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An axiomatization of Nesterenko's method and applications on Mahler functions II

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Abstract. Measures for the algebraic independence of the values of Mahler functions are given, where the arguments are complex algebraic numbers or well approximable real Liouville numbers. The proofs depend on the elimination-theoretic method of Nesterenko and Philippon.

1. Introduction and statement of the results

In his papers [Ne1], [Ne2], [Ne3], [Ne4] Nesterenko introduced an elimination-theoretic method to study algebraic independence of certain numbers, and he applied his method to the values of E-functions, the exponential function and Mahler functions. Philippon generalized Nesterenko's approach and proved a criterion for algebraic independence over algebraic number fields with an arbitrary valuation in [P1], and in [P2] he extended this criterion to diophantine domains. In [J] Jabbouri gave a sharpened quantitative version of Philippon's criterion for algebraic number fields with the usual absolute value. For a survey of the development and the results see [Br].

In our first paper [T1] we used Nesterenko's original method to derive a criterion for algebraic independence, which gives lower bounds for the number of algebraically independent quantities in a set of numbers. By application of the criterion earlier results about the algebraic independence of the values of Mahler functions obtained by Nishioka [Ni1] and Amou [A] were generalized.

In this paper we prove algebraic independence measures for the values of Mahler functions at algebraic points in $\mathbb{C}$ (this generalizes results of Nesterenko [Ne4], Becker [Be3] and Nishioka [Ni3]) or at well approximable transcendental (Liouville-) points in $\mathbb{R}$; almost nothing is known in the case of transcendental arguments apart from some results of Amou [A]. The measures are derived from a proposition, which forms a quantitative version of the criterion for algebraic independence stated in [T1].

For the statement of the theorems we need some preliminaries. Throughout the paper K denotes an algebraic number field and O_K the ring of integers in K. For algebraic α we define $\overline{|\alpha|}$, the house of α, as the maximum of the moduli of the conjugates of α. For a polynomial $P(z_1, \ldots, z_n, y_1, \ldots, y_m)$ with complex coefficients $\deg_{z_i} P$ is the degree of P in z_i, $\deg_{\mathbf{y}} P$ is the total degree in $\mathbf{y} = (y_1, \ldots, y_m)$, $\deg P$ is the total degree in all variables, and if the coefficients are algebraic, the height $H(P)$ is the maximum of the houses of the coefficients, and the length $L(P)$ is the sum of the houses of the coefficients. Positive constants independent of the parameters M, k, k_0, k_1 are denoted by $\gamma_0, \gamma_1, \ldots,$ if they occur within the proofs, or $c_0, c_1, \ldots, C_0, C_1, \ldots,$ if they occur in the whole paper.

Now we will state some independence measures for the values of Mahler functions. These are holomorphic functions on a neighbourhood of $0 \in \mathbb{C}$, and they satisfy a functional equation of the shape

$$f(z^p) = \frac{Q(z, f(z))}{R(z, f(z))}$$

with $p \in \mathbb{N}$, $p \geqslant 2$, and $Q, R \in K[z, y]$. Arithmetical properties of the values of such functions were first studied by Mahler in [M1], [M2] and [M3]. Surveys of the problems and of the historical development can be found in [K], [L], [LvdP], [M4].

The first independence measure for the values of Mahler functions (even in several variables) was given by Becker–Landeck in [Be2]; the measure was effective and best possible in $\log H$, but only partial effective in D, and the method of proof was an improvement of Mahler's classical method in [M3].

Nesterenko used his elimination–theoretic method in [Ne4] to derive an independence measure for Mahler functions in one variable, but (in this case) under weaker assumptions than Becker–Landeck, and the measure was effective and best possible in $\log H$, but ineffective in D, since no zero-order estimate for the auxiliary function was known. Some years later Nishioka proved a zero-order estimate in [Ni2], which was used by Becker [Be3] to give an effective independence measure. This measure was improved by Nishioka in [Ni3]. By an elementary zero-order estimate Wass [W] obtained a weaker independence measure.

In Theorem 1 we derive a general algebraic independence measure for Mahler functions, which includes most of the known results. In view of remark (b) after the proposition, on which the proof depends, we can restrict ourselves to the case of Mahler functions in one variable, where the algebraic independence of all values can be proved and a zero-order estimate is known.

THEOREM 1. *Let* $f_1, \dots, f_m : U_1(0) \to \mathbb{C}$ *be holomorphic functions on* $U_1(0) = \{z \in \mathbb{C} : |z| < 1\}$, *which are algebraically independent over* $\mathbb{C}(z)$ *and have a Taylor series expansion*

$$f_i(z) = \sum_{h \geq 0} f_{ih} z^h \quad (i = 1, \dots, m)$$

with $f_{ih} \in K$, *satisfying*

$$\overline{|f_{ih}|} \leqslant \exp(c_1(1 + h^L)), \quad E^{[c_1(1+h^L)]} f_{ih} \in O_K$$

for all $h \in \mathbb{N}_0$ *and* $i = 1, \dots, m$ *with* $L > 0$ *and* $E \in \mathbb{N}$.

Assume that $\mathbf{f}(z) = (f_1(z), \dots, f_m(z))$ *satisfies a functional equation of the shape*

$$a(z)\mathbf{f}(z^p) = \mathbf{P}(z, \mathbf{f}(z)),$$

where $p \geqslant 2$ *is an integer,* $\mathbf{P}(z, \mathbf{y}) = (P_1(z, \mathbf{y}), \dots, P_m(z, \mathbf{y}))$, $a(z) \in K[z]$, *and* $P_i(z, \mathbf{y}) \in K[z, \mathbf{y}]$ *are polynomials with* $t = \max_{i=1,\dots,m}(\deg_{\mathbf{y}} P_i) < p^{1/m}$ *and* $(m-1)((L+1)\mu - m)\log t < \log p$, *where* $\mu = 1 + m + m^2 \log t/(\log p - m \log t)$.

Suppose that $\alpha \in U_1(0)\backslash\{0\}$ *is algebraic and* $a(\alpha^{p^k}) \neq 0$ *for all* $k \in \mathbb{N}_0$.

Then for any D, $H \in \mathbb{N}$ *and any polynomial* $R \in \mathbb{Z}[\mathbf{y}]\backslash\{0\}$ *with* $\deg R \leqslant D$, $H(R) \leqslant R$ *the inequality*

$$|R(\mathbf{f}(\alpha))| > \exp\left(-c_2 D^\mu \left(D^{((L+1)\mu - m)\mu_1/\mu_2} + \left(\frac{\log H}{D}\right)^{\mu_1/\mu_3}\right)\right)$$

holds with a constant $c_2 \in \mathbb{R}_+$ *depending only on* α, $f_1, \dots, f_m$, *and* K, *where* $\mu_1 = \log p + (m-1)\mu \log t$, $\mu_2 = \log p - (m-1)((L+1)\mu - m)\log t$, $\mu_3 = \log p - \log t$.

COROLLARY. *Let* $f_1, \dots, f_m : U_1(0) \to \mathbb{C}$ *be holomorphic and algebraically independent over* $\mathbb{C}(z)$ *with a Taylor series expansion*

$$f_i(z) = \sum_{h \geq 0} f_{ih} z^h \quad (i = 1, \dots, m)$$

with algebraic coefficients f_{ih}. *Assume that* $\mathbf{f}(z)$ *satisfies the functional equation*

$$\mathbf{f}(z) = A(z)\mathbf{f}(z^p) + \mathbf{B}(z)$$

with $p \in \mathbb{N}$, $p \geqslant 2$, $\mathbf{B}(z) \in K[z]^m$ *and a non-singular* $m \times m$*-matrix* $A(z)$ *with entries in* $K[z]$.

Suppose that $\alpha \in U_1(0)\backslash\{0\}$ *is algebraic and* $\det A(\alpha^{p^k}) \neq 0$ *for all* $k \in \mathbb{N}_0$.

Then for any $D, H \in \mathbb{N}$, $\varepsilon \in \mathbb{R}_+$ *and any polynomial* $R \in \mathbb{Z}[\mathbf{y}] \backslash \{0\}$ *with* $\deg R \leqslant D$, $H(R) \leqslant H$ *the inequality*

$$|R(\mathbf{f}(\alpha))| > \exp(-c_2 D^m (D^{2+\varepsilon} + \log H))$$

holds with a constant $c_2 \in \mathbb{R}_+$ *depending on* $\alpha, f_1, \ldots, f_m, K$, *and* ε.

Proof. Similar to the proof of Lemma 1 in [Be1] we can show that there exists an algebraic number field K' with $f_{ih} \in K'$ and

$$\overline{|f_{ih}|} \leqslant \exp(\gamma_0 \log(h+1)), \quad E^{[\log(h+1)]+1} f_{ih} \in O_{K'}$$

for $h \in \mathbb{N}_0$ and $i = 1, \ldots, m$ with $E \in \mathbb{N}$. Hence the Taylor coefficients satisfy the assumption of Theorem 1 for all $L > 0$ with a suitable constant $c_1(L)$. Since $t = 1$, all other assumptions of Theorem 1 are satisfied, $\mu = 1 + m$, $\mu_1 = \mu_2 = \mu_3 = \log p$, and Theorem 1 yields the assertion. □

REMARKS. (a) The results of Nesterenko, Becker, Nishioka and Wass, which have already been obtained, dealt with the case $t = 1$ of Theorem 1.

(b) The sharpest independence measure for the values of Mahler functions was given by Nishioka in [Ni3]. Under the assumptions of the corollary she proved

$$|R(\mathbf{f}(\alpha))| > \exp(-cD^m(D^2 \log(D+1) + \log H)).$$

This estimate can be derived from the proposition analogous to the proof of Theorem 1, but with respect to the sharp bounds for the Taylor coefficients given in the proof of the corollary (this yields sharper bounds for the coefficients of the auxiliary function, hence a better estimate for $\Theta(k, N)$ resp. $\Phi_2(k, N)$ and so a sharper bound for $|R(\mathbf{f}(\alpha))|$).

(c) The first effective independence measure obtained by Becker [Be3] was (under the assumptions of the corollary)

$$|R(\mathbf{f}(\alpha))| > \exp(-cD^m(D^{m+2} + \log H)).$$

This can be deduced from the proposition similar to the proof of Theorem 1, but with a fixed value for N, i.e. $N_0 = N_1$.

Almost all algebraic independence results for Mahler functions were obtained for the values at algebraic points. In [T1] the algebraic independence at special transcendental points, which can be well approximated by fractions, was proved under certain technical assumptions. If α is an arbitrary point in $U_1(0)$, Amou [A] showed that $\operatorname{tr}\deg_{\mathbb{Q}} \mathbb{Q}(\alpha, f_1(\alpha), \ldots, f_m(\alpha)) \geqslant [(m+1)/2]$. The only quantitative result at transcendental points, due to Amou [A], is an independence measure for α and $f_1(\alpha)$, if two algebraically independent Mahler functions are given and α, $f_2(\alpha)$ are algebraically dependent. In view of remark (b) after the proposition it is worthless to look for independence measures in the general case of arbitrary transcendental arguments, but the next theorem deals with the situation at well approximable points, which is much better.

THEOREM 2. *Let* $f_1, \ldots, f_m : U_1(0) \to \mathbb{C}$ *be holomorphic functions, which are algebraically independent over* $\mathbb{C}(z)$ *and have a Taylor series expansion*

$$f_i(z) = \sum_{h \geq 0} f_{ih} z^h \quad (i = 1, \ldots, m)$$

with algebraic coefficients.

Assume that $\mathbf{f}(z) = (f_1(z), \ldots, f_m(z))$ *satisfies the functional equation*

$$a(z)\mathbf{f}(z^p) = A(z)\mathbf{f}(z) + \mathbf{B}(z),$$

where $p \geq 2$ *is an integer,* $a(z) \in K[z]$, $\mathbf{B}(z) \in K[z]^m$, *and* $A(z)$ *is a* $m \times m$*-matrix with entries in* $K[z]$.

Suppose that the real number α *with* $0 < |\alpha| < 1$ *is well approximable in the following way: there exist* $\tau \in \mathbb{R}_+$ *and an infinite sequence* $(p_l/q_l)_{l \in \mathbb{N}}$ *of reduced fractions with strictly increasing denominators such that* $\tau > 2m + 2$ *and for all* $l \in \mathbb{N}$

$$0 < \left|\alpha - \frac{p_l}{q_l}\right| \leq \exp(-c_3(\log|q_l|)^\tau)$$

and $\log|q_l| \leq \varphi(\log|q_{l-1}|)$ *with an increasing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$.

