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The Picard group and subintegrality in positive characteristic

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Introduction

We prove some analogues in positive characteristic of the following two recent results, which relate the Picard group, invertible modules and subintegrality/seminormalization:

(0.1) ([2, Theorem 3.6]). Let A be a G-algebra, i.e. a graded commutative ring $A = \oplus_{d \geq 0} A_d$, with A_0 a field and A finitely generated as an A_0-algebra. Assume that $\mathrm{char}(A_0) = 0$ and that A is reduced. Then there exists a functorial isomorphism Θ_A: $\mathrm{Pic}(A) \to {}^+A/A$ of groups, where ${}^+A$ denotes the seminormalization of A.

(0.2) ([7, Main Theorem (5.6)] together with [6, Theorem (2.3)]). Let $A \subseteq B$ be an extension of rings containing a field of characteristic zero. Assume that this extension is subintegral (in the sense of Swan [8]). Then there exists a natural isomorphism $\xi_{B/A}: B/A \to \mathscr{I}(A, B)$ of groups, where $\mathscr{I}(A, B)$ is the group of invertible A-submodules of B.

In (0.2) if we let A be a reduced G-algebra with $B = {}^+A$ then $\mathscr{I}(A, B) = \mathrm{Pic}(A)$ [7, (2.5)]. So, writing $\xi_A = \xi_{B/A}$, (0.2) gives an isomorphism ξ_A: ${}^+A/A \to \mathrm{Pic}(A)$. This isomorphism differs from Θ_A^{-1} of (0.1) by the group automorphism of ${}^+A/A$ induced by the negative Euler derivation of ${}^+A$, i.e. by the map which is multiplication by $-d$ on the homogeneous component of degree d [7, §7].

As they stand, both (0.1) and (0.2) are false in characteristic $p > 0$. For, while the group B/A is always killed by p, the groups $\mathrm{Pic}(A)$ and $\mathscr{I}(A, B)$ are not killed by p in general. As a specific example (cf. also [7, §7] and [3]) one can take $A = k + t^{p+1}k[t] \subseteq B = k[t]$, where k is a field of characteristic $p > 0$ and t is an indeterminate. Then $\mathscr{I}(A, B) = \mathrm{Pic}(A)$ [7, (2.5)], and this group is not killed by p. For a proof see Example (3.6).

Our main results are, however, of a positive nature. What we show is that if A and B are graded (or if we are in a somewhat more general situation) then there are natural filtrations on $\mathrm{Pic}(A)$, $\mathscr{I}(A, B)$ and B/A

such that corresponding to (0.1) and (0.2) there exist functorial isomorphisms between the associated grades $\operatorname{gr}\operatorname{Pic}(A)$, $\operatorname{gr}\mathscr{I}(A, B)$ and $\operatorname{gr}(B/A)$. Moreover, if B is graded then $\operatorname{gr}(B/A) = B/A$. In characteristic zero these filtration degenerate, the associated grades coincide with the original groups and the isomorphisms are the same as those given by (0.1) and (0.2).

We state our results more precisely now in the graded case. (For more general statements, including the local case, see Section 4.) Let $A = \oplus_{d \geqslant 0} A_d$ be a positively graded ring with A_0 a field. We do not assume that A is finitely generated as an A_0-algebra. Put

$$p = \begin{cases} \infty, & \text{if } \operatorname{char}(A_0) = 0 \\ \operatorname{char}(A_0), & \text{if } \operatorname{char}(A_0) > 0. \end{cases}$$

Let $B = \oplus_{d \geqslant 0} B_d$ be a positively graded ring such that A is a graded subring of B. Put $F_0 B = B$ and $F_i B = A + \Sigma_{d \geqslant p^i} B_d$ for $i \geqslant 1$. Then $F = (F_i B)_{i \geqslant 0}$ is a decreasing filtration on B consisting of A-subalgebras of B. Writing $F_i \mathscr{I}(A, B) = \mathscr{I}(A, F_i B)$, we get a decreasing filtration $(F_i \mathscr{I}(A, B))_{i \geqslant 0}$ of subgroups on $\mathscr{I}(A, B)$ with associated graded

$$\operatorname{gr}\mathscr{I}(A, B) = \bigoplus_{i \geqslant 0} F_i \mathscr{I}(A, B)/F_{i+1}\mathscr{I}(A, B).$$

Note that if $\operatorname{char}(A_0) = 0$ then $F_i B = A$ for $i \geqq 1$ whence $\operatorname{gr}\mathscr{I}(A, B) = \mathscr{I}(A, B)$.

As a special case, suppose A is reduced and has only finitely many minimal primes, and let $B = {}^+A$. By [4] ${}^+A$ is positively graded and contains A as a graded subring. Writing

$$F_i \operatorname{Pic}(A) = \ker(\operatorname{Pic}(A) \to \operatorname{Pic}(F_i\, {}^+A)),$$

we get a decreasing filtration $(F_i \operatorname{Pic}(A))_{i \geqslant 0}$ of subgroups on $\operatorname{Pic}(A)$ with associated graded

$$\operatorname{gr}\operatorname{Pic}(A) = \bigoplus_{i \geqslant 0} F_i \operatorname{Pic}(A)/F_{i+1} \operatorname{Pic}(A).$$

Note that if $\operatorname{char}(A_0) = 0$ then $\operatorname{gr}\operatorname{Pic}(A) = F_0 \operatorname{Pic}(A) = \ker(\operatorname{Pic}(A) \to \operatorname{Pic}({}^+A))$.

