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The generalized Thue inequality

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Abstract. Let $F(\underline{x}) = F(x, y)$ be a form in $\mathbb{Z}[x, y]$ of degree $r \geq 3$ and without multiple factors. A generalization of the classical Thue inequality $|F(\underline{x})| \leq h$ is the inequality $|F(\underline{x})| \leq h|\underline{x}|^\gamma$ where $|\underline{x}|$ is the maximum norm. When $\gamma < r - 2$ this inequality has only finitely many solutions in integers. The present paper deals with upper bounds for the number of such solutions.

1. Introduction

As is well known, a *Thue equation*

$$F(x, y) = h$$

has only finitely many solutions in integers. Here F is a form of degree $r \geq 3$ with coefficients in \mathbb{Z} and without multiple factors, and $h \in \mathbb{Z}$. Upper bounds $B_1(r, h)$ for the number of solutions which depend only on r and h but are independent of the coefficients of F were given by Evertse [2] and then by Bombieri and Schmidt [1]. Clearly the *Thue inequality*

$$|F(x, y)| \leq h \tag{1.1}$$

also has only finitely many solutions. Upper bounds $B_2(r, h)$ for the number of solutions were given by Schmidt [7] and by Thunder [9], [10]. It is an immediate consequence of Roth's Theorem that a *generalized Thue inequality*

$$|F(x, y)| \leq h|\underline{x}|^\gamma \tag{1.2}$$

where $|\underline{x}| = \max(|x|, |y|)$ and where $\gamma < r - 2$, has only finitely many solutions. The obvious question whether there is a bound $B_3(r, \gamma, h)$ for the number of solutions of (1.2) has a negative answer, as may be seen as follows.

Given two forms F, G as above, write $F \sim G$ if there is a transformation $T \in \text{SL}(2, \mathbb{Z})$ with $F(\underline{x}) = G(T\underline{x})$; here we use the notation $\underline{x} = (x, y)$. We will show that given $\gamma > 0$ and given a form G there is a constant $c_1(G, \gamma) > 0$ and there are infinitely many forms $F \sim G$ such that the inequality $|F(\underline{x})| \leq |\underline{x}|^\gamma$

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has at least $c_1(G, \gamma)H(F)^{2\gamma/r^2}$ primitive solutions. Here a pair (x, y) is called *primitive* if $\gcd(x, y) = 1$, and $H(F)$ is the maximum modulus of the coefficients of F .

For natural k let T be the map $(x, y) \mapsto (X, Y)$ with $X = kx + y$, $Y = (k - 1)x + y$, and set $F(\underline{X}) = G(T^{-1}\underline{X})$. Then $F \sim G$ and $H(F) \ll_G k^r$, with the implicit constant in \ll depending only on G . We have $|G(\underline{x})| \ll_G |\underline{x}|^r$ for $\underline{x} \in \mathbb{R}^2$, and therefore when k is large, the inequality $|G(\underline{x})| \leq k^\gamma$ will have $\gg_{G,\gamma} k^{2\gamma/r}$ integer solutions \underline{x} . In fact it will have $\gg_{G,\gamma} k^{2\gamma/r}$ primitive solutions with $xy > 0$, and these solutions will have $|G(\underline{x})| \leq |kx + y|^\gamma$. When \underline{x} is such a solution and $\underline{X} = T\underline{x}$, then \underline{X} is again primitive and $|F(\underline{X})| = |G(\underline{x})| \leq |kx + y|^\gamma = |\underline{X}|^\gamma$. Thus $|F(\underline{X})| \leq |\underline{X}|^\gamma$ has $\gg_{G,\gamma} k^{2\gamma/r} \gg_{G,\gamma} H(F)^{2\gamma/r^2}$ primitive solutions.

Now suppose that F is a form of degree r with $s + 1$ nonzero coefficients in \mathbb{Z} :

$$F(x, y) = \sum_{i=0}^s a_i x^{r_i} y^{r-r_i}, \tag{1.3}$$

where $0 = r_0 < r_1 < \dots < r_s = r$. We will not need to assume that F has no multiple factors. As we saw in [6], new methods can be used for the Thue inequality when F is ‘‘sparse,’’ i.e., when $r > 2s$. It turns out that the analogous condition $r - \gamma > 2s$ works for the generalized Thue inequality. In what follows, set

$$\rho = r - \gamma. \tag{1.4}$$

THEOREM. *Let F be of the type (1.3), ρ a number with*

$$2s < \rho \leq r, \tag{1.5}$$

and $\gamma = r - \rho$. Then the number of primitive solutions of the generalized Thue inequality (1.2) is

$$\leq c_2(r, \rho)h^\kappa,$$

where

$$\kappa = \max(2/\rho, 1/(\rho - 2s)). \tag{1.6}$$

When $4s \leq \rho \leq r$, the number of primitive solutions is

$$\ll c_3(r, s, \rho)h^{2/\rho}$$

with an absolute constant in \ll and

$$c_3(r, s, \rho) = s^{1+(r/\rho)} \exp(\rho^{-1}(12r + 4rs\rho^{-1} \log s + 3200s \log^3 r)).$$

