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J. H. M. STEENBRINK

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Monodromy and weight filtration for smoothings of isolated singularities

Dedicated to Frans Oort on the occasion of his 60th birthday

J.H.M. STEENBRINK
Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands

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Abstract. We investigate the connection between monodromy and weight filtration for one-parameter smoothings of isolated singularities. We give a formula for the signature of the intersection form in terms of the Hodge numbers of the vanishing cohomology.

Key words: singularity, mixed Hodge structure, monodromy, weight filtration

1. Introduction

Let V be a finite dimensional vectorspace and let N be a nilpotent endomorphism of V. Then for each integer n there exists a unique decreasing filtration $W = W(N, n)$ of V such that $N(W_i) \subset W_{i-2}$ for each i and the induced map $N^i: Gr^W_{n+i} \to Gr^W_{n-i}$ is an isomorphism for all i.

If $F: Z \to \mathbf{C}$ is a flat projective morphism with smooth generic fiber, then associated to the critical value 0 we have a limit mixed Hodge structure $H^n(Z_F)$ whose weight filtration is equal to $W(N, n)$ where N is the logarithm of the unipotent part of the monodromy transformation T around 0.

A similar situation arises in the case of an isolated hypersurface singularity $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ and its vanishing cohomology $\tilde{H}^n(X_{f,0})$. Again we have a monodromy operator T, but now the description of the weight filtration is slightly more complicated: write

$$H^n(X_{f,0}) = H^n(X_{f,0})_1 \oplus H^n(X_{f,0})_{\neq 1}, \tag{1}$$

where $H^n(X_{f,0})_1$ (resp. $H^n(X_{f,0})_{\neq 1}$) is the subspace on which T acts with eigenvalue 1 (resp. eigenvalues $\neq 1$). Then $W = W(N, n+1)$ on $H^n(X_{f,0})_1$ and $W = W(N, n)$ on $H^n(X_{f,0})_{\neq 1}$.

In this note we deal with the case of the weight filtration on the vanishing cohomology of a one-parameter smoothing of an isolated singularity. Part of the results were announced in [9] with a short indication of proof. In this general case the decomposition (1) has to be replaced by a suitable decomposition of $Gr^W H^n(X_{f,0})$.

We also give precise results about the polarizations on these summands and express the index of the intersection form (in the even-dimensional case) in terms of Hodge numbers. This generalizes and simplifies [8] Theorem 4.11 and [9] Theorem 2.23. The main tool in our proof is a strong globalization theorem for one-parameter smoothings of isolated singularities, in the spirit of the Appendix of [4].

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2. Monodromy and weight filtration

Let (X', x) be an isolated singularity of a complex space of pure dimension $n+1$, and $f\colon (X', x) \to (\mathbf{C}, 0)$ a holomorphic function germ. Suppose that $X := f^{-1}(0)$ has an isolated singularity at x. We let $X'_{f,x}$ denote the Milnor fibre of f at x. We first sharpen a globalization theorem due to Looijenga [4]:

THEOREM 1. *Let $f\colon (X', x) \to (\mathbf{C}, 0)$ be a smoothing of an isolated singularity of pure dimension n. Then there exists a flat projective morphism $F\colon Z \to \mathbf{C}$, a point $z \in Z_0$ and an isomorphism $h\colon (X', x) \to (Z, z)$ such that $F \circ h = f$ and F is smooth along $Z_0 \setminus \{z\}$ and such that the restriction mapping $H^n(Z_F, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective. Here Z_F denotes the generic fibre of F.*

