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On an extension of a theorem of Tunnell

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1. Introduction

Let K be a quadratic extension of a non-Archimedean local field k of characteristic $\neq 2$. Then it is a theorem of Tunnell [Tu] in odd residue characteristic, and proved recently by Saito [S] in general, that one can describe which characters of K^* appear in an irreducible admissible representation of $GL(2, k)$ or in an irreducible representation of D_k^*, where D_k is the unique quaternion division algebra over k, in terms of certain epsilon factors. If the representation of $GL(2, k)$ comes from a character of K^* via the construction of the Weil representation, cf. [J-L, Theorem 4.6], then the representation decomposes into two irreducible representations when restricted to $GL(2, k)^+ = \{x \in GL(2, k) \mid \det(x) \in NK^*\}$ where NK^* is the subgroup of k^* of index 2 consisting of norms from K^*; similarly for D_k^* for which we denote the corresponding subgroup of index 2 by D_k^{*+}. Clearly K^* is contained both in $GL(2, k)^+$, and in D_k^{*+}, and it is the purpose of this note to generalise Tunnell's theorem to describe which characters of K^* appear in these two representations of $GL(2, k)^+$, and of D_k^{*+}. For a discrete series representation π of $GL(2, k)$, we let π' denote the representation of D_k^* associated by Jacquet-Langlands to π.

We now state Tunnell's theorem, and our generalisation, more precisely.

THEOREM 1.1. (Tunnell). *Let π be an irreducible admissible infinite dimensional representation of $GL(2, k)$ with central character ω_π and let σ_π be the associated two-dimensional representation of the Weil-Deligne group of k. Let χ be a character of K^* such that $\chi|_{k^*} = \omega_\pi$. Let ψ be an additive character of k and x_0 an element of K such that $tr(x_0) = 0$. Define an additive character ψ_0 of K by $\psi_0(x) = \psi(tr[(-xx_0/2)])$. Then the epsilon factor $\varepsilon(\sigma_\pi|_K \otimes \chi^{-1}, \psi_0)$ is independent of the choice of ψ and x_0, and takes the*

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value 1 *if and only if* χ *appears in* π, *and takes the value* -1 *if and only if* χ *appears in* π'.

REMARK. It is customary, as Tunnell himself did, to use $\psi_K(x) = \psi(\mathrm{tr}\, x)$ instead of the character ψ_0 that we used. If one uses ψ_K, then Tunnell's theorem says that a character χ of K^* as before appears in a representation π of $GL(2, k)$ if and only if $\varepsilon(\sigma_\pi|_K \otimes \chi^{-1}, \psi_K) \cdot \omega_\pi(-1) = 1$. The character ψ_K has been changed to ψ_0 with the purpose of eliminating the factor $\omega_\pi(-1)$, and as we shall see in the next paragraph, when σ_π is a sum of two characters, the resulting two epsilon factors with respect to ψ_0 still take values in $\{\pm 1\}$; both of these are important to our extension of Tunnell's theorem. The introduction of the "extra factor" of $-1/2$ in the definition of ψ_0 is to make the later formulae a little easier.

If the representation π of $GL(2, k)$ comes from a character θ of K^*, the representation σ_π of the Weil group is induced from the character θ of K^*, cf. [J-L, p. 396]. Therefore $\sigma_\pi|_{K^*} = \theta + \bar{\theta}$ where $\bar{\theta}$ is the character $\bar{\theta}(x) = \theta(\bar{x})$ (where $x \to \bar{x}$ is the non-trivial automorphism of K over k). Therefore the epsilon factor $\varepsilon(\sigma_\pi|_{K^*} \otimes \chi^{-1}, \psi_0)$ considered in Tunnell's theorem factorises as $\varepsilon(\sigma_\pi|_{K^*} \otimes \chi^{-1}, \psi_0) = \varepsilon(\theta\chi^{-1}, \psi_0) \cdot \varepsilon(\bar{\theta}\chi^{-1}, \psi_0)$. We check that both $\varepsilon(\theta\chi^{-1}, \psi_0)$, and $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0)$ take values in $\{\pm 1\}$ (and here it is important that ψ_0 is trivial on k). By the Galois invariance of the epsilon factor, $\varepsilon(\theta\chi^{-1}, \psi_0) = \varepsilon(\bar{\theta}\bar{\chi}^{-1}, \bar{\psi}_0) = \varepsilon(\bar{\theta}\bar{\chi}^{-1}, \psi_0(-x))$, and by the condition on central characters, $(\theta\chi^{-1})|_{k^*} = \omega_{K/k}$ where $\omega_{K/k}$ is the quadratic character of k^* associated by the classfield theory to K, and therefore $\bar{\theta}\bar{\chi}^{-1} = \theta^{-1}\chi$. It follows that