In particular when $\rho \geq s \log s$, $r \geq s \log^3 s$, then $s \log^3 r \ll r$ and $c_3 \leq s(c_4s)^{r/\rho}$ with an absolute constant c_4 , so that the number of primitive solutions is $\ll s(c_4s)^{r/\rho}h^{2/\rho}$. When $\rho = r$, we recover the bound $\ll s^2h^{2/r}$ of [6, below (1.9)], at least for primitive solutions. A special case of (1.2) is

$$|ax^r - by^r| \leq h|\underline{x}|^{r-\rho}.$$

The number of primitive solutions, assuming $4 \leq \rho \leq r$, is $\leq c_4^{r/\rho}h^{2/\rho}$.

Note that we have to restrict ourselves to primitive solutions. For there certainly are forms F for which the Thue equation $F(\underline{x}) = 1$ has a solution \underline{x}_0 with $|\underline{x}_0|$ arbitrarily large. Then $\underline{x} = t\underline{x}_0$ will have $|F(\underline{x})| \leq |\underline{x}|^\gamma$ precisely when $|t|^r \leq |\underline{x}_0|^\gamma|t|^\gamma$, i.e., when $|t|^\rho \leq |\underline{x}_0|^\gamma$. The number of choices for t cannot be bounded in terms of r, s, ρ .

Let $f(x, y)$ be a polynomial of total degree $\gamma < r - 2s$. Suppose f has coefficients of modulus $\leq M$. The diophantine equation

$$F(x, y) = f(x, y) \tag{1.7}$$

yields (1.2) with $h = \binom{\gamma + 2}{2} M$. When $\underline{x} = t\underline{x}_0$ with x_0 primitive, then also \underline{x}_0 satisfies (1.2), so that the number of possibilities for \underline{x}_0 is estimated by our Theorem. Once \underline{x}_0 is fixed, (1.7) gives an algebraic equation for t of degree r , hence with at most r solutions t . Hence the number of solutions of (1.7) is

$$\leq c_5(r)M^\kappa,$$

with κ given by (1.6). Again, under suitable conditions on r, s, ρ , good explicit bounds may be given.

Mahler [4] gave an asymptotic formula for the number $N_F(h)$ of solutions of the Thue inequality (1.1). He established that $N_F(h) \sim A_F h^{2/r}$ as $h \rightarrow \infty$, where A_F is the area of the region of $\underline{x} \in \mathbb{R}^2$ with $|F(\underline{x})| \leq 1$. We expect that for $0 \leq \gamma < r - 2$ there is an analogous formula for the number $N_{F,\gamma}(h)$ of solutions of the generalized Thue inequality (1.2):

$$N_{F,\gamma}(h) \sim A_{F,\gamma} h^{2/\rho} \quad \text{as } h \rightarrow \infty,$$

where $A_{F,\gamma}$ is the area of the plane region $|F(\underline{x})| \leq |\underline{x}|^\gamma$. This should hold generally, i.e., for forms F not necessarily of the sparse type (1.3); but good error estimates are more likely for sparse forms.

2. The Plan of the Paper

We will follow [6] very closely – our task will be to show that the method developed in that paper for Thue inequalities extends to generalized Thue inequalities.

The Mahler height of a form

$$F(x, y) = a_0(x - \alpha_1 y) \dots (x - \alpha_r y) \quad (2.1)$$

is

$$M(F) = |a_0| \prod_{i=1}^r \max(1, |\alpha_i|).$$

It has the properties that $M(F(x, y)) = M(F(y, x))$, and $M(FG) = M(F)M(G)$. The Mahler height $M(\alpha)$ of an algebraic number α is the height of its homogenized defining polynomial (chosen to have coprime coefficients in \mathbb{Z}). If F as above has coefficients in \mathbb{Z} , each $M(\alpha_i) \leq M(F)$.

Set

$$R = e^{800 \log^3 r}, \quad (2.2)$$

$$C = (2r^{1/2} M(F))^r h R, \quad (2.3)$$

$$Y_L = C^{2/(\rho-2)}, \quad Y_S = Y_0^{1/(\rho-2s)}, \quad (2.4)$$

with

$$Y_0 = (e^6 s)^r R^{2s} h. \quad (2.5)$$

Then

$$Y_S^\rho > Y_0 > (rs)^{2s} (4e^3 s)^r h \quad (2.6)$$

since $R > rs$.