Proof. If $n = 0$ then f is finite, hence projective. So in the sequel we suppose that $n \geqslant 1$. We follow the proof of [4]. Let Y be an affine variety of dimension $n+1$ with a unique singular point y and P a regular function on Y such that the germ $f\colon (X', x) \to (\mathbf{C}, 0)$ is biholomorphic to $P\colon (Y, y) \to \mathbf{C}$. The existence of Y such that $(X', x) \simeq (Y, y)$ follows from work of Artin [1] and Hironaka [2], and the existence of a polynomial P with the desired properties follows from finite determinacy for germs with isolated singularities, due to Mather and Looijenga [4]. We assume Y to be embedded in affine N-space such that $y = 0$. Let $\mathbf{m}$ denote the ideal of regular functions on Y vanishing at y. Fix a positive integer k such that all germs $P + g$ for $g \in \mathbf{m}^k$ are analytically isomorphic to P. Let Z' denote the projective closure of Y. We may assume that $Z' \setminus \{y\}$ and $Z' \setminus Y = Z'_\infty$ are smooth.

Choose a sufficiently general (to be made precise below) homogeneous polynomial g of degree $d \geqslant k$ sufficiently big and let $Q = P + g$. Let $Z = \{(\xi, t) \in Z' \times \mathbf{C} \mid \xi_0^d Q(\xi_1/\xi_0, \ldots, \xi_N/\xi_0) = t\xi_0^d\}$. We embed Y in Z as the graph of Q and let $z = (y, 0)$. The projection F of Z onto the second factor provides a globalization of f. We will show that we can choose g in such a way that it has the desired properties. First we require that g defines a smooth hypersurface in $\mathbf{P}^{N-1}$ which is transverse to Z'_∞ and that z is the only critical point of F on $F^{-1}(0)$.

We fix a good Stein representative $f\colon X' \to \Delta$ for the germ f in the sense of [3] Chapter 2.B. Write $\Omega^{\cdot}_f = \Omega^{\cdot}_{X'}/df \wedge \Omega^{\cdot -1}_{X'}$. By [3] Theorem 8.7, the sheaf $\mathcal{H}^n f_*(\Omega^{\cdot}_f)$ is coherent. Let $\omega_Y = j_*\Omega^{n+1}_{Y\setminus\{y\}}$ where $j\colon Y \setminus \{y\} \to Y$ is the inclusion map. Put $Y_t = Q^{-1}(t)$. First observe that for $t \neq 0$ sufficiently small the restriction map $H^n(Y_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective. This follows from the specialization sequence

$$H^n(Y_0, \mathbf{C}) \to H^n(Y_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C}) \to H^{n+1}(Y_0, \mathbf{C}),$$

(here we use that F has no critical point at infinity) and the fact that for an affine variety of dimension n the cohomology groups are zero in degrees $> n$. Moreover, for such t there is a natural map $\rho\colon\ H^0(Y, \omega_Y) \to H^0(Y_t, \Omega^n_{Y_t}) \to \mathcal{H}^n f_*(\Omega^\cdot_f)(t)$ which is the composition of the map $\eta \mapsto$ the restriction to Y_t of η/dP and the restriction to $X'_{f,x}$. Then ρ is the composition of two surjections, hence surjective. (The second map is surjective as $H^n(Y_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective.) Choose $\eta_1, \dots, \eta_r \in H^0(Y, \omega_Y)$ whose images generate $\mathcal{H}^n f_*(\Omega^\cdot_f)(t)$ for all $t \neq 0$ sufficiently small. If g is a small perturbation of f, they will still generate $\mathcal{H}^n g_*(\Omega^\cdot_g)(t)$ for all $t \neq 0$ sufficiently small, again by Looijenga's coherence theorem.

There exists $l \in \mathbf{N}$ such that $\eta_1, \dots, \eta_r$ extend to sections of $\omega_{Z'}(lZ_\infty)$. Let $D = Z'_\infty \cap Z'_0 = Z'_\infty \cap Z'_t$. Then $\eta_1/dQ, \dots, \eta_r/dQ$ extend to sections of $\Omega^n_{Z_t}((l-d)D)$. So if $d \geqslant l$ the map $H^0(Z_t, \Omega^n_{Z_t}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective. Then a fortiori $H^n(Z_t, \mathbf{C}) \to H^n(X'_{f,x}, \mathbf{C})$ is surjective.