$$\varepsilon(\theta\chi^{-1}, \psi_0) = \varepsilon(\theta^{-1}\chi, \psi_0(-x)) = \omega_{K/k}(-1)\varepsilon(\theta^{-1}\chi, \psi_0),$$

and since $\varepsilon(V, \psi_0) \cdot \varepsilon(V^*, \psi_0) = (\det V)(-1)$ for any representation V of the Weil group of k, $\varepsilon(\theta\chi^{-1}, \psi_0)^2 = 1$. Similarly $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0)^2 = 1$. Therefore the set of characters χ of K^* with $(\chi^{-1} \cdot \theta)|_{k^*} = \omega_{K/k}$ and $\varepsilon(\sigma_\pi|_{K^*} \otimes \chi^{-1}, \psi_0) = 1$ is exactly the set of characters χ of K^* such that $\varepsilon(\theta\chi^{-1}, \psi_0)$ and $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0)$ are either both 1 or are both -1.

We now state our generalisation of Tunnell's theorem.

THEOREM 1.2. *Let* π *be an irreducible admissible representation of* $GL(2, k)$ *associated to a character* θ *of* K^*. *Fix embeddings of* K^* *in* $GL(2, k)^+$ *and in* D_k^{*+} *(there are two conjugacy classes of such embeddings in general), and choose an additive character* ψ *of* k, *and an element* x_0 *of* K^* *with* $\mathrm{tr}(x_0) = 0$. *Then the respresentation* π *of* $GL(2, k)$ *decomposes as* $\pi = \pi_+ \oplus \pi_-$ *when restricted to* $GL(2, k)^+$, *and the representation* π' *of* D_k^* *decomposes as* $\pi' = \pi'_+ \oplus \pi'_-$ *when restricted to* D_k^{*+}, *such that for a* *character* χ *of* K^* *with* $(\chi \cdot \theta^{-1})|_{k^*} = \omega_{K/k}$, χ *appears in* π_+ *if and only if* $\varepsilon(\theta\chi^{-1}, \psi_0) = \varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = 1$, χ *appears in* π_- *if and only if* $\varepsilon(\theta\chi^{-1}, \psi_0) = \varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = -1$, χ *appears in* π'_+ *if and only if* $\varepsilon(\theta\chi^{-1}, \psi_0) = 1$ *and* $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = -1$, *and* χ *appears in* π'_- *if and only if* $\varepsilon(\theta\chi^{-1}, \psi_0) = -1$ *and* $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = 1$.

REMARK. Theorem 1.2 is also true, and easy to prove, for $GL(2, \mathbb{R})$, but as $D_{\mathbb{R}}^{*+} = D_{\mathbb{R}}^*$, it does not make sense in this case.

The possibility of such a generalisation of Tunnell's theorem was suggested by M. Harris whom the author thanks heartily. Analogous factorisation of the epsilon factors, though not covering this case, has been conjectured to exist very generally in [G-P]. The author also wishes to thank the referee for his comments which have helped improve the exposition.

