G. A. CAMERA
J. GIMENEZ

The nonlinear superposition operator acting on Bergman spaces


<http://www.numdam.org/item?id=CM_1994__93_1_23_0>
The nonlinear superposition operator
acting on Bergman spaces

G. A. CÁMERA & J. GIMÉNEZ
Department of Mathematics, I.V.I.C., Caracas, Venezuela

Received 30 August 1992; accepted in revised form 20 June 1993

1. Introduction

Let \( \Delta \) denote the unit disc \( \{ z : |z| < 1 \} \) in the complex plane and let \( H(\Delta) \) denote the space of analytic functions in \( \Delta \) with the topology of the uniform convergence in compact subsets of \( \Delta \). Given a function \( f : \mathbb{C} \to \mathbb{C} \) we associate to it the operator \( F_f \) defined by

\[
F_f(u)(z) = f(u(z)), \quad u \in H(\Delta).
\]

This operator is known as the autonomous nonlinear superposition (or composition) operator \[1\]. If \( A \) and \( B \) are linear subspaces of \( H(\Delta) \) and \( F_f(u) \in B \) whenever \( u \in A \) we shall say that \( F_f \) acts from \( A \) to \( B \). It is easy to see that if \( F_f \) acts from \( H(\Delta) \) to \( H(\Delta) \), then \( f \) must be an entire function and conversely. In this case mere action implies the continuity and the boundedness of the operator \[2\]. That mere action implies continuity has already been proved for various spaces of real functions, for instance \( L^p \) spaces \[6\] and Sobolev spaces \[7\]. Necessary and sufficient conditions have been given in \[2\] in order that \( F_f \) acts from \( H^p \) to \( H^q \), \( 0 < p, q \leq +\infty \), where \( H^p \) denotes the classical Hardy space in the unit disc. It is also true in this case that mere action implies continuity \[2\]. If \( N \) denotes the Nevanlinna space of functions in \( H(\Delta) \) of bounded characteristic then the actions from \( \bigcup_{p < q} H^q \) to \( N \) and from \( N \) to \( N \) have been studied in \[3\].

In this note we shall consider the problem of action and continuity between the Bergman space \( B_p \) defined by

\[
B_p = \{ u \in H(\Delta) : u \in L^p(dx \, dy) \}, \quad 0 < p < \infty.
\]

The space \( B_{\infty} \) is the usual one of bounded analytic functions in \( \Delta \). The topology in these spaces is given by the metric induced (when \( p \geq 1 \)) by

\[
\| u \|_{B_p} = \left( \frac{1}{\pi} \int \int_{\Delta} |u(z)|^p \, dx \, dy \right)^{1/p}.
\]
If \( p < 1 \) the topology induced by the metric \( \|u\|_{B_p}^p \) is used. We also consider the action between \( B_p \) and the Hardy space \( H^q \) and vice versa. For functions in \( H^p \) we use the standard notation

\[
M_p(r, u) = \left( \frac{1}{2\pi} \int_0^{2\pi} |u(r e^{i\theta})|^p \, d\theta \right)^{1/p} \quad \text{and} \quad \|u\|_p = \lim_{r \to 1} M_p(r, u).
\]

The symbol \( BN \) (which stands for Bergman-Nevanlinna) shall denote the set of functions \( u \) in \( H(\Delta) \) such that

\[
\int_\Delta \log^+ |u(z)| \, dx \, dy < \infty.
\]

Clearly \( H^p \subset B_p \) and \( B_p \subset BN \) for all \( p \). Finally, we study the action between Hardy functions and Bergman-Nevanlinna functions.

We would like to thank the referee for his helpful comments.

2. The action in \( B_p \)

We shall need the following lemma.

**Lemma 1.** Let \( 0 < p < \infty \). If \( u \in B_p \) then

\[
|u(z)| \leq \frac{\|u\|_{B_p}}{(1 - |z|)^{2/p}}, \quad z \in \Delta.
\]

**Proof.** This is an easy consequence of the subharmonicity of \( |u|^p \).

Next we are ready to prove the following result. In what follows the symbol \( \lceil s \rceil \) denotes the integer part of \( s \).

