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# Multicanonical systems of elliptic surfaces in small characteristics*

*Dedicated to Frans Oort on the occasion of his 60th birthday*

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**Abstract.** We determine the smallest integer $M$ such that the multicanonical system $|mK_S|$ gives the structure of elliptic surface for any elliptic surface $S$ with Kodaira dimension $\kappa(S) = 1$ and for any integer $m \geqslant M$. We also investigate the structure of elliptic surfaces of type $(p, p, \ldots, p)$.

## 0. Introduction

Let $k$ be an algebraically closed field of characteristic $p$, and $f\colon S \to C$ an elliptic surface over $k$ with a nonsingular complete curve $C$. As is well known, for an elliptic surface $S$ with Kodaira dimension $\kappa(S) = 1$, there exists a positive integer $m$ such that the multicanonical system $|mK_S|$ gives a unique structure of elliptic surface. In this paper, we consider the following problem.

PROBLEM. *Find the smallest integer $M$ such that the multicanonical system $|mK_S|$ gives the structure of elliptic surface for any elliptic surface $S$ with Kodaira dimension $\kappa(S) = 1$ and for any integer $m \geqslant M$.*

For any *analytic* elliptic surface $f\colon S \to C$ with $\kappa(S) = 1$ over the field $\mathbb{C}$ of complex numbers, Iitaka showed that for any $m \geqslant 86$, the multicanonical system $|mK_S|$ gives a unique structure of elliptic surface $f\colon S \to C$. He also showed that the number 86 is best possible, that is, there exists an example of an elliptic surface S with $\kappa(S) = 1$ such that $|85K_S|$ doesn't give the structure of elliptic surface. In the paper [4], we showed that for any *algebraic* elliptic

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surface $f\colon S \to C$ with $\kappa(S) = 1$ over $k$ and for any $m \geqslant 14$, the multicanonical system $|mK_S|$ gives a unique structure of elliptic surface, and that the number 14 is best possible if $p = 0$ or $p \geqslant 5$. In this paper, we show, first, that in case $p = 3$, the number 14 is best possible. Secondly, we show that in case $p = 2$, for any $m \geqslant 12$ $|mK_S|$ gives a unique structure of elliptic surface and that the number 12 is best possible. To prove these results, we use the theory of purely inseparable coverings obtained by p-closed rational vector fields (cf. [2] and [5]), and investigate the structure of elliptic surfaces of type $(p, p, \dots, p)$ (for the definition, see Section 2). In the final section we give some examples of wild fibres of elliptic surfaces constructed by our method.

## 1. Preliminaries

In this section, we recall some basic facts on rational vector fields and elliptic surfaces. For a non-singular complete algebraic variety $X$ of dimension $n$ defined over an algebraically closed field $k$ of characteristic $p > 0$, we use the following notation:

| | |
|---|---|
| $\mathcal{O}_X$: | the structure sheaf of $X$, |
| $K_X$: | a canonical divisor on $X$, |
| $\Phi_{\lvert mK_X\rvert}$: | the rational mapping associated with the multicanonical system $\lvert mK_X\rvert$, |
| $\kappa(X)$: | the Kodaira dimension of $X$, |
| $c_n(X)$: | the $n$th Chern number of $X$, |
| $\chi(X, \mathcal{O}_X) = \sum_{i=0}^{n}(-1)^i \dim_k H^i(X, \mathcal{O}_X)$, | |
| $\mathrm{Pic}^0(X)$: | the Picard scheme of $X$, |

For divisors $E_1$ and $E_2$ on $X$, $E_1 \sim E_2$ (resp. $E_1 \equiv E_2$) means that $E_1$ is linearly (resp. numerically) equivalent to $E_2$. Sometimes, a Cartier divisor and the associated invertible sheaf will be identified. For a group $G$ and an element $\sigma$ of $G$, we denote by $\langle\sigma\rangle$ the subgroup generated by $\sigma$.

Let $D$ be a non-zero rational vector field on $X$. $D$ is called a $p$-closed vector field if there exists a rational function $f$ on $X$ such that $D^p = fD$. In particular, $D$ is called multiplicative if $D^p = D$ , and $D$ is called additive if $D^p = 0$. Let $\{\mathrm{Spec}A_i\}_{i\in I}$ be an affine open covering of $X$. We set $A_i{}^D = \{\alpha \in A_i \mid D(\alpha) = 0\}$. Then, $X^D$ is defined by $\bigcup_{i\in I} \mathrm{Spec}A_i{}^D$. It is well known that $X^D$ is normal. If $D$ is $p$-closed, then the canonical projection $\pi_D\colon X \to X^D$ is a purely inseparable morphism of degree $p$. Conversely, if $\pi\colon X \to Y$ is a finite purely inseparable morphism of degree $p$ with a normal variety $Y$, then there exists a rational $p$-closed vector field $D$ such that $\pi = \pi_D$ and $Y = X^D$. Moreover, $X^D$ is non-singular if and only if $D$ is divisorial (cf. [5, Section 1]). For a $p$-closed vector field $D$, we denote by $(D)$ the divisor associated with $D$. The following lemma is a special case in Rudakov and Shafarevich [5, Section 1, Proposition 1].

LEMMA 1.1. *Let $X$ be a non-singular complete algebraic surface over $k$, and $D$ a divisorial $p$-closed rational vector field on $X$. Let $\pi$: $X \to X^D$ be the natural morphism. For an irreducible curve $C$ on $X$, $C'$ denotes the image of $C$ in $X^D$. If $C$ is an integral curve for $D$, then $\pi_* C = pC'$ and $\pi^* C' = C$. If $C$ is not an integral curve for $D$, then $\pi_* C = C'$ and $\pi^* C' = pC$.*

For details of these facts, see [5].