We will distinguish *large*, *medium* and *small* solutions to (1.2). Writing $\underline{x} = (x, y)$, $|\underline{x}| = \max(|x|, |y|)$, $\langle \underline{x} \rangle = \min(|x|, |y|)$, a solution will be called

$$\begin{array}{ll} \text{large} & \text{if } |\underline{x}| > Y_L, \\ \text{medium} & \text{if } |\underline{x}| \leq Y_L \text{ and } \langle \underline{x} \rangle \geq Y_S, \\ \text{small} & \text{if } \langle \underline{x} \rangle < Y_S. \end{array}$$

PROPOSITION 1. *The number of primitive large solutions is $\leq c_6(s, r, \rho)$. When $\rho \geq 4$, this number is*

$$\ll s \left(\frac{\log r}{\log \rho} \right)^2 \left(1 + \frac{\log \log r}{\log \rho} \right). \quad (2.7)$$

PROPOSITION 2. *The number of primitive medium solutions is*

$$\ll \frac{s^2 r^2}{\rho(\rho - 2)}(1 + r^{-2} \log h). \tag{2.8}$$

PROPOSITION 3. *The number of small solutions is $\leq c_7(s, r, \rho)h^\kappa$ with κ given by (1.6). When $\rho \geq 4s$, the number of small solutions is $\ll c_3 h^{2/\rho}$ with $c_3 = c_3(r, s, \rho)$ as in the Theorem.*

The Theorem follows from these propositions since the bound (2.7) is

$$\ll s(\log r / \log \rho)^3 \ll s \exp(\rho^{-1}r),$$

and the bound in (2.8) is

$$\ll s^2(r/\rho)^2 h^{2/\rho} \ll s^2 \exp(\rho^{-1}r)h^{2/\rho}.$$

3. Large Solutions

LEMMA 1. *For every $\underline{x} = (x, y)$ with (1.2) and $y \neq 0$ there is an α_i (as given in (2.1)) with*

$$\min \left(1, \left| \alpha_i - \frac{x}{y} \right| \right) \leq (2^{r/2} M(F))^r h |\underline{x}|^{-\rho}.$$

Proof. This lemma corresponds to Lemma 4 of [6] and the proof is the same. In fact one just has to recall (1.4) and to substitute in Lemma 1 of [1], which essentially is already in Lewis and Mahler [3].

LEMMA 2. *There is a subset S of the set $\{\alpha_1, \dots, \alpha_r\}$ of cardinality $|S| \leq 6s + 4$ such that for every \underline{x} with (1.2) and $y \neq 0$ there is an $\alpha_i \in S$ with*

$$\min \left(1, \left| \alpha_i - \frac{x}{y} \right| \right) \leq C |\underline{x}|^{-\rho}. \tag{3.1}$$

Proof. This corresponds to Lemma 8 of [6] and is deduced in exactly the same way.

Now if $|\underline{x}| \geq Y_L$, say $y \geq Y_L$, we have from (2.4) that the minimum in (3.1) is $< y^{-(\rho+2)/2}$, and therefore

$$\left| \alpha_i - \frac{x}{y} \right| < y^{-(\rho+2)/2}.$$

Observe that $y \geq Y_L = C^{2/(\rho-2)} > M(F)^{2r/(\rho-2)} \geq M(F) \geq M(\alpha_i)$. But in [8] it was pointed out that the number of solutions of $\left| \alpha_i - \frac{x}{y} \right| < y^{-\rho}$ with $y > M(\alpha_i)$

is $\leq c_8(r, \rho)$, and when $\rho \geq 3$, it is in fact $\ll (\log r / \log \rho)^2(1 + \log \log r / \log \rho)$. If we apply this with $(\rho + 2)/2$ in place of ρ and note that $\log((\rho + 2)/2) \gg \log \rho$, we see that for fixed α_i our number of solutions is under an analogous bound. After multiplication by $|S| \ll s$ we obtain the estimates of Proposition 1.

4. Medium Solutions

Given (1.3), let P_i for $0 \leq i \leq s$ be the point in the plane with coordinates $(r_i, -\log |a_i|)$. In [6] we defined the Newton Polygon to be the “lower boundary” of the convex hull of P_0, \dots, P_s , and denoted its vertices by $P_{i(0)}, P_{i(1)}, \dots, P_{i(\ell)}$; here $1 \leq \ell \leq s$. Also $\sigma(i, j)$ for $i \neq j$ was the slope of the segment P_iP_j . Further for $0 < j \leq \ell$ we set $\sigma(i(j)) = \sigma(i(j - 1), i(j))$ and for $0 \leq j < \ell$ we set $\sigma^+(i(j)) = \sigma(i(j), i(j + 1))$, so that $\sigma(i(j)), \sigma^+(i(j))$ are the slopes of the segments of the Newton polygon to the left and to the right of $P_{i(j)}$. For each root α of $f(x) = F(x, 1)$ we defined integers $k(\alpha), K(\alpha)$ having $0 \leq k(\alpha) < K(\alpha) \leq \ell$. Also, $H = H(f)$ was the maximum modulus of the coefficients a_i , and q an integer with $|a_q| = H$.