As an analogue of (0.1) we prove

(4.5) THEOREM. *Let A be a reduced positively graded ring with A_0 a field. Assume that A has only finitely many minimal primes. Then there exists a natural isomorphism $\xi_A \colon {}^+A/A \to \operatorname{grPic}(A)$ of groups, where ${}^+A$ is the seminormalization of A. If* $\operatorname{char}(A_0) = 0$ *then* $\operatorname{grPic}(A) = \operatorname{Pic}(A) =$

$\mathscr{I}(A, {}^+A)$ and ξ_A coincides with the isomorphism $\xi_{{}^+A/A}$ given by (0.2) *and differs from the isomorphism Θ_A^{-1} of* (0.1) *by the group automorphism of ${}^+A/A$ induced by the negative Euler derivation of ${}^+A$.*

More generally, as an analogue of (0.2) in the graded case we prove

(4.4) THEOREM. *Let $A \subseteq B$ be a subintegral extension of positively graded rings with A_0 a field and $A_0 = B_0$. Then there exists a natural isomorphism $\xi_{B/A}: B/A \to \mathrm{gr}\mathscr{I}(A, B)$ of groups. If* $\mathrm{char}(A_0) = 0$ *then* $\mathrm{gr}\mathscr{I}(A, B) = \mathscr{I}(A, B)$ *and $\xi_{B/A}$ coincides with the isomorphism given by* (0.2).

We prove (4.4) and (4.5) in Section 4 as immediate corollaries to the more general (and more technical) result (4.1). In order to prove (4.1) we first prove in Theorem (2.6) a special case of (4.1). Then, to deduce the general case from this special case, we need the following result which we prove in Section 3:

(3.3) THEOREM. *Let $A \subseteq B$ be a subintegral extension. Then for all rings C with $A \subseteq C \subseteq B$ the sequence $1 \to \mathscr{I}(A, C) \to \mathscr{I}(A, B) \to \mathscr{I}(C, B) \to 1$ of natural maps is exact.*

1. Notation

All rings are assumed to be commutative with 1 and all homomorphisms unitary.

$\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the sets of integers, natural numbers and rational numbers, respectively.

For a ring A, A^* denotes the group of units of A. For an extension $A \subseteq B$ of rings, $\mathscr{I}(A, B)$ denotes the group of all invertible A-submodules of B. For properties of this group see [7, §2].

For the notion and properties of subintegrality see [8].

2. A special case

The results of this section may be compared with those of [3].

By an *admissible extension* we mean a triple $(A \subseteq B, \mathfrak{b}, n)$, where $A \subseteq B$ is an extension of rings, $\mathfrak{b}$ is an ideal of B and $n \in \mathbb{N} \cup \{\infty\}$ such that $B = A + \mathfrak{b}$, $\mathfrak{b}^n \subseteq A$ and every positive integer less than n is a unit in A. Here we make the convention that $\mathfrak{b}^\infty = 0$ for every ideal (resp. element) $\mathfrak{b}$ of a ring B.

Note that giving an admissible extension $(A \subseteq B, B, \infty)$ is equivalent to giving the extension $A \subseteq B$ of $\mathbb{Q}$-algebras. Apart from this, an elementary example of an admissible extension is $(A \subseteq B, tB, p)$, where $B = k[t]$ with k a field of characteristic $p > 0$ and t an indeterminate, and $A = k + t^pB$.

Let $E = (A \subseteq B, \mathfrak{b}, n)$ be an admissible extension. Let x be an element of an overring R of B. In case $n = \infty$, we assume that R is x-adically complete. Put

$$e_n(x) = \sum_{i=0}^{n-1} x^i/i! \quad \text{and} \quad \log_n(1 + x) = \sum_{i=1}^{n-1} (-1)^{i+1}x^i/i.$$

Note that $e_\infty(x) = e^x$ and $\log_\infty(1 + x) = \log(1 + x)$.

Let T be an indeterminate. For $b \in \mathfrak{b}$ define $I_E(b)$ to be the $A[T]$-submodule of $A[b][T]$ given by

$$I_E(b) = A[[T]](e_n(bT), b^nT^n) \cap A[b][T].$$

Note that if $n = \infty$ then $I_E(b) = A[[T]]e^{bT} \cap A[b][T]$ which is the module $I(b)$ defined in [7] in the case of an extension $A \subseteq B$ of $\mathbb{Q}$-algebras. It was shown in [7] that if the extension $A \subseteq B$ of $\mathbb{Q}$-algebras is subintegral then $I(b) \in \mathscr{I}(A[T], B[T])$, and that the map $B \to \mathscr{I}(A[T], B[T])$ given by $b \mapsto I(b)$ is a homomorphism of groups with A contained in its kernel. This map was used to construct the isomorphism $\xi_{B/A}: B/A \to \mathscr{I}(A, B)$ of (0.2). The remainder of this section is devoted to proving the corresponding results for admissible extensions with $n < \infty$. Note that if $n < \infty$ then the extension $A \subseteq B$ is always subintegral in view of the assumptions $B = A + \mathfrak{b}$ and $\mathfrak{b}^n \subseteq A$.

Let $E = (A \subseteq B, \mathfrak{b}, n)$ be an admissible extension. For $b \in \mathfrak{b}$ let $J_E(b)$ denote the A-submodule of B generated by $e_n(b)$ and b^n. Let $\mathscr{C}$ be the conductor of the extension $A \subseteq B$. Note that $\mathfrak{b}^n \subseteq \mathscr{C}$.

(2.1) LEMMA. *Assume that $n < \infty$. Then:*

(1) $J_E(b) = Ae_n(b) + \mathfrak{b}^n = Ae_n(b) + \mathscr{C}$ *for every* $b \in \mathfrak{b}$.

(2) $J_E(b) \in \mathscr{I}(A, B)$ *for every* $b \in \mathfrak{b}$, *and the map* $J_E: \mathfrak{b} \to \mathscr{I}(A, B)$ *is a homomorphism of groups with* $A \cap \mathfrak{b} \subseteq \ker(J_E)$. *Moreover, for a fixed* n, J_E *is functorial in* E.