Now, let $f$: $S \to C$ be an elliptic surface defined over $k$ with a non-singular complete curve $C$. Throughout this paper, we assume that no fibers of $f$ contain exceptional curves of the first kind. Let $\mathcal{T}$ be the torsion part of $R^1 f_* \mathcal{O}_S$. Since $C$ is a non-singular curve, we have

$$R^1 f_* \mathcal{O}_S \simeq \mathcal{L} \oplus \mathcal{T}, \tag{1.1}$$

where $\mathcal{L}$ is an invertible sheaf. By $m_i E_i$ ($i = 1, 2, \dots, n$), we denote the multiple singular fibers of $f$: $S \to C$ with multiplicities $m_i$. We have the canonical divisor formula,

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^{n} a_i E_i, \tag{1.2}$$

where $a_i$'s are integers such that $0 \leqslant a_i \leqslant m_i - 1$, and where

$$-\deg \mathcal{L} = \chi(S, \mathcal{O}_S) + t \quad \text{with } t = \text{length } \mathcal{T}. \tag{1.3}$$

We have

$$K_S^2 = 0. \tag{1.4}$$

By Noether's formula and (1.4), we have

$$\chi(S, \mathcal{O}_S) = c_2(X)/12 \geqslant 0.$$

We set $P_i = f(E_i)$. Then, the following conditions are equivalent.

(i) $\mathcal{T}_{P_i} = 0$,
(ii) $\dim_k H^0(m_i E_i, \mathcal{O}_{m_i E_i}) = 1$,
(iii) $\operatorname{ord} \mathcal{O}_S(E_i)|_{E_i} = m_i$.

In this case, the multiple fiber $m_i E_i$ is called a tame fiber, and we have $a_i = m_i - 1$. If a multiple fiber is not tame, it is called a wild fiber. Namely, a multiple fiber $m_i E_i$ is wild if and only if one of the following equivalent conditions is satisfied.

(i) $\mathcal{T}_{P_i} \neq 0$,
(ii) $\dim_k H^0(m_i E_i, \mathcal{O}_{m_i E_i}) \geqslant 2$,

(iii) $1 \leqslant \operatorname{ord} \mathcal{O}_S(E_i)|_{E_i} \leqslant m_i - 1$.

For these facts, see Bombieri and Mumford [1]

Finally, we recall results on the multicanonical systems of elliptic surfaces in [3] and [4]. Using the notation above, we have

$$H^0(S, \mathcal{O}_S(mK_S)) \simeq H^0\left(C, \mathcal{O}_C\left(mK_C - m\mathcal{L} + \sum_{i=1}^{n}[ma_i/m_i]P_i\right)\right),$$

where $[x]$ is the integral part of a rational number $x$. We set

$$W_m = mK_C - m\mathcal{L} + \sum_{i=1}^{n}[ma_i/m_i]P_i.$$

We denote by $g$ the genus of $C$. Then, $\kappa(S)$ is equal to 1 if and only if

$$2g - 2 + \chi(S, \mathcal{O}_S) + t + \sum_{i=1}^{n} a_i/m_i > 0. \tag{1.5}$$

If $\deg W_m \geqslant 2g + 1$, then $W_m$ is very ample, therefore, $\Phi_{|mK_S|}$ gives a unique structure of the elliptic surface. Hence, we must look for the lowest bound $M$ for $m$ such that under the condition (1.5), $\deg W_m \geqslant 2g + 1$ holds for any $m \geqslant M$ (cf. [3] and [4]).

## 2. Elliptic Surfaces of Type (p,p,...,p)

In this section, let $k$ be an algebraically closed field of characteristic $p > 0$. Let $f\colon S \to \mathbb{P}^1$ be an elliptic surface with $\chi(S, \mathcal{O}_S) = 0$ over the projective line $\mathbb{P}^1$. Then, as is easily seen (cf. [4]), $f\colon S \to \mathbb{P}^1$ has no singular fibers except multiple fibers $m_iE_i$ $(i = 1, 2, \ldots, n)$ with elliptic curves $E_i$. Put $\nu_i = \operatorname{ord}\mathcal{O}_S(E_i)|_{E_i}$.

DEFINITION 2.1 (cf. [4, Definition 3.1]). The elliptic surface $f\colon S \to \mathbb{P}^1$ as above is called of type $(m_1, m_2, \ldots, m_n \mid \nu_1, \nu_2, \ldots, \nu_n)$. The elliptic surface $f\colon S \to \mathbb{P}^1$ as above is called of type $(m_1, m_2, \ldots, m_n)$ if $\nu_i = m_i$ for all $i$.

In particular, the multiple fibers of an elliptic surface of type $(m_1, m_2, \ldots, m_n)$ are tame. Let $\widetilde{C}$ be a non-singular complete algebraic curve over $k$. The following lemma is an easy exercise of an elementary calculus.

LEMMA 2.2. *Let $\Delta$ be a rational vector field on $\widetilde{C}$.*

(i) *If $\Delta$ is multiplicative, then the zero points of $\Delta$ are simple.*

(ii) *If $\Delta$ has at least one simple zero point, then $\Delta$ is not additive.*

Let $E$ be an ordinary elliptic curve. Then, there exists a non-zero regular vector field $\delta$ on $E$ such that $\delta^p = \delta$. Let $\Delta$ be a non-zero multiplicative rational vector field on $\widetilde{C}$, and $n$ the number of zero points of $\Delta$. Since $\deg \Delta = 2 - 2g$, the sum of orders of poles of $\Delta$ is equal to $n - 2 + 2g$. We set $\widetilde{S} = E \times \widetilde{C}$. We extend $\delta$ and $\Delta$ naturally to vector fields on $\widetilde{S}$, which we denote by the same letters. We denote by $D$ the vector field on $\widetilde{S}$ defined by $D := \delta + \Delta$. Set

$$S = \widetilde{S}^D \quad \text{and} \quad C = \widetilde{C}^\Delta.$$

We have an elliptic surface

$$f\colon\ S \longrightarrow C, \tag{2.1}$$

where $f$ is the morphism induced from the projection $pr$: $\widetilde{S} = E \times \widetilde{C} \to \widetilde{C}$ (cf. [2]).