**Theorem 1.** Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function. Then \( f \) acts from \( B_p \) to \( B_q \), \( 0 < p, q \leq \infty \) if and only if \( f \) is a polynomial of degree less than or equal to \( \left\lfloor \frac{p}{q} \right\rfloor \).

**Proof.** If \( f \) is a polynomial of degree \( n \leq \left\lfloor \frac{p}{q} \right\rfloor \) then \( f \circ u \in B_q \), \( \forall u \in B_p \). In fact, it is enough to see that if \( k \leq \left\lfloor \frac{p}{q} \right\rfloor \), \( k \in \mathbb{N} \), then \( u^k \in B_q \). This is true since

\[
\|u^k\|_{B_q} = \left( \frac{1}{\pi} \int_\Delta |u(z)|^{kq} \, dx \, dy \right)^{1/q} = \left( \left( \frac{1}{\pi} \int_\Delta (|u(z)|^k \, dx \, dy \right)^{1/kq} \right)^k
\]
Next, we assume that $F_f$ acts from $B_p$ to $B_q$, $0 < p, q < \infty$. Let $\varepsilon > 0$ and define $u_{\varepsilon}^{(1)}(z) = \left(\frac{1}{1 - z} - \frac{1}{2}\right)^{2/p + \varepsilon}$. Clearly $u_{\varepsilon}^{(1)}$ belongs to $B_p$. Therefore $f \circ u_{\varepsilon}^{(1)} \in B_q$ and by Lemma 1 one can write

$$|f(u_{\varepsilon}^{(1)}(z))| \leq \frac{\|f \circ u_{\varepsilon}^{(1)}\|_{B_q}}{(1 - |z|)^{2/q}} \quad z \in \Delta. \quad (2.1)$$

Set $w_1 = u_{\varepsilon}^{(1)}(z)$. Assume first that $p < 1$ and take $\varepsilon < 1 - p$. Then the range set of $u_{\varepsilon}^{(1)}$ is $\mathbb{C} \setminus 0$. Given $w_1 \in \mathbb{C} \setminus 0$ let $z \in \Delta$ such that $w_1 = u_{\varepsilon}^{(1)}(z)$ and

$$|z| = \frac{|w_1^{p + \varepsilon/2} - \frac{1}{2}|}{|w_1^{p + \varepsilon/2} + \frac{1}{2}|}.$$

Thus, from (2.1) we get

$$|f(w_1)| \leq \frac{\|f \circ u_{\varepsilon}^{(1)}\|_{B_q}}{(1 - |w_1^{p + \varepsilon/2} - \frac{1}{2}|)^{2/q}} \leq \frac{\|f \circ u_{\varepsilon}^{(1)}\|_{B_q}}{(|w_1^{p + \varepsilon/2} + \frac{1}{2}| - |w_1^{p + \varepsilon/2} - \frac{1}{2}|)^{2/q}}.$$

If $w_1$ is such that $|w_1| > e^{1/2(p + \varepsilon)}$ then $z_1 = w_1^{p + \varepsilon/2}(-\pi \leq \text{Arg } w_1 < \pi)$ satisfies $\text{Re } z_1 > e^{1/4} \cos(-\pi/2(p + \varepsilon)) > 0$. In fact, since $\text{Arg } w_1 \geq -\pi$ then $((p + \varepsilon)/2) \text{Arg } w_1 \geq -\pi/2(p + \varepsilon)$. On the other hand

$$\text{Re } z_1 = |w_1|^{p + \varepsilon/2} \cos\left(\frac{p + \varepsilon}{2} \text{Arg } w_1\right) > |w_1|^{p + \varepsilon/2} \cos\left(-\frac{\pi}{2}(p + \varepsilon)\right).$$

Hence, if $|w_1| > e^{1/2(p + \varepsilon)}$ then $\text{Re } z_1 > e^{1/4} \cos(\pi/2(p + \varepsilon)) > 0$. Therefore, one can find a positive constant $c$ (depending on $\varepsilon$ and $p$) such that

$$|w_1^{p + \varepsilon/2} + \frac{1}{2}| - |w_1^{p + \varepsilon/2} - \frac{1}{2}| > c, \quad |w_1| > e^{1/2(p + \varepsilon)}.$$

If we use this inequality in (2.2) we obtain

$$|f(w_1)| \leq C(p, q, \varepsilon)|w_1|^{p + \varepsilon/q + 2^{-2/q}}.$$
for all $w_i$ such that $|w_i| > e^{1/2(p+\varepsilon)}$. Thus $f$ is a polynomial of degree less than or equal to $p + \varepsilon/q$. By letting $\varepsilon \to 0$ we obtain the desired result when $p < 1$.