Proof. (1) Put $J = J_E(b)$. Since $J \subseteq Ae_n(b) + \mathfrak{b}^n \subseteq Ae_n(b) + \mathscr{C}$, it is enough to prove that $\mathscr{C} \subseteq J$. If $m \geqq n$ then $b^m\mathscr{C} = b^{m-n}\mathscr{C}b^n \subseteq J$. We show now by descending induction on m that $b^m\mathscr{C} \subseteq J$ for all $m \geqq 0$. Let $m \geqslant 0$ and let $y \in b^m\mathscr{C}$. Then $ye_n(b) = y + z$ with $z \in \Sigma_{i \geqslant 1} yb^iA \subseteq \Sigma_{i \geqslant 1} b^{m+i}\mathscr{C}$ which is contained in J by induction. Therefore, since $ye_n(b) \in J$ we get $y \in J$, proving that $b^m\mathscr{C} \subseteq J$ for all $m \geqslant 0$. In particular, taking $m = 0$, we get $\mathscr{C} \subseteq J$.

(2) We have $e_n(b)e_n(c) \equiv e_n(b+c) \pmod{\mathfrak{b}^n}$ for all $b, c \in \mathfrak{b}$. Therefore, since $\mathfrak{b}^n \subseteq J_E(b+c)$ by (1), we get $J_E(b)J_E(c) \subseteq J_E(b+c)$. In particular, $J_E(b)J_E(-b) \subseteq J_E(0) = A$. We claim that $J_E(b)J_E(-b) = A$. For this, upon writing $K = J_E(b)J_E(-b)$, it is enough to show that $1 \in K$. Since $1 = e_n(0) \equiv e_n(b)e_n(-b) \pmod{\mathfrak{b}^n}$, there exists $y \in \mathfrak{b}^n$ such that $1 + y \in K$. So $\mathfrak{b}^n(1 + y) \subseteq K$. Further, since by (1) $\mathfrak{b}^n \subseteq J_E(x)$ for every $x \in \mathfrak{b}$, we have $\mathfrak{b}^n y \subseteq \mathfrak{b}^{2n} \subseteq K$. Therefore $\mathfrak{b}^n \subseteq K$. Now, since $1 + y \in K$, we get $1 \in K$, and our claim is proved. Thus $J_E(b) \in \mathscr{I}(A, B)$ with $J_E(b)^{-1} = J_E(-b)$. Now, as seen above we have $J_E(b)J_E(c) \subseteq J_E(b+c)$ for all $b, c \in \mathfrak{b}$. Therefore

$$(J_E(b)J_E(c))^{-1} = J_E(-b)J_E(-c) \subseteq J_E(-b-c) = J_E(b+c)^{-1}$$

whence $J_E(b)J_E(c) \supseteq J_E(b+c)$. Thus $J_E(b)J_E(c) = J_E(b+c)$, proving that J_E is a homomorphism of groups. If $a \in A \cap \mathfrak{b}$ then $J_E(-a) \subseteq A$ whence $A = J_E(-a)J_E(a) \subseteq J_E(a) \subseteq A$, showing that $J_E(a) = A$, i.e. $a \in \ker(J_E)$. The last assertion is clear. □

Put $E[T] = (A[T] \subseteq B[T], \mathfrak{b}[T], n)$. Then $E[T]$ is again an admissible extension. So if $n < \infty$ then by the above lemma we have the homomorphism

$$J_{E[T]}\colon \mathfrak{b}[T] \to \mathscr{I}(A[T], B[T]) \quad \text{with} \quad A[T] \cap \mathfrak{b}[T] \subseteq \ker(J_{E[T]}).$$

(2.2) PROPOSITION. *Suppose $n < \infty$. Then $I_E(b) = J_{E[T]}(bT)$ for every $b \in \mathfrak{b}$.*

Proof. Let $b \in \mathfrak{b}$. Since $\mathscr{C}$ is contained in the conductor of the extension $A[T] \subseteq B[T]$, we have $\mathscr{C} \subseteq J_{E[T]}(bT)$ by (2.1) (1). Now, let $h \in I_E(b)$. Then $h = fe_n(bT) + gb^nT^n$ with $f, g \in A[[T]]$ and $h \in A[b][T]$. Write

$$f = \sum_{i \geqslant 0} f_i T^i, \quad g = \sum_{i \geqslant 0} g_i T^i \quad \text{and} \quad h = \sum_{i \geqslant 0} h_i T^i$$

with f_i, $g_i \in A$, $h_i \in A[b]$ and $h_i = 0$ for all $i > m$ for some nonnegative integer m. We claim that $f_j \in \mathscr{C}$ for all $j \geqslant m + n$. To see this, let r be an integer with $0 \leqslant r \leqslant n - 1$ and let $j \geqslant m + n - r$. Letting $f_i = g_i = 0$ for $i < 0$, we have $0 = h_j = g_{j-n}b^n + \Sigma_{k=0}^{n-1} f_{j-k}b^k/k!$ whence

$$0 \equiv f_j b^r \left(\mathrm{mod}\ Ag_{j-n}b^{r+n} + \sum_{k=1}^{n-1} Af_{j-k}b^{r+k} \right).$$

Therefore, since $b^s \in \mathscr{C}$ for all $s \geqslant n$, it follows by descending induction on r that $f_j b^r \in \mathscr{C}$ for all (j, r) with $0 \leqslant r \leqslant n - 1$ and $j + r \geqslant m + n$. Our claim is proved by taking $r = 0$ in this assertion. Now, let

$$f' = \sum_{i=0}^{m+n-1} f_i T^i \quad \text{and} \quad f'' = f - f'.$$

Then $f'' \in \mathscr{C}[[T]]$ whence

$$h - f'e_n(bT) = f''e_n(bT) + gb^nT^n \in A[b][T] \cap \mathscr{C}[[T]]$$
$$= \mathscr{C}[T] \subseteq J_{E[T]}(bT).$$