THEOREM 2.3. *Under the notation as above, the singular fibers of the elliptic surface in* (2.1) *consist of* $n$ *tame multiple fibers* $pE_i$ $(i = 1, 2, \dots, n)$ *with multiplicity* $p$, *where* $E_i$*'s are elliptic curves. The canonical divisor of* $S$ *is given by*

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^{n}(p-1)E_i, \tag{2.2}$$

*where the divisor* $\mathcal{L}$ *is given as in* (1.2) *with deg* $\mathcal{L} = 0$.

*Proof.* Let $\widetilde{P}_i$ $(i = 1, 2, \dots, n)$ be zero points of $\Delta$. We have a diagram

$$\begin{tikzcd} S & E \times \widetilde{C} = \widetilde{S} \arrow[l, "\pi"'] \arrow[d, "pr"] \\ C \arrow[from=1-1, d, "f"'] & \widetilde{C} \arrow[l, "F"'] \end{tikzcd}$$

where $F$ (resp. $\pi$) is the natural morphism given by $\Delta$ (resp. $D$). $F$ is nothing but the Frobenius morphism. By Lemma 1.1, multiple fibers of $f$ exist only over $P_i = F(\widetilde{P}_i)$ $(i = 1, 2, \dots, n)$. We set $pr^*(\widetilde{P}_i) = \widetilde{E}_i$ $(\simeq E)$ and $f^*(P_i) = m_i E_i$ $(i = 1, 2, \dots, n)$. Since $\pi^* \circ f^*(P_i) = pr^* \circ F^*(P_i)$ and $\pi^*(E_i) = \widetilde{E}_i$ by Lemma 1.1, we have

$$m_i \widetilde{E}_i = p\widetilde{E}_i.$$

Therefore, we have $m_i = p$. The canonical divisor formula of $S$ is given by

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^{n} a_i E_i$$

with integers $a_i$ $(0 \leqslant a_i \leqslant p-1)$. Since $\pi$ is radicial, we have $c_2(S) = c_2(\widetilde{S}) = 0$. Since $\chi(S, \mathcal{O}_S) = c_2(S)/12 = 0$, we have

$$-\deg \mathcal{L} = \chi(S, \mathcal{O}_S) + t = t,$$

where $t$ is the length of the torsion part of $R^1f_*\mathcal{O}_S$. Since $K_{\tilde{S}} \sim \pi^*K_S + (p-1)(D)$, we have

$$K_{\tilde{S}} \equiv pr^*(p\{(2g-2)+t\}P) + \sum_{i=1}^{n} a_i\widetilde{E}_i - (p-1)(n-2+2g)pr^*(P)$$

with a point $P$ of $\widetilde{C}$. Taking the degrees of both sides, we have

$$2g-2 = p(2g-2+t) + \sum_{i=1}^{n} a_i - (p-1)(n-2+2g).$$

Therefore, we have

$$pt + \sum_{i=1}^{n} a_i = n(p-1). \tag{2.3}$$

Suppose that there exist some wild fibers of $f$. We may assume that $pE_i$ $(i = 1,2,\ldots,\ell)$ are wild with $\ell \geqslant 1$, and $pE_i$ $(i = \ell+1,\ldots,n)$ are tame. Then, we have $t \geqslant \ell$, and $a_i = p-1$ $(i = \ell+1,\ldots,n)$. Since $a_i \geqslant 0$ for all $i$, the left-hand side of (2.3) is greater than the right-hand side of (2.3). A contradiction. Therefore, all multiple fibers $pE_i$ $(i = 1,\ldots,n)$ are tame, that is, we have $t = 0$ and $a_i = p-1$ $(i = 1,\ldots,n)$. Hence, the canonical divisor formula of S is given by

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^{n}(p-1)E_i \quad \text{with } \deg \mathcal{L} = 0.$$

□

As a special case of Theorem 2.3, we have the following result.

COROLLARY 2.4. *Under the same notation as in Theorem* 2.3 *except for* $\widetilde{C} = \mathbb{P}^1$, *the elliptic surface* $f\colon S \to \mathbb{P}^1$ *constructed as in* (2.1) *is of type* $(p,p,\ldots,p)$ *with* $n$ *multiple fibers. The canonical divisor* $K_S$ *of* $S$ *is given by*

$$K_S \sim f^*(-2P) + \sum_{i=1}^{n}(p-1)E_i.$$

The following lemma is well known. We give here an easy direct proof for the reader's convenience.

LEMMA 2.5. *Let* $f\colon S \to \mathbb{P}^1$ *be an elliptic surface without degenerate fibers. Then, S is isomorphic to* $E \times \mathbb{P}^1$ *with an elliptic curve* $E$, *and* $f$ *is given by the projection* $pr\colon E \times \mathbb{P}^1 \to \mathbb{P}^1$.

*Proof.* Since $K_S \sim f^*(-2P)$ for a point $P$ of $\mathbb{P}^1$, we have the geometric genus $p_g(S) = 0$ and $\kappa(S) = -\infty$. Therefore, the Picard scheme of $S$ is reduced and we have $\dim_k H^1(S, \mathcal{O}_S) = 1$ by $\chi(S, \mathcal{O}_S) = 0$. Therefore, the dimension of the Albanese variety $\mathrm{Alb}(S)$ of $S$ is equal to one . Since $\kappa(S) = -\infty$, $S$ is a ruled surface. Using the argument by the Stein factorization, we see that $g$: $S \to \mathrm{Alb}(S)$ gives the structure of ruled surface. We set $E = \mathrm{Alb}(S)$. Let $G$ be a general fiber of $g$. Then, $G^2 = 0$. Since $G$ is isomorphic to $\mathbb{P}^1$, the vertual genus of $G$ is equal to 0. Therefore, we have $(G \cdot K_S) = -2$. Therefore we have

$$(G \cdot f^{-1}(P)) = 1. \tag{2.4}$$

By the universality of fiber product, we have the morphism

$$(f, g)\colon\ S \longrightarrow \mathbb{P}^1 \times E.$$

By (2.4), we see that $(f, g)$ is an isomorphism. The rest is clear. □

THEOREM 2.6. *Let $f$: $S \to \mathbb{P}^1$ be an elliptic surface of type $(p, p, \dots, p)$. Then, there exist an ordinary elliptic curve $E$ and a rational multiplicative vector field $D = \delta + \Delta$ on $E \times \mathbb{P}^1$ such that*

$$S \simeq (E \times \mathbb{P}^1)^D, \quad \mathbb{P}^1 \simeq (\mathbb{P}^1)^\Delta$$

