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A factorization theorem for the polar of a curve with two branches

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Introduction

Let $f \in \mathbb{C}[[X, Y]]$ be a reduced formal series over the complex field $\mathbb{C}$ (i.e. $f = \Pi_1^d f_i$ where the f_i's are irreducible and $f_i \neq f_j$ if $i \neq j$), and let $h \in \mathbb{C}[[X, Y]]$ be a regular parameter (i.e. h defines a nonsingular plane algebroid curve). The polar of f with respect to h, $P(f, h)$, is the algebroid curve defined by:

$$J(f, h) = \det \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial h} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} \end{pmatrix} = 0$$

Examples by Pham show that the topological type of $P(f, h)$ depends on the analytic type of C, the curve defined by $f = 0$, and not only on its topological type, even for h transversal to f. However we may wonder what information of $P(f, h)$ depends on the topological type of C. Roughly speaking:

Assuming the topological type of C fixed,
What can we say about the topological type of $P(f, h)$?

The best results about this question have been obtained by Lê, Michel and Weber ([LMW], [LMW2]) for the case in which h is transversal to f, by using topological methods: let $\pi: X \mapsto \mathbb{C}^2$ be the canonical resolution of the germ C, E the exceptional divisor of π. Then, $\tilde{P}$, the strict transform of $P(f, h)$, does not meet the strict transform of C, and one can determine (not completely) the components of E which meet $\tilde{P}$. As a consequence, they also compute the set of *polar quotients*, that is, the set

$$\left\{ \frac{(f, \varphi)}{m(\varphi)} \,\middle|\, \varphi \text{ is an irreducible component of } J(f, h) \right\}$$

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where (f, φ) denotes the intersection multiplicity of f with φ and $m(\varphi)$ is the multiplicity of φ. Polar quotients were introduced by Teissier and depend on the topological type of C only ([T]). Similar computations have been made by Kuo and Lu in [KL] and by Steenbrink and Zucker in [SZ].

When f is irreducible, the first methods used to study the problem above were of arithmetical nature. Merle in [M] proves that if h is transversal to f, then $J(f, h)$ can be factorized as $\Gamma_1 \cdots \Gamma_g$, where g is the number of Puiseux pairs of f and each Γ_i is a product of branches with constant polar quotient. The polar quotient corresponding to branches of Γ_i and the multiplicity of Γ_i can be explicitly computed in terms of the minimal set of generators of the semigroup of values of f (the factorization Theorem is also proved using the arithmetical properties of this semigroup). The i-th polar quotient, corresponding to a branch in Γ_i, is equal to the *coefficient of contact*, $(f, \psi)/m(\psi)$, for a curve ψ of genus $i-1$, having maximal contact with f, that is, ψ has the maximal possible intersection multiplicity with f among the curves with $i-1$ Puiseux pairs. From this, one finds the set of free infinitely near singular points that a branch of Γ_i and the curve f have in common and, as consequence, one obtains the results in [LMW2] in a more precise form. Notice that the first $i-1$ Puiseux pairs of a branch of Γ_i are equal to the corresponding ones for f. Ephraim in [E] extends Merle's result to any regular h, by using the *(f, h)-sequence* instead of the minimal set of generators. Finally, both results are generalized by Granja ([G]) to a larger class of curves which includes the polars as a particular case. We want to remark that the only data used in the results above are the intersection multiplicities of $J(f, h)$ with f and h, and no other properties of polars.

The arithmetical description of the semigroup of values of a curve with several branches given in [D], (and in [B] for the case $d = 2$) induces us to try to obtain similar results for the polar of a non irreducible f using the arithmetic of the semigroup and taking as only data the intersection multiplicities of $J(f, h)$ with the branches $f_1, \ldots, f_d$ of f and h, that is

$$(J(f,\ h),\ f_1),\ (J(f,\ h),\ f_2), \ldots, (J(f,\ h),\ f_d),\ (J(f,\ h),\ h).$$

In this paper we carry out this program for the case of two branches ($d = 2$) and h not necessarily transversal to f. Unfortunately this is not possible for more than two branches ($d \geqslant 3$) as the example (4.12) shows. So, for $d \geqslant 3$ further properties of $J(f,\ h)$ are necessary in order to give a "good" factorization or in order to refine the results in [LMW2].

The main results proved here go in parallel with the ones proved by Merle and Ephraim. There is also an interpretation in terms of the resolution process for the singularity of $f = 0$, included in section 4. Taking into account that the topological type of a curve singularity is an equivalent data to the semigroup of values (or to the resolution process), the topological type of $P(f, h)$ has some restrictions imposed by that one of C, as well as some "similarities" with it.

More precisely, assume for the sake of simplicity that h is transversal to the curve with two branches C defined by $f = f_1 f_2 = 0$ (in the paper there is no restriction for h). Denote by $S \subseteq \mathbb{Z}_+^2$ the semigroup of values of f and for any $\varphi \in \mathbb{C}[[X, Y]]$ let $I(f, \varphi) = ((f_1, \varphi)/m(\varphi), (f_2, \varphi)/m(\varphi))$. If φ is an irreducible component of $J = J(f, h)$ we will say that $I(f, \varphi)$ is a polar multi-quotient of f. Then J has a factorization

$$J = \Gamma_1 \cdots \Gamma_q \cdot \Gamma_{q+1}^1 \cdots \Gamma_s^1 \cdot \Gamma_{q+1}^2 \cdots \Gamma_t^2 \cdot D$$

with the following properties:

The numbers s and t are the number of Puiseux pairs of the branches defined by f_1 and f_2 respectively. Each factor is a product of branches of J with constant polar multi-quotient. The polar multi-quotient corresponding to one of the factors, Γ, its multiplicity and the value in S of Γ — that is, the ordered pair of natural numbers $((\Gamma, f_1), (\Gamma, f_2))$ — can be explicitly computed in terms of certain set of elements of S, called the set of values of the maximal contact. This set plays a role similar to that of the minimal set of generators in the case of an irreducible f (notice that the semigroup of values of a curve with several branches is not finitely generated, but can be "determined" by a finite number of elements, see [D]).

Given $\varphi \in \Gamma_i^1$, $i \geqslant q + 1$, the first coordinate of $I(f, \varphi)$, $(f_1, \phi)/m(\varphi)$ is the i-th polar quotient of the branch f_1 (the second one can be easily determined in terms of the first one and (f_1, f_2)), so we can determine in a precise form the set of free infinitely near singular points in common for φ and f_1. Similar results are true for the irreducible components of Γ_i^2, $i \geqslant q + 1$. The irreducible components, φ, of the first q factors, Γ_i $(1 \leqslant i \leqslant q)$, satisfy

$$(f_1, \varphi)/m(\varphi) = (f_2, \varphi)/m(\varphi)$$

and this rational number is equal to the i-th polar quotient of f_1 (or f_2) so we can make a similar interpretation in terms of the infinitely near singular points. Finally, D corresponds, in general, to the irreducible factors of J which have in common with f the set of common infinitely near singular points of f_1 and f_2. In other words, the branches φ of D become transversal to f exactly in the same step of the resolution procedure in which f_1 and f_2 become transversal. There are some cases in which the existence of such a D cannot be guaranteed because its components are, from the point of view of arithmetic, undistinguishable from the components of Γ_q or Γ^i_{q+1}; see Theorems (3.11), (3.12) and (4.9) for a precise statement taking into account all the possible cases.

Now, let $\pi: X \to \mathbb{C}^2$ be the minimal embedded resolution for the germ C, C' the total transform of C and E the exceptional divisor. We can translate the results above in these terms, obtaining the components of E appearing in the resolution procedure for $P = P(f, h)$ (not completely, because only the free infinitely near singular points are determined) or, in other words, the irreducible components E_i of E such that $E_i \cap \tilde{P} \neq \emptyset$, $\tilde{P}$ being the strict transform of P by π. Moreover, as a consequence, the polar multi-quotients are in $1-1$ correspondence with the "rupture divisors" of E (that means irreducible components F of E such that $\# F \cap \overline{(C' - F)} \geqslant 3$). In fact, the set of polar multi-quotients can be realized as the set $\{I(f, \pi(\xi_F))\}$, where F belongs to the set of "rupture divisors" and ξ_F is the germ of a smooth curve in X with normal crossings with E at a point of F.

Notice that the point of view of this paper is rather different to the one of Casas in [Ca]. He proves that if f is a "general" element in the set of curves with a prefixed topological type, then the equisingularity type of the generic polar can be completely determined.

Briefly, the contents of the different sections are as follows:

In section 1 we fix the notation and prove some results involving the coefficient of contact (in the sense of Hironaka) of two branches f and g with respect to h, that is, the rational number $(f, g)/(g, h)$. The main arithmetical properties needed in the following sections are stated.

In section 2 we prove a factorization theorem for curves g such that (f, g) (for f irreducible) belongs to an *enlarged Apery basis* of f with respect to h (see (2.2) for the definition of such *enlarged* Apery basis). These results are needed in order to prove a factorization theorem (at the beginning of the third section) for the polar of a curve with d branches in a very special case named here *diagonal case* (d branches with the same equisingularity type and high intersection multiplicity between pairs of branches), and also provides an important tool for the proof of the factorization in the two branch case.

Section 3 is devoted to proving the main results, namely the factorization of $J(f, h)$, for a curve f with two branches, into packages with constant polar multi-quotient.

Finally, the last part is devoted to the comparison with the results in [LMW2] when h is transversal to f and to extend these geometrical interpretations to arbitrary h. At the end of this part we include a counterexample of the factorization theorems when one takes $d = 3$ instead of $d = 2$.

1. Preliminary facts

Throughout all this paper k will denote an algebraically closed field and $h \in k[[X, Y]]$ an irreducible regular parameter, that is, h defines a nonsingular plane algebroid curve over the ground field k.

(1.1) Let C be an irreducible algebroid plane curve defined by an irreducible formal series $f \in k[[X, Y]]$. Denote by $S(C)$, $S(f)$, or simply S if confusion is not possible, the semigroup of values of C, that is, the set

$$\begin{aligned} S &= \{(f, g) \mid g \in k[[X, Y]], g \notin (f)\} \\ &= \{v(g) \mid g \in \mathcal{O} = k[[X, Y]]/(f), g \neq 0\}, \end{aligned}$$

(f, g) denoting the intersection multiplicity between f and g, and v the normalized valuation corresponding to C, that is, the valuation associated to the valuation ring $\overline{\mathcal{O}}$, normalization of $\mathcal{O}$ in its field of fractions.

(1.2) The maximal contact values of C, $\bar{\beta}_0, \ldots, \bar{\beta}_g$ (see [Z], [C]) can be defined as the minimal set of generators of S in the following way:

$$\bar{\beta}_0 := \min(S - \{0\}),$$

and for $i \geqslant 1$,

$$\begin{aligned} \bar{\beta}_i &= \min\left\{\gamma \in S \,\middle|\, \gamma \notin \sum_{j=0}^{i-1} \mathbb{Z}_+ \bar{\beta}_j\right\} \\ &= \min\{\gamma \in S \mid \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_{i-1}, \gamma) < \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_{i-1})\}. \end{aligned}$$

It is well known (see [Z], [C]) that, if $\{\beta_0, \ldots, \beta_{g'}\}$ denotes the set of characteristic exponents of C, then $g = g'$, $e_i = \gcd(\beta_0, \ldots, \beta_i) = \gcd(\bar{\beta}_0, \ldots, \bar{\beta}_i)$,

$$\bar{\beta}_0 = \beta_0, \quad \bar{\beta}_1 = \beta_1, \qquad \text{and}$$

$$\bar{\beta}_{i+1} = \frac{e_{i-1}}{e_i} \bar{\beta}_i + \beta_{i+1} - \beta_i \qquad \forall\, i \geqslant 1.$$

These equalities provide the equivalence between the two different sets of exponents.

(1.2.1) REMARK. Recall that in characteristic 0 the set of characteristic exponents can be easily defined in terms of a Puiseux expansion for f: Assume that x is transversal to f, denote by n the multiplicity of f and let $y = \Sigma_{i \geqslant 0} a_i x^{i/n}$ a Puiseux expansion for f. Then $\beta_0 = n$ and, recursively, β_i is the minimum integer k such that $a_k \neq 0$ and $\gcd(\beta_0, \ldots, \beta_{i-1}, k) < \gcd(\beta_0, \ldots, \beta_{i-1})$. The integer g is usually called the "genus" of the curve singularity given by f and $\bar{\beta}_i$ is the highest possible contact of f with an irreducible curve of genus $i - 1$. In particular $\bar{\beta}_1$ is the highest possible contact of f with a smooth curve, $\bar{\beta}_1 = \max\{(f, g) \mid g \in k[[X, Y]] \text{ regular}\}$. When the characteristic of k is positive, same properties hold, now in terms of Hamburger-Noether expansions instead of Puiseux expansions (see Campillo [C]).

(1.3) Let m be the intersection multiplicity between f and h: $m = v(h) = (f, h) \in S$. Then $\bar{\beta}_0 \leqslant m \leqslant \bar{\beta}_1$, and the *m-sequence*, $v_0, \ldots, v_s$, of S is defined as follows (see [A], [E], [P]):

$$v_0 = m; \; d_0 = v_0$$

and for $i \geqslant 1$

$$v_i = \min\{\gamma \in S \mid \gcd(d_{i-1}, \gamma) < d_{i-1}\}; \quad d_i = \gcd(d_{i-1}, v_i).$$

This procedure stops when we find $s \in \mathbb{N}$ such that $d_s = 1$. So, $\{v_0, \ldots, v_s\} \subset S$ and $d_0 > d_1 > \cdots > d_s = 1$. Note that the $\bar{\beta}_0$*-sequence* (obtained taking h transversal to f) is exactly the set of values of the maximal contact. The m-sequence can be seen as a system of generators for S *with respect* to m (or h). Associated with the *m-sequence* $v_0, \ldots, v_s$ we define the natural numbers

$$M_i = \frac{d_{i-1}}{d_i} \quad \text{and} \quad \delta_i = \frac{v_0}{d_{i-1}} \quad \text{for } i = 1, \ldots, s.$$

(1.4) Depending on the different possibilities for m, namely, $m = k\bar{\beta}_0$ $(1 \leqslant k \leqslant [\bar{\beta}_1/\bar{\beta}_0])$ or $m = \bar{\beta}_1$, the relationship between the *m-sequence* and the minimal set of generators of S are as follows:

(a) If $m = v_0 = \beta_0$ ($\Leftrightarrow v_1 > v_0$), then $s = g$ and $v_i = \bar{\beta}_i$ for $i = 1, \ldots, g$.

(*b*) If $v_0 = m = k\bar{\beta}_0$ for some integer $k > 1$ ($\Leftrightarrow d_1 = v_1$) then $s = g + 1$, $v_1 = \bar{\beta}_0$ and $v_i = \bar{\beta}_{i-1}$ for $i = 2, \ldots, g + 1$. Note that, as consequence,

$$d_1 = \bar{\beta}_0 = e_0, \quad d_i = e_{i-1} \quad \text{for } i \geqslant 2,$$

$$M_1 = k, \quad M_i = N_{i-1} = \frac{e_{i-2}}{e_{i-1}} \quad \text{for } i \geqslant 2,$$

$$\delta_1 = 1, \quad \delta_2 = k, \quad \text{and} \quad \delta_i = k\,\frac{\bar{\beta}_0}{e_{i-2}} \quad \text{for } i \geqslant 2.$$

(c) If $v_0 = m = \bar{\beta}_1 (\Leftrightarrow d_1 < v_1 < v_0)$, then $s = g$ and $v_1 = \bar{\beta}_0$, $v_i = \bar{\beta}_i$ for $i = 2, \ldots, g$. As consequence,

$$d_0 = \bar{\beta}_1, \; d_i = e_i \quad \text{for } i \geqslant 1,$$

$$M_1 = \frac{\bar{\beta}_1}{e_1}, \quad M_i = N_i \quad \text{for } i \geqslant 2,$$

$$\text{and } \delta_i = \frac{\bar{\beta}_1}{e_{i-1}} \quad \text{for } i \geqslant 1.$$

The minimal set of generators of the semigroup of a plane curve, $\{\bar{\beta}_0, \ldots, \bar{\beta}_g\}$, satisfies some well-known and important properties. The main ones for our purposes are

$$N_i\bar{\beta}_i \in \langle \bar{\beta}_0, \ldots, \bar{\beta}_{i-1} \rangle \quad \text{and} \quad N_i\bar{\beta}_i < \bar{\beta}_{i+1} \qquad i \geqslant 1.$$

These properties characterize the subsemigroups of $\mathbb{N}$ which are the semigroup of values for some plane algebroid curve. The first one is related to the property of complete intersection for a monomial curve, see Azevedo [Az], Herzog [He] and Angermüller [An].

Similar properties can be easily proved, using computations above, for m-sequences, namely:

(i) $M_i v_i \in \langle v_0, \ldots, v_{i-1} \rangle := \Sigma_0^{i-1}\, \mathbb{Z}_+ v_j$ for $i \geqslant 1$.

(ii) $M_i v_i < v_{i+1}$ for $i \geqslant 1$.

(iii) If $\alpha \in S$, then α can be written in a unique way as $\alpha = \Sigma_{i=0}^{s} \alpha_i v_i$ with $\alpha_0 \geqslant 0$ and $0 \leqslant \alpha_i < M_i$ for $1 \leqslant i \leqslant s$.

(1.5) The *contact pair*, $(f \mid g)$, for two irreducible branches f and g, is defined in [D, (3.3)] in terms of the Hamburger-Noether expansions of f and g. This pair of integers, $(f \mid g) = (q, c)$, can be also characterized in the following way:

Denote by $\bar{\beta}'_0, \ldots, \bar{\beta}'_{g'}$, $e'_0, \ldots, e'_{g'}$ the numbers defined in (1.2) for the curve g and let t be the minimum integer such that

$$(f, g) \leqslant \min\{e'_t\bar{\beta}_{t+1}, e_t\bar{\beta}'_{t+1}\} = p(t).$$

(Note that this integer always exists, setting $\overline{\beta}_{g+1} = \overline{\beta}_{g'+1} = \infty$ if it is necessary.) Denote by l_t (resp. l'_t) the integer part of $(\overline{\beta}_{t+1} - N_t\overline{\beta}_t)/e_t$ (resp. of $(\overline{\beta}'_{t+1} - N'_t\overline{\beta}'_t)/e'_t$). Then,

— If $(f, g) < p(t)$, there exists an integer c, $0 < c \leqslant \min\{l_t, l'_t\}$, such that

$$(f, g) = e'_{t-1}\overline{\beta}_t + ce_te'_t = e_{t-1}\overline{\beta}'_t + ce_te'_t.$$

In this case $q = t$ and $(f \mid g) = (t, c)$.

— If $(f, g) = p(t)$ and $e'_t\overline{\beta}_{t+1} \neq e_t\overline{\beta}'_{t+1}$ then $(q, c) = (t, \infty)$. (In [D] this case appears as $(t, \min\{l_t + 1, l'_t + 1\})$). Recall that, if $\xi = e'_t\overline{\beta}_{t+1} < e_t\overline{\beta}'_{t+1}$ then $l_t \leqslant l'_t$ and if $l_t < l'_t$ then $\xi = e'_t\overline{\beta}_{t+1} < e_t\overline{\beta}'_{t+1}$ ([D]).

— Finally, if $(f, g) = p(t)$ and $e'_t\overline{\beta}_{t+1} = e_t\overline{\beta}'_{t+1}$, $(q, c) = (t + 1, 0)$.

Note that, if we set

$$\rho = \max\{n \mid \overline{\beta}_0\overline{\beta}'_i = \overline{\beta}'_0\overline{\beta}_i \text{ for } i = 1, \ldots, n\},$$

the possible intersection multiplicities between two branches with the same singularity types as f and g (that is, the same maximal contact values or the same characteristic exponents) are the elements in the set

$$\{\xi(q, c) \mid 0 \leqslant q \leqslant \rho;\ 0 \leqslant c \leqslant \min\{l_q, l'_q\}\} \cup \{\xi(\infty)\},$$

where $\xi(q, c) = e'_{q-1}\overline{\beta}_q + ce_qe'_q = e_{q-1}\overline{\beta}'_q + ce_qe'_q$ and $\xi(\infty) = \min\{e'_\rho\overline{\beta}_{\rho+1}, e_\rho\overline{\beta}'_{\rho+1}\}$. These elements correspond one to one to the set of possible contact pairs:

$$\{(q, c) \mid 0 \leqslant q \leqslant \rho;\ 0 \leqslant c \leqslant \min\{l_q, l'_q\}\} \cup \{(\rho, \infty)\}.$$

Moreover, (taking the lexicographic order in $(\mathbb{N} \cup \{\infty\})^2$),

$$\xi(q, c) < \xi(q', c') \Leftrightarrow (q, c) < (q', c').$$

(1.5.1) REMARK. Suppose k of characteristic 0 and x transversal to f and g, the contact pair can be characterized in terms of Puiseux expansions for f and g as follows: Denote by n (resp. n') the multiplicity of f (resp. of g) and let $y = \Sigma_{i\geqslant 0} a_ix^{i/n}$, $y' = \Sigma_{i\geqslant 0} a'_ix^{i/n'}$ be Puiseux expansions for f and g respectively.

Assume that $(f \mid g) = (q, c)$ with $c < \infty$. Then there exists a n'-th root of the unit, ω, such that $a_i = a'_i\omega^i$ for any i such that $i/n < \gamma$, where

$\gamma = (\beta_q + ce_q)/n = (\beta'_q + ce'_q)/n'$ and, moreover, γ is the greatest exponent with these conditions. If $c = \infty$, the same property holds with $\gamma = \min\{\beta_{q+1}/n, \beta'_{q+1}/n'\}$; in this case $\beta_{q+1}/n \neq \beta'_{q+1}/n'$. These properties permit, in the characteristic 0 case, to define the contact pair and to prove in an easy way the properties above for it (see [Z2]). As in the Remark (1.2.1) the positive characteristic case can be stated in a similar way using Hamburger-Noether expansions ([D]).

(1.6) Similar statements to those obtained in (1.5) can be given taking the *m-sequences* for f and g with respect to a fixed regular parameter h instead of the sets of values of the maximal contact. Denote by $v'_0, \dots, v'_t$, $d'_0, \dots, d'_t, \dots$ the corresponding data for g with respect to h and let r be the minimum integer such that

$$(f, g) \leqslant \min\{d'_r v_{r+1}, d_r v'_{r+1}\} = t(r).$$

Then we can define the contact pair of f and g with respect to h, $(f \mid g)_h$, and also prove the following:

(1.7) LEMMA-DEFINITION. *With notations as above*:

(1) $M_i = M'_i$, $d_0 v'_i = d'_0 v_i$ *for* $1 \leqslant i \leqslant r$ *and* $\delta_i = \delta'_i$ *for* $1 \leqslant i \leqslant r + 1$.

(2) *If* $(f, g) < t(r)$ *then there exists an integer* d *with*

$$0 < d \leqslant \left[\frac{v_{r+1} - M_t v_r}{d_r}\right]$$

such that

$$(f, g) = d'_{r-1} v_r + d d_r d'_r = d_{r-1} v'_r + d d_r d'_r.$$

In this case define $(f \mid g)_h = (r, d)$.

(3) *If* $(f, g) = t(r)$ *and* $d'_r v_{r+1} \neq d_r v'_{r+1}$ *then* $\delta'_{r+2} \neq \delta_{r+2}$, *and we define* $(f \mid g)_h = (r, \infty)$.

(4) *If* $(f, g) = t(r)$ *and* $d'_r v_{r+1} = d_r v'_{r+1}$, *then* $M_{r+1} = M'_{r+1}$, $\delta'_{r+2} = \delta_{r+2}$ *and* $d_0 v'_{r+1} = d'_0 v_{r+1}$. *In this case, define* $(f \mid g)_h = (r + 1, 0)$.

(5) *The relationship between* $(f \mid g) = (q, c)$ *and* $(f \mid g)_h$ *is as follows*:
If $q \geqslant 1$, *then*

$$\begin{cases} (f \mid g)_h = (q, c) \text{ if } \gcd(v_0, v_1) < v_1 \\ (f \mid g)_h = (q + 1, c) \text{ if } \gcd(v_0, v_1) = v_1 \end{cases}$$

If $q = 0$, *then*:

$$\begin{cases} (f \mid g)_h = (0,\ c) \text{ if } v_0 < v_1 \text{ and } v'_0 < v'_1 \\ (f \mid g)_h = (1,\ c-k) \text{ if } v_0 = kv_1 \text{ and } v'_0 = kv'_1 \\ (f \mid g)_h = (0,\ \infty) \text{ otherwise.} \end{cases}$$

Proof. The proof of (1) is a simple computation taking into account the relationship between the m-sequence and the values of the maximal contact given in (1.4) and (1.5). For the rest of statements, note first that in the case in which h is transversal to f and g the result is trivial, because $(f \mid g)_h = (f \mid g)$. So, assume that h is not transversal and denote $(f \mid g) = (q, c)$. We shall consider three different cases, depending on q and c.

(A) $q \geqslant 1$.

In this case, as h is non-singular, $(f \mid h) = (g \mid h) < (f \mid g)$. Thus, there are two possibilities. The first one is $(f, h) = k\bar{\beta}_0$, $(g \mid h) = k\bar{\beta}'_0$ with $1 \leqslant k \leqslant l_0 = l'_0$. In this case one finds $(f \mid g)_h = (q + 1, c)$ and the result is a consequence of (1.5). In the other case, one must have $(f, h) = \bar{\beta}_1$, $(g, h) = \bar{\beta}'_1$ and then $(f \mid g)_h = (q, c)$, so the result is trivial, since $d_0 v'_1 = \bar{\beta}_1 \bar{\beta}'_0 = e'_0 \bar{\beta}_1 = e_0 \bar{\beta}'_1 = v_1 d'_0$.

(B) $q = 0$, $c = \infty$.

Without loss of generality we can assume that $(f, g) = e'_0 \bar{\beta}_1$, so $l_0 \leqslant l'_0$. The different possibilities for h are the following:

— $(f,\ h) = k\bar{\beta}_0$, $(g,\ h) = k\bar{\beta}'_0$ with $1 < k \leqslant l_0$. One finds $r = 1$ and $(f, g) = e'_0 \bar{\beta}_1 = v_2 d'_1 < v'_2 d_1 = \bar{\beta}'_1 e'_0$. So, $(f \mid g)_h = (1, \infty)$ and the lemma follows.

— $(f,\ h) = \bar{\beta}_1$. Thus either $(g,\ h) = k\bar{\beta}'_0$ with $l_0 < k \leqslant l'_0$ or $(g,\ h) = \bar{\beta}'_1$ and in any case, $v'_1 = \bar{\beta}'_0$, $(f,\ g) = e'_0 \bar{\beta}_1 = v'_1 d_0 < v_1 d'_0$. So, we have $r = 0$ and $(f \mid g)_h = (0, \infty)$.

(C) $q = 0$, $c \leqslant \min\{l_0,\ l'_0\}$.

In this case, $(f,\ g) = c\bar{\beta}_0 \bar{\beta}'_0$ and depending on h we have the following possibilities:

— $(f,\ h) = k\bar{\beta}_0$, $(g,\ h) = k\bar{\beta}'_0$ with $1 < k \leqslant c$. Then $d_0 v'_1 = k\bar{\beta}_0 \bar{\beta}'_0 = d'_0 v_1 \leqslant (f, g) < \min\{d_1 v'_2, d'_1 v_2\}$ and $(f \mid g)_h = (1,\ c - k)$.

— $(f, h) = c\bar{\beta}_0$ and $(g,\ h) > c\bar{\beta}'_0$. Then $v'_1 = \bar{\beta}'_0$ and $c\bar{\beta}_0 \bar{\beta}'_0 = d_0 v'_1 < d'_0 v_1$, so, $(f \mid g)_h = (0, \infty)$.

— $(g,\ h) = c\bar{\beta}'_0$ and $(f,\ h) > c\bar{\beta}_0$. This is the same case as above with f and g interchanged.

(1.8) REMARK. Let r be as in (1.6), the minimum integer such that $(f, g) \leqslant t(r)$. By the Lemma above we have

$$\frac{(f, g)}{(g, h)} \leqslant \frac{v_{r+1}}{\delta_{r+1}}$$

and, r is the minimum integer such that the inequality above holds.

Moreover, d_r divides (f, g) except, at most, in the case $(f, g) = d'_r v_{r+1} < d_r v'_{r+1}$, that is, except when:

$$\frac{(f, g)}{(g, h)} = \frac{v_{r+1}}{\delta_{r+1}}.$$

As Consequence,

$$\frac{(f, g)}{(g, h)} < \frac{v_t}{\delta_t} \Rightarrow d_{t-1} \mid (f, g). \tag{1.8.1}$$

Finally, if $\dfrac{(f, g)}{(g, h)} > \dfrac{v_n}{\delta_n}$, then, since

$$\frac{v_1}{\delta_1} < \frac{v_2}{\delta_2} < \cdots < \frac{v_s}{\delta_s},$$

must be $r \geqslant n$ and so $\delta_{n+1} = \delta'_{n+1}$.

(1.9) Let f, g, $p \in k[[X, Y]]$ be irreducibles, keep the notations above for the different invariants associated to f and g and denote by $\bar{\beta}''_0, \ldots, e''_0, \ldots$, the corresponding ones for the curve defined by p. It is well-known that at least two of the numbers

$$\frac{(f, g)}{m(f)m(g)}, \frac{(f, p)}{m(f)m(p)}, \frac{(g, p)}{m(g)m(p)}$$

are equal, the third one being greater or equal than the repeated one ([P1]). In other words,

$$\frac{(f, g)}{m(g)} > \frac{(f, p)}{m(p)} \Rightarrow \frac{(g, p)}{m(g)} = \frac{(f, p)}{m(f)}.$$

This fact can be easily generalized changing the multiplicities of the curves for the intersection multiplicities with h. More precisely:

(1.10) PROPOSITION. *Let f, g, $p \in k[[X, Y]]$ be irreducibles. Assume*

$$\frac{(f, g)}{(g, h)} > \frac{(f, p)}{(p, h)}.$$

Then,

$$\frac{(f, p)}{(f, h)} = \frac{(g, p)}{(g, h)}.$$

Proof. Consider the set

$$\left\{\frac{(f, g)}{e_0 e_0'}, \frac{(f, h)}{e_0}, \frac{(g, h)}{e_0'}\right\}$$

By (1.9) this set has at most two elements, so we have the following possibilities:

(A) $$\frac{(f, g)}{e_0 e_0'} = \frac{(g, h)}{e_0'} \leqslant \frac{(f, h)}{e_0}.$$

By hypothesis $(p, h)(f, g) > (f, p)(g, h)$ and hence $(p, h) > (f, p)(g, h)(f, g)^{-1} = (f, p)e_0^{-1}$. Using (1.9) for p, h and f, we find

$$\frac{(p, h)}{e_0''} > \frac{(f, p)}{e_0 e_0''} = \frac{(f, h)}{e_0} \geqslant \frac{(g, h)}{e_0'}$$

so, $(f, p) = (f, h)e_0''$ and by (1.9) again $(g, p)e_0' = (g, h)e_0' e_0''$, that is

$$\frac{(g, p)}{(g, h)} = e_0' = \frac{(f, p)}{(f, h)}.$$

(B) $$\frac{(f, g)}{e_0 e_0'} = \frac{(f, h)}{e_0} < \frac{(g, h)}{e_0'}.$$

As above, by hypothesis $(p, h) > (f, p)(g, h)(f, g)^{-1} > (f, p)e_0^{-1}$, and then

$$\frac{(p, h)}{e_0''} > \frac{(f, p)}{e_0 e_0''} = \frac{(f, h)}{e_0} = \frac{(f, g)}{e_0 e_0'}.$$

As consequence, $(f, p) = e_0''(f, h)$ and $(f, p) = (f, g)e_0''(e_0')^{-1}$, so $(p, h) > (g, h)e_0''(e_0')^{-1}$ and by (1.9)

$$\frac{(p, h)}{e_0''} > \frac{(g, h)}{e_0'} = \frac{(g, p)}{e_0' e_0''}.$$

(C) $$\frac{(f, h)}{e_0} = \frac{(g, h)}{e_0'} < \frac{(f, g)}{e_0 e_0'}.$$

Assume $(f, g)e_0'' \leqslant (f, p)e_0'$; using (1.9) for f, p and h we obtain

$$\frac{(p, h)}{e_0''} = \frac{(f, h)}{e_0} = \frac{(g, h)}{e_0'}.$$

But in this case,

$$\frac{(f, p)}{(p, h)} = \frac{(f, p)e_0'}{(g, h)e_0''} \geqslant \frac{(f, g)}{(g, h)}$$

which gives a contradiction. As consequence $(f, g)e_0'' > (f, p)e_0'$ and $(f, p)e_0' = (g, p)e_0$. This equality, together with $(f, h)e_0' = (g, h)e_0$ leads to the result.

(1.10) REMARK. Obviously, the proposition above is equivalent to say that in the set

$$\left\{\frac{(f, g)}{(g, h)(f, h)}, \frac{(f, p)}{(f, h)(p, h)}, \frac{(p, g)}{(p, h)(g, h)}\right\}$$

there are at most two different elements, being the repeated element the minimum of the set.

2. Apery basis and curves with contact in it

Let $f \in k[[X, Y]]$ be an irreducible series, $h \in k[[X, Y]]$ irreducible and non-singular and $m = (f, h)$. We keep the notations of section 1.

(2.1) DEFINITION. The Apery basis of S with respect to m is the ordered set

$$A_m = \{\gamma \in S \mid \gamma - m \notin S\} = \{a_0 < a_1 < \cdots < a_{m-1}\}.$$

The Apery basis has been treated by several authors ([Ap], [An], [A], [P], ...) and some of the facts that we will use below can be found in these references.

Let $\{v_0, \ldots, v_s\}$ be the *m-sequence* of S, then one can compute the Apery basis in the following way: If k is an integer with $0 \leqslant k \leqslant m - 1$, k can be written in a unique way as $k = \Sigma_1^s \alpha_i \delta_i$ with α_i integers such that $0 \leqslant \alpha_i < M_i$ $(1 \leqslant i \leqslant s)$. Then,

$$k = \sum_1^s \alpha_i \delta_i \Leftrightarrow a_k = \sum_1^s \alpha_i v_i. \tag{2.1.1}$$

(2.2) DEFINITION. Let $\xi \geqslant d_{s-1} v_s = \eta$ be a natural number. The *enlarged Apery basis* of S with respect to m and ξ is the ordered set $A_{m,\xi} = \{a_l \mid l \in \mathbb{N}\}$, where, if $l = pm + k$ with $0 \leqslant k < m$ we define $a_l := p\xi + a_k$.

(2.3) REMARK. We have enlarged the Apery basis in such a way that the equivalence (2.1.1) is conserved. In fact, if $l \in \mathbb{N}$, l can be written in a unique way as $l = pm + \Sigma_1^s \alpha_i \delta_i$ with $0 \leqslant \alpha_i < M_i$ $(1 \leqslant i \leqslant s)$ and

$$l = pm + \sum_1^s \alpha_i \delta_i \Leftrightarrow a_l = p\xi + \sum_1^s \alpha_i v_i.$$

In the following, we shall prove some facts for the enlarged Apery basis which are in general, known for the Apery basis. Unless otherwise specified, ξ will be a natural number with $\xi \geqslant \eta = d_{s-1} v_s$, and the elements a_l, $(l \in \mathbb{N})$, will be the elements of the enlarged Apery basis of S with respect to m and ξ.

(2.4) LEMMA. *Let $l_1, \ldots, l_t$ be natural numbers and $l = \Sigma_1^t l_i$. Then, $\Sigma_1^t a_{l_i} \leqslant a_l$. Moreover, if $l_i < m$ for all $i = 1, \ldots, t$ and $l = pm + k$ with $k < m$ then $\Sigma_1^t a_{l_i} \leqslant p\eta + a_k$.*

Proof. We will use induction on t. In the case $t = 1$ there is nothing to prove. So, assume $t > 1$, put $l' = \Sigma_1^{t-1} l_i = qm + v$ with $v < m$ and assume that $\Sigma_1^{t-1} a_{l_i} \leqslant q\xi + a_v$ (resp. $\leqslant q\eta + a_v$ if $l_i < m$ for any i).

If $l_t = rm + u$ with $u < m$, then

$$\sum_1^t a_{l_i} \leqslant q\xi + a_v + r\xi + a_u = (q + r)\xi + a_v + a_u.$$

Thus, the problem is reduced to the computation of $a_v + a_u$ when $v, u < m$. Assume $u = \Sigma_1^s u_i \delta_i$, $v = \Sigma_1^s v_i \delta_i$ with $u_i, v_i < M_i$, $(1 \leqslant i \leqslant s)$. Then we can write:

$$\begin{aligned} u_1 + v_1 &= \varepsilon_1 M_1 + \gamma_1 \\ u_2 + v_2 + \varepsilon_1 &= \varepsilon_2 M_2 + \gamma_2 \\ \dots & \quad \dots \\ u_s + v_s + \varepsilon_{s-1} &= \varepsilon_s M_s + \gamma_s \end{aligned}$$

where $\gamma_i < M_i$ and $\varepsilon_i \in \{0, 1\}$ for $i = 1, \dots, s$. We may compute $u + v$ and $a_u + a_v$:

$$\begin{aligned} u + v &= \sum_1^s (u_i + v_i)\delta_i = \sum_1^s (\varepsilon_i M_i + \gamma_i - \varepsilon_{i-1})\delta_i \\ &= \sum \gamma_i\delta_i + \sum (\varepsilon_i M_i - \varepsilon_{i-1})\delta_i = \sum \gamma_i\delta_i + \sum (\varepsilon_i \delta_{i+1} - \varepsilon_{i-1}\delta_i) \\ &= \sum \gamma_i \delta_i + \varepsilon_s m. \end{aligned}$$

Note that $k = \Sigma \gamma_i\delta_i$, as $\Sigma \gamma_i \delta_i < m$ and $l \equiv \Sigma \gamma_i \delta_i \pmod{m}$.

On the other hand:

$$\begin{aligned} a_v + a_u &= \sum (v_i + u_i)v_i = \sum(\varepsilon_i M_i + \gamma_i - \varepsilon_{i-1})v_i \\ &= \sum \gamma_i v_i + \Sigma (\varepsilon_i M_i - \varepsilon_{i-1})v_i = a_k + \sum_1^{s-1} (M_i v_i - v_{i+1})\varepsilon_i + \varepsilon_s M_s v_s \\ &= a_k + \varepsilon_s \eta + \sum (M_i v_i - v_{i+1})\varepsilon_i \leqslant a_k + \varepsilon_s \eta. \end{aligned}$$

This computation provides:

$$\sum_1^t a_{l_i} \leqslant (q + r + \varepsilon_s)\xi + a_k = p\xi + a_k$$

as $q + r + \varepsilon_s = p$. Note that $v + u < m \Leftrightarrow \varepsilon_s = 0$.

(2.5) REMARK. This lemma above generalizes the well-known fact that $a_v + a_u \leqslant a_{u+v}$ if $u + v < m$ (see the references in (2.1)). Note that we have also proved that:

If $\xi > \eta$, $a_v + a_u = a_{u+v} \Leftrightarrow \varepsilon_i = 0$, $1 \leqslant i \leqslant s$; in particular $u + v < m$.

If $\xi = \eta$, $a_v + a_u = a_{u+v} \Leftrightarrow \varepsilon_i = 0$, $1 \leqslant i \leqslant s - 1$.

These facts can be easily generalized to the case of an arbitrary number, t, of summands:

Let $l_1, \ldots, l_t \in \mathbb{N}$ and $l = \Sigma_1^t l_i$. Assume $l_i = \Sigma_{j=1}^s \alpha_j^i \delta_j + p_i m$ with $\alpha_j^i < M_j$ for $1 \leqslant j \leqslant s$ and $1 \leqslant i \leqslant t$ and $l = \Sigma_{r=1}^s \alpha_r \delta_r$ with $\alpha_r < M_r$ for $1 \leqslant r \leqslant s$. