$. Note that r' and s' are partitions, that $\sum_i (r'_i + s'_i)$ is equal to $\sum_i e_i$ and that $f(r', s')$ is strictly smaller than $f(r, s)$. □

5. Examples

The aim of this section is to give examples that show that the bounds in Theorem 3.3 and Corollary 3.4 are sharp, in a sense that will become clear in the examples. The examples we construct here will play an important role in Section 6. We give our examples over the field $K := \mathbf{C}((q))$ of formal Laurent series over the complex numbers with its usual valuation, but it is easy to get similar examples in mixed characteristic, or equal characteristic $p > 0$.

The building stones of our examples are the following. For each integer $n \geqslant 1$ we let E_n be the so-called Tate elliptic curve "$\mathbf{G}_m / q^{n\mathbf{Z}}$" over K as described in [2, §VII] or in [11, §6] (E_n is obtained from the analytic family of elliptic curves over the punctured unit disc with coordinate q whose fibres are the $\mathbf{C}^*/q^{n\mathbf{Z}}$, by base change from the field of finite tailed convergent Laurent series to K). It is well known that the special fibre of the Néron model of E_n over $D := \mathbf{C}[[q]]$ is an extension of $\mathbf{Z}/n\mathbf{Z}$ by the multiplicative group.

For each prime l and integer $r \geqslant 0$ we define the ring

$$\Lambda_{l,r} := \mathbf{Z}[x]/(f_{l,1} \cdots f_{l,r}),$$

where as before $f_{l,i}$ is the polynomial whose roots are the roots of unity of order l^i. When $l > 2$, we let $A_{l,r}$ be an abelian variety over $\mathbf{C}$ obtained as follows: we choose an isomorphism of $\mathbf{R}$-algebras between $\Lambda_{l,r} \otimes \mathbf{R}$ and a product of a number of copies of $\mathbf{C}$ and define $A_{l,r} := (\Lambda_{l,r} \otimes \mathbf{R})/\Lambda_{l,r}$ (it is well known that the trace form on $\Lambda_{l,r}$ implies the existence of a polarization). The first three examples will be isogenous to twists of products of copies of E_n and of $A_{l,r,K}$. Of course Lemma 4.8 tells us how to cook up the required examples.

EXAMPLE 5.1. Let $d \geqslant 0$ and let G be any finite abelian group that can be generated by d elements. Then $G \cong \oplus_{i=1}^{d} \mathbf{Z}/n_i\mathbf{Z}$, say. For $A_K := \prod_{i=1}^{d} E_{n_i}$ one has $\Phi = \Phi^3 = \oplus_{i=1}^{t} \mathbf{Z}/n_i\mathbf{Z}$ and we have $d = t = \dim(A_K)$.

EXAMPLE 5.2. Now consider parts 2 and 4 of Theorem 3.3. Let l be a prime. For $i \geqslant 1$ we let $B_{l,i}$ be the abelian variety $E_1 \otimes \Lambda_{l,i}$ over $\mathbf{C}$; i.e., $B_{l,i}$ is a direct sum of copies of E_1, indexed by some $\mathbf{Z}$-basis of $\Lambda_{l,i}$, and $\Lambda_{l,i}$ acts on $B_{l,i}$ according to its action on itself. In particular, multiplication by x in $\Lambda_{l,i}$ induces an automorphism σ of $B_{l,i}$. Note that σ has order l^i. Let $C_{l,i}$ be the twist of $B_{l,i,K}$ over $K(q^{1/l^i})$ by σ; i.e., $C_{l,i}$ is the quotient of the K-scheme $B_{l,i,K} \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K(q^{1/l^i}))$ by the group $\mathrm{Gal}(K(q^{1/l^i})/K) = \mathbf{Z}/l^i\mathbf{Z}$ (here we choose a root of unity of order l^i) which acts by $a \mapsto \sigma^a$ on the first factor and via its natural action on the second factor.

We will now compute the group of connected components Ψ of the Néron model of $C_{l,i}$ over D, using (2.4). First of all we have $\mathrm{T}_l(E_1(K^s)) = \mathbf{Z}_l(1) \oplus \mathbf{Z}_l$, with I acting via its quotient $\mathbf{Z}_l(1)$ in the following way: an element of I with image a in $\mathbf{Z}_l(1)$ acts as multiplication by the matrix $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$. By construction, $\mathrm{T}_l(C_{l,i}(K^s)) = \mathrm{T}_l(E_1(K^s)) \otimes \Lambda_{l,i}$ and an element in I with image a in $\mathbf{Z}_l(1)$ acts as $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right) \otimes x^a$. Since $C_{l,i}$ has $\tilde{a} = t = 0$, we have $\Psi_l^1 = \Psi_l^2$ and $\Psi_l^3 = 0$. The filtration $\mathbf{Z}_l(1) \subset \mathrm{T}_l(E_1(K^s)$ induces the filtration $\widetilde{W}_l \subset \mathrm{T}_l(C_{l,i}(K^s))$. It follows that Ψ_l is the cokernel of $\left(\begin{smallmatrix}x-1 & x\\ 0 & x-1\end{smallmatrix}\right)$ and that Ψ_l/Ψ_l^1 and Ψ_l^2/Ψ_l^3 are both isomorphic to $\Lambda_{l,i}/(x-1) = \mathbf{Z}/l^i\mathbf{Z}$. An analogous computation shows that $\Psi = \Psi_l$. One can show that Ψ is isomorphic to $\mathbf{Z}/l^i\mathbf{Z} \oplus \mathbf{Z}/l^i\mathbf{Z}$ if $l > 2$ and to $\mathbf{Z}/2^{i+1}\mathbf{Z} \oplus \mathbf{Z}/2^{i-1}\mathbf{Z}$ if $l = 2$.

Let G be a finite abelian group of l-power order, say with invariant $a = (a_1, a_2, \ldots)$. Then for $A_K := \prod_{i \geqslant 1} C_{l,a_i}$ we have $\Phi_l/\Phi_l^1 \cong \Phi_l^2/\Phi_l^3 \cong G$ and $\delta_l(G) = t_l = \dim(A_K)$. We remark that abelian varieties over K that are isogeneous to A_K provide examples with Φ_l/Φ_l^1 not isomorphic to Φ_l^2/Φ_l^3.

EXAMPLE 5.3. For $l > 2$ prime and $i \geqslant 0$ we let $D_{l,i}$ be the abelian variety over K obtained by twisting $A_{l,i,K}$ over $K(q^{1/l^i})$ by the automorphism σ of $A_{l,i}$ which is induced from the multiplication by x in $\Lambda_{l,i}$. Then we have $\mathrm{T}_l(D_{l,i}(K^s)) = \mathrm{T}_l(A_{l,i}(\mathbf{C})) = \Lambda_{l,i} \otimes \mathbf{Z}_l$, and an element in I with image a in $\mathbf{Z}_l(1)$ acts as x^a. Let Ψ_l denote the group of connected components of attached to $D_{l,i}$. In this case we have $\tilde{t} = a = 0$, hence $\Psi_l = \Psi_l^1$ and $\Psi_l^2 = 0$. By (2.4) we have $\Psi_l = \Lambda_{l,i}/(x-1) = \mathbf{Z}/l^i\mathbf{Z}$.

Suppose now that $l \neq 2$. Let G be a finite abelian group of l-power order, say with invariant $a = (a_1, a_2, \ldots)$. Then for $A_K := \prod_{i \geqslant 1} D_{l,a_i}$ we have $\Phi_l = \Phi_l^1$, $\Phi_l^2 = 0$, $\Phi_l^1/\Phi_l^2 \cong G$ and $\delta_l(G) = 2a_l = 2\dim(A_K)$. The case $l = 2$ is a little bit different because $m_{\mathrm{a},2,1}$ is always even.

EXAMPLE 5.4. Let l be prime and let $r > 0$ and $s > 0$ be positive integers. We will construct an abelian variety A_K with $t = a = 0$, $\tilde{t} = l^r - 1$, $\tilde{a} = (l^{r+s} - l^r)/2$ and $\Phi = \Phi_l \cong \mathbf{Z}/l^{2r+s}\mathbf{Z}$. It follows from Theorem 3.3 that in such an example Φ_l/Φ_l^1 and Φ_l^2 are cyclic of order l^r, that Φ_l^1/Φ_l^2 is cyclic of order l^s and that Φ_l/Φ_l^2 and Φ_l^1 are cyclic of order l^{r+s}. Hence this example shows that, as far as the exponent is concerned, the two-fold extension Φ_l/Φ_l^3 can be arbitrary.

As in the previous examples, $f_{l,i}$ will denote the polynomial whose roots are the roots of unity of order l^i, and $\Lambda_{l,r}$ is the ring $\mathbf{Z}[x]/(f_{l,1}\cdots f_{l,r})$. Let $\Lambda_{l,r,s} := \mathbf{Z}[x]/(f_{l,r+1}\cdots f_{l,r+s})$. Let $D_{l,r,s}$ be an abelian variety over K obtained by replacing $\Lambda_{l,i}$ by $\Lambda_{l,r,s}$ and q^{1/l^i} by $q^{1/l^{r+s}}$ in the construction of $D_{l,i}$ in Example 5.3. Let $C_{l,r}$ be as in Example 5.2. Our example A_K will be isogeneous to $C_{l,r} \times D_{l,r,s}$. Let $V := \mathrm{T}_l((C_{l,r} \times D_{l,r,s})(K^s)) \otimes \mathbf{Q}$. Then V is a $\mathbf{Q}_l$-vector space with an action of $I = \mathrm{Gal}(K^s/K)$. We have an isomorphism of $\mathbf{Q}_l$-vector spaces with I-action

$$\Lambda_{l,r} \otimes \mathbf{Q}_l \;\oplus\; \Lambda_{l,r,s} \otimes \mathbf{Q}_l \;\oplus\; \Lambda_{l,r} \otimes \mathbf{Q}_l \;\xrightarrow{\sim}\; V \tag{5.4.1}$$

such that an element of I with image a in $\mathbf{Z}_l(1)$ acts via

$$\begin{pmatrix} x^a & 0 & ax^a \\ 0 & x^a & 0 \\ 0 & 0 & x^a \end{pmatrix}. \tag{5.4.2}$$

Let

$$V = V^0 \supset V^1 \supset V^2 \supset V^3 = 0 \tag{5.4.3}$$

be the filtration (2.7) on V. Then V^2 is simply the first term in (5.4.1) and V^1 is the sum of the first two terms. For any $\mathbf{Z}_l$-lattice M in V let $M^i := M \cap V^i$. To get our example A_K, it suffices to find an I-invariant $\mathbf{Z}_l$-lattice M in V such that M^1 and M/M^2 are isomorphic, as $\mathbf{Z}_l[I]$-modules, to $\Lambda_{l,r+s} \otimes \mathbf{Z}_l$, where an element of I with image a in $\mathbf{Z}_l(1)$ acts on $\Lambda_{l,r+s} \otimes \mathbf{Z}_l$ as x^a. Namely, since M is I-invariant, M is the l-adic Tate module of an abelian variety A_K which is isogeneous to $C_{l,r} \times D_{l,r,s}$; for A_K one has Φ_l/Φ_l^2 and Φ_l^1 cyclic of order l^{r+s}, hence Φ_l cyclic of order l^{2r+s} by Lemma 4.9.

Let us now try to find such a M. Note that we have canonical projections $\Lambda_{l,r+s} \to \Lambda_{l,r}$ and $\Lambda_{l,r+s} \to \Lambda_{l,r,s}$ which induce an embedding $\Lambda_{l,r+s} \otimes \mathbf{Z}_l \subset V^1$. We will first show that we only have to look among the sublattices M with $M^1 = \Lambda_{l,r+s} \otimes \mathbf{Z}_l$. Namely, if M is an I-invariant $\mathbf{Z}_l$-lattice in V of the type we are looking for, then for a suitable element of the form $v = (a, b, a)$ of V^* (here we consider V as a $\mathbf{Q}_l$-algebra and V^* denotes the group of units of V) vM is isomorphic to M as $\mathbf{Z}_l[I]$-module and has $(vM)^1 = \Lambda_{l,r+s} \otimes \mathbf{Z}_l$. Since the $\mathbf{Z}_l[I]$-module structure determines the filtration, we also have $(vM)/(vM)^2 \cong M/M^2$.

From now on we only consider M with $M^1 = \Lambda_{l,r+s} \otimes \mathbf{Z}_l$. Such M are determined by their image in V/M^1. So we look for an I-invariant torsion free $\mathbf{Z}_l$-submodule N of $V/M^1 = V^1/M^1 \oplus \Lambda_{l,r} \otimes \mathbf{Q}_l$ whose image in $\Lambda_{l,r} \otimes \mathbf{Q}_l$ is a lattice and for whose associated M we have $M/M^2 \cong \Lambda_{l,r+s} \otimes \mathbf{Z}_l$. It follows that such a N is isomorphic, via the canonical projection, to its image in $\Lambda_{l,r} \otimes \mathbf{Q}_l$. Lemma 4.8 implies that this image is isomorphic to $\Lambda_{l,r} \otimes \mathbf{Z}_l$, hence of the form $z{\cdot}\Lambda_{l,r} \otimes \mathbf{Z}_l$ for some z in $(\Lambda_{l,r} \otimes \mathbf{Q}_l)^*$. We conclude that N is of the form $\mathrm{im}(\alpha)$, where

$$\alpha\colon \Lambda_{l,r} \otimes \mathbf{Z}_l \longrightarrow V^1/M^1 \oplus \Lambda_{l,r} \otimes \mathbf{Q}_l, \qquad a \mapsto (\phi(a), za) \tag{5.4.4}$$

with $\phi\colon \Lambda_{l,r} \otimes \mathbf{Z}_l \to V^1/M^1$ a morphism of $\mathbf{Z}_l$-modules, and $z \in (\Lambda_{l,r} \otimes \mathbf{Q}_l)^*$. For a given pair (ϕ, z), let $N_{\phi,z}$ denote the image of the corresponding α.

Let us first study what it means for (ϕ, z) that $N_{\phi,z}$ is I-invariant. Using that $N_{\phi,z}$ is I-invariant if and only if it is invariant under the matrix in (5.4.2) with a replaced by 1, one easily sees that $N_{\phi,z}$ is I-invariant if and only if

$$\forall a \in \Lambda_{l,r} \otimes \mathbf{Z}_l\colon\ \phi(xa) = x\phi(a) + \overline{(xza, 0)}. \tag{5.4.5}$$

To find out which (ϕ, z) satisfy (5.4.5), we write out everything in terms of the $\mathbf{Z}_l$-basis $(1, x, \ldots, x^{l^r-2})$ of $\Lambda_{l,r} \otimes \mathbf{Z}_l$. Let $y = (y_1, y_2)$ be in V^1 such that $\phi(1) = \overline{y}$. One then checks that

$$\phi(x^i) = x^i\overline{y} + ix^i\,\overline{(z, 0)} \quad \text{for } 0 \leqslant i \leqslant l^r - 2. \tag{5.4.6}$$

Applying (5.4.5) with $a = x^{l^r-2}$, and using that $\sum_{i=0}^{l^r-1} x^i = 0$ in $\Lambda_{l,r}$, gives

$$(xg_r'(x)z, g_r(x)y_2) \in \Lambda_{l,r+s} \otimes \mathbf{Z}_l \tag{5.4.7}$$

where $g_r = f_{l,1} \cdots f_{l,r}$ and g_r' is the derivative of g_r. The conclusion is that $N_{\phi,z}$ is I-invariant if and only if ϕ is given by (5.4.6) and (y, z) satisfies (5.4.7). For a given such pair (y, z), let $M_{y,z}$ denote the lattice M in V corresponding to $N_{\phi,z}$.

It remains now to be seen that there exist (y, z) satisfying (5.4.7) such that $M_{y,z}/M_{y,z}^2$ is isomorphic to $\Lambda_{l,r+s} \otimes \mathbf{Z}_l$, or, equivalently, such that $(M_{y,z}/M_{y,z}^2)_I$ is cyclic. In order to have a useful description of $M_{y,z}$, we lift ϕ to V^1 as follows: let $\widetilde{\phi}\colon \Lambda_{l,r} \otimes \mathbf{Z}_l \to V^1$ be the morphism of $\mathbf{Z}_l$-modules such that

$$\widetilde{\phi}\colon x^i \mapsto x^i y + ix^i(z, 0) \quad \text{for } 0 \leqslant i \leqslant l^r - 2. \tag{5.4.8}$$

Then we have an isomorphism of $\mathbf{Z}_l$-modules:

$$\begin{aligned} \beta\colon \Lambda_{l,r+s}\otimes \mathbf{Z}_l \,\oplus\, \Lambda_{l,r}\otimes\mathbf{Z}_l &\xrightarrow{\sim} M\subset V, \\ (a,b)&\mapsto (a,0)+(\tilde{\phi}(b),zb) \end{aligned} \tag{5.4.9}$$

Let τ be an element of I with image 1 in $\mathbf{Z}_l(1)$. Then τ acts on V by the matrix in (5.4.2) with $a=1$. One computes that in order to make β invariant under I, one must let τ act on the source of β in (5.4.9) by

$$\tau\colon (a,b)\mapsto \Big(xa+(xzb,0)+x\tilde{\phi}(b)-\tilde{\phi}(xb),xb\Big). \tag{5.4.10}$$

Using this formula, we can study $M_{y,z}/M_{y,z}^2$. Recall that $\Lambda_{l,r+s}\otimes\mathbf{Z}_l$ is the image in V^1 of the sum of the two canonical projections from $\Lambda_{l,r+s}\otimes\mathbf{Z}_l$ to $\Lambda_{l,r}\otimes\mathbf{Z}_l$ and $\Lambda_{l,r,s}\otimes\mathbf{Z}_l$. It follows that

$$M_{y,z}/M_{y,z}^2\cong N:=\Lambda_{l,r,s}\otimes\mathbf{Z}_l \,\oplus\, \Lambda_{l,r}\otimes\mathbf{Z}_l \tag{5.4.11}$$

with τ acting on N by

$$\tau\colon (a,b)\mapsto (xa+x\overline{\phi}(b)-\overline{\phi}(xb),xb) \tag{5.4.12}$$

where $\overline{\phi}\colon \Lambda_{l,r}\otimes\mathbf{Z}_l\to V^1/V^2=\Lambda_{l,r,s}\otimes\mathbf{Q}_l$ is $\tilde{\phi}$ composed with the projection $V^1\to V^1/V^2$; we have $\overline{\phi}(x^i)=x^iy_2$ for $0\leqslant i\leqslant l^r-2$.

Note that $\Phi_l/\Phi_l^2\cong N/(\tau-1)N$. Hence Φ_l/Φ_l^2 is cyclic if and only if the endomorphism $\tau-1$ of the $\mathbf{F}_l$-vector space $N\otimes\mathbf{F}_l$ has corank 1. Now $N\otimes\mathbf{F}_l$ is the direct sum of $\mathbf{F}_l[\varepsilon]/(\varepsilon^{l^{r+s}-l^r})$ and $\mathbf{F}_l[\varepsilon]/(\varepsilon^{l^r-1})$, with $\varepsilon=x-1$. The matrix of $\tau-1$ with respect to the direct sum of the bases $(1,\varepsilon,\ldots,\varepsilon^{l^{r+s}-l^r-1})$ and $(1,\varepsilon,\ldots,\varepsilon^{l^r-2})$ is of the form

$$\left(\begin{array}{cccc|cccc} 0 & & & & & & & \\ 1 & 0 & & & & A & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ \hline & & & & 0 & & & \\ & & & & 1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & 0 \end{array}\right). \tag{5.4.13}$$

It follows that $\tau-1$ has corank 1 if and only if the upper right coefficient of A is not zero, or, equivalently, if and only if there exists b in $\Lambda_{l,r}\otimes\mathbf{Z}_l$ such that $x\overline{\phi}(b)-\overline{\phi}(xb)$ is a unit in $\Lambda_{l,r,s}\otimes\mathbf{Z}_l$. A computation shows that $x\overline{\phi}(x^{l^r-2})-\overline{\phi}(x^{l^r-1})=g_r(x)y_2$. Now recall that we are free to choose $y=(y_1,y_2)$ in V^1 and z in $(\Lambda_{l,r}\otimes\mathbf{Q}_l)^*$ as long as (y,z) satisfies (5.4.7). Note that $g_r(x)$ and $g_r'(x)$ are units in $\Lambda_{l,r,s}\otimes\mathbf{Q}_l$ and $\Lambda_{l,r}\otimes\mathbf{Q}_l$, respectively. Hence we can choose $y_2=g_r(x)^{-1}$ and $z=x^{-1}g_r'(x)^{-1}$.

EXAMPLE 5.5. Our final example is the analog of Example 5.4 in the case $\tilde{a} = 0$. More precisely, let l be a prime and $r \geqslant 0$ an integer. Then there exists an abelian variety A_K with $t = \tilde{a} = 0$, $\tilde{t} = l^r - 1$ and $\Phi = \Phi_l \cong \mathbf{Z}/l^{2r}\mathbf{Z}$.

Let $C_{l,r}$ be as in Example 5.2. The abelian variety A_K can be found in the isogeny class of $C_{l,r}$ in the same way as used in Example 5.4. In this case the construction is somewhat easier, since the filtration on V has only two steps ($V^1 = V^2$), so we leave the details to the reader. Let us just mention that all formulas up to (5.4.10) remain valid (in adapted form), and after (5.4.10) one shows that $M/(\tau - 1)M$ can be cyclic with the same method as used to show that $N/(\tau - 1)N$ can be cyclic.

6. Classification of the $\Phi_{(p)}$

The aim of this section is to prove the following theorem.

THEOREM 6.1. *Let D be a strictly henselian discrete valuation ring of residue characteristic $p \geqslant 0$. Let G be a finite commutative group of order not divisible by p. For each prime $l \neq p$, let $m_l := (m_{l,1}, m_{l,2}, \ldots)$ be the partition corresponding to the l-part G_l of G (i.e., $G_l \cong \oplus_{i\geqslant 1}\mathbf{Z}/l^{m_{l,i}}\mathbf{Z}$ and $m_{l,1} \geqslant m_{l,2}\cdots$). Let d, t, a and u be non-negative integers such that $d = t + a + u$. Then there exists an abelian variety over the field of fractions of D, of dimension d, toric rank t, abelian rank a and unipotent rank u which has $\Phi_{(p)} \cong G$, if and only if*

$$u \geqslant \sum_{l\neq p}\sum_{i\geqslant t+1}\left(\frac{l^{\lfloor m_{l,i}/2\rfloor} + l^{\lceil m_{l,i}/2\rceil}}{2} - 1\right) \tag{6.1.1}$$

where for any real number x, $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the largest (resp. smallest) integer $\leqslant x$ (resp. $\geqslant x$).

Proof. We will start by showing that if A_K is as indicated in the theorem, then (6.1.1) holds. Let $l \neq p$ be a prime. Let f_l be the map from the set of partitions to $\mathbf{R}$ defined by

$$f_l(m) = \sum_{i\geqslant 1}\left(\frac{l^{\lfloor m_i/2\rfloor} + l^{\lceil m_i/2\rceil}}{2} - 1\right). \tag{6.1.2}$$

Then f_l is strictly increasing for the partial ordering in which $a \geqslant b$ if and only if $a_i \geqslant b_i$ for all i. One easily sees that f_l is increasing for the lexicographical ordering on the set of partitions of a fixed number, but we won't use that. Consider the filtration

$$\Phi_l \supset \Phi_l^1 \supset \Phi_l^3 \supset 0 \tag{6.1.3}$$

induced by (2.1). Theorem 3.3 shows that

$$2(t_l - t + a_l - a) \geqslant \delta_l(\Phi_l/\Phi_l^1) + \delta_l(\Phi_l^1/\Phi_l^3). \tag{6.1.4}$$

Let n_l be the invariant of Φ_l/Φ_l^3. Lemma 4.11 shows that for all $i \geqslant 1$ we have $n_{l,i} \geqslant m_{l,i+t}$, or, in the terminology of the proof of that lemma, that $n_l \geqslant d^t(m_l)$ in the partial ordering. Lemma 4.12 says that

$$\delta_l(\Phi_l/\Phi_l^1) + \delta_l(\Phi_l^1/\Phi_l^3) \geqslant 2f_l(n_l). \tag{6.1.5}$$

It follows that

$$2(t_l - t + a_l - a) \geqslant 2f_l(n_l). \tag{6.1.6}$$

Summing over all $l \neq p$ and dividing by 2 gives (6.1.1).

It remains to show that all groups G satisfying (6.1.1) can occur as the $\Phi_{(p)}$ of an abelian variety A_K over the field of fractions K of D of dimension d, toric rank t, abelian rank a and unipotent rank u. It is sufficient to show that all groups G satisfying

$$u = \left\lceil \left(\sum_{l \neq p} \sum_{i \geqslant t+1} \left(\frac{l^{\lfloor m_{l,i}/2 \rfloor} + l^{\lceil m_{l,i}/2 \rceil}}{2} - 1 \right) \right) \right\rceil \tag{6.1.7}$$

occur in such a way, since one can replace A_K by the product of A_K with an abelian variety B_K which has unipotent reduction and trivial group of connected components.

Let us first suppose that $K = \mathbf{C}((q))$. Let d, t, a, u and G be as in the theorem, and suppose that they satisfy (6.1.7). We have $G \cong \oplus_{i \geqslant 1} \mathbf{Z}/n_i\mathbf{Z}$ with $n_i \geqslant 1$ and $n_{i+1} | n_i$ for all i. Let B_K be of the type described in Example 5.1: it has dimension t, completely toric reduction and group of connected components $\mathbf{Z}/n_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/n_t\mathbf{Z}$. The abelian variety A_K we are constructing will be of the form

$$A_K = B_K \times \prod_{l \neq p} C_{K,l} \quad \text{with}$$

$$\dim(C_{K,l}) = \left\lceil \left(\sum_{i \geqslant t+1} \left(\frac{l^{\lfloor m_{l,i}/2 \rfloor} + l^{\lceil m_{l,i}/2 \rceil}}{2} - 1 \right) \right) \right\rceil{}' \tag{6.1.8}$$

and such that all $C_{K,l}$ have unipotent reduction. Note that in fact such an A_K has unipotent rank u, since for $l \neq 2$ the function f_l defined above has integer values. For $l \neq 2$ we define

$$C_{K,l} = \prod_{i > t} C_{K,l,i} \tag{6.1.9}$$

where $C_{K,l,i}$ is the abelian variety defined as follows. If $m_{l,i} \neq 1$ is odd, then $C_{K,l,i}$ is the abelian variety constructed in Example 5.4 with $r = (m_{l,i} - 1)/2$ and $s = 1$. If $m_{l,i} = 1$, then $C_{K,l,i}$ is the abelian variety constructed in Example 5.3 with

$i = 1$. If $m_{l,i}$ is even, then $C_{K,l,i}$ is the abelian variety constructed in Example 5.18 with $r = m_{l,i}/2$. Note that the group of connected components of the reduction of $C_{K,l,i}$ is cyclic of order $l^{m_{l,i}}$. For $l = 2$ the construction of $C_{K,l}$ is a bit different. Let $r \geqslant 0$ be maximal such that $m_{2,i} > 1$ for all $i \leqslant r$. Then $C_{K,2}$ will be of the form

$$C_{K,2} = D_{K,2} \times \prod_{t<i\leqslant r} C_{K,2,i} \tag{6.1.10}$$

where $C_{K,2,i}$ is defined as $C_{K,l,i}$ but with l replaced by 2, and where $D_{K,2}$ is as follows. Let v be the number of $i > t$ such that $m_{2,i} = 1$. If v is even we let $D_{K,2}$ be the product of $v/2$ elliptic curves which have unipotent reduction and group of connected components isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. If v is odd we let $D_{K,2}$ be the product of $(v-1)/2$ elliptic curves with unipotent reduction and group of connected components isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and one elliptic curve with unipotent reduction and group of connected components cyclic of order 2. One verifies easily that A_K has all the desired properties.

To finish the proof of the theorem, we have to show that similar examples exist over any strictly henselian discrete valuation ring D with residue characteristic p. Since our examples are products of the examples of Section 5, it suffices to show that the examples in Section 5 exist over D. Since we do not suppose D complete, we cannot use a Tate curve "$\mathrm{G}_m/q^{\mathbf{Z}}$" with q a uniformizer of D. Instead we can use any elliptic curve E over K which has toric reduction and trivial group of connected components. Then I acts on the Tate module $\mathrm{T}_l(E(K^{\mathrm{s}}))$ through its quotient $\mathbf{Z}_l(1)$ and for a suitable choice of a $\mathbf{Z}_l$-basis of $\mathrm{T}_l(E(K^{\mathrm{s}}))$, an element of I with image a in $\mathbf{Z}_l(1)$ acts as $\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)$. It follows that Examples 5.1 and 5.2 with E_1 replaced by E still work. To make Example 5.3 work over D, it is enough to show that for all $l \neq p$ and $r > 0$ such that $l^r > 2$, there exists an abelian scheme over D of relative dimension $l^{r-1}(l-1)/2$ and with an action by $\mathbf{Z}[x]/(f_{l,r})$. Once one has these abelian schemes, the constructions of Section 5 can be carried out over D. The fact that such abelian schemes exist is a consequene of the theory of abelian varieties of "CM-type". Fix an l and r as above. The moduli scheme over $\mathbf{Z}[1/l]$ of abelian schemes of the desired type, with a suitable polarization and l-power level structure, is finite etale and not empty. Another way to prove the desired existence is to consider isogeny factors over $\mathbf{Q}(\zeta_{l^r})$ of the Jacobian of the Fermat curve of degree l^r. Yet another way to construct these abelian varieties is to use elliptic curves with an action by the ring of integers of an imaginary quadratic subfield of $\mathbf{Q}(\zeta_{l^r})$. □

7. Further Remarks and Questions

Although Theorem 6.1 gives a complete classification of the prime-to-p parts of the groups of connected components of special fibres of Néron models with some fixed invariants, there are still questions left. For example, it is clear that the groups of connected components Φ have functorial additional structure coming from the fact that the category of abelian varieties has an involution: every abelian variety has its dual. More precisely, suppose that Φ comes from the abelian variety A_K. Let A'_K be the dual of A_K and denote its group of connected components by Φ'. Then there are several pairings with values in $\mathbf{Q}/\mathbf{Z}$, conjecturally perfect and the same up to sign, between Φ and Φ'; see [5, §§1.2, 1.3, 11.2], [9], [7, §3] and [10]. Let us note by the way that the pairing given in [12, Proposition 3.3] cannot exist, since it is supposed to have values in $(\mathbf{Q}/\mathbf{Z})(1)$; the mistake in the proof is that the direct sum decomposition in the unique displayed formula in it is not unique. Anyway, for each of the remaining pairings we get a filtration

$$\Phi_{(p)} = \Phi'^{4,\perp}_{(p)} \supset \Phi'^{3,\perp}_{(p)} \supset \Phi'^{2,\perp}_{(p)} \supset \Phi'^{1,\perp}_{(p)} \supset \Phi'^{0,\perp}_{(p)} = 0. \tag{7.1}$$

It would be interesting to know the common refinement of this filtration with (2.1). Also, it would be of interest to prove that the various pairings are the same up to a determined sign. Some relations between the two filtrations (2.1) and (7.1) on the l-part for $l \neq p$ can be found in [7, Theorem 3.21], under the hypothesis that A_K has a polarization of degree prime to l.

Let us consider the functor from the category of abelian varieties over K to the category of finite abelian groups which associates to each abelian variety the group of connected components of the special fibre of its Néron model. A rather vague question one can ask is through what categories of abelian groups endowed with some extra structure this functor factors. We have seen for example that there is a filtration of four steps on the prime-to-p part, but as we have just remarked that is certainly not all there is.

Lorenzini has shown [7, Theorem 3.22], under the hypothesis that there is a polarization of degree prime to l, that $\Phi'^{2,\perp}_l$ is the prime-to-p part of the kernel of the map from Φ to the group of connected components of A_L, where $K \to L$ is any extension over which A_K has semi-stable reduction. It would be interesting to generalize this. Even in the case in which A_K acquires semi-stable reduction after a tamely ramified extension $K \to L$, when the theory of [3] applies, I have not been able to give a description of the filtration (2.1) in terms of the special fibre of the Néron model of A_L with its action of $\mathrm{Gal}(L/K)$.

The p-part of Φ remains difficult. For example, one expects a bound for its order in terms of the dimension of A_K if the toric part of the reduction is zero, but even in the case of potentially good reduction I don't know of any such bound (of course, if A_K is the Jacobian of a curve with a rational point, the usual bound, i.e., the bound we have when k is of characteristic zero, holds, since one can apply Winters's theorem [14]). In a forthcoming article [4] one can find a generalization of a result of McCallum (unpublished) which says that in the case of potentially good reduction the p-part is annihilated by the degree of any extension after which one obtains semi-stable reduction, but not in general by the exponent of the Galois group of such an extension. Work in progress by Bosch and Xarles, using a rigid analytic uniformization of Néron models, seems to imply that there is a four-step functorial filtration on the whole of Φ, for which three of the four successive quotients can be described in terms of the inertia group acting on the character group of the toric part of the semi-stable reduction. The remaining successive quotient comes from an abelian variety, obtained by Raynaud's extension, which has potentially good reduction. This part is still a mystery.

References

1. Bosch, S., Lütkebohmert, W. and Raynaud, M.: *Néron Models,* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge, Band 21, Springer-Verlag, 1990.
2. Deligne, P. and Rapoport, M.: Les schémas de modules de courbes elliptiques, in *Modular Functions of One Variable II,* Lecture Notes in Math. 349, Springer-Verlag, 1975, pp. 143–316.
3. Edixhoven, B.: Néron models and tame ramification, *Compositio Mathematica* **81** (1992), 291–306.
4. Edixhoven, B., Liu, Q. and Lorenzini, D.: The p-part of the group of components. To be submitted for publication.
5. Grothendieck, A.: Modèles de Néron et monodromie, *Lecture Notes in Math.* 288, Séminaire de Géométrie 7, Exposé IX, Springer-Verlag, 1973.
6. Lenstra, H. W. and Oort, F.: Abelian varieties having purely additive reduction, *J. Pure Appl. Algebra* **36** (1985), 281–298.
7. Lorenzini, D.: On the group of components of a Néron model, *J. Reine Angew. Math.* **445** (1993), 109–160.
8. MacDonald, I. G.: *Symmetric Functions and Hall Polynomials,* Oxford, Clarendon Press, Oxford, 1979.
9. Milne, J. S.: *Arithmetic Duality Theorems,* Perspectives in Math. 1, Academic Press, New York, 1986.
10. Moret-Bailly, L.: *Pinceaux de Variétés Abéliennes,* Astérisque 129, Société Mathématique de France, 1985.
11. Mumford, D.: An analytic construction of degenerating abelian varieties over complete rings, *Compositio Mathematica* **24**(3) (1972), 239–272.
12. Oort, F.: Good and stable reduction of abelian varieties, *Manuscripta Math.* **11** (1974), 171–197.
13. Tate, J.: Algorithm for determining the type of a singular fibre in an elliptic pencil, in *Modular Functions of One Variable IV,* Lecture Notes in Math. 476, Springer-Verlag, 1975, pp. 33–52.
14. Winters, G.: *On the Existence of Certain Families of Curves,* Am. J. Math. **96** (1974), 215–228.