2. Two theorems on epsilon factors

We will assume that the reader is familiar with the basic properties of the epsilon factor, $\varepsilon(\sigma, \psi)$, associated to a finite dimensional complex representation σ of Weil group of k, and an additive character ψ of k. We refer to Tate's article [Ta] as our general reference on epsilon factors; our convention for the epsilon factor are the one used by Langlands, and in the notation of [Ta], it is $\varepsilon_L(\sigma, \psi) = \varepsilon_D(\sigma\|^{1/2}, \psi, dx)$ where dx is the Haar measure on k self-dual for Fourier transform with respect to ψ. We, however, do want to recall two theorems about epsilon factors which will be crucial to our calculations; the first due to Deligne [D, Lemma 4.1.6] describes how epsilon factor changes under twisting by a character of small conductor, and the second is a theorem of Frohlich and Queyrut [F-Q, Theorem 3].

THEOREM 2.1. *Let* α *and* β *be two multiplicative characters of a local field* K *such that* $\text{cond}(\alpha) \geqslant 2\,\text{cond}(\beta)$. *For an additive character* ψ *of* K, *let* y *be an element of* K *such that* $\alpha(1 + x) = \psi(xy)$ *for all* $x \in K$ *with* $\text{val}(x) \geqslant \frac{1}{2}\text{cond}(\alpha)$ *if conductor of* α *is positive; if conductor of* α *is* 0, *let* $y = \pi_k^{-\text{cond}(\psi)}$ *where* π_k *is a uniformising parameter of* k. *Then*

$$\varepsilon(\alpha\beta, \psi) = \beta^{-1}(y)\varepsilon(\alpha, \psi).$$

THEOREM 2.2. *Let* K *be a separable quadratic extension of a local field* k, *and* ψ *an additive character of* k. *Let* ψ_K *be the additive character of* K *defined by* $\psi_K(x) = \psi(\text{tr}\, x)$. *Then for any character* χ *of* K^* *which is trivial* *on* k^*, *and any* $x_0 \in K^*$ *with* $\mathrm{tr}(x_0) = 0$

$$\varepsilon(\chi, \psi_K) = \chi(x_0).$$

3. The main lemma

Here is the main lemma used in the proof of our theorem; it may be of some independent interest. It will be proved here only for local field of odd residue characteristic.

LEMMA 3.1. *Let* K *be a quadratic extension of a local field* k. *Let* ψ *be an additive character of* k, *and* $x_0 \in K^*$ *such that* $\mathrm{tr}(x_0) = 0$. *Define an additive character* ψ_0 *of* K *by* $\psi_0(x) = \psi(\mathrm{tr}[-xx_0/2])$. *Then*

$$\varepsilon(\omega_{K/k}, \psi) \frac{\omega_{K/k}\left(\dfrac{x - \bar{x}}{x_0 - \bar{x}_0}\right)}{\left\|\dfrac{(x - \bar{x})^2}{x\bar{x}}\right\|_k^{1/2}} = \sum_{\substack{\varepsilon(\chi, \psi_0) = 1 \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x),$$

where, as is usual, the summation on the right is by partial sums over all characters of K^* *of conductor* $\leqslant n$.

Proof. As already observed in the introduction, for characters χ of K^* with $\chi|_{k^*} = \omega_{K/k}$, $\varepsilon(\chi, \psi_0) = \pm 1$. For an element r of k^*, and character χ of K^* as before, $\varepsilon(\chi, \psi_0(rx)) = \omega_{K/k}(r)\varepsilon(\chi, \psi_0)$. The equation

$$\sum_{\substack{\varepsilon(\chi, \psi_0) = 1 \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x) + \sum_{\substack{\varepsilon(\chi, \psi_0) = -1 \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x) = 0,$$

can therefore be written as,

$$\sum_{\substack{\varepsilon(\chi, \psi_0) = 1 \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x) = \omega_{K/k}(r) \sum_{\substack{\varepsilon(\chi, \psi_0(rx)) = 1 \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x).$$

From this, it follows that once the lemma is proved for one choice of the pair (x_0, ψ), it is true for any other. We will choose the additive character ψ to have conductor 0, and x_0 to be a unit if K is an unramified extension and a uniformising parameter if K is ramified. It is also clear that once the lemma is true for $x \in K^*$, it is true for any rx for $r \in k^*$.