Now, we assume that $p \geq 1$. Let
\[
S_1 = \left\{ w_1 : \frac{-\pi}{2(p+\varepsilon)} \leq \text{Arg} \ w_1 < \frac{\pi}{2(p+\varepsilon)} \right\}.
\]
If $w_1 \in S_1$ we choose $z_1$ such that $w_1 = z_1^{2/p+\varepsilon}$ with $-\pi/4 < \text{Arg} \ z_1 < \pi/4$. Thus $|(w_1^{p+\varepsilon/2} + \frac{1}{2}) - (w_1^{p+\varepsilon/2} - \frac{1}{2})|^{2/q} \geq c > 0$, for some constant $c$. Combining this inequality with (2.2) we obtain
\[
|f(w_1)| \leq \frac{\|f \circ u_1^{(1)}\|_{B_q}}{c} (|w_1|^{p+\varepsilon/2} + \frac{1}{2})^{2/q}.
\]
(2.3)

Let
\[
w_2 \in S_2 = \left\{ w_2 : \frac{-\pi}{2(p+\varepsilon)} \leq \text{Arg} \ w_2 < \frac{3\pi}{2(p+\varepsilon)} \right\}.
\]
and
\[
u_2^{(2)}(z) = \left( \frac{1}{1 - z} - \frac{1}{2} \right)^{2/p+\varepsilon} e^{i\pi/p+\varepsilon}.
\]
Clearly $u_2^{(2)} \in B_p$. By hypothesis $f \circ u_2^{(2)} \in B_q$ and by Lemma 1
\[
|f(u_2^{(2)}(z))| \leq \frac{\|f \circ u_2^{(2)}\|_{B_q}}{(1 - |z|)^{2/q}}, \quad z \in \Delta.
\]
(2.4)

Given $w_2 \in S_2$ we choose $z_2$ such that $|z_2| > 1$ and $w_2 = z_2^{2/p+\varepsilon} e^{i\pi/p+\varepsilon}$ with $-\pi/4 \leq \text{Arg} \ z_2 \leq \pi/4$. From (2.4) we obtain
\[
|f(w_2)| \leq \frac{\|f \circ u_2^{(2)}\|_{B_q}}{(1 - |w_2|^{p+\varepsilon/2} e^{i\pi/2} + \frac{1}{2})^{2/q}}
\]
\[
= \frac{\|f \circ u_2^{(2)}\|_{B_q} |w_2|^{p+\varepsilon/2} e^{i\pi/2} + \frac{1}{2})^{2/q}}{(1 - |w_2|^{p+\varepsilon/2} e^{i\pi/2} + \frac{1}{2})^{2/q}}.
\]
(2.5)

Since $-\pi/4 \leq \text{Arg} \ z_2 \leq \pi/4$ then
Hence, from (2.5) we obtain

$$|f(w_2)| \leq c^{-1} \| f \circ u_e^{(2)} \|_{B_q} |w_2^{p+\varepsilon/2} e^{-i\pi/2} + \frac{1}{2}|^{1/2/q}.$$ 

By repeating the same argument $n$ times, where $n$ is such that $A = C \setminus \bigcup_{i=1}^{n} S_i$, we obtain

$$|f(w)| \leq c_1(|w|^{p+\varepsilon/2} + \frac{1}{2})^{1/q}, \quad w \in C \setminus A,$$

where $c_1$ depends on $f$, $\varepsilon$ and $q$. This proves that $f$ is an entire function of order at most $p + \varepsilon/q$. Letting $\varepsilon$ tend to zero gives the desired result for $p \geq 1$.

3. Continuity of $F_f$

We shall prove in this section that if $F_f$ acts from $B_p$ to $B_q$ then it is necessarily continuous. We also prove local Lipschitzness.

First of all let us prove the following lemma.