Then for any D, $H \in \mathbb{N}$ *and any polynomial* $R \in \mathbb{Z}[\mathbf{y}] \setminus \{0\}$ *with* $\deg R \leq D$, $H(R) \leq H$ *the inequality*

$$|R(\mathbf{f}(\alpha))| > \exp(-c_4 \varphi(c_5(D^{(2m+2)/(\tau-(2m+2))} + \psi(D, H)))^\tau)$$

holds with positive constants c_4, c_5, c_6 *depending only on* α, $f_1, \ldots, f_m$, K, *and*

$$\psi(D, H) = c_6 \min\left\{(D^m \log H)^{1/(\tau-m)}, \left(\frac{\log H}{D^{m+2}}\right)^{1/(m+2)}\right\}.$$

REMARK. Examples for Liouville numbers which satisfy the assumptions of the theorem can be given in the following way:

Suppose α is a gap series of the form

$$\alpha = \sum_{h \geq 0} a_h g^{-e(h)} \tag{1}$$

or a continued fraction

$$\alpha = [a_0; a_h g^{e(h)}]_{h=1}^{\infty} \tag{2}$$

with $g \in \mathbb{N}$, $g \geqslant 2$, $a_h \in \{1, \ldots, g-1\}$ for $h \in \mathbb{N}$, $a_0 \in \{0, \ldots, g-1\}$ in (1) or $a_0 \in \{-1, 0\}$ in (2) (this implies $\alpha \in]-1, 1[$) and

$$e(h) = \tau^{\tau^h}$$

with $\tau \in \mathbb{N}$, $\tau > 2m + 2$.

In (1) we put $p_l/q_l = \Sigma_{h=0}^{l} a_h g^{-e(h)}$. Then an easy calculation shows

$$0 < \left|\alpha - \frac{p_l}{q_l}\right| < g^{1-e(l+1)} \leqslant \exp(-\gamma_0 \log q_{l+1})$$

and $\log q_l \leqslant \gamma_1 (\log q_{l-1})^{\tau}$, since $g^{e(l)-2} \leqslant q_l \leqslant g^{e(l)}$. Similar estimates hold for the continued fractions in (2), but some inequalities for continued fractions are necessary (see e.g. [Bu]).

The proofs of Theorems 1 and 2 depend on the following proposition, which is a quantitative version of the algebraic independence criterion stated in [T1]. The proof of the proposition is originally an axiomatization of Nesterenko's elimination theoretic method, as it is used in [Be3], [Ni3], or [T1].

PROPOSITION. *Suppose* $\omega \in \mathbb{C}^m$. *Then there exists a constant* $C_1 = C_1(\omega, K) \in \mathbb{R}_+$ *with the following property:*

If there exist functions $\Psi_1, \Psi_2, \Phi_1, \Phi_2, \Lambda, \Theta : \mathbb{N}^2 \to \mathbb{R}_+$, *positive integers* k_1, N_0, N_1 *with* $N_0 < N_1$, *for each* $N \in \{N_0, \ldots, N_1\}$ *a positive integer* $k_0(N) < k_1$, *polynomials* $Q_{k,N} \in O_K[y_1, \ldots, y_m]$ *for* $N \in \{N_0, \ldots, N_1\}$ *and* $k \in \{k_0(N), \ldots, k_1\}$ *such that the following assumptions are satisfied for positive integers D, H and all* $N_0 \leqslant N \leqslant N_1$ *and* $k_0(N) \leqslant k \leqslant k_1$:

(i) $\Psi_1, \Psi_2, \Phi_1, \Phi_2, \Lambda$ *are increasing in each variable and*
  (a) $\Phi_2(k, N) \geqslant \max\{\Phi_1(k, N), \Theta(k, N)\}$,
  (b) $1 < \Psi_1(k+1, N)/\Psi_2(k, N) \leqslant \Lambda(k, N)$,
  (c) $\Psi_1(k_0(N+1), N+1) \leqslant \Psi_1(k_1, N)$,
(ii) $\Phi_1(k, N) \geqslant C_1$, $\Psi_2(k, N) \geqslant C_1 \Phi_2(k, N)$,
(iii) (a) $\deg Q_{k,N} \leqslant \Phi_1(k, N)$,
  (b) $\log H(Q_{k,N}) \leqslant \Theta(k, N)$,
  (c) $\exp(-\Psi_1(k, N)) \leqslant |Q_{k,N}(\omega)| \leqslant \exp(-\Psi_2(k, N))$,
(iv) $\Psi_2(k_1, N_1) \geqslant C_1 \Phi_1(k_1, N_1)^{m-1} \Lambda(k_1, N_1)^{m-1} \Psi_1(k_0(N_0), N_0) D$,
(v) $\Psi_2(k_1, N_1) \geqslant C_1 \Phi_1(k_1, N_1)^{m-1} \Lambda(k_1, N_1)^{m}$
  $\times (\Phi_1(k_1, N_1) \log H + \Phi_2(k_1, N_1) D)$,

then for all polynomials $R \in \mathbb{Z}[y_1, \ldots, y_m] \backslash \{0\}$ *with* $\deg R \leqslant D, H(R) \leqslant H$ *the inequality*

$$|R(\boldsymbol{\omega})| > \exp(-C_2 \Psi_2(k_1, N_1))$$

holds with a constant $C_2 = C_2(\boldsymbol{\omega}, K) \in \mathbb{R}_+$.

REMARKS. (a) The proposition can easily be derived from Jabbouri's criterion [J]. This was pointed out by one of the referees.

(b) In fact, the assumptions of the propositon imply the algebraic independence of $\omega_1, \ldots, \omega_m$. This can be seen by the application of Theorem 1 in [T1]. So it is worthless to try to derive algebraic independence measures, if it is impossible to prove the algebraic independence of $\omega_1, \ldots, \omega_m$ by this method.

(c) To derive algebraic independence measures for $\omega_1, \ldots, \omega_m$, one has to construct auxiliary polynomials with suitable bounds for height, degree, and absolute value at the point $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$, and to choose the parameters k_1, N_1 with respect to D and $\log H$ in such a way, that the assumptions of Theorem 1 are satisfied.

2. Proof of Theorem 1

The first step in the proof is the construction of the auxiliary function with certain bounds. Therefore we need some notations:

$$f_i(z)^j = \sum_{h \geqslant 0} f_{ih}^{(j)} z^h, \quad f_{ih}^{(j)} = \sum_{\substack{l_1, \ldots, l_j \in \mathbb{N}_0 \\ l_1 + \cdots + l_j = h}} f_{il_1} \cdots f_{il_j},$$

$$\mathbf{f}(z)^{\mathbf{j}} = f_1(z)^{j_1} \cdots f_m(z)^{j_m} = \sum_{h \geqslant 0} f_h^{(\mathbf{j})} z^h,$$

$$f_h^{(\mathbf{j})} = \sum_{l_1 + \cdots + l_m = h} f_{1l_1}^{(j_1)} \cdots f_{ml_m}^{(j_m)}.$$

LEMMA 1. *Under the assumptions of Theorem* 1 *the Taylor series coefficients of* $f_i(z)^j$ *and* $\mathbf{f}(z)^{\mathbf{j}}$ *satisfy*

(i) $\overline{|f_{ih}^{(j)}|} \leqslant \exp(c_7 j\,(1 + h^L)), \quad E^{[c_7 j(1+h^L)]} f_{ih}^{(j)} \in O_K,$

(ii) $\overline{|f_h^{(\mathbf{j})}|} \leqslant \exp(c_8 |\mathbf{j}|(1 + h^L)), \quad E^{[c_8 |\mathbf{j}|(1+h^L)]} f_h^{(\mathbf{j})} \in O_K.$

Proof. This follows from the representation of $f_{ih}^{(j)}$ and $f_h^{(\mathbf{j})}$ under the observation that the number of $(l_1, \ldots, l_j) \in \mathbb{N}_0^j$ with $l_1 + \cdots + l_j = h$ is bounded by $(h + 1)^j$. □

LEMMA 2. *For $N \in \mathbb{N}$, $N \geqslant c_9$, and $f_1, \ldots, f_m$ as in Theorem* 1, *there exists a polynomial $R_N \in O_K[z, \mathbf{y}] \setminus \{0\}$ with*

(i) $\deg_{\mathbf{y}} R_N \leqslant N$, $\deg_z R_N \leqslant N$,
(ii) $H(R_N) \leqslant \exp(c_{10} N^{1+(1+m)L})$,
(iii) $\nu_0 = c_{11} N^{m+1} \leqslant \nu = \operatorname{ord}_{z=0} R_N(z, \mathbf{f}(z)) \leqslant c_{12} N^{\mu}$,

where $\mu = 1 + m + m^2 \log t / (\log p - m \log t)$.

Proof. We put

$$R_N(z, \mathbf{y}) = \sum_{\substack{\mathbf{m} \in \mathbb{N}_0^m \\ |\mathbf{m}| \leqslant N}} \sum_{n=0}^{N} r(\mathbf{m}, n) \mathbf{y}^{\mathbf{m}} z^n.$$

With respect to the Taylor series expansion of the functions f_i we can write

$$R_N(z, \mathbf{f}(z)) = \sum_{h \geqslant 0} \left(\sum_{|\mathbf{m}| \leq N} \sum_{n \leqslant \min(h,N)} r(\mathbf{m}, n) f_{h-n}^{(\mathbf{m})} \right) z^h =: \sum_{h \geq 0} \beta_h z^h. \tag{3}$$

The requirement $\beta_h = 0$ for $0 \leqslant h < c_{11} N^{m+1}$ yields a system of $[c_{11} N^{m+1}] + 1$ linear equations in $\binom{N+m}{m}(N+1)$ unknowns $r(\mathbf{m}, n)$. After multiplication with $E^{[c_8 N(1+[c_{11}N^{m+1}]^L)]}$ the coefficients of the system are algebraic integers, whose houses can be bounded by Lemma 1. Then Lemma 2 follows from Siegel's lemma (see e.g. Hilfssatz 31 in [S]) except for the right inequality in (iii), which is a consequence of the zero-order estimate of Nishioka in [Ni3]. □

LEMMA 3. *Suppose that R_N is defined as in Lemma* 2, *β_h as in* (3), *and $f_1, \ldots, f_m$ satisfy the assumptions of Theorem* 1. *Then*

(i) $|\beta_h| \leqslant \exp(c_{13} N(\nu_0^L + h^{\min(1,L)}))$,
(ii) $\overline{|\beta_h|} \leqslant \exp(c_{13} N(\nu_0^L + h^L))$,
(iii) $E^{[c_{13} N(1+h^L)]} \beta_h \in O_K$.

Proof. Since $f_1, \ldots, f_m$ converge in $U_1(0)$, the Taylor series coefficients satisfy

$$|f_{ih}| \leqslant \exp(\gamma_0(1 + h^{\min(1,L)})),$$

and an estimate as in Lemma 1 yields

$$|f_h^{(\mathbf{j})}| \leqslant \exp(\gamma_1 |\mathbf{j}| (1 + h^{\min(1,L)})).$$

Now (i) follows from (3) by standard estimates, and (ii), (iii) follow by Lemma 1. □

LEMMA 4. *Suppose that R_N is defined as in Lemma* 2, $\alpha \in U_1(0)\setminus\{0\}$, *and* $k \in \mathbb{N}$ *satisfies* $p^k \geqslant c_{14} N v^L$. *Then*

$$\exp(-c_{15} v p^k) < |R_N(\alpha^{p^k}, \mathbf{f}(\alpha^{p^k}))| < \exp(-c_{16} v p^k).$$

Proof. According to (3) we have

$$R_N(\alpha^{p^k}, \mathbf{f}(\alpha^{p^k})) = \beta_v \alpha^{v p^k} \left(1 + \sum_{h \geqslant 1} \frac{\beta_{h+v}}{\beta_v} \alpha^{h p^k}\right).$$

Lemma 3 and the fundamental inequality for algebraic numbers imply

$$\left|\frac{\beta_{h+v}}{\beta_v}\right| \leqslant \exp(\gamma_0 N(v^L + h^{\min(1,L)})),$$

and then we get under the assumption on k

$$\left|\sum_{h \geqslant 1} \frac{\beta_{h+v}}{\beta_v} \alpha^{h p^k}\right| \leqslant \exp(\gamma_0 N v^L) \sum_{h \geqslant 1} \exp(\gamma_0 N h^{\min(1,L)} - \gamma_1 h p^k)$$
$$\leq \exp(\gamma_0 N v^L - \gamma_2 p^k) < \tfrac{1}{2}.$$

Now the assertion follows with $|\alpha|^{v p^k} = \exp(-\gamma_1 v p^k)$ and $\exp(-\gamma_3 N v^L) \leqslant |\beta_v| \leqslant \exp(\gamma_4 N v^L)$. □

By iteration of the functional equation we define the polynomials $P_i^{(k)}(z, \mathbf{y}) \in K(z)[\mathbf{y}]$ by

$$f_i(z^{p^k}) = P_i^{(k)}(z, \mathbf{f}(z)) \quad (i = 1, \ldots, m)$$

and put $\mathbf{P}^{(k)} = (P_1^{(k)}, \ldots, P_m^{(k)}) \in (K(z)[\mathbf{y}])^m$. By induction we get

$$a^{(k)}(z) P_i^{(k)}(z, \mathbf{y}) \in K[z, \mathbf{y}] \quad (i = 1, \ldots, m)$$

with

$$a^{(k)}(z) = \prod_{j=0}^{k-1} a(z^{p^j})^{t^{k-1-j}}.$$

LEMMA 5. *For* $i = 1, \ldots, m$ *and* $k \in \mathbb{N}$ *the following assertions hold*:

(i) $\deg_z P_i^{(k)}, \deg_z a^{(k)} \leqslant c_{17} p^k, \deg_{\mathbf{y}} P_i^{(k)} \leqslant t^k$,
(ii) $\log H(P_i^{(k)}), \log H(a^{(k)}) \leqslant c_{18}(t^k + k)$,

(iii) *if* $BP_i \in O_K[z, \mathbf{y}]$ *for* $i = 1, \ldots, m$ *and* $Ba \in O_K[z]$ *with* $B \in \mathbb{N}$, *then*

$$B^{[c_{19}(t^k+k)]}P_i^{(k)} \in O_K[z, \mathbf{y}], \quad B^k a^{(k)} \in O_K[z].$$

Proof. The identity

$$\mathbf{P}^{(k)}(z, \mathbf{y}) = \mathbf{P}^{(k-1)}(z^p, \mathbf{P}^{(1)}(z, \mathbf{y})) = \mathbf{P}^{(1)}(z^{p^{k-1}}, \mathbf{P}^{(k-1)}(z, \mathbf{y}))$$

(notice that $a(z)\mathbf{P}^{(1)}(z, \mathbf{y}) = \mathbf{P}(z, \mathbf{y})$) implies inductively

$$\begin{aligned}
&\deg_{\mathbf{y}} P_i^{(k)} \leqslant \deg_{\mathbf{y}} P_i \max_{j=1,\ldots,m} (\deg_{\mathbf{y}} P_j^{(k-1)}) \leqslant t^k,\\
&\deg_z P_i^{(k)} \leqslant p^{k-1} \deg_z P_i + \deg_{\mathbf{y}} P_i \max_{j=1,\ldots,m} (\deg_z P_j^{(k-1)})\\
&\qquad \leqslant \gamma_0 \sum_{l=0}^{k-1} t^l p^{k-1-l} \leqslant c_{17} p^k,\\
&\deg_z a^{(k)} \leqslant \gamma_1 \sum_{l=0}^{k-1} p^l t^{k-1-l} \leqslant c_{17} p^k,\\
&H(P_i^{(k)}) \leqslant L(P_i^{(k)}) \leqslant L(P_i^{(k-1)}) \max_{j=1,\ldots,m} \{1, L(P_j)\}^{\deg_{\mathbf{y}} P_i^{(k)}}\\
&\qquad\qquad \leqslant \exp(c_{18}(t^k + k)),\\
&H(a^{(k)}) \leqslant L(a^{(k)}) \leqslant L(a)^k \leqslant \exp(c_{18}(t^k + k))
\end{aligned}$$

(notice that $t = 1$ is possible) and (iii) follows similarly. □

Let A denote a positive integer such that $A\alpha$ is an algebraic integer. Now we define polynomials $Q_{k,N}$ for $k, N \in \mathbb{N}$ by

$$Q_{k,N}(\mathbf{y}) = A^{2[c_{17}p^k]N + Np^k} B^{2[c_{19}(t^k+k)]N} a^{(k)}(\alpha)^N R_N(\alpha^{p^k}, \mathbf{P}^{(k)}(\alpha, \mathbf{y})).$$

The polynomials satisfy

$$\begin{aligned}
&Q_{k,N} \in O_{K(\alpha)}[\mathbf{y}],\\
&\deg Q_{k,N} \leqslant Nt^k = \Phi_1(k, N),\\
&\log H(Q_{k,N}) \leqslant c_{20} N p^k = \Theta(k, N) = \Phi_2(k, N),\\
&\exp(-c_{21} \nu p^k) \leqslant |Q_{k,N}(\mathbf{f}(\alpha))| \leqslant \exp(-c_{22} \nu p^k)
\end{aligned}$$

for $N \geqslant N_0(\alpha, f_1, \ldots, f_m, K)$ and $k \in \mathbb{N}$ with $p^k \geqslant c_{23} N \nu^L$, where

$$c_{11} N^{m+1} \leqslant \nu \leqslant c_{12} N^{\mu}, \ \mu = 1 + m + m^2 \log t / (\log p - m \log t).$$

Hence we put $\Psi_1(k, N) = c_{21}\nu p^k$, $\Psi_2(k, N) = c_{22}\nu p^k$, $\Lambda(k, N) = (c_{21}/c_{22})p$ and fix $k_0(N) \in \mathbb{N}$ by $p^{k_0(N)} \geqslant c_{12}^L c_{23} N^{1+L\mu} > p^{k_0(N)-1}$. In the last step of the proof we have to satisfy the inequalities

$$p^{k_1} \geqslant \gamma_0 N_1^{(1+L)\mu - m}, \tag{4}$$

$$N_1^{m+1} p^{k_1} \geqslant \gamma_1 N_1^{m-1} t^{(m-1)k_1} N_0^{\mu} p^{k_0(N_0)} D, \tag{5}$$

$$N_1^{m+1} p^{k_1} \geqslant \gamma_2 N_1^{m-1} t^{(m-1)k_1} (N_1 t^{k_1} \log H + N_1 p^{k_1} D) \tag{6}$$

for sufficiently large constants $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}_+$ and a suitable choice of k_1 and N_1 with respect to D and $\log H$. Since

$$c_{21} p^{k_1} N^{m+1} \geqslant \gamma_3 N_1^{(1+L)\mu - m} N^{m+1} \geqslant \gamma_4 (N+1)^{\mu+1+L\mu} \geqslant c_{21}(N+1)^{\mu} p^{k_0(N+1)},$$

(4) implies (i)(c) of the proposition. Further (5) implies (iv), (6) implies (v). Therefore we define

$$N_1 = [\gamma_5 D t^{(m-1)k_1}] + 1$$

with

$$k_1 = \left[\frac{1}{\mu_1} \log\left(D^{((L+1)\mu - m)\mu_1/\mu_2} + \left(\frac{\log H}{D}\right)^{\mu_1/\mu_3}\right) + \gamma_6\right]$$

and suitable constants $\gamma_5, \gamma_6 \in \mathbb{R}_+$, where $\mu_1 = \log p + (m-1)\mu \log t$, $\mu_2 = \log p - (m-1)((L+1)\mu - m)\log t$, $\mu_3 = \log p - \log t$.