We now fix a character $\tilde{\omega}_{K/k}$ of K^* which extends the character $\omega_{K/k}$ of k^* in the following way. If K is an unramified extension of k, then we let $\tilde{\omega}_{K/k}$ be trivial on the units, and take the value -1 on any uniformising parameter of K^*. If K is a ramified extension of k with maximal compact subrings $\mathcal{O}_K$ and $\mathcal{O}_k$, and uniformising parameters π_K and π_k respectively, then $(\mathcal{O}_K/\pi_K)^* \cong (\mathcal{O}_k/\pi_k)^*$. We use this isomorphism to extend $\omega_{K/k}$ to $\mathcal{O}_K^*$ ($\omega_{K/k}$ is trivial on $1 + \pi_k\mathcal{O}_k$ in the odd residue characteristic), and then extend to K^* arbitrarily in one of the two possible ways. The character $\tilde{\omega}_{K/k}$ of K^* has conductor 0 if K is an unramified extension of k, and has conductor 1 if K is ramified. We will apply Theorem 2.1 to the characters, in the notation of that theorem, $\alpha = \chi \cdot \tilde{\omega}_{K/k}^{-1}$ and $\beta = \tilde{\omega}_{K/k}$. The hypothesis of that theorem will be satisfied if either K is unramified or if $\operatorname{cond}(\chi) \geqslant 2$. It is easy to see that if K is ramified, then any character χ of K^* which extends the character $\omega_{K/k}$ of k^*, has either even conductor, or has conductor 1; if the conductor is 1, χ is either $\tilde{\omega}_{K/k}$, or is $\tilde{\omega}_{K/k} \cdot \mu$ where μ is the unramified character of K^* taking the value -1 at π_K. Since μ is unramified with $\mu(\pi_K) = -1$, and $\tilde{\omega}_{K/k}$ has conductor 1, $\varepsilon(\tilde{\omega}_{K/k} \cdot \mu, \psi_0) = -\varepsilon(\tilde{\omega}_{K/k}, \psi_0)$. It follows that exactly one of $\tilde{\omega}_{K/k}$ or $\tilde{\omega}_{K/k} \cdot \mu$ has its epsilon factor 1. We now use Theorems 2.1 and 2.2 to calculate $\varepsilon(\chi, \psi_0)$ where χ has conductor $\geqslant 2$ if K is a ramified extension. We let y_χ denote an element of K such that for all $x \in K$ with $\operatorname{val}(x) \geqslant \operatorname{cond}(\chi)/2$, $\chi \cdot \tilde{\omega}_{K/k}(1 + x) = \psi_K(y_\chi \cdot x)$ where ψ_K is the character $\psi_K(x) = \psi(\operatorname{tr} x)$. With this notation, we have

$$\begin{aligned}
\varepsilon(\chi, \psi_0) &= \chi(-x_0/2)\varepsilon(\chi, \psi_K) \\
&= \chi(-x_0/2)\varepsilon(\chi \cdot \tilde{\omega}_{K/k}^{-1} \cdot \tilde{\omega}_{K/k}, \psi_K) \\
&= \chi(-x_0/2)\varepsilon(\chi \cdot \tilde{\omega}_{K/k}^{-1}, \psi_K) \cdot \tilde{\omega}_{K/k}^{-1}(y_\chi) \\
&= \chi(-x_0/2) \cdot (\chi \cdot \tilde{\omega}_{K/k}^{-1})(x_0) \cdot \tilde{\omega}_{K/k}^{-1}(y_\chi) \\
&= \chi(-x_0^2/2) \cdot \tilde{\omega}_{K/k}^{-1}(x_0)\tilde{\omega}_{K/k}^{-1}(y_\chi) \\
&= \tilde{\omega}_{K/k}(-x_0/2)\tilde{\omega}_{K/k}^{-1}(y_\chi).
\end{aligned}$$

If K is unramified over k, then x_0 has been chosen to be a unit, and therefore $\tilde{\omega}_{K/k}(x_0) = 1$. In this case y_χ can be taken to be $\pi_k^{-\operatorname{cond}(\chi)}$, therefore we find $\varepsilon(\chi, \psi_0) = (-1)^{\operatorname{cond}(\chi)}$. If $x = a_0 + a_1\pi_k + \cdots + a_r\pi_k^r + \cdots$ where $a_i \in \mathcal{O}_K$, and r is the largest positive integer such that $a_i \in \mathcal{O}_k$ for all $i < r$, the lemma reduces to

$$(-1)^r q^r = \sum_{\substack{2 \mid \operatorname{cond} \chi \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x)$$

where q is the cardinality of the residue field of k. This is easy to verify, and we omit the proof, and turn our attention to the more difficult case of ramified extension.