**Lemma 2.** If $u_k \to u$ in $B_p$, $n \in \mathbb{N}$ and $n \leq \left[ \frac{p}{q} \right]$, then $u^{\ast}_k \to u^n$ in $B_q$.

**Proof.** The proof is similar to the analogue lemma given in [2]. We give it here somewhat simplified. The case $n = 1$ is obvious. Let us assume that $n > 1$. The functions $u^{\ast}_k$ and $u^n$ belong to $B_q$. In fact, since $nq \leq p$ then for every $u \in B_p$ we have

$$\|u^n\|_{B_q} \leq \|u\|^n_{B_p}.$$ 

On the other hand,

$$\|u^{\ast}_k - u^n\|_{B_q} = \left( \frac{1}{\pi} \int_{\Delta} \int_{\Delta} |u^{\ast}_k - u^n|^q \, dx \, dy \right)^{1/q}$$

$$= \left( \left( \frac{1}{\pi} \int_{\Delta} \int_{\Delta} |u^{\ast}_k - u^n|^{1/nq} \, dx \, dy \right)^{1/nq} \right)^n$$

$$\leq \left( \frac{1}{\pi} \int_{\Delta} \int_{\Delta} |u^{\ast}_k - u^n|^{p/n} \, dx \, dy \right)^{n/p}$$

$$= \left( \frac{1}{\pi} \int_{\Delta} \int_{\Delta} |u^{\ast}_k - u|^{p/n} |u_k - u|^{n-1} + \cdots + u^{n-1}|^{p/n} \, dx \, dy \right)^{\frac{n}{p}}.$$
Now we use Hölder inequality
\[ \left| \int fg \right| \leq \left( \int f^r \right)^{1/r} \left( \int g^s \right)^{1/s}, \]
with \( f = |u_k - u|^{p/n}, \ g = |u_k^{n-1} + \cdots + u^{n-1}|^{p/n}, r = n > 1 \) and
\[ \frac{1}{n} + \frac{1}{s} = 1 \left( s = \frac{n}{n - 1} \right), \]
and obtain
\[ \|u_k^n - u^n\|_{B_q} \leq \left( \frac{1}{\pi} \int \int_{\Delta} |u_k - u|^p \, dx \, dy \right)^{1/p} \]
\[ \left( \frac{1}{\pi} \int \int_{\Delta} |u_k^{n-1} + \cdots + u^{n-1}|^{p/n-1} \, dx \, dy \right)^{n-1/p} \]
\[ \leq c \|u_k - u\|_{B_p} \sum_{l=0}^{n-1} \|u_k^{n-1-l}u_l\|_{B_{p/n-1}}, \quad (3.1) \]
where \( c \) is a constant. Again by a refined version of Hölder inequality we get that \( u_k^{n-1-l}u_l \in B_{p/n-1}, \ l = 0, 1, \ldots, n - 1 \) and
\[ \|u_k^{n-1-l}u_l\|_{B_{p/n-1}} \leq \|u_k^{n-1-l}\|_{B_{p/n-1}} \|u_l\|_{B_{p/n}} = u_k^l \|u\|_{B_p} \quad (3.2) \]
This inequality implies that all summands on the right-hand side of (3.1) are bounded (for all \( k \)). Hence \( u_k^n \to u^n \) in \( B_q \) as required.

**THEOREM 2.** If \( F_f \) acts from \( B_p \) to \( B_q \) then it is necessarily continuous, bounded and locally Lipschitz.

**Proof.** Since \( F_f \) acts from \( B_p \) to \( B_q \) then, by Theorem 1, \( f \) is a polynomial of degree \( n \leq \left\lfloor \frac{p}{q} \right\rfloor \). Set \( f(z) = a_n z^n + \cdots + a_0 \). Let \( u_k(z) \to u(z) \), as \( k \to \infty \), in \( B_p \).