LEMMA 3. *Suppose (1.2) holds with $|x| \leq |y|$, $y \neq 0$. Let α be a root of $f(x)$ with*

$$|x - \alpha y| = \min_{1 \leq j \leq r} |x - \alpha_j y|. \tag{4.1}$$

Suppose that $q < i(K)$ where $K = K(\alpha)$. Then there is a u , $1 \leq u \leq i(K)$, with

$$\left| \alpha - \frac{x}{y} \right| < \frac{1}{H^{(1/u)-(1/r)}} \left(\frac{(rs)^{2s}(2e^3s)^r h}{|y|^\rho} \right)^{1/u}. \tag{4.2}$$

Proof. Our lemma corresponds to Lemma 15 in [6]. The only difference in the proof is that in Lemma 10 of [6], h is to be replaced by $h|\underline{x}|^\gamma = h|y|^\gamma = h|y|^{r-\rho}$, and therefore the conclusion of that Lemma is true with $|y|^\rho$ in place of $|y|^r$.

LEMMA 4. *Suppose (1.2) holds with $|x| \leq |y|$ and*

$$|y|^\rho \geq 2^r(rs)^{2s}h. \tag{4.3}$$

Let α be a root of $f(x)$ with (4.1). Suppose that $i(k) < q$ where $k = k(\alpha)$. Then there is a v , $1 \leq v \leq s - i(k)$, with

$$\left| \alpha^{-1} - \frac{y}{x} \right| < \frac{1}{H^{(\rho/rv)-(1/r)}} \left(\frac{(rs)^{2s}(4e^3s)^r h}{|x|^\rho} \right)^{1/v}. \tag{4.4}$$

Proof. The lemma corresponds to Lemma 16 of [6]. In analogy to (8.8) of that paper we now have

$$\left| \alpha - \frac{x}{y} \right| < \left(\frac{2^r (rs)^{2s} h}{\Delta^* |y|^\rho} \right)^{1/v}$$

with

$$\Delta^* = \Delta^*(\alpha, v) = |a_{i(k)}| |\alpha|^{r_{i(k)} - v}.$$

As in [6] we have $\log(|\alpha|^v \Delta^*(\alpha, v)) > 0$, so that

$$\left| \alpha - \frac{x}{y} \right| < |\alpha| \left(\frac{2^r (rs)^{2s} h}{|y|^\rho} \right)^{1/v} \leq |\alpha|$$

by (4.3), and therefore $|x| < 2|\alpha y|$. We may infer that

$$\begin{aligned} \left| \alpha^{-1} - \frac{y}{x} \right| &= \left| \frac{y}{\alpha x} \right| \left| \alpha - \frac{x}{y} \right| < \left(\frac{2^r (rs)^{2s} h}{\Delta^* |\alpha^v y^{\rho-v} x^v|} \right)^{1/v} \\ &< \left(\frac{4^r (rs)^{2s} h}{|a_{i(k)}| |\alpha|^{-(\rho-v-r_{i(k)})} |x|^\rho} \right)^{1/v} \\ &= \left(\frac{4^r (rs)^{2s} h}{\Gamma(\alpha, v) |x|^\rho} \right)^{1/v} \end{aligned} \tag{4.5}$$

with

$$\Gamma(\alpha, v) = |a_{i(k)}| |\alpha|^{-(\rho-v-r_{i(k)})},$$

so that

$$\log \Gamma(\alpha, v) = -(\rho - v - r_{i(k)}) \log |\alpha| + \log |a_{i(k)}|. \tag{4.6}$$

Case 1. $\rho - v - r_{i(k)} \geq 0$. We proceed as in [6]. The number $k = k(\alpha)$ has $\sigma^+(i(k)) > \log |\alpha| - \log(e^3 s)$, and on the other hand $\sigma(i(k), q) \geq \sigma^+(i(k))$ by $q > i(k)$ and by convexity considerations. Therefore

$$\begin{aligned} \log \Gamma(\alpha, v) &\geq -(\rho - v - r_{i(k)}) \sigma(i(k), q) + \log |a_{i(k)}| - r \log(e^3 s) \\ &= -(\rho - v - r_q) \sigma(i(k), q) + \log |a_q| - r \log(e^3 s), \end{aligned}$$