In the rest of the proof we will assume K to be a ramified extension of k, χ a character of K^* of conductor $2f \geqslant 2$ with $\chi|_{k^*} = \omega_{K/k}$. Our job is to calculate $\tilde{\omega}_{K/k}(y_\chi)$ where $y_\chi \in K^*$ has the property that

$$\chi \cdot \tilde{\omega}_{K/k}^{-1}(1 + x) = \psi_K(x \cdot y_\chi) \quad \text{for all } x \in \pi_K^{(\text{cond }\chi)/2}\mathcal{O}_K.$$

Since ψ is supposed to have conductor 0, y_χ looks like $\pi_K^{-(2f+1)}a_0(\chi) + \pi_K^{-2f}a_1(\chi) + \cdots$ with $a_0(\chi) \in \mathcal{O}_k^*$, and $a_i(\chi) \in \mathcal{O}_k$. As x_0 is this time chosen to be π_K, $\varepsilon(\chi, \psi_0) = \omega_{K/k}(-\pi_K^{2f+2}a_0(\chi)/2)$. Since $x_0 = \pi_K$ is supposed to have trace 0, $N(\pi_K) = -\pi_K^2$, and hence $\omega_{K/k}(\pi_K^2) = \omega_{K/k}(-1)$. Therefore $\varepsilon(\chi, \psi_0) = \omega_{K/k}((-1)^f a_0(\chi)/2)$. Since $\omega_{K/k}((-1)^f a_0(\chi)/2)$ is clearly 1 or -1 depending on whether $(-1)^f a_0(\chi)/2$ is a square in the finite field $\mathcal{O}_k/\pi_k$ or not, it is clear that out of $2(q^f - q^{f-1})$ characters χ of K^* with $\chi|_k^* = \omega_{K/k}$ and of conductor $2f$, exactly $q^f - q^{f-1}$ have $\varepsilon(\chi, \psi_0) = 1$. We are now ready to evaluate $\Sigma\chi(x)$ where the summation is over the characters χ of K^* with $\varepsilon(\chi, \psi_0) = 1$ and $\chi|_{k^*} = \omega_{K/k}$ at an element $x = 1 + a\pi_K^{2r-1} + \cdots$, where $a \in \mathcal{O}_k^*$.

$$\begin{aligned}\sum_{\substack{\varepsilon(\chi,\psi_0)=1\\ \chi|_{k^*}=\omega_{K/k}}} \chi(x) &= 1 + \sum_{\substack{\varepsilon(\chi,\psi_0)=1\\ \chi|_{k^*}=\omega_{K/k}\\ 2\leqslant \text{cond}(\chi)<2r}} \chi(x) + \sum_{\substack{\varepsilon(\chi,\psi_0)=1\\ \chi|_{k^*}=\omega_{K/k}\\ \text{cond}(\chi)=2r}} \chi(x) + \sum_{\substack{\varepsilon(\chi,\psi_0)=1\\ \chi|_{k^*}=\omega_{K/k}\\ \text{cond}(\chi)>2r}} \chi(x)\\ &= 1 + \sum_{i=1}^{r-1}(q^i - q^{i-1}) + 2q^{r-1}\sum_{(-1)^r a_0(\chi)/2\in(\mathcal{O}_k/\pi_k)^{*2}}\\ &\quad\times \psi_K(\pi_K^{-2}aa_0(\chi)) + 0\\ &= q^{r-1} + 2q^{r-1}\sum_{(-1)^r a_0(\chi)\in(\mathcal{O}_k/\pi_k)^{*2}} \psi(\pi_K^{-2}aa_0(\chi))\end{aligned}$$

The terms with $\text{cond}(\chi) > 2r$ add up to zero because if $\varepsilon(\chi, \psi_0) = 1$ then for any character ν of K^* of conductor $2r$ with $\nu|_{k^*} = 1$, $\varepsilon(\chi \cdot \nu, \psi_0) = 1$ also.