*and such that $f$ is the natural morphism from $(E \times \mathbb{P}^1)^D$ to $(\mathbb{P}^1)^\Delta$. Here, $\delta$ (resp. $\Delta$) is a non-zero regular (resp. rational) multiplicative vector field on $E$ (resp. $\mathbb{P}^1$).*

*Proof.* Let $pE_i$ $(i = 1, 2, \dots, n)$ be the multiple fibers of $f$: $S \to \mathbb{P}^1$. Since the multiple fiber $pE_i$ is tame by assumption, $\mathcal{O}_S(E_i)|_{E_i}$ is of order $p$. Since $\mathcal{O}_S(E_i)|_{E_i}$ is an element of $\mathrm{Pic}^0(E_i)$, we see that $E_i$ is an ordinary elliptic curve.

First, we show that, taking the base change by the Frobenius morphism $F$: $\mathbb{P}^1 \to \mathbb{P}^1$, and taking the normalization of the fiber product, we have the following diagram:

$$\begin{tikzcd} S & \tilde{S} \arrow[l, "\pi"'] \arrow[d, "\tilde{f}"] \\ \mathbb{P}^1 \arrow[from=u, "f"'] & \mathbb{P}^1 \arrow[l, "F"'] \end{tikzcd}, \tag{2.5}$$

where $\pi$ is a purely inseparable finite morphism of degree $p$, and where $\tilde{f}$: $\tilde{S} \to \mathbb{P}^1$ is an elliptic surface without multiple fibers. Take a wild fiber $pE_i$, and set $P_i = f(pE_i)$. Let $x$ be a local parameter at $P_i$, and let $V$ be a small affine open neighborhood of $P_i$ such that $x - x(P)$ is a local parameter at any point $P$ of $V$. We consider the invertible sheaf $\mathcal{O}_S(E_i)|_{f^{-1}(V)}$ on $f^{-1}(V)$. Since $pE_i$ is linearly equivalent to zero on $f^{-1}(V)$, we see that $\mathcal{O}_S(E_i)|_{f^{-1}(V)}$ is of order $p$. Take an affine open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of $f^{-1}(V)$. Let $x_\lambda$ be a local equation of $E_i$ on $U_\lambda(\lambda \in \Lambda)$. Then, there exists a unit $u_\lambda$ on $U_\lambda$ such that

$$f^*(x) = u_\lambda x_\lambda^p \quad \text{on each } U_\lambda \tag{2.6}$$

The transition functions $f_{\lambda\mu} = x_\lambda/x_\mu$ on $U_\lambda \cap U_\mu$ $(\lambda, \mu \in \Lambda)$ give the invertible sheaf $\mathcal{O}_S(E_i)|_{f^{-1}(V)}$. Using

$$f_{\lambda\mu}^p = x_\lambda^p/x_\mu^p = (x/u_\lambda)/(x/u_\mu) = u_\mu/u_\lambda,$$

we can construct the non-trivial purely inseparable covering of $f^{-1}(V)$ of degree $p$ defined by

$$u_\lambda = z_\lambda^p \quad \text{on } U_\lambda, \qquad z_\mu = f_{\lambda\mu}z_\lambda \quad \text{on } U_\lambda \cap U_\mu. \tag{2.7}$$

We denote this covering by $\rho$: $\widetilde{f^{-1}(V)} \to f^{-1}(V)$. Now, we consider a regular 1-form $\omega$ on $f^{-1}(V)$ defined by

$$\omega = \mathrm{d}u_\lambda/u_\lambda \quad \text{on } U_\lambda\ (\lambda \in \Lambda).$$

Since the cocycle $\{f_{\lambda\mu}\}$ is non-trivial, we see that $\omega$ is not zero. By (2.7), we have $\rho^*\omega = 0$. By (2.6), we have

$$\rho^* f^*(\mathrm{d}x/x) = \rho^*(\mathrm{d}u_\lambda/u_\lambda) = \rho^*\omega = 0.$$

Therefore, there exists an element $y$ of the function field $k(\widetilde{f^{-1}(V)})$ such that $\rho^*f^*(x) = y^p$. This means that $k(\mathbb{P}^1) = k(x)$ is not algebraically closed in $k(\widetilde{f^{-1}(V)})$, and by the base change by the Frobenius morphism, the fiber product $W$ is birationally equivalent to $\widetilde{f^{-1}(V)}$:

$$\begin{tikzcd} f^{-1}(V) \arrow[d, "f"'] & W \arrow[l, "\theta"'] \arrow[d, "g"] & \widetilde{f^{-1}(V)} \arrow[l, "\eta"'] \\ V & V \arrow[l, "F"'] & \end{tikzcd}, \tag{2.8}$$

where $\rho = \theta \circ \eta$. Set $\tilde{f} = g \circ \eta$ and take the point $\widetilde{Q}$ such that $F(\widetilde{Q}) = Q$. Since $\mathcal{O}_S(E_i)|_{E_i}$ is of order $p$, the restriction of this covering $\rho$: $\widetilde{f^{-1}(V)} \to f^{-1}(V)$ on $E_i$ is non-trivial. Therefore, $\widetilde{f^{-1}(V)}$ is non-singular on $\tilde{f}^{-1}(Q)$ (cf. [4, p. 312]), and $\tilde{f}^{-1}(Q)$ is a regular fiber. Since the fiber of $f$ is an elliptic curve or a multiple elliptic curve, we get the diagram (2.5) by (2.8).