Since $N_0, k_0(N_0)$ depend only on $\alpha, f_1, \ldots, f_m, K$, the choice of N_1 yields

$$N_1^2 \left(\frac{p}{t^{m-1}}\right)^{k_1} \geqslant \gamma_7 D,$$

and this guarantees (5). From the definition of k_1 the inequality

$$k_1 \geqslant \frac{1}{\log p - \log t} \log\left(\frac{\log H}{D}\right) + \gamma_8$$

follows, and this implies

$$p^{k_1}(N_1 t^{-mk_1} - \gamma_9 D t^{-k_1}) \geqslant \gamma_{10} D p^{k_1} t^{-k_1} \geqslant \gamma_{11} \log H,$$

hence (6). Furthermore

$$k_1 \geqslant \frac{1}{\log p - (m-1)((L+1)\mu - m)\log t} \log(D^{(L+1)\mu - m}) + \gamma_{12},$$

and (by replacing N_1 in (4)) this proves (4). Now all assumptions of the proposition are satisfied, and with

$$\begin{aligned}\Psi_2(k_1, N_1) &\leqslant \gamma_{13} N_1^{\mu} p^{k_1} \leqslant \gamma_{14} D^{\mu} (t^{(m-1)\mu} p)^{k_1} \\ &\leqslant \gamma_{15} D^{\mu} \left(D^{((L+1)\mu - m)\mu_1/\mu_2} + \left(\frac{\log H}{D}\right)^{\mu_1/\mu_3} \right)\end{aligned}$$

the assertion follows from the proposition. □

3. Proof of Theorem 2

As shown in [Be3], the Taylor series coefficients of $f_1, \ldots, f_m$ have the properties

$$\overline{|f_{ih}|} \leqslant \exp(c_{24}(1+h)), \quad E^{[c_{24}(1+h)]} f_{ih} \in O_K$$

for $i = 1, \ldots, m$ and $h \in \mathbb{N}_0$ with $E \in \mathbb{N}$ (notice that $L = 1$). In the same way as in Lemma 1 we can show

$$\overline{|f_h^{(\mathbf{j})}|} \leqslant \exp(c_{25}(|\mathbf{j}| + h)), \quad E^{[c_{25}(|\mathbf{j}| + h)]} f_h^{(\mathbf{j})} \in O_K$$

for $\mathbf{j} \in \mathbb{N}_0^n$ and $h \in \mathbb{N}_0$ (since the number of $(l_1, \ldots, l_j) \in \mathbb{N}_0^j$ with $l_1 + \cdots + l_j = h$ is bounded by $\binom{h+j-1}{j-1} \leqslant 2^{h+j}$). Analogous to Lemma 2 we construct a polynomial $R_N \in O_K[z, \mathbf{y}]$ for $N \geqslant c_{26}$ with

$$\begin{aligned}&\deg_{\mathbf{y}} R_N \leqslant N, \quad \deg_z R_N \leqslant N, \\ &H(R_N) \leqslant \exp(c_{27} N^{m+1}), \\ &\nu_0 = c_{11} N^{m+1} \leqslant \nu = \operatorname{ord}_{z=0} R_N(z, \mathbf{f}(z)) \leqslant c_{12} N^{m+1},\end{aligned}$$

and for $k \in \mathbb{N}$ with $p^k \geqslant c_{28} N^{m+1}$ the inequalities

$$\exp(-c_{29} \nu p^k) \leqslant |R_N(\alpha^{p^k}, \mathbf{f}(\alpha^{p^k}))| \leqslant \exp(-c_{30} \nu p^k)$$

hold (with a proof analogous to Lemmas 3 and 4). The iteration of the functional equation yields

$$a^{(k)}(z)\mathbf{f}(z^{p^k}) = A^{(k)}(z)\mathbf{f}(z) + \mathbf{B}^{(k)}(z)$$

with

$$a^{(k)}(z) = \prod_{i=0}^{k-1} a(z^{p^i}), \qquad A^{(k)}(z) = \prod_{i=0}^{k-1} A(z^{p^i}),$$
$$\mathbf{B}^{(k)}(z) = \sum_{i=0}^{k-1} a^{(i)}(z)\left(\prod_{j=i+1}^{k-1} A(z^{p^j})\right)\mathbf{B}(z^{p^i}),$$

and similar to the construction in the proof of Theorem 1 via Lemma 5 we define polynomials

$$R_{k,N}(z, \mathbf{y}) = B^{Nk}a^{(k)}(z)^N R_N(z^{p^k}, (A^{(k)}(z)\mathbf{y} + \mathbf{B}^{(k)}(z))/a^{(k)}(z)),$$

where $B \in \mathbb{N}$ is a common denominator of the coefficients of $a(z)$ and of the entries of $A(z)$ and $\mathbf{B}(z)$, i.e., after multiplication with B the coefficients of $a(z)$ and of the entries of $A(z)$, $\mathbf{B}(z)$ are algebraic integers. Since $\exp(-\gamma_0 p^k) \leqslant |a^{(k)}(\alpha)| \leqslant \exp(\gamma_1 k)$, we have the following inequalities for $R_{k,N} \in O_K[z, \mathbf{y}]$:

$$\deg_{\mathbf{y}} R_{k,N} \leqslant N, \quad \deg_z R_{k,N} \leqslant c_{31}Np^k,$$
$$H(R_{k,N}) \leqslant \exp(c_{32}(N^{m+1} + Nk)),$$
$$\exp(-c_{33}\nu p^k) \leqslant |R_{k,N}(\alpha, \mathbf{f}(\alpha))| \leqslant \exp(-c_{34}\nu p^k).$$

LEMMA 6. *Suppose that* $p^k \geqslant c_{35}N^{m+1}$ *and* $\xi \in \mathbb{C}$ *with*

$$|\alpha - \xi| \leqslant \tfrac{1}{2}\exp(-c_{36}Np^k - c_{34}\nu p^k).$$

Then

$$\exp(-c_{37}\nu p^k) \leqslant |R_{k,N}(\xi, \mathbf{f}(\alpha))| \leqslant \exp(-c_{38}\nu p^k).$$

Proof. Application of the mean value theorem yields

$$\begin{aligned}|R_{k,N}(\alpha, \mathbf{f}(\alpha)) - R_{k,N}(\xi, \mathbf{f}(\alpha))| &\leqslant \deg_z R_{k,N} H(R_{k,N})(1+\eta)^{\deg_z R_{k,N}}\left(1+\sum_{i=1}^{m}|f_i(\alpha)|\right)^{\deg_{\mathbf{y}} R_{k,N}}|\alpha-\xi| \\ &\leqslant \exp(c_{36}Np^k)|\alpha-\xi|\end{aligned}$$

with some $\eta \in U_1(0)$. Now we have

$$\begin{aligned}|R_{k,N}(\xi, \mathbf{f}(\alpha))| &\geqslant |R_{k,N}(\alpha, \mathbf{f}(\alpha))| - |R_{k,N}(\alpha, \mathbf{f}(\alpha)) - R_{k,N}(\xi, \mathbf{f}(\alpha))| \\ &\geqslant \tfrac{1}{2}\exp(-c_{34}\nu p^k),\end{aligned}$$

and this yields the left-hand side of the asserted inequality. The right-hand side follows analogous. □

Now we take for ξ one of the approximating fractions p_l/q_l and put $q = \max\{|p_l|, |q_l|\}$; since $\alpha \in]-1, 1[$, we may suppose $|q_l| = q$ for sufficiently large l. Then we define polynomials $Q_{k,N} \in O_K[\mathbf{y}]$ for $k, N \in \mathbb{N}$ and $N \geqslant c_{26}$ by

$$Q_{k,N}(\mathbf{y}) = q^{[c_{31}Np^k]}R_{k,N}(p_l/q_l, \mathbf{y}),$$

and for

$$p^k \geqslant c_{35}N^{m+1}, \tag{7}$$

$$|\alpha - p_l/q_l| \leqslant \exp(-c_{39}\nu p^k), \tag{8}$$

$$N^m \geqslant c_{40}\log q \tag{9}$$

they have the properties

$$\deg Q_{k,N} \leqslant N =: \Phi_1(k, N),$$

$$\log H(Q_{k,N}) \leqslant c_{41}Np^k \log q =: \Theta(k, N) = \Phi_2(k, N),$$

$$\exp(-c_{42}\nu p^k) \leqslant |Q_{k,N}(\mathbf{f}(\alpha))| \leqslant \exp(-c_{43}\nu p^k),$$

where $c_{11}N^{m+1} \leqslant \nu \leqslant c_{12}N^{m+1}$; hence put $\Psi_1(k, N) = c_{42}\nu p^k$, $\Psi_2(k, N) = c_{43}\nu p^k$, $\Lambda(k, N) = (c_{42}/c_{43})p$. To fulfill the assumptions of the proposition we choose p_l/q_l in such a way that

$$\log|q_l| \geqslant \gamma_1(D^{(2m+2)/(\tau-(2m+2))} + \psi(D, H)) > \log|q_{l-1}|$$

with $\psi(D, H) = \gamma_2 \min\{(D^m \log H)^{1/(\tau - m)}, (\log H/D^{m+2})^{1/(m+2)}\}$, and fix N_0, $k_0(N_0)$, k_1, N_1 in the following way:

$$N_0 = c_{39}(\log q)^{1/m},$$
$$p^{k_0(N)} \geqslant c_{35}N^{m+1} > p^{k_0(N)-1},$$
$$N_1 = [2\gamma_3 D \log q] + 1,$$
$$k_1 = \left[\frac{1}{\log p} \log\left((D \log q)^{m+1} + \frac{\log H}{D \log q}\right) + \gamma_4\right]$$

with suitable constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}_+$.

The defintion of N_0 ensures the validity of (9), the choice of $k_0(N)$ guarantees (7), and (8) follows by some calculation: for $\gamma_5 \leqslant \gamma_1/2$ we have

$$(\log q)^{\tau-(2m+2)} - \gamma_5 D^{2m+2} \geqslant \gamma_6 \max\{(\log q)^{\tau-(2m+2)}, D^{2m+2}\},$$

and this implies for $\psi(D, H) = \gamma_2(D^m \log H)^{1/(\tau-m)}$

$$\begin{aligned}(\log q)^{2m+2}((\log q)^{\tau-(2m+2)} - \gamma_5 D^{2m+2}) &\geqslant \gamma_6(\log q)^{\tau}\\ &\geqslant \gamma_7(\log q)^m \psi(D, H)^{\tau-m}\\ &\geqslant \gamma_8(D \log q)^m \log H;\end{aligned}$$

similar in the case $\psi(D, H) = \gamma_2(\log H)^{1/(m+2)}/D$:

$$\begin{aligned}(\log q)^{2m+2}((\log q)^{\tau-(2m+2)} - \gamma_5 D^{2m+2}) &\geqslant \gamma_6(D \log q)^{2m+2}\\ &\geqslant \gamma_9(D \log q)^m \psi(D, H)^{m+2} D^{m+2}\\ &\geqslant \gamma_{10}(D \log q)^m \log H.\end{aligned}$$

Hence

$$(\log q)^{\tau} \geqslant \gamma_{11}(D \log q)^{m+1}\left((D \log q)^{m+1} + \frac{\log H}{D \log q}\right),$$

and, since we have chosen the constants in the right way,

$$\begin{aligned}\left|\alpha - \frac{p_l}{q_l}\right| &\leqslant \exp(-c_3(\log q)^{\tau})\\ &\leqslant \exp\left(-\gamma_{12}(D \log q)^{m+1}\left((D \log q)^{m+1} + \frac{\log H}{D \log q}\right)\right)\\ &\leqslant \exp(-\gamma_{13}N_1^{m+1}p^{k_1}),\end{aligned}$$

which implies (8). Now we have to satisfy the assumptions (i)(c), (iv), (v) of the proposition; the others are obviously fulfilled. Since $k_1 > k_0(N_0)$ and

$$\begin{aligned} N_1^2 p^{k_1} &\geqslant \gamma_{14} D^2 (\log q)^2 p^{k_0(N_0)} \\ &\geqslant \gamma_{15} D N_0^{m+1} p^{k_0(N_0)}, \end{aligned}$$

we have

$$N_1^{m+1} p^{k_1} \geqslant \gamma_{16} N_1^{m-1} D N_0^{m+1} p^{k_0(N_0)},$$

whence (iv) is satisfied. Furthermore

$$p^{k_1}(N_1 - \gamma_3 D \log q) \geqslant \gamma_3 p^{k_1} D \log q \geqslant \gamma_3 \log H,$$