Observe that $\psi(\pi_k^{-1}x)$ is a non-trivial additive character on the finite field $\mathcal{O}_k/\pi_k$, and if ω denotes the unique non-trivial quadratic character of $(\mathcal{O}_k/\pi_k)^*$, then

$$\sum_{x\in(\mathcal{O}_k/\pi_k)^*} \omega(x)\psi(\pi_k^{-1}x) = \sqrt{q}\,\omega_{K/k}(\pi_k)\varepsilon(\omega_{K/k}, \psi),$$

or,

$$\sum_{\omega(x)=1} \psi(\pi_k^{-1}x) - \sum_{\omega(x)=-1} \psi(\pi_k^{-1}x) = \sqrt{q}\,\omega_{K/k}(\pi_k)\varepsilon(\omega_{K/k}, \psi).$$

On the other hand,

$$\sum_{\omega(x)=1} \psi(\pi_k^{-1}x) + \sum_{\omega(x)=-1} \psi(\pi_k^{-1}x) = -1.$$

Therefore for any $a \in (\mathcal{O}_k/\pi_k)^*$,

$$\sum_{\omega(x)=\omega(a)} \psi(\pi_k^{-1}x) = \frac{1}{2}(\sqrt{q}\,\omega_{K/k}(a \cdot \pi_k)\varepsilon(\omega_{K/k}, \psi) - 1).$$

Therefore,

$$\sum_{\substack{\varepsilon(\chi,\psi_0)=1 \\ \chi|_{k^*}=\omega_{K/k}}} \chi(x) = q^{r-1} + 2q^{r-1} \sum_{\omega((-1)^r a_0(\chi))=1} \psi(\pi_K^{-2} a \cdot a_0(\chi))$$

$$= q^{r-(1/2)}\omega_{K/k}(\pi_K^2 a(-1)^r) \cdot \varepsilon(\omega_{K/k}, \psi).$$

As $\omega_{K/k}(\pi_K^2) = \omega_{K/k}(-1)$, this is exactly what the lemma requires at $x = 1 + a\pi_K^{2r-1} + \cdots$ with $a \in \mathcal{O}_k^*$. This proves the identity of the lemma at units of K^*. It remains to check it at the uniformising parameter π_K. If χ is a character of even conductor with $\varepsilon(\chi, \psi_0) = 1$, then for the unramified character μ of K^* with $\mu(\pi_K) = -1$, $\varepsilon(\chi \cdot \mu, \psi_0) = 1$. It follows that the summation in the lemma reduces to just one term of conductor 1, which is easily checked to be equal to the left-hand side of the purported equality. This completes the proof of the lemma.

4. Proof of the main theorem

Before we begin the proof of our main theorem, we note the following lemma of Langlands ([L], Lemma 7.19).

LEMMA 4.1. *Let π (resp. π') be the representation of $GL(2, k)$ (resp. D_k^*) associated to a character θ of K^*. Then π restricted to $GL(2, k)^+ = \{x \in GL(2, k) | \det(x) \in NK^*\}$ and π' restricted to $D_k^{*+} = \{x \in D_k^* | \det(x) \in NK^*\}$ decompose into two irreducible representations. If we fix an additive character ψ of k, an element $x_0 \in K^*$ with $\mathrm{tr}(x_0) = 0$, and embeddings of K^* in $GL(2, k)^+$ and D_k^{*+}, then we can write the two irreducible components of π as π_+ and π_- with characters χ_+ and χ_-, and of π' as π'_+ and π'_- with characters χ'_+ and χ'_- such that on K^*,*

$$\chi_+ - \chi_- = \varepsilon(\omega_{K/k}, \psi) \frac{\omega_{K/k}\left(\dfrac{x-\bar{x}}{x_0-\bar{x}_0}\right)}{\left\|\dfrac{(x-\bar{x})^2}{x\bar{x}}\right\|_k^{1/2}} [\theta(x) + \theta(\bar{x})],$$

and,

$$\chi'_+ - \chi'_- = \varepsilon(\omega_{K/k}, \psi) \frac{\omega_{K/k}\left(\dfrac{x-\bar{x}}{x_0-\bar{x}_0}\right)}{\left\|\dfrac{(x-\bar{x})^2}{x\bar{x}}\right\|_k^{1/2}} [\theta(x) - \theta(\bar{x})].$$

REMARK 4.2 It is customary to use the lambda factor $\lambda(K/k, \psi)$ in the above lemma instead of $\varepsilon(\omega_{K/k}, \psi)$ that we have used. They are of course equal.

We are now ready to prove our main theorem (Theorem 1.2) which we recall again.

THEOREM 4.3. *Let π be an irreducible admissible representation of $GL(2, k)$ associated to a character θ of K^*. Fix embeddings of K^* in $GL(2, k)^+$ and in D_k^{*+} (there are two conjugacy classes of such embeddings in general), and choose an additive character ψ of k, and an element x_0 of K^* with $\mathrm{tr}(x_0) = 0$. Then the representation π of $GL(2, k)$ decomposes as $\pi = \pi_+ \oplus \pi_-$ when restricted to $GL(2, k)^+$, and the representation π' of D_k^* decomposes as $\pi' = \pi'_+ \oplus \pi'_-$ when restricted to D_k^{*+} such that for a character χ of K^* with $(\chi \cdot \theta^{-1})|_{k^*} = \omega_{K/k}$, χ appears in π_+ if and only if $\varepsilon(\theta\chi^{-1}, \psi_0) = \varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = 1$, χ appears in π_- if and only if $\varepsilon(\theta\chi^{-1}, \psi_0) = \varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = -1$, χ appears in π'_+ if and only if $\varepsilon(\theta\chi^{-1}, \psi_0) = 1$ and $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = -1$, and χ appears in π'_- if and only if $\varepsilon(\theta\chi^{-1}, \psi_0) = -1$ and $\varepsilon(\bar{\theta}\chi^{-1}, \psi_0) = 1$.*

Proof. The proofs for $GL(2, k)$ and D_k^* are completely analogous. We carry out the proof only for the case of $GL(2, k)$.

From Lemma 3.1,

$$\varepsilon(\omega_{K/k}, \psi) \frac{\omega_{K/k}\left(\dfrac{x-\bar{x}}{x_0-\bar{x}_0}\right)}{\left\|\dfrac{(x-\bar{x})^2}{x\bar{x}}\right\|_k^{1/2}} \cdot \theta(x) = \sum_{\substack{\varepsilon(\chi, \psi_0)=1 \\ \chi|_{k^*} = \omega_{K/k}}} \chi(x)\theta(x)$$

$$= \sum_{\substack{\varepsilon(\chi \cdot \theta^{-1})=1 \\ (\chi \cdot \theta^{-1})|_{k^*} = \omega_{K/k}}} \chi(x).$$

Therefore by Lemma 4.1,

$$\chi_+ - \chi_- = \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=1\\(\chi\cdot\theta^{-1})|_{k^*}=\omega_{K/k}}} \chi + \sum_{\substack{\varepsilon(\chi\cdot\bar{\theta}^{-1})=1\\(\chi\cdot\bar{\theta}^{-1})|_{k^*}=\omega_{K/k}}} \chi$$

$$= 2\sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=1}} \chi + \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=-1}} \chi + \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=-1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=1}} \chi.$$

As the sum of all characters χ with $(\chi\cdot\theta^{-1})|_{k^*} = \omega_{K/k}$ is zero, this reduces to

$$\chi_+ - \chi_- = \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=1}} \chi - \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=-1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=-1}} \chi.$$

By Tunnell's theorem,

$$\chi_+ + \chi_- = \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=1}} \chi + \sum_{\substack{\varepsilon(\chi\cdot\theta^{-1})=-1\\\varepsilon(\chi\cdot\bar{\theta}^{-1})=-1}} \chi.$$

The last two equations complete the proof of the theorem.