It follows that $h \in J_{E[T]}(bT)$. This proves that $I_E(b) \subseteq J_{E[T]}(bT)$. The other inclusion being trivial, the proposition is proved. □

(2.3) COROLLARY. *Let* $E = (A \subseteq B, \mathfrak{b}, n)$ *be an admissible extension. In case* $n = \infty$, *assume that the extension* $A \subseteq B$ *is subintegral. Then* $I_E(b) \in \mathscr{I}(A[T], B[T])$ *for every* $b \in \mathfrak{b}$, *and the map* $I_E: \mathfrak{b} \to \mathscr{I}(A[T], B[T])$ *is a homomorphism of groups with* $A \cap \mathfrak{b} \subseteq \ker(I_E)$.

Proof. For $n < \infty$ the assertion is immediate by the above proposition and the remark preceding it. For the case $n = \infty$ we have only to observe, as noted above, that I_E is just the restriction to $\mathfrak{b}$ of the homomorphism $I_{B/A}: B \to \mathscr{I}(A[T], B[T])$ of [7, (5.1)]. □

Now, as in [7, §5], let $\sigma: B[T] \to B$ be the B-algebra homomorphism given by $\sigma(T) = 1$. Then we have $\mathscr{I}(\sigma): \mathscr{I}(A[T], B[T]) \to \mathscr{I}(A, B)$. Let $\eta_E = \mathscr{I}(\sigma) \circ I_E: \mathfrak{b} \to \mathscr{I}(A, B)$. Then η_E is a homomorphism of groups with $A \cap \mathfrak{b} \subseteq \ker(\eta_E)$. If $n < \infty$ then for $b \in \mathfrak{b}$ we have by (2.2) $\eta_E(b) = \mathscr{I}(\sigma)(J_{E[T]}(bT)) = J_E(b)$. Thus

$$\eta_E = J_E \quad \text{in case } n < \infty. \tag{2.4}$$

Now, let $\xi_E: \mathfrak{b}/A \cap \mathfrak{b} \to \mathscr{I}(A, B)$ be the homomorphism induced by η_E. Since $B = A + \mathfrak{b}$, we have $\mathfrak{b}/A \cap \mathfrak{b} = B/A$ as A-modules. Thus we have the group homomorphism $\xi_E: B/A \to \mathscr{I}(A, B)$.

(2.5) REMARKS. (1) In the case $n < \infty$ we could have defined ξ_E via J_E (in view of (2.4)) without introducing the indeterminate T and specializing it to 1. Our reason for doing it via T is to make the construction compatible with the corresponding one in [7] in characteristic zero. Indeed we note that if $n = \infty$ then $\xi_E = \xi_{B/A}$ in the notation of [7, §5], i.e. ξ_E coincides with the map $\xi_{B/A}$ of (0.2). Also note that if $E = (A \subseteq B, \mathfrak{b}, \infty)$ is an admissible extension then so is $E' = (A \subseteq B, B, \infty)$ and in view of the equality $\mathfrak{b}/A \cap \mathfrak{b} = B/A$ the maps ξ_E and $\xi_{E'}$ coincide. Thus in the case $n = \infty$ the ideal $\mathfrak{b}$ is of no significance and the admissible extension

$(A \subseteq B, \mathfrak{b}, \infty)$ may be identified with the extension $A \subseteq B$ of $\mathbb{Q}$-algebras (which is assumed to be subintegral for ξ_E to exist).

(2) For a fixed n, ξ_E is functorial in E. This is clear from the functoriality of J_E in the case $n < \infty$, and it is [7, (5.3)] in the case $n = \infty$.

(2.6) THEOREM. *Let $E = (A \subseteq B, \mathfrak{b}, n)$ be an admissible extension. In case $n = \infty$, assume that the extension $A \subseteq B$ is subintegral. Then the homomorphism $\xi_E\colon B/A \to \mathscr{I}(A, B)$ is an isomorphism.*

Proof. The case $n = \infty$ is proved in (0.2). So we assume that $n < \infty$. In this case it is enough, in view of (2.4), to prove that $\ker(J_E) = A \cap \mathfrak{b}$ and that J_E is surjective. If $b \in \ker(J_E)$ then $A = J_E(b) = A(e_n(b), b^n)$ whence $e_n(b) \in A$. Therefore $\log_n(e_n(b)) \in A$ and, since $b \equiv \log_n(e_n(b)) \pmod{\mathfrak{b}^n}$, we get $b \in A$. This proves that $\ker(J_E) = A \cap \mathfrak{b}$. To prove the surjectivity of J_E consider the commutative diagram

$$\begin{array}{ccc}
\mathfrak{b} & \xrightarrow{J_E} & \mathscr{I}(A, B) \\
\lambda\downarrow & & \downarrow\mu \\
\mathfrak{b}' & \xrightarrow{J_{E'}} & \mathscr{I}(A', B')
\end{array}$$

where $A' = A/\mathfrak{b}^n$, $B' = B/\mathfrak{b}^n$, $\mathfrak{b}' = \mathfrak{b}/\mathfrak{b}^n$, $E' = (A' \subseteq B', \mathfrak{b}', n)$ and λ, μ are natural maps. Since μ is an isomorphism by [7, (2.6)] and λ is surjective, it is enough to prove that $J_{E'}$ is surjective, i.e. we may assume that $\mathfrak{b}^n = 0$. Let $\bar{A} = A/A \cap \mathfrak{b}$, $\bar{B} = B/\mathfrak{b}$ and consider the commutative diagram