and this implies

$$N_1^{m+1} p^{k_1} \geqslant \gamma_3 N_1^m (\log H + p^{k_1} D \log q),$$

hence (v). The last assumption (i)(c) is satisfied because

$$p^{k_1} \geqslant \gamma_{17} (D \log q)^{m+1} \geqslant \gamma_{18} N_1^{m+1},$$

which implies (by the definition of $k_0(N+1)$)

$$c_{42} N^{m+1} p^{k_1} \geqslant c_{42} (N+1)^{m+1} p^{k_0(N+1)}.$$

Finally the assumptions of the proposition are fulfilled, and we deduce

$$|R(\mathbf{f}(\alpha))| > \exp(-\gamma_{19} \Psi_2(k_1, N_1))$$

with

$$\begin{aligned} \Psi_2(k_1, N_1) &\leqslant \gamma_{20} v p^{k_1} \\ &\leqslant \gamma_{21} (\log q)^\tau \\ &\leqslant \gamma_{21} \varphi (\log |q_{l-1}|)^\tau \\ &\leqslant \gamma_{21} \varphi (\gamma_{22} (D^{(2m+2)/(\tau-(2m+2))} + \psi(D, H)))^\tau. \end{aligned}$$

□

4. Proof of the proposition

For notations, assumptions and the statement of Jabbouri's criterion, on which the proof depends, the reader is referred to [J]. With $n = m$, $k = m - 1$ and

$$C_0 = 2m \log(m + 1) + \log(1 + \|\omega\|_\infty)$$

we put

$$\sigma = 2\max\{C_0;\ \Lambda(k_1, N_1)\} \geqslant 1,$$
$$\delta = \Phi_1(k_1, N_1),$$
$$\tau = \max\left\{C_0\Phi_2(k_1, N_1);\ \frac{2\Psi_1(k_0(N_0), N_0)}{\sigma}\right\},$$
$$U = \frac{\Psi_2(k_1, N_1)\Lambda(k_1, N_1)}{\sigma}.$$

Thus $1 \leqslant \delta, \sigma, \tau, U$ and $U \geqslant 2\max\{\tau, \sigma^m\}$. If we now define the constant $C_1 = C_1(\omega, K)$ by

$$C_1 = 10 \cdot 54^m C_0^{m+2}(4[K:\mathbb{Q}] + m + 1)$$

and assume that the conditions of the proposition are satisfied with this value of C_1, then

$$\begin{aligned} U &\geqslant \Psi_2(k_1, N_1)/2C_0 \\ &\geqslant (4[K:\mathbb{Q}] + m + 1)(27\sigma)^m(\log H\delta^m + (\tau + \delta\log(m + 1))D\delta^{m-1}) \end{aligned}$$

holds, and assumption (f) of Jabbouri's criterion is fulfilled (notice that the heights in [J] are logarithmic).

Now suppose that $S \in \mathbb{N}$ satisfies

$$\frac{\tau}{\sigma^m} < S \leqslant \frac{U}{\sigma^m}.$$

Then we choose a pair (k, N) with $N_0 \leqslant N \leqslant N_1$ and $k_0(N) \leqslant k \leqslant k_1$ such that

$$\frac{\Psi_1(k, N)}{\sigma^{m+1}} < S - \frac{C_0\tau}{\sigma^{m+1}} \leqslant \frac{\Psi_2(k, N)\Lambda(k, N)}{\sigma^{m+1}}. \tag{10}$$

This is possible, since (i)(c) of the proposition together with

$$S\sigma^{m+1} - C_0\tau \geqslant \sigma\tau - C_0\tau \geqslant \Psi_1(k_0(N_0), N_0),$$
$$S\sigma^{m+1} - C_0\tau \leqslant \sigma U - C_0\tau \leqslant \sigma U = \Psi_2(k_1, N_1)\Lambda(k_1, N_1)$$

holds. For this value of S we fix the polynomial $Q_{k,N}$ with k, N chosen above. Obviously the assumptions (a), (b), (c) of Jabbouri's criterion are fulfilled. To verify assumption (e) for the polynomial $Q_{k,N}$, we apply the mean-value theorem to $Q_{k,N}$ with an arbitrary zero ζ of $Q_{k,N}$ and get for some $\xi \in \mathbb{C}^m$ lying between ω and ζ

$$\begin{aligned}\exp(-\Psi_1(k, N)) &\leqslant |Q_{k,N}(\omega)| = |Q_{k,N}(\omega) - Q_{k,N}(\zeta)| \\ &\leqslant \|\operatorname{grad} Q_{k,N}(\xi)\|_1 \|\omega - \zeta\|_\infty \\ &\leqslant \deg Q_{k,N} H(Q_{k,N})((m+1)(1 + \|\omega\|_\infty))^{\deg Q_{k,N}} \|\omega - \zeta\|_\infty \\ &\leqslant \exp(C_0\Phi_2(k, N))\|\omega - \zeta\|_\infty \\ &\leqslant \exp(C_0\tau)\|\omega - \zeta\|_\infty\end{aligned}$$

with C_0 as above, since without loss of generality $\|\xi\|_\infty \leqslant 1 + \|\omega\|_\infty$. Thus by (10) no zero of $Q_{k,N}$ lies in the open ball of radius $\exp(-S\sigma^{m+1}) \leqslant \exp(-\Psi_1(k, N) - C_0\tau)$ around ω, and this implies (e). Assumption (d) follows from

$$|Q_{k,N}(\omega)| \max\{1, \|\omega\|_\infty\}^{-\deg Q_{k,N}} \leqslant \exp(-\Psi_2(k, N))$$

and

$$\frac{\Psi_2(k, N)}{2} \geqslant \frac{\Psi_2(k, N)\Lambda(k, N)}{\sigma} \geqslant S\sigma^m - \frac{C_0\tau}{\sigma} \geqslant \frac{1}{2} S\sigma^m.$$

Hence the assumptions of the proposition imply that all conditions of Jabbouri's criterion are fulfilled, and we get for all polynomials $R \in \mathbb{Z}[y_1, \ldots, y_m] \backslash \{0\}$ with $\deg R \leqslant D$ and $H(R) \leqslant H$ the estimate

$$\log|R(\omega)| \geqslant -U > -\frac{\Psi_2(k_1, N_1)}{2C_0}. \qquad \square$$

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