REMARK 4.4 It is possible to reformulate Theorem 4.2 as follows. Define a character $\varepsilon_\chi: \mathbb{Z}/2 \times \mathbb{Z}/2 \to \{\pm 1\}$ by $\varepsilon_\chi(1,0) = \varepsilon(\theta\chi^{-1}, \psi_0)$ and $\varepsilon(0,1) = \varepsilon(\bar{\theta}\chi^{-1}, \psi_0)$. The value of ε_χ on $(1,1)$ is 1 or -1 depending on whether χ appears in the representation π of $GL(2,k)$ or in the representation π' of D_k^*. Corresponding to representations $\pi_+, \pi_-, \pi'_+, \pi'_-$, define characters $\varepsilon_+, \varepsilon_-, \varepsilon'_+, \varepsilon'_-$ by $\varepsilon_+ \equiv 1$, $\varepsilon_-(0,1) = -1$, $\varepsilon_-(0,1) = -1$, $\varepsilon'_+(1,0) = 1$, $\varepsilon'_+(0,1) = -1$, $\varepsilon'_-(1,0) = -1$, $\varepsilon'_-(0,1) = 1$. Let $\tilde{\pi}$ be any of the characters $\pi_+, \pi_-, \pi'_+, \pi'_-$, and $\varepsilon(\tilde{\pi})$ the corresponding character $\varepsilon_+, \varepsilon_-, \varepsilon'_+, \varepsilon'_-$. Then Theorem 4.2 can be reformulated to say that the multiplicity with which the character χ of K^* appears in any of the representations $\tilde{\pi}$ is

$$\langle \varepsilon(\tilde{\pi}), \varepsilon\rangle = \frac{1}{4}\sum \varepsilon(\tilde{\pi})(g)\varepsilon_\chi(g),$$

where the sum is over $g \in \mathbb{Z}/2 \times \mathbb{Z}/2$.

Finally, we note that if $\tilde{\pi}$ also denotes the restriction of $\tilde{\pi}$ to $SL(2,k)$ or $SL_1(D_k)$ as the case may be, and if we let χ^1 denote the restriction of χ to the subgroup K^1 of norm one elements of K^*, then the multiplicity $m(\tilde{\pi}, \chi^1)$ with which χ^1 appears in $\tilde{\pi}$ is

$$m(\tilde{\pi}, \chi^1) = \sum_\mu \langle \varepsilon(\tilde{\pi}), \varepsilon_{\chi\mu}\rangle$$

where the sum is over all the characters μ of K^* which are trivial on $K^1 \cdot k^*$.

We now make two remarks concerning the situation when a representation of $GL(2, k)$ is obtained from a quadratic field K but is restricted to L^* for $L \neq K$.

REMARK 4.5 Let $\pi = \pi_+ \oplus \pi_-$ be the decomposition of a representation π obtained from a character of K^* as a representation of $GL(2, k)^+$ as before. Let $L \neq K$ be a quadratic extension of k, and let $L' = \{l \in L^* | Nl \in NK^*\}$. Clearly, L' is contained in $GL(2, k)^+$. Since there is an element of L^* whose determinant does not lie in NK^*, and any such element permutes π_+ and π_-, any character of L' which appears in π_+ also appears in π_-. Since L' has index 2 in L^*, it follows that any character of L' appears with multiplicity $\leqslant 1$ in π_+ and π_-, and that the restriction to L' of a character θ of L^* appears in π_+ or π_- if and only if θ appears in π.

REMARK 4.6 Let $\pi = \pi_1 \oplus \pi_2 \oplus \pi_3 \oplus \pi_4$ be a representation of $GL(2, k)$ such that $\pi_1 \oplus \pi_i, i = 2, 3, 4$ is a representation of $GL(2, k)^+$ corresponding to three distinct quadratic fields K_2, K_3, K_4. Let $K'_2 = \{x \in K_2^* | Nx \in NK_3^*\}$. It follows from the previous remark that if a character of K'_2 appears in π_1 it does so with multiplicity 1, and then it also appears in π_2 with multiplicity 1 but does not appear in π_3 and π_4.

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