By Lemma 2.5, there exists an elliptic curve $E$ such that $\widetilde{S} \simeq E \times \mathbb{P}^1$ and such that $\tilde{f}$ is the second projection. Since $\pi^{-1}(E_i) \simeq E$, $E$ must be an ordinary elliptic curve. We fix a non-zero regular multiplicative vector field $\delta$ on $E$. Since

$\pi$ is a purely inseparable finite morphism of degree $p$, there exists a $p$-closed divisorial vector field $D$ on $E \times \mathbb{P}^1$ such that $S = (E \times \mathbb{P}^1)^D$ (cf. Rudakov and Shafarevich [5, Section 1]). Since $D$ is divisorial, by the argument in Ganong and Russell [2, 3.7.1], we may assume that $D$ is a non-zero $p$-closed vector field of the form

$$D = \delta + h(x)\frac{\partial}{\partial x},$$

where $x$ is a local coodinate of an affine line in $\mathbb{P}^1$. We set $\Delta = h(x)\frac{\partial}{\partial x}$. Since $D$ is $p$-closed and $\delta$ is multiplicative, we conclude that $D$ is multiplicative. Therefore, $\Delta$ is multiplicative. Hence, by Lemma 2.2, the zero points of $\Delta$ are simple. The rest is clear by Lemma 1.1. □

COROLLARY 2.7. *Let $k$ be an algebraically closed field of characteristic 2, and let $f\colon S \to \mathbb{P}^1$ be an elliptic surface of type $(2, 2, \ldots, 2)$ over $k$. Then, the number of multiple fibers is even.*

*Proof.* Under the notation similar to that in Theorem 2.6, the elliptic surface $f\colon S \to \mathbb{P}^1$ is expressed as

$$f\colon S \simeq (E \times \mathbb{P}^1)^D \longrightarrow (\mathbb{P}^1)^\Delta \simeq \mathbb{P}^1,$$

where $\Delta$ is a rational vector field on $\mathbb{P}^1$ whose zero points are all simple. The number of zero points of $\Delta$ is equal to the number of multiple fibers of $f\colon S \longrightarrow \mathbb{P}^1$ by Theorem 2.3. Using a local coordinate $x$ of an affine line in $\mathbb{P}^1$, we write $\Delta$ as

$$\Delta = h(x)\frac{\partial}{\partial x},$$

where $h(x)$ is a non-zero rational function on $\mathbb{P}^1$. Since $\Delta^2 = \Delta$, we have a differential equation in characteristic 2:

$$\frac{\partial}{\partial x}h(x) = 1.$$

Solving the differential equation in characteristic 2 on the affine line in $\mathbb{P}^1$, we have $h(x) = x + g(x)^2$ with an arbitrary rational function $g(x)$. Hence, the order of each pole of $h(x)$ is even. Let $y = 1/x$ be a local coordinate of the point at infinity. Then, we have

$$\frac{\partial}{\partial x} = -y^2\frac{\partial}{\partial y} = y^2\frac{\partial}{\partial y}.$$

Therefore the order of pole of $\Delta$ at the point at infinity is also even. Hence, the sum of the orders of poles of $\Delta$ is even. Since the degree of the divisor given by $\Delta$ is equal to 2, we conclude that the number of zero points of $\Delta$ is even. □

## 3. Characteristic 3

In this section, let $k$ be an algebraically closed field of charcteristic 3. Let $E$ be an ordinary elliptic curve, and $\delta$ a non-zero regular multiplicative vector field on $E$. We take two different points $a, b$ of order two of $E$, and consider two translations $T_a$ and $T_b$ defined by $a$ and $b$, respectively. Let $x$ be a local coordinate of an affine line in $\mathbb{P}^1$. We set

$$\Delta = (x^3 - 1/x)\frac{\partial}{\partial x}.$$

Then, we have $\Delta^3 = \Delta$ and $\Delta$ has four simple zero points on $\mathbb{P}^1$:

$$\Omega = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}.$$

We consider two automorphisms of $\mathbb{P}^1$ defined by

$$\sigma' \colon x \mapsto -x, \qquad \tau' \colon x \mapsto 1/x.$$

These automorphisms are of order 2 and induce permutations of the elements of $\Omega$. The vector field $\Delta$ is invariant under $\sigma'$ and $\tau'$. We set

$$D = \delta + \Delta$$

and

$$\sigma = T_a \times \sigma', \quad \tau = T_b \times \tau'.$$

Then, $D$ is a multiplicative vector field on $E \times \mathbb{P}^1$, and $\sigma, \tau$ are automorphisms of order 2 of $E \times \mathbb{P}^1$ which have no fixed points on $E \times \mathbb{P}^1$. $\sigma$ and $\tau$ commute with each other, and $D$ is invariant under $\sigma, \tau$. We set

$$S_1 = (E \times \mathbb{P}^1)^D, \qquad C_1 = (\mathbb{P}^1)^\Delta \quad (\simeq \mathbb{P}^1).$$

Then we have a diagram

$$\begin{tikzcd} S_1 \arrow[d, "f_1"'] & E \times \mathbb{P}^1 \arrow[l, "\pi_1"'] \arrow[d, "pr"] \\ C_1 & \mathbb{P}^1 \arrow[l, "F_1"] \end{tikzcd}, \tag{3.1}$$

where $\pi_1$ and $F_1$ are natural morphisms, and where $f_1$ is the morphism induced from the projection $pr$. We set

$$P_1 = F_1(1), \qquad P_2 = F_1(-1), \qquad P_3 = F_1(\sqrt{-1}),$$

$$P_4 = F_1(-\sqrt{-1}), \qquad P_5 = F_1(0), \qquad P_6 = F_1(\infty).$$

Then, $f_1$ has four tame multiple fibers

$$f_1^{-1}(P_i) = 3E_i \qquad (i = 1, 2, 3, 4)$$

with elliptic curves $E_i$. Automorphisms $\sigma$ and $\tau$ induce automorphisms $\sigma_1$ and $\tau_1$, respectively, of order 2 of $S_1$ which have no fixed points on $S_1$. Automorphisms $\sigma'$ and $\tau'$ induce automorphisms $\sigma_1'$ and $\tau_1'$, respectively, of order 2 of $C_1$. We have