By [9] we have the following exact sequences of mixed Hodge structures associated with the Milnor fibre $X'_{f,x}$ of f at 0:

$$0 \to H^{n+1}_{\{x\}}(X') \to H^n(X'_{f,x})_1 \xrightarrow{V} H^n_c(X'_{f,x})_1(-1) \to H^{n+2}_{\{x\}}(X') \to 0, \quad (2)$$

$$0 \to H^{n-1}(X'_{f,x}) \to H^n_{\{x\}}(X) \to H^n_c(X'_{f,x})$$

$$\xrightarrow{j} H^n(X'_{f,x}) \to H^{n+1}_{\{x\}}(X) \to H^{n+1}_c(X'_{f,x}) \to 0, \quad (3)$$

where the subscript 1 denotes the generalized eigenspace of T for the eigenvalue 1 and $jV = N = \log(T)$ (resp. $Vj = N_c = \log(T_c)$) on $H^n_c(X'_{f,x})_1$ (resp. $H^n(X'_{f,x})_1$). We recall

THEOREM 2.

$$Gr^W_i H^{n+1}_{\{x\}}(X') = 0 \quad \text{for } i \geqslant n+1; \quad (4)$$

$$Gr^W_i H^n_{\{x\}}(X) = 0 \quad \text{for } i \geqslant n; \quad (5)$$

$$Gr^W_i H^{n+2}_{\{x\}}(X') = 0 \quad \text{for } i \leqslant n+1; \quad (6)$$

$$Gr^W_i H^{n+1}_{\{x\}}(X) = 0 \quad \text{for } i \leqslant n. \quad (7)$$

See [9] Corollary 1.12. Both N and N_c map W_i to W_{i-2}.

THEOREM 3. *For all $i \geqslant 0$ the map*

$$N_c^i\colon Gr^W_{n+1+i}\,\mathrm{im}(V) \to Gr^W_{n+1-i}\,\mathrm{im}(V)$$

is an isomorphism.

REMARK 4. In the hypersurface case, i.e. when X' is smooth, the map V is an isomorphism and we recover [8] Corollary 4.9.

Proof. We choose a flat projective morphism $F\colon\ Z \to \mathbf{C}$, a point $z \in Z$ and an isomorphism $h\colon\ (X', x) \to (Z, z)$ such that $F \circ h = f$ and F is smooth along $Z_0 \setminus \{z\}$ as in Theorem 1. Let Z_F denote the generic fibre of F. Then one has the exact sequence of mixed Hodge structures

$$\to H^n(Z_0) \to H^n(Z_F) \to H^n(X'_{f,x}) \to 0, \tag{4}$$

where $H^n(Z_F)$ carries the limit mixed Hodge structure. There is a monodromy action T on this sequence, and T acts as the identity on $H^n(Z_0)$. We have the following sequence

$$H^n(Z_F)_1 \xrightarrow{k} H^n(X'_{f,x})_1 \xrightarrow{V} H^n_c(X'_{f,x})_1(-1) \xrightarrow{k^t} H^n(Z_F)_1(-1)$$

and $N = k^t \circ V \circ k$. As k is surjective, its transpose k^t is injective and defines an isomorphism of mixed Hodge structures $\mathrm{im}(V) \to \mathrm{im}(N)$ such that $k^t \circ N_c = N \circ k^t$. As $W = W(N, n)$ on $H^n(Z_F)_1$ we get that $W = W(N, n+1)$ on $\mathrm{im}(N)$. We conclude that $W = W(N_c, n+1)$ on $\mathrm{im}(V)$.