for clearly $\sigma(i(k), q) = (\log |a_q| - \log |a_{i(k)}|) / (r_{i(k)} - r_q)$. When $\rho - v - r_q \geq 0$, we observe that $\sigma(i(k), q) \leq 0$ by the choice of q and that

$$\log \Gamma(\alpha, v) \geq \log |a_q| - r \log(e^3 s). \tag{4.7}$$

When $\rho - v - r_q < 0$, we observe that $\sigma(i(k), q) \geq \sigma(0, q)$, and

$$\begin{aligned} \log \Gamma(\alpha, v) &\geq -(\rho - v - r_q)\sigma(0, q) + \log |a_q| - r \log(e^3 s) \\ &= (\rho - v - r_q)((\log |a_q| - \log |a_0|)/r_q) \\ &\quad + \log |a_q| - r \log(e^3 s) \\ &\geq ((\rho - v)/r_q) \log |a_q| - r \log(e^3 s). \end{aligned}$$

Since by hypothesis $\rho > 2s > v$, we obtain

$$\log \Gamma(\alpha, v) \geq ((\rho - v)/r) \log |a_q| - r \log(e^3 s). \tag{4.8}$$

By (4.7) this holds always in Case 1, and therefore

$$\Gamma(\alpha, v) \geq (e^3 s)^{-r} H^{(\rho-v)/r},$$

which in conjunction with (4.5) yields (4.4).

Case 2. $\rho - v - r_{i(k)} < 0$. Then $i(k) > 0$, since $v \leq s < \rho$. We claim that

$$\sigma^+(i(k)) < \log |\alpha| + \log(e^3 s). \tag{4.9}$$

This is certainly true if $\sigma(s) = \sigma(i(\ell)) < \log |\alpha| + \log(e^3 s)$, and otherwise $K = K(\alpha)$ was smallest with $\sigma^+(i(K)) \geq \log |\alpha| + \log(e^3 s)$ (see [6, Sect. 6]). But $k < K$, so that indeed (4.9) holds. Now (4.6) gives

$$\log \Gamma(\alpha, v) > -(\rho - v - r_{i(k)})(\sigma^+(i(k)) - \log(e^3 s)) + \log |a_{i(k)}|.$$

But

$$\sigma^+(i(k)) \geq \sigma(0, i(k)) = (\log |a_0| - \log |a_{i(k)}|)/r_{i(k)} \geq -\log |a_{i(k)}|/r_{i(k)}.$$

Thus

$$\log \Gamma(\alpha, v) > ((\rho - v)/r_{i(k)}) \log |a_{i(k)}| - r \log(e^3 s). \tag{4.10}$$

We observe that $0 < i(k) < q$, therefore $\sigma(0, i(k)) < \sigma(0, q)$, and

$$\begin{aligned} r_{i(k)}^{-1} \log |a_{i(k)}| &= r_{i(k)}^{-1} \log |a_0| - \sigma(0, i(k)) > r_q^{-1} \log |a_0| - \sigma(0, q) \\ &= r_q^{-1} \log |a_q| \geq r^{-1} \log |a_q|, \end{aligned}$$

which together with (4.10) gives (4.8) again, and therefore (4.4).

Now if

$$Y_S \leq |x| \leq |y|, \tag{4.11}$$

we have (4.3) by (2.6), and since either $q < i(K)$ or $q > i(k)$ will certainly hold, the conclusion of Lemma 3 or Lemma 4 will hold. Moreover, the right hand sides of (4.2), (4.4) will increase in u resp. v , so that we may replace u, v by s . Combining this with Lemma 7 of [6] we obtain the following lemma, which corresponds to Lemma 17 of [6].

LEMMA 5. *There is a set S of roots of $F(x, 1)$ and a set S^* of roots of $F(1, y)$, both of cardinality $\leq 6s + 4$, such that every solution of (1.2) with (4.11) either has*

$$\left| \alpha - \frac{x}{y} \right| < \frac{R}{H^{(1/s)-(1/r)}} \left(\frac{(rs)^{2s}(4e^3s)^r h}{|y|^\rho} \right)^{1/s} \tag{4.12}$$

for some $\alpha \in S$, or has

$$\left| \alpha^* - \frac{y}{x} \right| < \frac{R}{H^{(\rho/rs)-(1/r)}} \left(\frac{(rs)^{2s}(4e^3s)^r h}{|x|^\rho} \right)^{1/s} \tag{4.13}$$

for some $\alpha^* \in S^*$.

The medium solutions to (1.2) were those with $Y_S \leq \langle \underline{x} \rangle$, $|\underline{x}| \leq Y_L$. Without loss of generality we may restrict ourselves to solutions with

$$Y_S \leq |x| \leq |y| \leq Y_L.$$

We will estimate such solutions with (4.13) (the case (4.12) being easier since the exponent of H is better). We have

$$\left| \alpha^* - \frac{y}{x} \right| < K/(2|x|^{\rho/s}) \tag{4.14}$$

with

$$\begin{aligned} K &= 2R(rs)^2(4e^3s)^{r/s}h^{1/s}H^{(1/r)-(1/rs)} \\ &< R^2(e^5s)^{r/s}h^{1/s}H^{-\rho/2rs} \end{aligned} \tag{4.15}$$

by (2.2) and since $\rho > 2s$. Let $y_0/x_0, \dots, y_\nu/x_\nu$ be the solutions of (4.14) with $Y_S \leq x \leq Y_L$, ordered such that $x_0 \leq \dots \leq x_\nu$. Then for $0 \leq i < \nu$,

$$\frac{1}{x_i x_{i+1}} \leq \left| \alpha^* - \frac{y_i}{x_i} \right| + \left| \alpha^* - \frac{y_{i+1}}{x_{i+1}} \right| < \frac{1}{2}K(x_i^{-\rho/s} + x_{i+1}^{-\rho/s}) \leq Kx_i^{-\rho/s},$$

so that we have the ‘‘gap principle’’

$$x_{i+1} > K^{-1}x_i^{(\rho/s)-1} \geq K^{-1}Y_S^{(\rho/s)-2}x_i = K^{-1}Y_0^{1/s}x_i > e^{r/s}H^{\rho/2rs}x_i$$

by (2.4), (2.5), (4.15). Therefore

$$x_\nu > (e^{r/s} H^{\rho/2rs})^\nu,$$

$$\log x_\nu > \nu((\rho/2rs) \log H + (r/s)) > (\nu\rho/2rs)(\log H + r). \tag{4.16}$$

On the other hand, by the definitions (2.2), (2.3), (2.4) of R, C, Y_L , we get

$$\log Y_L = 2(\rho - 2)^{-1} \log C \ll (\rho - 2)^{-1}(r \log M(F) + r \log r + \log h).$$

But $M(F) \leq (r + 1)H$ according to Mahler [5], so that we obtain

$$\log Y_L \ll r(\rho - 2)^{-1}(\log H + \log r + \log h^{1/r}).$$

The same upper bound holds for $\log x_\nu$. Comparison with (4.16) yields

$$\nu \ll r^2 s \rho^{-1} (\rho - 2)^{-1} (1 + r^{-1} \log h^{1/r}).$$

Taking account of the summation over $\alpha^* \in S^*$ we obtain Proposition 2.

5. Small Solutions

LEMMA 6. *Let $p(y) = A_s y^{r_s} + \dots + A_1 y^{r_1} + A_0$ be a polynomial with real coefficients with $r = r_s > \dots > r_1 > r_0 = 0$ and with $|A_s| \geq 1$. Let $h > 0$ and $0 \leq \gamma < r - s$, $\rho = r - \gamma$ as in (1.4). Then the real numbers y with*

$$|p(y)| \leq h|y|^\gamma \tag{5.1}$$

make up a set of measure

$$\mu < 18(rs^2(\gamma + 1))^{s/\rho} h^{1/\rho}. \tag{5.2}$$

Given $x \geq 1$, the numbers y with (5.1) and $|y| > x$ make up a set of measure

$$\mu < 18(\gamma + 1)rs^2 h^{1/s} x^{1-(\rho/s)}. \tag{5.3}$$

This lemma corresponds to Lemma 19 of [6], but the role of the variables has been interchanged.

Proof. Define

$$p_i(y) = \sum_{j=i}^s (r_j - r_0)(r_j - r_1) \dots (r_j - r_{i-1}) A_j y^{r_j - r_i}, \quad (0 \leq i \leq s).$$

Then $p_0(y) = p(y)$,

$$p'_i(y) = p_{i+1}(y)y^{r_{i+1} - r_i - 1}, \quad (0 \leq i < s). \tag{5.4}$$

We now introduce a new parameter Z . We will initially concentrate on numbers y with

$$|y| \geq Z. \tag{5.5}$$

Set

$$g(y) = p'(y) - hs^2|y|^\gamma/Z. \tag{5.6}$$

Then $g(y)$ restricted to $y > 0$ (or to $y < 0$) is a polynomial of degree $r - 1$ if $\gamma \in \mathbb{Z}$, and in general is a linear combination of powers of y (or of $-y$), the highest power of y occurring being y^{r-1} . In fact it is a sum of at most $s + 1$ powers, therefore has at most $2s + 1$ real zeros. The same is true if the $-$ sign in (5.6) is replaced by $+$. Therefore there are at most $4s + 2$ real numbers y with $|p'(y)| = hs^2|y|^\gamma/Z$. The real numbers y with (5.5) and $|p'(y)| > hs^2|y|^\gamma/Z$ make up at most $4s + 4 \leq 8s$ intervals and half-lines. If y_1, y_2 with (5.1) lie in such an interval, we have on the one hand

$$|p(y_2) - p(y_1)| < h(|y_1|^\gamma + |y_2|^\gamma).$$

On the other hand, if, say, $0 < y_1 < y_2$, then

$$\begin{aligned} |p(y_2) - p(y_1)| &= \left| \int_{y_1}^{y_2} p'(y) dy \right| > hs^2 Z^{-1} \int_{y_1}^{y_2} y^\gamma dy \\ &= hs^2(\gamma + 1)^{-1} Z^{-1}(y_2^{\gamma+1} - y_1^{\gamma+1}) \\ &> hs^2(\gamma + 1)^{-1} Z^{-1}(y_2 - y_1)y_2^\gamma. \end{aligned}$$

Therefore $y_2 - y_1 < 2(\gamma + 1)s^{-2}Z$, so that our interval is of length $< 2(\gamma + 1)s^{-2}Z$. Thus if we neglect a set of measure $< 16(\gamma + 1)s^{-1}Z$, we may concentrate on numbers y with $|p'(y)| \leq hs^2|y|^\gamma/Z$, i.e., with

$$|p_1(y)y^{r_1-1}| \leq hs^2|y|^\gamma/Z. \tag{5.7}$$

We now repeat the argument with hs^2/Z in place of h and $q(y) = p_1(y)y^{r_1-1}$ in place of $p(y)$. If we neglect a further set of measure $< 16(\gamma + 1)s^{-1}Z$, we may suppose that $|q'(y)| \leq hs^4|y|^\gamma/Z^2$. Now $q'(y) = p_1'(y)y^{r_1-1} + (r_1 - 1)p_1(y)y^{r_1-2}$. The second summand here is of modulus $\leq (r_1 - 1)hs^2|y|^\gamma/Z^2$ by (5.5), (5.7), so that we obtain $|p_1'(y)y^{r_1-1}| \leq r_1hs^4|y|^\gamma/Z^2$, whence by (5.4),

$$|p_2(y)y^{r_2-2}| \leq r_1s^4h|y|^\gamma/Z^2.$$

We now deal with this in a manner analogous to (5.7). We have to replace p_1, r_1, h by $p_2, r_2 - 1, r_1s^2h/Z$. So if we neglect a further set of measure $< 16(\gamma + 1)s^{-1}Z$, we may suppose that

$$|p_3(y)y^{r_3-3}| \leq r_1(r_2 - 1)s^6h|y|^\gamma/Z^3.$$

And so on. The conclusion is that except for a set of measure $\leq s \cdot 16(\gamma + 1)s^{-1}Z = 16(\gamma + 1)Z$, the numbers y with (5.1), (5.5) have

$$|p_i(y)y^{r_i-i}| \leq r_1 \dots r_{i-1}s^{2i}h|y|^\gamma/Z^i, \quad (i = 1, \dots, s) \tag{5.8}$$

(where $r_1 \dots r_{i-1} = 1$ when $i = 1$). Incidentally, here we have used the fact that $p_i(y)y^{r_i-i}$ has degree $r - i$, so that its derivative is of degree $r - i - 1 \geq r - s > \gamma$ for $i < s$, and therefore the analogue of the function in (5.6) at the i th step of the argument is not zero.

We now apply (5.8) with $i = s$ and note that $p_s(y)$ is a constant of modulus $\geq r$. We get

$$Z^s|y|^{\rho-s} \leq r^{s-2}s^{2s}h. \tag{5.9}$$

If we also neglect y with $|y| \leq (\gamma + 1)Z$, then altogether we are neglecting a set of measure $< 18(\gamma + 1)Z$. Now (5.5) holds, and $|y| > (\gamma + 1)Z$ in conjunction with (5.9) yields $Z^\rho(\gamma + 1)^{\rho-s} < r^{s-2}s^{2s}h$. This is impossible if we choose $Z = Z_0 = (r^s s^{2s} h)^{1/\rho}(\gamma + 1)^{(s/\rho)-1}$. Therefore the numbers y with (5.1) constitute a set of measure $< 18(\gamma + 1)Z_0$, giving (5.2).

On the other hand when $|y| \geq x$, then (5.9) gives $Z^s < r^s s^{2s} h x^{s-\rho}$. This is impossible if we choose $Z = Z_1 = r s^2 h^{1/s} x^{1-(\rho/s)}$. Therefore the numbers y with (5.1), (5.5) and $|y| \geq x$ constitute a set of measure $< 16(\gamma + 1)Z_1$. The interval $|y| < Z_1$ (the complement of (5.5)) has measure $2Z_1$, so that we get altogether $< 18(\gamma + 1)Z_1$, i.e., (5.3).

We now turn to the proof of Proposition 3. The problem is to estimate the number of solutions with $\langle \underline{x} \rangle < Y_S$. We may suppose that $|x| \leq |y|$ and $|x| < Y_S$. This number is $\sum z(x)$ over x with $|x| < Y_S$, where $z(x)$ is the number of integers y with (1.2) and $|x| \leq |y|$. Given x , the number of real y with $F(x, y) = \pm|y|^\gamma$ or with $y = \pm x$ is $\leq 4s + 7$ (e.g. $\leq s + 1$ solutions of $F(x, y) = |y|^\gamma$ with $y > 0$, and the same for $y < 0$ or $F(x, y) = -|y|^\gamma$, plus 3 solutions with $y = \pm x$ or $y = 0$). Thus the real numbers y with $|x| \leq |y|$, $|F(x, y)| \leq h|y|^\gamma$ make up at most $2s + 4$ intervals. The number $z(x)$ then is $\leq \mu(x) + 2s + 4$, where $\mu(x)$ is the total measure of these intervals. The number of small solutions then is

$$\leq (2s + 4) \sum_{\substack{x \\ |x| < Y_S}} 1 + \sum_x \mu(x). \tag{5.10}$$

The first summand here is

$$\ll sY_S \ll c_9(r, s, \rho)h^{1/(\rho-2s)},$$

with

$$c_9(r, s, \rho) = s((e^6 s)^r R^{2s})^{1/(\rho-2s)}$$

by (2.4), (2.5). In the case when $\rho \geq 4s$ we obtain

$$\ll c_9(r, s, \rho)h^{2/\rho},$$

with

$$c_9(r, s, \rho) = s((e^6 s)^r R^{2s})^{1/(\rho-2s)} \ll s^a e^{12r/\rho} R^{4s/\rho},$$

where

$$a = 1 + r(\rho - 2s)^{-1} = 1 + r\rho^{-1} + 2rs/\rho(\rho - 2s) \leq 1 + r\rho^{-1} + 4rs\rho^{-2}.$$

Thus

$$\begin{aligned} c_9(r, s, \rho) &\ll s^{1+(r/\rho)} \exp(\rho^{-1}(12r + 3200s \log^3 r + 4rs\rho^{-1} \log s)) \\ &= c_3(r, s, \rho). \end{aligned}$$

The second summand in (5.10) is $\Sigma_1 + \Sigma_2$, with Σ_1, Σ_2 respectively a sum over $|x| \leq c_{10}h^{1/\rho}$, $|x| > c_{10}h^{1/\rho}$ where

$$c_{10} = ((\gamma + 1)rs^2)^{s/\rho}.$$

By (5.2) of Lemma 6

$$\sum_1 \ll c_{10}((\gamma + 1)rs^2)^{s/\rho} h^{2/\rho} = ((\gamma + 1)rs^2)^{2s/\rho} h^{2/\rho}.$$

By (5.3) of Lemma 6

$$\sum_2 \ll (\gamma + 1)rs^2 h^{1/s} \sum_{x > c_{10}h^{1/\rho}} x^{1-(\rho/s)}.$$

The sum on the right is

$$\begin{aligned} &\ll (c_{10}h^{1/\rho})^{1-(\rho/s)} + (\rho s^{-1} - 2)^{-1} (c_{10}h^{1/\rho})^{2-(\rho/s)} \\ &\ll \rho(\rho - 2s)^{-1} c_{10}^{2-(\rho/s)} h^{(2/\rho)-(1/s)}, \end{aligned}$$

so that

$$\sum_2 \ll (\gamma + 1)(\rho - 2s)^{-1} \rho r s^2 c_{10}^{2-(\rho/s)} h^{2/\rho} = c_{11}(r, s, \rho)h^{2/\rho},$$

with

$$c_{11}(r, s, \rho) = \rho(\rho - 2s)^{-1} (rs^2(\gamma + 1))^{2s/\rho}.$$

When $\rho \geq 4s$ we have

$$c_{11}(r, s, \rho) \ll (rs)^{4s/\rho} \leq r^{8s/\rho} = \exp(\rho^{-1} \cdot 8s \log r) \leq c_3(r, s, \rho).$$

Combining our results we see that the total number of small solutions to (1.2) is $\ll c_6(r, s, \rho)h^\kappa$ with $\kappa = \max(2/\rho, 1/(\rho - 2s))$, and a certain constant c_6 , and it is $\ll c_3(r, s, \rho)h^{2/\rho}$ when $\rho \geq 4s$.

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