$$\tau_1(E_1) = E_1, \qquad \tau_1(E_2) = E_2, \qquad \tau_1(E_3) = E_4, \qquad \tau_1(E_4) = E_3,$$

$$\tau_1'(P_1) = P_1, \qquad \tau_1'(P_2) = P_2, \qquad \tau_1'(P_3) = P_4, \qquad \tau_1'(P_4) = P_3,$$

and

$$\sigma_1(E_1) = E_2, \qquad \sigma_1(E_2) = E_1, \qquad \sigma_1(E_3) = E_4, \qquad \sigma_1(E_4) = E_3,$$

$$\sigma_1'(P_1) = P_2, \qquad \sigma_1'(P_2) = P_1, \qquad \sigma_1'(P_3) = P_4, \qquad \sigma_1'(P_4) = P_3.$$

The automorphisms $\sigma_1'$ has two fixed points $P_5$ and $P_6$ on $C_1$. Automorphisms $\sigma_1$ and $\tau_1$ commute with each other. We set

$$S_2 = S_1/\langle\sigma_1\rangle, \qquad C_2 = C_1/\langle\sigma_1'\rangle \quad (\simeq \mathbb{P}^1).$$

Then, we have a diagram

$$\begin{tikzcd} S_2 & S_1 \arrow[l, "\pi_2"'] \arrow[d, "f_1"] \\ C_2 \arrow[from=u, "f_2"'] & C_1 \arrow[l, "F_2"'] \end{tikzcd}, \tag{3.2}$$

where morphisms $\pi_2, F_2$ and $f_2$ are similar to those in (3.1). We have

$$F_2(P_1) = F_2(P_2) \quad \text{and} \quad F_2(P_3) = F_2(P_4).$$

We set

$$Q_1 = F_2(P_1), \qquad Q_3 = F_2(P_3), \qquad Q_5 = F_2(P_5), \qquad Q_6 = F_2(P_6).$$

Then, $f_2$ has four tame multiple fibers

$$f_2^{-1}(Q_1) = 3F_1, \qquad f_2^{-1}(Q_3) = 3F_3,$$

$$f_2^{-1}(Q_5) = 2F_5, \qquad f_2^{-1}(Q_6) = 2F_6.$$

$\tau_1$ (resp. $\tau_1'$) induces an automorphism $\tau_2$ (resp. $\tau_2'$) of order 2 of $S_2$ (resp. $C_2$). $\tau_2$ has no fixed points on $S_2$. We have

$$\tau_2(F_1) = F_1, \qquad \tau_2(F_3) = F_3, \qquad \tau_2(F_5) = F_6, \qquad \tau_2(F_6) = F_5,$$

$$\tau_2'(Q_1) = Q_1, \qquad \tau_2'(Q_3) = Q_3, \qquad \tau_2'(Q_5) = Q_6, \qquad \tau_2'(Q_6) = Q_5.$$

We set

$$S = S_2/\langle\tau_2\rangle, \qquad C = C_2/\langle\tau_2'\rangle \quad (\simeq \mathbb{P}^1).$$

Then, we have a diagram

$$\begin{tikzcd} S \arrow[d, "f"'] & S_2 \arrow[l, "\pi_3"'] \arrow[d, "f_2"] \\ C & C_2 \arrow[l, "F_3"] \end{tikzcd}, \tag{3.3}$$

where morphisms $\pi_3, F_3$ and $f$ are similar to those in (3.1). We have

$$F_3(Q_5) = F_3(Q_6).$$

We set

$$R_1 = F_3(Q_1), \qquad R_3 = F_3(Q_3), \qquad R_5 = F_3(Q_5).$$

Then, $f$ has three tame multiple fibers:

$$f^{-1}(R_1) = 6G_1, \qquad f^{-1}(R_3) = 6G_3, \qquad f^{-1}(R_5) = 2G_5,$$

where $G_1, G_3$ and $G_5$ are ordinary elliptic curves. Hence, we have in characteristic 3 an elliptic surface of type $(2,6,6)$. For this elliptic surface, $|mK_S|$ gives a unique structure of the elliptic surface for any $m \geqslant 14$, and $|13K_S|$ does not give the structure of elliptic surface. Hence, using results in [4, Section 5], we have the following theorem.

THEOREM 3.1. *Let $k$ be an algebraically closed field of characteristic* 3. *If $S$ is an elliptic surface defined over $k$ with $\kappa(S) = 1$, then $\Phi_{|mK_S|}$ gives a unique structure of elliptic surface for any $m \geqslant 14$. Moreover, the number* 14 *is best possible.*

## 4. Characteristic 2

In this section, let $k$ be an algebraically closed field of characteristic 2. First, we prove that there doesn't exist an elliptic surface of type (2, 6, 6) in characteristic 2.

Suppose there exists an elliptic surface $f\colon S \to \mathbb{P}^1$ of type (2, 6, 6). We denote by $2E_1, 6E_2, 6E_6$ the multiple fibers, and we set $P_i = f(E_i)$ $(i = 1,2,3)$. We consider the invertible sheaf $L = \mathcal{O}_S(2E_2 - 2E_3)$. Since $L^{\otimes 3} \simeq \mathcal{O}_S$, we have an étale covering $\widetilde{S}$ of $S$ of degree 3 by a standard method. We denote by E the fiber on a general point $P$ of $\mathbb{P}^1$. Since the restriction of $L$ on $E$ (resp. $E_1$) is trivial, we see that the étale covering restricted on $E$ (resp. $E_1$) splits into three pieces of $E$ (resp. $E_1$). Hence, taking the normalization of the base curve $\mathbb{P}^1$ in the function field of $\widetilde{S}$, we have a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & \widetilde{S} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ \mathbb{P}^1 & \xleftarrow{\phi} & \mathbb{P}^1 \end{array},$$

where $\pi$ is the étale covering of degree 3 which we constructed above, and where $\phi$ is the morphism of degree 3 given by the normalization which ramifies only at $P_2$ and $P_3$. Since $\phi^{-1}(P_1)$ consists of three points, and $\phi^{-1}(P_2)$ (resp. $\phi^{-1}(P_3)$) consists of one point, the elliptic surface $\tilde{f}\colon \widetilde{S} \to \mathbb{P}^1$ has five multiple fibers with multiplicity 2. Since $2E_1$ (resp. $6E_2$, resp. $6E_3$) is tame, the order of the normal bundle $\mathcal{O}_S(E_1)|_{E_1}$ (resp. $\mathcal{O}_S(E_2)|_{E_2}$, resp. $\mathcal{O}_S(E_3)|_{E_3}$) is two (resp. six, resp. six). Using these facts, we see that all multiple fibers of $\tilde{f}$ are tame. Hence, the elliptic surface $\tilde{f}\colon \widetilde{S} \to \mathbb{P}^1$ is of type $(2,2,2,2,2)$. This contradicts Corollary 2.7. Hence, there doesn't exist an elliptic surface of type (2, 6, 6).

Now, we construct an elliptic surface of type (2,5,10) in characteristic 2. Let $E$ be an ordinary elliptic curve, and $\delta$ a non-zero regular multiplicative vector field on $E$. We take a point $a$ of order 5 of $E$, and consider the translation $T_a$ given by $a$. Let $x$ be a local coordinate of an affine line in $\mathbb{P}^1$. We consider a rational vector field given by

$$\Delta = (x - 1/x^4)\frac{\partial}{\partial x}$$

on $\mathbb{P}^1$. We have $\Delta^2 = \Delta$. Let $\zeta$ be a primitive fifth root of unity. We take the automorphism $\sigma'$ of order 5 defined by

$$\sigma'\colon\ x \longmapsto \zeta x.$$

Then, $\Delta$ is invariant under $\sigma'$. We set

$$D = \delta + \Delta, \qquad \sigma = T_a \times \sigma'.$$

Then, $D$ is a rational multiplicative vector field on $E \times \mathbb{P}^1$, and $\sigma$ is an automorphism of order 5 of $E \times \mathbb{P}^1$ which has no fixed points on $E \times \mathbb{P}^1$. $D$ is invariant under $\sigma$. As in Section 3, we have an elliptic surface $\tilde{f}\colon (E\times\mathbb{P}^1)^D \to (\mathbb{P}^1)^\Delta \simeq \mathbb{P}^1$ of type $(2,2,2,2,2,2)$. On the elliptic surface, we have the automorphism $\tilde{\sigma}$ of order 5 which is induced from $\sigma$. Then, taking the quotient by $\langle\tilde{\sigma}\rangle$, we have an elliptic surface

$$f\colon\ S \longrightarrow \mathbb{P}^1$$

of type $(2,5,10)$. Since $|mK_S|$ gives a unique structure of the elliptic surface for any $m \geqslant 12$, and $|11K_S|$ doesn't give the structure of elliptic surface. Hence, by careful calculations as in [4, Section 5] in characteristic 2, we have the following theorem.

THEOREM 4.1. *Let $k$ be an algebraically closed field of characteristic* 2. *If $S$ is an elliptic surface defined over $k$ with $\kappa(S) = 1$, then $\Phi_{|mK_S|}$ gives a unique structure of elliptic surface for any $m \geqslant 12$. Moreover, the number* 12 *is best possible.*

## 5. Examples of Multiple Fibers

In this section, let $k$ be an algebraically closed field of characteristic $p > 0$. We give some examples of elliptic surfaces which have multiple fibers obtained by vector fields. We denote by $x$ a local coordinate of an affine line in $\mathbb{P}^1$.

EXAMPLE 5.1. We fix a supersingular elliptic curve $E$, and a non-zero regular additive vector field $\delta$ on $E$. Let $\Delta$ be a rational vector field on $\mathbb{P}^1$ given by

$$\Delta = x^{pn}\frac{\partial}{\partial x}$$

with a positve integer $n$. We set $D = \delta + \Delta$. This gives an additive vector field on $E \times \mathbb{P}^1$. We set $S = (E \times \mathbb{P}^1)^D$ and $C = (\mathbb{P}^1)^\Delta$. $C$ is isomorphic to $\mathbb{P}^1$, and we have a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & E \times \mathbb{P}^1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle pr} \\ \mathbb{P}^1 & \xleftarrow{F} & \mathbb{P}^1 \end{array},$$

where $F$, $\pi$ are natural morphisms, and where $f$ is the morphism induced from the projection $pr$. The morphism $F$ is nothing but the Frobenius morphism. We set $P_0 = F(0)$, $f^{-1}(P_0) = pG_0$ and $pr^{-1}(\infty) = E_\infty$. Then we have

$$(D) = -(pn - 2)E_\infty.$$

The elliptic surface $f\colon S \to \mathbb{P}^1$ has only one multiple fiber $f^{-1}(P_0) = pG_0$. Since $G_0$ is a supersingular elliptic curve, the Picard variety of $G_0$ is also supersingular, therefore, it has no points of order $p$. Therefore, the normal bundle $\mathcal{O}_S(G_0)|_{G_0}$ is trivial. Hence, $f^{-1}(P_0)$ is a wild fiber. The canonical divisor formula of $S$ is given by a form

$$K_S \sim f^*((t-2)P) + aE_0$$

with non-negative integers $t$, $a$ $(0 \leqslant a \leqslant p-1)$ and a point $P$ of $\mathbb{P}^1$. Let $\widetilde{P}$ be the point of $\mathbb{P}^1$ such that $F(\widetilde{P}) = P$. Since we have

$$K_{E\times\mathbb{P}^1} \sim \pi^*K_S + (p-1)(D) \tag{5.1}$$

and

$$\pi^*K_S \sim pr^{-1}((t-2)p\widetilde{P}) + a(pr^{-1}(0)), \tag{5.2}$$

taking the degrees of both sides in (5.1) and (5.2), we have

$$-2 = p(t-2) + a - (p-1)(pn-2).$$

Therefore, we have

$$a = p(-t + (p-1)n). \tag{5.3}$$

Since $0 \leqslant a \leqslant p-1$, we conclude

$$a = 0 \quad \text{and} \quad t = (p-1)n.$$

Hence, the canonical divisor of $S$ is given by

$$K_S \sim f^{-1}(((p-1)n-2)P).$$

It is easy to see that if $p = 2$ and $n = 1$, then $S$ is a ruled surface over an elliptic curve ($\kappa(S) = -\infty$), if either $p = 2, n = 2$, or $p = 3, n = 1$, then $S$ is a hyperelliptic suface ($\kappa(S) = 0$), and otherwise, $S$ is an elliptic surface with $\kappa(S) = 1$ (see also [2]).

EXAMPLE 5.2. We give here an example of an elliptic surface with one tame fiber and one wild fiber. We fix an ordinary elliptic curve $E$, and a non-zero regular multiplicative vector field $\delta$ on $E$. Let $\Delta$ be a rational vector field on $\mathbb{P}^1$ given by

$$\Delta = (x^p - x)\frac{\partial}{\partial x}.$$

It is easy to see that $\Delta$ is multiplicative. We take a point $a$ of order $p$ of $E$, and consider the translation $T_a$ given by $a$. We consider the automorphism of order $p$ of $\mathbb{P}^1$ defined by

$$\sigma' \colon x \longmapsto x + 1.$$

We set $D = \delta + \Delta$ and $\sigma = T_a \times \sigma'$. Then $D$ is a $p$-closed rational vector field of $\mathbb{P}^1 \times E$, and $\sigma$ is an automorphism of order $p$ of $\mathbb{P}^1 \times E$ which has no fixed points on $\mathbb{P}^1 \times E$. $D$ is invariant under $\sigma$. We set $\widetilde{S} = (E \times \mathbb{P}^1)^D$ and $\widetilde{C} = (\mathbb{P}^1)^\Delta$ ($\simeq \mathbb{P}^1$). Then, the elliptic surface $\tilde{f}\colon \widetilde{S} \to \widetilde{C}$ has $p$ tame multiple fibers with multiplicity p, where $\tilde{f}$ is the natural morphism. We denote the multiple fibers by $p\widetilde{E}_i$ $(i = 1, 2, \ldots, p)$. The automorphism $\sigma$ (resp. $\widetilde{\sigma}'$) induces an automorphism $\widetilde{\sigma}$ (resp. $\widetilde{\sigma}'$) of order $p$ of $\widetilde{S}$; $\widetilde{\sigma}$ is fixed-point-free. The group $\langle\widetilde{\sigma}\rangle$ acts transitively on the set of multiple fibers. We set $S = \widetilde{S}/\langle\widetilde{\sigma}\rangle$ and $C = \widetilde{C}/\langle\widetilde{\sigma}'\rangle$. Then, we have an elliptic surface $f\colon S \to C \simeq \mathbb{P}^1$ and a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & \widetilde{S} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ C & \xleftarrow{\phi} & \widetilde{C} \end{array},$$

where $\phi$ and $\pi$ are natural morphisms, and where $\pi$ is an étale morphism of degree $p$. The elliptic surface $f\colon S \to C \simeq \mathbb{P}^1$ has two multiple fibers of multiplicity p: one comes from multiple fibers of $\tilde{f}$ which is tame, and the other comes from the fixed point of $\tilde{\sigma}'$. We denote by $pE_0$ the former tame multiple fiber, and by $pE_\infty$ the later multiple fiber. We take a point $P$ of $C$, and a point $\widetilde{P}$ of $\widetilde{C}$ such that $\phi(\widetilde{P}) = P$. Then, the canonical divisor formula of $S$ is given by

$$K_S \sim f^{-1}((t-2)P) + (p-1)E_0 + aE_\infty$$

with non-negative integers $t, a$ $(0 \leqslant a \leqslant p-1)$. We set $P_\infty = \tilde{f}(E_\infty)$, and denote by $\widetilde{P}_\infty$ the unique point of $\widetilde{C}$ such that $\phi(\widetilde{P}_\infty) = P_\infty$. We set $\widetilde{E}_\infty = \tilde{f}^{-1}(\widetilde{P}_\infty)$. This fiber is a regular fiber. Since $\pi$ is étale, we have $K_{\tilde{S}} \sim \pi^* K_S$. Therefore, by the canonical divisor formula, we have

$$\begin{aligned} &\tilde{f}^{-1}(-2\widetilde{P}) + (p-1)\widetilde{E}_1 + \ldots + (p-1)\widetilde{E}_p \\ &\quad \sim \tilde{f}^{-1}((t-2)pP) + (p-1)\widetilde{E}_1 + \ldots + (p-1)\widetilde{E}_p + a\widetilde{E}_\infty. \end{aligned}$$

Therefore, taking the degrees of both sides, we have

$$-2 = (t-2)p + a.$$

Since $0 \leqslant a \leqslant p-1$, we have

$$a = p-2 \quad \text{and} \quad t = 1.$$

Hence, we have the canonical divisor formula

$$K_S \sim f^{-1}(-P) + (p-1)E_0 + (p-2)E_\infty.$$

This gives an example of an elliptic surface which has one tame fiber and one wild fiber. Incidently, if $p = 2$, then $S$ is a ruled surface over an elliptic curve ($\kappa(S) = -\infty$), if $p = 3$, then $S$ is a hyperelliptic surface ($\kappa(S) = 0$, $3K_S \sim 0$), and otherwise, $S$ is an elliptic surface with $\kappa(S) = 1$.

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