It follows that $Gr^W(\mathrm{im}(V))$ is completely determined by the kernel of N_c on $\mathrm{im}(V)$. In order to determine this kernel, observe that (4) implies that $\ker(V)$ has weights $\leqslant n$ and that (7) implies that $\mathrm{coker}(j)$ has weights $\geqslant n+1$. Hence $\ker(V) \subset \mathrm{im}(j)$. So we have the exact sequence

$$0 \to \ker(j) \to \ker(N_c) \xrightarrow{j} \ker(V) \to 0 \tag{5}$$

and hence $\ker(N_c)$ has weights $\leqslant n$. By considering the action of N_c on the exact sequence

$$0 \to \mathrm{im}(V) \to H^n(X'_{f,x})_1(-1) \to H^{n+2}_{\{x\}}(X') \to 0$$

we obtain the exact sequence

$$0 \to \ker(N_c;\ \mathrm{im}(V)) \to \ker(N_c)(-1) \to W_{n+2}H^{n+2}_{\{x\}}(X') \to 0$$

and hence $\ker(N_c;\ \mathrm{im}(V)) = W_{n+1}(\ker(N_c)(-1))$. So from (5) we obtain

LEMMA 5. *We have the exact sequence of mixed Hodge structures*

$$0 \to \ker(j)(-1) \to \ker(N_c;\ \mathrm{im}(V)) \xrightarrow{j} W_{n+1}(\ker(V)(-1)) \to 0$$

THEOREM 6. *Regarding the map* $H^n_c(X'_{f,x}) \xrightarrow{j} H^n(X'_{f,x})$ *we have that*

$$N^i : Gr^W_{n+i}\,\mathrm{im}(j) \to Gr^W_{n-i}\,\mathrm{im}(j)$$

is an isomorphism for all $i \geqslant 0$, *i.e.* $W = W(N, n)$ *on* $\mathrm{im}(j)$.

Proof. Choose a globalization $F\colon\ Z \to \mathbf{C}$ of f as in the proof of Theorem 2. Then j is factorized as

$$H^n_c(X'_{f,x}) \xrightarrow{k^t} H^n(Z_F) \xrightarrow{k} H^n(X'_{f,x}).$$

Let $P^n(Z_F) = \ker(L\colon\ H^n(Z_F) \to H^{n+2}(Z_F))$ denote the primitive cohomology. Here L is the cup product with the hyperplane class. As a general hyperplane does not pass through the point x, the image of k^t is contained in $P^n(Z_F)$.

We have the nondegenerate pairing S on $P^n(Z_F)$, given by

$$S(x, y) = (-1)^{n(n-1)/2} \int_{Z_F} x \wedge y.$$

It is $(-1)^n$-symmetric, $W_\alpha = (W_{2n-1-\alpha})^\perp$ and $S(Nx, y) + S(x, Ny) = 0$. Moreover $N^\alpha : Gr^W_{n+\alpha}P^n(Z_F) \to Gr^W_{n-\alpha}P^n(Z_F)$ is an isomorphism for all $\alpha \geqslant 0$. If $P_{n+\alpha} := \ker(N^{\alpha+1} : Gr^W_{n+\alpha}P^n(Z_F) \to Gr^W_{n-\alpha-2}P^n(Z_F))$, the form $(x, y) \mapsto S(Cx, N^\alpha\overline{y})$ is hermitian positive definite on $P_{n+\alpha}$ by [7], Lemma 6.25.

Let $Q_\alpha = Gr^W_{n-\alpha}\ker(k) \subset Gr^W_{n-\alpha}P^n(Z_F)$. Then $Gr^W_{n+\alpha}\,\mathrm{im}(j) \simeq (Q_\alpha)^\perp$ as $Gr^W_{n+\alpha}\ker(j) = 0$. Therefore,

$$Gr^W_{n-\alpha}\,\mathrm{im}(j) \simeq N^\alpha(Q_\alpha)^\perp / Q_\alpha \cap N^\alpha(Q_\alpha)^\perp$$

so we have to show that

$$Q_\alpha \cap N^\alpha(Q_\alpha)^\perp = (0).$$

Clearly, $Q_\alpha \subset N^\alpha P_{n+\alpha}$ as $N = 0$ on $\ker(k)$. So let $x \in N^\alpha(Q_\alpha)^\perp \cap Q_\alpha$. Write $x = N^\alpha x'$ with $x' \in P_{n+\alpha} \cap (Q_\alpha)^\perp$. Then $S(Cx', N^\alpha\overline{x'}) = 0$ hence $x = 0$.

THEOREM 7. (i) *For all* $i > 0$ *the map*

$$V \circ N^{i-1}\colon\ Gr^W_{n+i}H^n(X'_{f,x})_1 \to Gr^W_{n-i}H^n_c(X'_{f,x})_1$$

is an isomorphism;

(ii) *for all* $i \geqslant 0$ *the map*

$$N^i \circ j\colon\ Gr^W_{n+i}H^n_c(X'_{f,x}) \to Gr^W_{n-i}H^n(X'_{f,x})$$

is an isomorphism.

Proof. For $i > 0$ we have $Gr^W_{n+i}\ker(V) = 0$ so

$$Gr^W_{n+i}H^n(X'_{f,x})_1 \simeq Gr^W_{n+i}\operatorname{im}(V).$$

This space is mapped isomorphically to $Gr^W_{n-i+2}\operatorname{im}(V)$ by N^{i-1} according to Theorem 3. As $\operatorname{coker}(V)$ has weights $\geqslant n+2$, we have

$$Gr^W_{n-i+2}\operatorname{im}(V) \simeq Gr^W_{n-i}H^n_c(X'_{f,x})_1.$$

This proves (i). Ones proves (ii) similarly using Theorem 6 instead of Theorem 3.

3. Primitive decomposition

Let V be a finite dimensional vector space and N a nilpotent endomorphism of V, n an integer and $W = W(N,n)$. Then we have the following decomposition of $Gr^W(V)$. Recall that $N^i\colon Gr^W_{n+i}(V) \to Gr^W_{n-i}(V)$ is an isomorphism for all $i \geqslant 0$. Put

$$P_{n+i} = \ker(N^{i+1}\colon Gr^W_{n+i}(V) \to Gr^W_{n-i-2}(V))$$

for $i \geqslant 0$ and 0 else. Then we have the primitive decomposition

$$Gr^W_\alpha(V) \simeq \bigoplus_{i\geqslant 0} N^i P_{\alpha+2i}.$$

We will give an analogous but more subtle decomposition of $Gr^W H^n(X'_{f,x})_1$ and $Gr^W H^n_c(X'_{f,x})_1$ (we use the same notation as in the preceding section). This was first mentioned in [6] and proved by Saito in a letter to the author. Define

$$B_{n+i} = \ker(N_c^{i+1}\colon Gr^W_{n+i}H^n_c(X'_{f,x})_1 \to Gr^W_{n-i-2}H^n_c(X'_{f,x})_1)$$

for $i \geqslant 0$ and 0 else, and

$$A_{n+i} = \ker(N^i\colon Gr^W_{n+i}H^n(X'_{f,x})_1 \to Gr^W_{n-i}H^n(X'_{f,x})_1)$$

for $i > 0$ and 0 else. By Theorem 7 B_{n+i} is mapped isomorphically to $Gr^W_{n-i}\ker(V)$ by $N^i \circ j$ and A_{n+i} is mapped isomorphically to $Gr^W_{n-i}\ker(j)$ by $V \circ N^{i-1}$.

THEOREM 8. *We have*

$$Gr^W_\alpha H^n_c(X'_{f,x})_1 = \bigoplus_{i\geqslant 0} N^i B_{\alpha+2i} \oplus \bigoplus_{i\geqslant 0} V N^i A_{\alpha+2+2i}$$

and

$$Gr^W_\alpha H^n(X'_{f,x})_1 = \bigoplus_{i\geqslant 0} N^i j B_{\alpha+2i} \oplus \bigoplus_{i\geqslant 0} N^i A_{\alpha+2i}.$$

Proof. Define a graded vectorspace C by $C_{2\alpha} = 0$ and

$$C_{2\alpha+1} = Gr^W_{\alpha+1} H^n(X'_{f,x})_1 \oplus Gr^W_\alpha H^n_c(X'_{f,x})_1.$$

Define an endomorphism λ of degree -2 of C as $\lambda(x, y) = (j(y), V(x))$. From Theorem 7 we obtain that for all $i \geqslant 0$ the map $\lambda^i : C_{2n+i} \to C_{2n-i}$ is an isomorphism. Hence, if $D_{2n+i} = \ker(\lambda^{i+1}\colon C_{2n+i} \to C_{2n-i-2})$ for $i \geqslant 0$ and 0 else, then we have that the map $\lambda^\alpha\colon D_{2n+i} \to C_{2n+i-\alpha}$ is injective for $\alpha \leqslant 2i$ and else the zero map. We obtain the primitive decomposition

$$C_\alpha = \bigoplus_{i \geqslant 0} \lambda^i D_{\alpha+2i}.$$

Finally observe that $D_{2n+2i+1} = A_{n+i+1} \oplus B_{n+i}$.

REMARK 9. The previous theorem leads to the decomposition

$$Gr^W H^n(X'_{f,x})_1 = A \oplus B$$

with $B = \bigoplus_\alpha \bigoplus_{i \geqslant 0} N^i j B_{\alpha+2i}$ and $A = \bigoplus_\alpha \bigoplus_{i \geqslant 0} N^i A_{\alpha+2i}$. We have $W = W(N, n)$ on B and $W = W(N, n+1)$ on A. Similarly we have

$$Gr^W H^n_c(X'_{f,x})_1 = A' \oplus B'$$

with $B' = \bigoplus_\alpha \bigoplus_{i \geqslant 0} N^i B_{\alpha+2i}$ and $A' = \bigoplus_\alpha \bigoplus_{i \geqslant 0} V N^i A_{\alpha+2+2i}$. These are decompositions as graded mixed Hodge structures. We have $W = W(N_c, n)$ on B' and $W = W(N_c, n-1)$ on A'. The maps $V : A \to A'(-1)$ and $j : B' \to B$ are isomorphisms. Observe that $A = 0$ if and only if (X, x) is a rational homology manifold and that $B = 0$ if and only if (X', x) is a rational homology manifold.

See also [5] for the case of isolated complete intersection singularities.

We finally want to indicate how one can polarize the mixed Hodge structures $Gr^W H^n(X'_{f,x})$ and $Gr^W H^n_c(X'_{f,x})$. For the part of these on which the monodromy acts with eigenvalues $\neq 1$, we can use the global case, and these mixed Hodge structures are polarized by N. So let us consider the eigenvalue 1 part.

By Remark 9 it suffices to define polarizations on the Hodge structures A_i and B_i, i.e. on the graded quotients of the local cohomology groups.

Define the pairing

$$\langle \cdot, \cdot \rangle\colon H^n(X'_{f,x}) \otimes H^n_c(X'_{f,x}) \to \mathbf{C}$$

by

$$\langle \omega, \eta \rangle := (-1)^{n(n-1)/2} \int_{X'_{f,x}} \omega \wedge \eta.$$

THEOREM 10. *The form* $(x, y) \mapsto \langle j(x), N^i y\rangle$ *polarizes* B_{n+i} *for all* $i \geqslant 0$. *The form* $(x, y) \mapsto \langle x, VN^{i-1}y\rangle$ *polarizes* A_{n+i} *for all* $i \geqslant 1$.

Proof. Fix a globalization $F\colon Z \to \mathbf{C}$ as in Theorem 1. We have the inclusion $k^t\colon Gr^W_{n+i}H^n_c(X'_{f,x})_1 \to Gr^W_{n+i}P^n(Z_F)_1$; observe that $\langle k(z), \eta\rangle = S(z, k^t(\eta))$ for $\eta \in H^n_c(X'_{f,x})$ and $z \in H^n(Z_F)_1$.

Let $i \geqslant 0$. For $0 \neq \xi \in B_{n+i}$ we have $N_c^{i+1}\xi = 0$ hence $k^t(\xi) \in P_{n+i}$. This implies that $\langle Cj(\xi), N^i(\overline{\xi})\rangle = S(Ck^t\xi, N^i(\overline{k^t\xi})) > 0$.

Let $i \geqslant 1$; then the map $k\colon Gr^W_{n+i}P^n(Z_F)_1 \to Gr^W_{n+i}H^n(X'_{f,x})_1$ is an isomorphism, as k is surjective and $\ker k = \operatorname{im}(H^n(Z_0) \to H^n(Z_F))$ is of weight $\leqslant n$. Let $\eta \in A_{n+i}$ and $z \in P_{n+i}$ such that $\eta = k(z)$, then $N^i\eta = 0$ implies that $N^i z \in \ker(k) \subset \ker(N)$ so $N^{i+1}z = 0$. Hence again $z \in P_{n+i}$. So if $z \neq 0$ we have $\langle C\eta, VN^{i-1}\overline{\eta}\rangle = \langle Ck(z), VN^{i-1}\overline{k(z)}\rangle = S(Cz, N^i\overline{z}) > 0$.

As an application we consider the intersection form h on $H^n_c(X'_{f,x}, \mathbf{R})$ given by $h(\omega, \eta) = \int_{X'_{f,x}} \omega \wedge \eta = (-1)^{n(n-1)/2}\langle j(\xi), \eta\rangle$. Clearly its null space is equal to $\ker(j)$. In the case that n is even, h is a symmetric bilinear form, and we will compute its index in terms of the Hodge numbers

$$h^{pq} = \dim Gr^p_F Gr^W_{p+q} H^n(X'_{f,x}, \mathbf{C}).$$

Note that if $h^{pq}_c = \dim Gr^p_F Gr^W_{p+q} H^n_c(X'_{f,x}, \mathbf{C})$ then $h^{pq}_c = h^{n-p,n-q}$.

THEOREM 11. *Let n be even. Then the index* $\sigma(h)$ *of h is given by*

$$\sigma(h) = \sum_{p+q=n} (-1)^p \left(h^{pq} + 2\sum_{i\geqslant 1}(-1)^i h^{p+i,q+i} \right).$$

Proof. First note that $W_{n-1}H^n_c(X'_{f,x})$ is an isotropic subspace of h which contains its null space. Moreover the orthogonal complement of $W_{n-1}H^n_c(X'_{f,x})$ with respect to h is equal to $W_n H^n_c(X'_{f,x})$. Therefore h induces a symmetric bilinear form h' on $Gr^W_n H^n_c(X'_{f,x})$ such that $\sigma(h') = \sigma(h)$. We extend h' to a hermitian form on $Gr^W_n H^n_c(X'_{f,x}, \mathbf{C})$. Let

$$\tilde{B}_{n+i} = \ker(N_c^{i+1}\colon Gr^W_{n+i}H^n_c(X'_{f,x}) \to Gr^W_{n-i-2}H^n_c(X'_{f,x})).$$

Then we have the decomposition

$$Gr^W_n H^n_c(X'_{f,x}, \mathbf{C}) = \bigoplus_{i\geqslant 0} \bigoplus_{p+q=n} N^i \tilde{B}^{p+i,q+i}_{n+2i} \oplus \bigoplus_{i\geqslant 1} \bigoplus_{p+q=n} VN^{i-1} A^{p+i,q+i}_{n+2i}$$

which is orthogonal with respect to h'. It follows from Theorem 10 that h' is definite on each of these summands, and its sign on $N^i \tilde{B}^{p+i,q+i}_{n+2i}$ and $VN^{i-1} A^{p+i,q+i}_{n+2i}$ is equal to $(-1)^{p+i}$ (note that $C = (-1)^{p+n/2}$ on these summands). Finally observe that

$$\dim \tilde{B}_{n+2i}^{p+i,q+i} = h_c^{p+i,q+i} - h_c^{p-i-1,q-i-1} = h^{p-i,q-i} - h^{p+i+1,q+i+1}$$

and

$$\dim A_{n+2i}^{p+i,q+i} = h^{p+i,q+i} - h^{p-i,q-i}.$$

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