$$\begin{array}{ccccccc}
A^* & \longrightarrow & B^* & \xrightarrow{\alpha} & \mathscr{I}(A, B) & \xrightarrow{\beta} & \mathrm{Pic}(A) \\
\downarrow & & \downarrow & & \downarrow\mu & & \downarrow\nu \\
(\bar{A})^* & \longrightarrow & (\bar{B})^* & \xrightarrow{\bar{\alpha}} & \mathscr{I}(\bar{A}, \bar{B}) & \xrightarrow{\bar{\beta}} & \mathrm{Pic}(\bar{A})
\end{array}$$

of natural maps with exact rows [7, (2.4)]. Since $B = A + \mathfrak{b}$, we have $\bar{A} = \bar{B}$ so that $\mathscr{I}(\bar{A}, \bar{B}) = 1$ whence $\nu\beta$ is trivial. Further, since $\mathfrak{b}$ is nilpotent, ν is an isomorphism [1, Ch. III, (2.12)]. Therefore β is trivial. So α is surjective. Therefore if $I \in \mathscr{I}(A, B)$ then $I = Au$ for some $u \in B^*$. Write $u = a + y$ with $a \in A, y \in \mathfrak{b}$. Then $a \in A^*$. Multiplying u by a^{-1} we may assume that $u = 1 + y$. Let $b = \log_n(1 + y)$. Then $b \in \mathfrak{b}$ and $J_E(b) = Ae_n(b) = A(1 + y) = I$. This proves the surjectivity of J_E. □

3. The map $\varphi(A, C, B)$

Let $A \subseteq C \subseteq B$ be extensions of rings. Then we have $\mathscr{I}(A, C) \subseteq \mathscr{I}(A, B)$. Denote by $\varphi(A, C, B)$ or simply φ the natural map $\mathscr{I}(A, B) \to \mathscr{I}(C, B)$ which is given by $\varphi(I) = CI$. The sequence $1 \to \mathscr{I}(A, C) \to \mathscr{I}(A, B) \xrightarrow{\varphi} \mathscr{I}(C, B)$

is always exact. For if $I \in \mathscr{I}(A, B)$ and $\varphi(I) = C$ then $\varphi(I^{-1}) = C$ whence both I and I^{-1} are A-submodules of C showing that $I \in \mathscr{I}(A, C)$. We show in Theorem (3.3) below that φ is surjective if the extension $A \subseteq B$ is subintegral. However, first we give an example to show that φ is not surjective in general:

(3.1) EXAMPLE. If $\mathrm{Pic}(A) = \mathrm{Pic}(B) = 0$ and $\mathrm{Pic}(C) \neq 0$ then φ is not surjective. This is clear from the commutative diagram

$$\begin{array}{ccccc} \mathscr{I}(A, B) & \longrightarrow & \mathrm{Pic}(A) & & \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \\ \mathscr{I}(C, B) & \longrightarrow & \mathrm{Pic}(C) & \longrightarrow & \mathrm{Pic}(B) \end{array}$$

with the row exact [7, (2.4)]. As a specific example, one could let $B = k[t]$, $C = k[t^2, t^3]$ and $A = k[t^2]$, where k is a field and t is an indeterminate. In fact, in this case one can verify directly that $\mathscr{I}(A, B) = 1$ and that $I = C(1 + t, t^2)$ is a nontrivial element of $\mathscr{I}(C, B)$ (with $I^{-1} = C(1 - t, t^2)$). □

(3.2) LEMMA. *Let* $\mathfrak{n}$ *be a nilpotent ideal of* B *and let* $\bar{A} = A/A \cap \mathfrak{n}$, $\bar{B} = B/\mathfrak{n}$ *and* $\bar{C} = C/C \cap \mathfrak{n}$. *If* $\varphi(\bar{A}, \bar{C}, \bar{B})$ *is surjective then so is* $\varphi(A, C, B)$.

Proof. We have the commutative diagram

$$\begin{array}{ccccc} \mathscr{I}(A, B) & \xrightarrow{\varphi} & \mathscr{I}(C, B) & \xleftarrow{\alpha} & B^* \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\rho} & & \\ \mathscr{I}(\bar{A}, \bar{B}) & \xrightarrow{\bar{\varphi}} & \mathscr{I}(\bar{C}, \bar{B}) & & \end{array}$$

where $\varphi = \varphi(A, C, B)$, $\bar{\varphi} = \varphi(\bar{A}, \bar{C}, \bar{B})$ and μ, ρ, α are natural maps. We claim:

(1) μ is surjective;
(2) $\ker(\rho) \subseteq \mathrm{im}(\alpha)$.

Grant these for the moment, and assume that $\bar{\varphi}$ is surjective. Let $I \in \ker(\rho)$. Then by (2) $I = Cu$ for some $u \in B^*$ whence $I = \varphi(Au) \in \mathrm{im}(\varphi)$. Thus $\ker(\rho) \subseteq \mathrm{im}(\varphi)$ and now it follows, in view of (1), that φ is surjective. As for the claims, (1) follows by chasing the commutative diagram

$$\begin{array}{ccccccc} B^* & \longrightarrow & \mathscr{I}(A, B) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & \mathrm{Pic}(B) \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\theta} \\ (\bar{B})^* & \longrightarrow & \mathscr{I}(\bar{A}, \bar{B}) & \longrightarrow & \mathrm{Pic}(\bar{A}) & \longrightarrow & \mathrm{Pic}(\bar{B}) \end{array}$$

of natural maps in which the rows are exact [7, (2.4)] and, $\mathfrak{n}$ being nilpotent, λ is surjective and ν, θ are isomorphisms [1, Ch. III, (2.12)]. To prove (2) consider the commutative diagram

$$\begin{array}{ccccc} B^* & \xrightarrow{\alpha} & \mathscr{I}(C,B) & \xrightarrow{\beta} & \operatorname{Pic}(C) \\ & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} \\ & & \mathscr{I}(\bar{C},\bar{B}) & \longrightarrow & \operatorname{Pic}(\bar{C}) \end{array}$$

of natural maps with the row exact. Since σ is an isomorphism, we get $\ker(\rho) \subseteq \ker(\beta) = \operatorname{im}(\alpha)$. □

(3.3) THEOREM. *Let $A \subseteq B$ be a subintegral extension. Then for all rings C with $A \subseteq C \subseteq B$ the sequence $1 \to \mathscr{I}(A, C) \to \mathscr{I}(A, B) \xrightarrow{\varphi} \mathscr{I}(C, B) \to 1$ of natural maps is exact.*

Proof in the excellent case. As noted above, we only need to prove the surjectivity of $\varphi = \varphi(A, C, B)$. We prove this first in the case when A is an excellent ring of finite Krull dimension, using induction on $\dim(A)$. For a ring R, write $\bar{R} = R_{\text{red}} = R/\text{nil}(R)$. Note that, since the extension $A \subseteq B$ is subintegral, so is $\bar{A} \subseteq \bar{B}$ whence $\bar{B}$ is contained in the total quotient ring of $\bar{A}$ by [8, 4.1]. Let $\bar{\varphi} = \varphi(\bar{A}, \bar{C}, \bar{B})$. If $\dim(A) = 0$ then $\bar{A}$ is its own total quotient ring so that $\bar{A} = \bar{C} = \bar{B}$ and trivially $\bar{\varphi}$ is surjective in this case. Hence φ is surjectivfe by (3.2). Now, let $\dim(A) > 0$. Since $\bar{A}$ is excellent, $\bar{B}$ is a finite $\bar{A}$-module. Therefore the conductor $\mathscr{C}$ of the extension $\bar{A} \subseteq \bar{B}$ contains a nonzero divisor of $\bar{A}$. Let $A' = \bar{A}/\mathscr{C}$, $B' = \bar{B}/\mathscr{C}$, $C' = \bar{C}/\mathscr{C}$ and $\varphi' = \varphi(A', C', B')$. The extension $A' \subseteq B'$ is subintegral and A' is excellent with $\dim(A') < \dim(\bar{A}) = \dim(A)$. So φ' is surjective by induction. Now, by [7, (2.6)] we can identify $\mathscr{I}(\bar{A}, \bar{B}) = \mathscr{I}(A', B')$, $\mathscr{I}(\bar{C}, \bar{B}) = \mathscr{I}(C', B')$ and $\bar{\varphi} = \varphi'$. Thus $\bar{\varphi}$ is surjective whence φ is surjective by (3.2). This proves the theorem in the case when A is excellent with $\dim(A)$ finite.

To prove the general case, we need another

(3.4) LEMMA. *Let $k \subseteq A \subseteq B$ be extensions of rings such that $A \subseteq B$ is subintegral. Let H be a finite subset of B. Then there exists a finitely generated k-subalgebra A' of A such that the extension $A' \subseteq A'[H]$ is subintegral.*

Proof. Since H is finite, it follows from [8, 2.8] that there exist $t_1, \dots, t_m \in B$ such that $t_i^2, t_i^3 \in A[t_1, \dots, t_{i-1}]$ for every i, $1 \leqslant i \leqslant m$, and $H \subseteq A[t_1, \dots, t_m]$. Let $A' = k[S]$, where S is any finite subset of A such that (1) for each $b \in H$, b is a polynomial in $t_1, \dots, t_m$ with coefficients in S; (2) for each i, $1 \leqslant i \leqslant m$, t_i^2, t_i^3 are polynomials in $t_1, \dots, t_{i-1}$ with coefficients in S; (3) $t_1^2, t_1^3 \in S$. Then the extension $A' \subseteq A'[t_1, \dots, t_m]$ is subintegral and $A'[H] \subseteq A'[t_1, \dots, t_m]$. Therefore $A' \subseteq A'[H]$ is also subintegral. □

(3.5) REMARK. The above lemma generalizes [6, (2.1)].

Proof of Theorem (3.3) *in the general case.* We have to show that φ is surjective. Let $I \in \mathscr{I}(C, B)$. Choose $x_1, \ldots, x_r \in I$, $y_1, \ldots, y_r \in I^{-1}$ such that $x_1 y_1 + \cdots + x_r y_r = 1$. Then $I = (x_1, \ldots, x_r)C$. Let

$$H = \{x_i y_j \mid 1 \leqslant i, j \leqslant r\} \cup \{x_1, \ldots, x_r, y_1, \ldots, y_r\}.$$

Let k be the natural image of $\mathbb{Z}$ in A. By (3.4) there exists a finitely generated k-subalgebra A' of A such that the extension $A' \subseteq A'[H]$ is subintegral. Let $B' = A'[H] \subseteq B$, let $C' = A'[x_i y_j \mid 1 \leqslant i, j \leqslant r] \subseteq C \cap B'$ and let I' (resp. J') be the C'-submodule of B' generated by $x_1, \ldots, x_r$ (resp. $y_1, \ldots, y_r$). Then $I'J' = C'$ whence $I' \in \mathscr{I}(C', B')$. Now, being a $\mathbb{Z}$-algebra of finite type, A' is excellent with $\dim(A') < \infty$. Therefore by the case already proved there exists $J' \in \mathscr{I}(A', B')$ such that $I' = C'J'$. Now, $AJ' \in \mathscr{I}(A, B)$ and we have $I = \varphi(AJ')$. □

(3.6) EXAMPLE (cf. Introduction). Let $A = k + t^q k[t] \subseteq B = k[t]$, where k is a field of characteristic $p > 0$, t is an indeterminate and q is any integer $\geqslant p + 1$. Then $\mathscr{I}(A, B) = \operatorname{Pic}(A)$ and this group is not killed by p whence, in particular, it is not isomorphic to the group B/A.

Proof. We have $\mathscr{I}(A, B) = \operatorname{Pic}(A)$ by [7, (2.5)]. Let $C = k + t^p B$. Then $A \subseteq C \subseteq B$, and $E = (C \subseteq B, tB, p)$ is an admissible extension. Let $J = J_E(t) = C(e_p(t), t^p)$. Then $J \in \mathscr{I}(C, B)$ by (2.1). By (3.3) there exists $I \in \mathscr{I}(A, B)$ such that $J = CI$. Since $C \subseteq k + t^2 B$ and $e_p(t) \notin k + t^2 B$, we have that I is not contained in $k + t^2 B$. It follows that I^p is not contained in A; in particular $I^p \neq A$, i.e. I is not killed by p. □

4. Main results

Let $A \subseteq B$ be a subintegral extension of rings. Suppose B is equipped with a decreasing filtration $\mathscr{F} = (\mathscr{F}_i B)_{i \geqslant 0}$ of A-subalgebras such that $\mathscr{F}_0 B = B$. Note then that, since $A \subseteq B$ is subintegral, so are the extensions $A \subseteq \mathscr{F}_i B$ and $\mathscr{F}_{i+1} B \subseteq \mathscr{F}_i B$ for all i. Put

$$\operatorname{gr}_{\mathscr{F}}(B) = \bigoplus_{i \geqslant 0} \mathscr{F}_i B / \mathscr{F}_{i+1} B$$

and

$$\operatorname{gr}_{\mathscr{F}}(\mathscr{I}(A, B)) = \bigoplus_{i \geqslant 0} \mathscr{F}_i \mathscr{I}(A, B) / \mathscr{F}_{i+1} \mathscr{I}(A, B),$$

where $\mathscr{F}_i \mathscr{I}(A, B) = \mathscr{I}(A, \mathscr{F}_i B)$.

Note that $\mathscr{F}_0\mathscr{I}(A, B) = \mathscr{I}(A, B)$ and $\bigcap_{i\geq 0}\mathscr{F}_i\mathscr{I}(A, B) = \mathscr{I}(A, \bigcap_{i\geq 0}\mathscr{F}_i B)$. This was pointed out by Leslie Roberts.

Let $n \in \mathbb{N} \cup \{\infty\}$. Let $\mathfrak{b} = \{\mathfrak{b}_i\}_{i\geq 0}$ be a sequence with $\mathfrak{b}_i$ an ideal of $\mathscr{F}_i B$. We say the filtration $\mathscr{F}$ is *n-admissible* with *associated sequence* $\mathfrak{b}$ if $(\mathscr{F}_{i+1}B \subseteq \mathscr{F}_i B, \mathfrak{b}_i, n)$ is an admissible extension for every $i \geq 0$.

Assume that $\mathscr{F}$ is n-admissible with associated sequence $\mathfrak{b}$. Put $E_i = (\mathscr{F}_{i+1}B \subseteq \mathscr{F}_i B, \mathfrak{b}_i, n)$. By (2.6) we have the functorial isomorphisms

$$\xi_i' : \mathscr{F}_i B/\mathscr{F}_{i+1}B \to \mathscr{I}(\mathscr{F}_{i+1}B, \mathscr{F}_i B) \qquad (*)$$

of groups, where we have written ξ_i' for ξ_{E_i}. Further, by (3.3) we have the natural isomorphisms

$$\bar{\varphi}_i : \mathscr{I}(A, \mathscr{F}_i B)/\mathscr{I}(A, \mathscr{F}_{i+1}B) \to \mathscr{I}(\mathscr{F}_{i+1}B, \mathscr{F}_i B) \qquad (**)$$

induced by $\varphi_i = \varphi(A, \mathscr{F}_{i+1}B, \mathscr{F}_i B)$. Combining (*) and (**) we get the isomorphisms

$$\xi_i : \mathscr{F}_i B/\mathscr{F}_{i+1}B \to \mathscr{I}(A, \mathscr{F}_i B)/\mathscr{I}(A, \mathscr{F}_{i+1}B)$$

where $\xi_i = \bar{\varphi}_i^{-1}\xi_i'$. Writing $\xi_{B/A,\mathscr{F},\mathfrak{b}} = \bigoplus_{i\geq 0}\xi_i$, to show the dependence of this map on $(A \subseteq B, \mathscr{F}, \mathfrak{b})$ we get an isomorphism

$$\xi_{B/A,\mathscr{F},\mathfrak{b}} : \mathrm{gr}_{\mathscr{F}}(B) \to \mathrm{gr}_{\mathscr{F}}(\mathscr{I}(A, B))$$

of groups which is, in view of (2.5) (2), functorial in $(A \subseteq B, \mathscr{F}, \mathfrak{b})$ for a fixed n. If $\mathrm{gr}_{\mathscr{F}}(B/A)$ denotes the associated graded A-module for the filtration on B/A induced by $\mathscr{F}$ then, since $A \subseteq \mathscr{F}_i B$ for every i, we have $\mathrm{gr}_{\mathscr{F}}(B/A) = \mathrm{gr}_{\mathscr{F}}(B)$. So we can rewrite the isomorphism as

$$\xi_{B/A,\mathscr{F},\mathfrak{b}} : \mathrm{gr}_{\mathscr{F}}(B/A) \to \mathrm{gr}_{\mathscr{F}}(\mathscr{I}(A, B)).$$

To summarize, we have

(4.1) THEOREM. *Let $n \in \mathbb{N} \cup \{\infty\}$ and let $A \subseteq B$ be a subintegral extension such that B is equipped with an n-admissible filtration $\mathscr{F}$ of A-subalgebras with associated sequence $\mathfrak{b}$. Then the map*

$$\xi_{B/A,\mathscr{F},\mathfrak{b}} : \mathrm{gr}_{\mathscr{F}}(B/A) \to \mathrm{gr}_{\mathscr{F}}(\mathscr{I}(A, B))$$

is a functorial isomorphism of graded abelian groups. If $n = \infty$ and $\mathscr{F}_i B = A$ for all $i \geq 1$ then $\mathrm{gr}_{\mathscr{F}}(B/A) = B/A$, $\mathrm{gr}_{\mathscr{F}}(\mathscr{I}(A, B)) = \mathscr{I}(A, B)$ and $\xi_{B/A,\mathscr{F},\mathfrak{b}}$ coincides with the isomorphism $\xi_{B/A}$ given by (0.2).

Proof. The first part is proved above. If $\mathscr{F}_i B = A$ for all $i \geqslant 1$ then in the above notation we have $\xi_{B/A,\mathscr{F},\mathfrak{b}} = \xi_0 = \xi'_0 = \xi_{E_0}$, with $E_0 = (A \subseteq B, \mathfrak{b}_0, n)$. Therefore the last assertion follows from (2.5) (1). □

(4.2) COROLLARY. *Let $A \subseteq B$ be a subintegral extension satisfying the following condition*:

A contains a field and $B = A + \mathfrak{m}$ for some proper ideal $\mathfrak{m}$ of B. (*C*)

Let $p = \infty$ if $\text{char}(A) = 0$, *and* $p = \text{char}(A)$ *otherwise, and let $\mathscr{F} = (\mathscr{F}_i B)_{i \geqslant 0}$ with $\mathscr{F}_0 B = B$ and $\mathscr{F}_i B = A + \mathfrak{m}^{p^i}$ for $i \geqslant 1$. (Recall that $\mathfrak{m}^\infty = 0$ by our convention.) Then there exists a functorial isomorphism $gr_{\mathscr{F}}(B/A) \to$* $\text{gr}_{\mathscr{F}}(\mathscr{I}(A, B))$.

Proof. The filtration $\mathscr{F}$ is p-admissible with associated sequence $\{\mathfrak{m}^{p^i}\}_{i \geqslant 0}$, where we let $\infty^0 = 1$. □

(4.3) REMARK. Note that (4.2) applies in a natural way to the following two cases of a subintegral extension $A \subseteq B$:

(1) A is an equicharacteristic local ring. In this case B is local and A and B have the same residue field. Therefore (C) holds with $\mathfrak{m}$ equal to the maximal ideal of B.

(2) $A = \oplus_{d \geqslant 0} A_d \subseteq B = \oplus_{d \geqslant 0} B_d$ with A_0 a field. Then B_0 has only one prime ideal, say $\mathfrak{p}$, with $A_0 = B_0/\mathfrak{p}$ (in particular, if B_0 is reduced then $A_0 = B_0$). In this case (C) holds with $\mathfrak{m} = \mathfrak{p} + \oplus_{d \geqslant 1} B_d$.

Now, in the graded case $A = \oplus_{d \geqslant 0} A_d \subseteq B = \oplus_{d \geqslant 0} B_d$ we consider a variant of the filtration given in the above remark, namely the filtration $F = (F_i B)_{i \geqslant 0}$ described in the Introduction. We assume that A_0 is a field, the extension $A \subseteq B$ is subintegral and $A_0 = B_0$. Let $p = \infty$ if $\text{char}(A_0) = 0$, and $p = \text{char}(A_0)$ otherwise. Let $\mathfrak{q} = \{\mathfrak{q}_i\}_{i \geqslant 0}$ be the sequence defined by $\mathfrak{q}_0 = \Sigma_{d \geqslant 1} B_d$ and $\mathfrak{q}_i = \Sigma_{d \geqslant p^i} B_d$ for $i \geqslant 1$. Then, since $B_0 = A_0$, the filtration F is given by $F_i B = A + \mathfrak{q}_i$, and it is p-admissible with associated sequence $\mathfrak{q}$. Therefore we have the isomorphism $\xi_{B/A,F,\mathfrak{q}}: \text{gr}_F(B/A) \to \text{gr}_F(\mathscr{I}(A, B))$ given by (4.1). Put

$$M_0 = \sum_{d=0}^{p-1} (B_d/A_d) \quad \text{and} \quad M_i = \sum_{d=p(i)}^{p(i+1)-1} (B_d/A_d) \quad \text{for } i \geqslant 1,$$

where $p(i) = p^i$. Then $B/A = \oplus_{i \geqslant 0} M_i$. Further, we have $M_i \subseteq F_i(B/A)$ and the natural map $F_i(B/A) \to F_i(B/A)/F_{i+1}(B/A)$ induces an isomorphism $\zeta_i: M_i \to F_i(B/A)/F_{i+1}(B/A)$ for every i, whereby we get a natural isomorphism $\zeta = \oplus_{i \geqslant 0} \zeta_i: B/A \to \text{gr}_F(B/A)$. Writing $\xi_{B/A}$ for $\xi_{B/A,F,\mathfrak{q}} \circ \zeta$ and $\text{gr}\,\mathscr{I}(A, B)$ for $\text{gr}_F(\mathscr{I}(A, B))$ we get the isomorphism $\xi_{B/A}: B/A \to \text{gr}\,\mathscr{I}(A, B)$.