Then
\[ F_f(u_k)(z) - F_f(u)(z) = a_n (u_k^n(z) - u^n(z)) + \cdots + a_1 (u_k(z) - u(z)). \]

Thus
\[ \|F_f(u_k) - F_f(u)\|_{B_q} \leq C(\|u_k^n - u^n\|_{B_q} + \cdots + \|u_k - u\|_{B_q}) \to 0 \quad \text{as} \ k \to \infty \]
by Lemma 2. The boundedness of \( F_f \) comes from the inequality
\[ \|F_f(u)\|_{B_q} \leq C(\|u\|_{B_p}^n + \cdots + |a_1|\|u\|_{B_p} + |a_0|), \quad u \in B_p \]

which can be deduced from \( \|u^n\|_{B_q} \leq \|u\|_{B_p}^n \), for all \( u \in B_p \) and all \( n \leq \left\lfloor \frac{p}{q} \right\rfloor \).

In order to prove that \( F_f \) is locally lipschitz we must see that if \( u, v \in B(0, R) \subset B_p \) then there exists a constant \( C = C(p, q, R, f) \) such that
\[ \|F_f(u) - F_f(v)\|_{B_q} \leq C\|u - v\|_{B_p}. \]

On the one hand,
\[ \|F_f(u) - F_f(v)\|_{B_q} \leq C(\|u^n - v^n\|_{B_q} + \cdots + \|u - v\|_{B_q}). \]

On the other hand, in the same way that we deduced (3.1) and (3.2) we obtain
\[ \|u^n - v^n\|_{B_q} \leq C_1\|u - v\|_{B_p} \left( \sum_{l=0}^{n-1} \|u^{n-1-l} - v^{n-1-l}\|_{B_{pl-1}} \right) \leq C_1\|u - v\|_{B_p} \left( \sum_{l=0}^{n-1} \|u\|_{B_p}^{n-1-l} \|v\|_{B_p}^l \right) \leq C(p, q, R, f)\|u - v\|_{B_p}, \]

for all \( n \leq \left\lfloor \frac{p}{q} \right\rfloor \).

4. The action from \( B_p \) to the Bergman-Nevanlinna space

The Bergman-Nevanlinna space is defined by
\[ \text{BN} = \left\{ u \in H(\Delta) : \|u\|_{BN} = \frac{1}{\pi} \int_{\Delta} \log^+ |u(z)| \, dx \, dy < \infty \right\}. \]

It is easy to see that \( B_p \subset \text{BN}, \forall p > 0 \). The following result is an easy consequence of Theorem 1.

COROLLARY 1. Let \( f \) be an entire function such that \( F_f \) acts from \( \text{BN} \) to \( B_q \), \( 0 < q \leq \infty \). Then \( f \) is constant.

Proof. In particular \( F_f \) acts from \( B_p \) to \( B_q \), for all \( p > 0 \). Then from Theorem 1 we conclude that \( f \) is a polynomial of degree at most \( \left\lfloor \frac{p}{q} \right\rfloor \).

Taking \( p \) less than \( q \) one obtains the desired conclusion.

LEMMA 3. If \( u \in \text{BN} \) then
\[ \log^+ |u(z)| \leq \frac{\|u\|_{BN}}{(1 - |z|)^2}, \quad z \in \Delta. \]
This lemma is a consequence of the subharmonicity of \( \log^+ |u| \).

**THEOREM 3.** Let \( f \) be an entire function. Then \( F_f \) acts from \( \bigcup_{p<q} B_q \) to \( B_N \) \((0 < p < \infty)\) if and only if \( f \) has order at most \( p \).

*Proof.* Assume that \( F_f \) acts from \( \bigcup_{p<q} B_q \) to \( B_N \). Let \( \varepsilon > 0 \). The functions

\[
w_1 = u_\varepsilon(z) = \left( \frac{1}{1 - z} - \frac{1}{2} \right)^{2/p + \varepsilon}
\]

belong to \( \bigcup_{p<q} B_q \). Hence, \( f \circ u_\varepsilon \in B_N \) and so, by Lemma 3

\[
|f(w_1)| \leq e^{C/(1-|z|)^2}. \quad (4.1)
\]

If \( p < 1 \) we take \( \varepsilon \) so that \( p + \varepsilon < 1 \). Then the range of \( u_\varepsilon \) is \( \mathbb{C} \setminus 0 \). Thus, given \( w_1 \in \mathbb{C} \setminus 0 \) we take \( z \in \Delta \) such that

\[
|z| = \left| \frac{w_1^{p+\varepsilon/2} - \frac{1}{2}}{w_1^{p+\varepsilon/2} + \frac{1}{2}} \right|.
\]

From (4.1) we get

\[
\log |f(w_1)| \leq \frac{C}{(1 - |z|)^2}
\]

\[
= \frac{C}{\left( 1 - \left| \frac{w_1^{p+\varepsilon/2} - \frac{1}{2}}{w_1^{p+\varepsilon/2} + \frac{1}{2}} \right| \right)^2}
\]

\( \leq \frac{C|w_1^{p+\varepsilon/2} + \frac{1}{2}|^2}{(|w_1^{p+\varepsilon/2} + \frac{1}{2}| - |w_1^{p+\varepsilon/2} - \frac{1}{2}|)^2}. \quad (4.2)
\]

If \( |w_1| > e^{1/2(p+\varepsilon)} \) then \( z_1 = w_1^{p+\varepsilon/2}(-\pi \leq \text{Arg} \ w < \pi) \) satisfies

\[
\text{Re} \ z_1 > e^{1/4} \cos \left( -\frac{\pi}{2} (p + \varepsilon) \right) > 0.
\]

Therefore, there is a positive constant \( C \) such that

\[
|w_1^{p+\varepsilon/2} + \frac{1}{2}| - |w_1^{p+\varepsilon/2} - \frac{1}{2}| > C, \ |w_1| > e^{1/2(p+\varepsilon)}. \quad (4.3)
\]

Combining (4.2) and (4.3) we obtain

\[
\log |f(w_1)| \leq C_1|w_1^{p+\varepsilon/2} + \frac{1}{2}|^2,
\]

for a suitable positive constant \( C_1 \) and all \( w_1 \) such that \( |w_1| > e^{1/2(p+\varepsilon)} \). This
shows that \( f \) is of order at most \( p + \varepsilon \). Letting \( \varepsilon \) tend to zero permits us to conclude that \( f \) is of order at most \( p \).

Next, we assume that \( p \geq 1 \). In this case we argue as in Theorem 1 and obtain from (4.2) that

\[
\log |f(w)| = O(|w|^{p+\varepsilon})
\]

for all \( w \) outside a ball. Hence \( f \) is an entire function of order at most \( p \).

Let us suppose now that \( p > 1 \). In this case we argue as in Theorem 1 and obtain from (4.2) that

\[
\log^+ M(r, f) \leq r^{p+\varepsilon} + C, \quad \forall r \geq 0.
\]

To prove that \( f \circ u \in BN \) we write

\[
\int_{\Delta} \int \log^+ |f(u(z))| \, dx \, dy = \int_0^1 r \, dr \int_0^{2\pi} \log^+ |f(u(r e^{i\theta}))| \, d\theta
\]

\[
\leq \int_0^1 r \, dr \int_0^{2\pi} \log^+ M(|u(r e^{i\theta})|, f) \, d\theta \leq \int_0^1 r \, dr \int_0^{2\pi} |u(r e^{i\theta})|^{p+\varepsilon} \, d\theta + \pi C
\]

\[
= \int_{\Delta} \int |u(z)|^{p+\varepsilon} \, dx \, dy + \pi C < \infty,
\]

since \( u \in B_{p+\varepsilon} \).

**THEOREM 4.** Let \( f \) be an entire function of order less than \( p \) or of order \( p \) and finite type \( 0 < p < \infty \). Then \( F_f \) acts from \( B_p \) to \( BN \).

**Proof.** We may assume that \( f \) is of order \( p \) and finite type \( \sigma - \delta > 0 \). Hence, there is a constant \( C \) such that

\[
\log^+ M(r, f) \leq \sigma r^p + C, \quad r \geq 0.
\]

If \( u \in B_p \) then

\[
\int_{\Delta} \int \log^+ |f(u(z))| \, dx \, dy = \int_0^1 r \, dr \int_0^{2\pi} \log^+ |f(u(r e^{i\theta}))| \, d\theta
\]

\[
\leq \sigma \int_{\Delta} \int |u(z)|^p \, dx \, dy + \pi C < \infty,
\]

as required.
5. Transforming Hardy functions into Bergman functions and vice versa

We shall begin by stating the following classical results by Hardy and Littlewood [5].

**LEMMA 4.** Let \( u \) be analytic in \( \Delta \) and

\[
M_p(r, u) \leq \frac{C}{(1 - r)^\beta}, \quad 0 < p < \infty, \quad \beta \geq 0.
\]

Then there is a constant \( K = K(p, \beta) \) such that

\[
M_{q_1}(r, u) \leq \frac{KC}{(1 - r)^{\beta + 1/p - 1/q_1}}, \quad p < q_1 \leq \infty.
\]

A proof of this result can be found in [4, p. 84].

**LEMMA 5.** If \( 0 < p < q_1 \leq \infty \), \( u \in H^p \), \( \beta \geq p \), and \( \alpha = 1/p - 1/q_1 \) then

\[
\int_0^1 (1 - r)^{\alpha - 1} M_{q_1}(r, f)^{\lambda} \, dr < \infty.
\]

The reader can find a proof of this result in [4, p. 87].

The next result shows that one cannot transform Bergman functions into Hardy functions by means of nonlinear superposition. In case \( p = \infty \) it is trivial that \( F_f \) acts for any \( f \).

**THEOREM 5.** Let \( f \) be an entire function. If \( p \neq \infty \) then \( F_f \) acts from \( B_p \) to \( H^q \) if and only if \( f \) is constant.

*Proof.* If \( F_f \) acts from \( B_p \) to \( H^q \) then, by Theorem 1, \( f \) is a polynomial. Now we get the desired conclusion by noting that for a non-constant polynomial \( f \) it is not true that \( f \circ u \in H^q \), \( \forall u \in B_p \). If this were true then the zeros of all Bergman functions would have to satisfy the Blaschke condition, and this is false.

**THEOREM 6.** Let \( f \) be an entire function. Then \( F_f \) acts from \( H^p \) to \( B_q \) if and only if \( f \) is a polynomial of degree at most \( \left[ \frac{2p}{q} \right] \).

If \( p = \infty \) then \( F_f \) acts from \( H^p \) to \( B_q \) for any \( f \). In the proof of this theorem we shall rule out this case.

**COROLLARY 2.** The operator \( F_f \) acts from \( H^p \) to \( B_p \) if and only if \( f \) is a polynomial of degree one or two.
Proof of Theorem 6. The proof that $f$ must be a polynomial of degree less than or equal to $\left\lfloor \frac{2p}{q} \right\rfloor$ can be done as in Theorem 1. Let us assume now that $f$ is a polynomial of degree $n \leq \left\lfloor \frac{2p}{q} \right\rfloor$. We shall prove that if $u \in H^p$ then $u^n \in B_q$. Let us suppose, first of all, that $n < \frac{2p}{q}$. Then

$$
\iint_{\Delta} |u''(z)|^q \, dx \, dy = \int_0^1 r \, dr \int_0^{2\pi} |u(r e^{i\theta})|^n q \, d\theta \\
\leq 2\pi \int_0^1 r \, dr \left( \frac{1}{2\pi} \int_0^{2\pi} |u(r e^{i\theta})|^{2p} \, d\theta \right)^{nq/2p} \\
= 2\pi \int_0^1 r M_{2p}(r, u)^n r \, dr.
$$

Now, using Lemma 4 with $\beta = 0$, $q_1 = 2p$ we obtain

$$M_{2p}(r, u) \leq \frac{C}{(1 - r)^{1/p - 1/2p}} = \frac{C}{(1 - r)^{1/2p}},$$

for some constant $C$. Combining this inequality with (5.1) one gets

$$\iint_{\Delta} |u''(z)|^q \, dx \, dy \leq 2\pi C \int_0^1 \frac{r \, dr}{(1 - r)^{nq/2p}} < \infty.$$  

Thus $u^n \in B_q$, as required.

Next we assume that $n = \frac{2p}{q}$. In this case we use Lemma 5 with $q_1 = 2p$, $\alpha = 1/2p$, and $\lambda = 2p$ to conclude that

$$\int_0^1 M_{2p}(r, u)^{2p} \, dr < \infty$$

as required.

6. The action from $H^p$ to $B^N$.

If $p = \infty$ and $f$ is any entire function then $F_f$ acts from $H^\infty$ to $H^\infty$ and consequently it acts from $H^\infty$ to $B^N$. When $p < \infty$ we have the following result.
THEOREM 7. Let \( f : \mathbb{C} \to \mathbb{C} \). Then \( F_f \) acts from \( \bigcup_{p<q} H^q \) to \( BN \) if and only if \( f \) is an entire function of order at most 2p.

Proof. Let us assume that \( F_f \) acts from \( \bigcup_{p<q} H^q \) to \( BN \). Clearly \( f \) must be an entire function. On the other hand, given \( \varepsilon > 0 \), the function

\[
u_\varepsilon(z) = \left\{ \frac{1}{1 - z} - \frac{1}{2}\right\}^{1/p + \varepsilon}\]

belongs to \( \bigcup_{p<q} H^q \). Using these functions and Lemma 3 and arguing as in Theorem 1 we deduce that \( f \) has order at most 2p. Conversely, let us suppose that \( f \) has order at most 2p. Let \( u \in \bigcup_{p<q} H^q \) and \( \varepsilon > 0 \) such that \( u \in H^{p+\varepsilon} \). Next we take a constant \( C \) such that

\[
\log^+ M(r, f) \leq r^{2(p+\varepsilon)} + C, \quad \forall r \geq 0,
\]

and use this inequality to get

\[
\iint_\Delta \log^+ |f(u(z))| \, dx \, dy \leq \iint_\Delta |u(z)|^{2(p+\varepsilon)} \, dx \, dy + \pi C < \infty,
\]

since \( u^2 \in B_{p+\varepsilon} \) in view of Corollary 2.

As a corollary we have

COROLLARY 3. If \( u \in N \), \( u(z) \neq 0 \) in \( \Delta \), and \( \log u(z) \neq 0 \) in \( \Delta \), then \( e^{(\log u)^2 - \varepsilon} \in BN \), \( \forall \varepsilon, 0 < \varepsilon \leq 2 \).

Proof. For \( \varepsilon = 0 \) the result breaks down as it is shown by the example

\[
u(z) = \exp\left\{\frac{1 + z}{1 - z}\right\},
\]

To prove the corollary we proceed as follows. Since \( u \) does not vanish in \( \Delta \) then \( \log u \) is analytic there. Moreover \( \log |u| \in h^1 \), where \( h^1 \) is the space of harmonic functions in \( \Delta \) which satisfy the Riesz-Herglotz representation. This can be seen from the following relations

\[
\int_0^{2\pi} \log |u(r e^{i\theta})| \, d\theta = \int_0^{2\pi} \log^+ |u(r e^{i\theta})| \, d\theta + \int_0^{2\pi} \log^- |u(r e^{i\theta})| \, d\theta
\]

\[
= 2 \int_0^{2\pi} \log^+ |u(r e^{i\theta})| \, d\theta - 2\pi \log |u(0)|,
\]

and the fact that \( u \in N \). Then \( \log u \in H^p \), \( \forall p < 1 \). Thus \( (\log u)^{2+\varepsilon} \in \bigcup_{1/2 < q} H^q \).
Since \( f(z) = e^z \) is an entire function of order 1 then, by Theorem 7, \( e^{(\log u)^2} \in BN \), as required.

Finally, we have

**Theorem 8.** Let \( f \) be an entire function of order less than \( 2p \) or of order \( 2p \) and finite type. Then \( F_f \) acts from \( H^p \) to \( BN \).

**Proof.** We may assume that \( f \) is of order \( 2p \) and finite type \( \sigma - \delta > 0 \). There is a constant \( C \) such that

\[
\log M(r, f) \leq \sigma r^{2p} + C, \quad r \geq 0.
\]

If \( u \in H^p \) we get from the last inequality

\[
\int_A \log^+ |f(u(z))| \, dx \, dy \leq \sigma \int_A |u(z)|^{2p} \, dx \, dy + \pi C < \infty,
\]

since \( u^2 \in B_p \) in view of Corollary 2.

### 7. Some open questions

We finish this article by posing some questions. (1) If \( F_f \) acts from \( B_p \) to \( BN \) then, in particular, it acts from \( \bigcup_{p < q} B_q \) to \( BN \) and, by Theorem 3, it has order at most \( p \). In case that \( f \) has order \( p \) is it true that \( f \) has finite type? (2) One may ask the corresponding question for the action between \( H^p \) and \( BN \).

### References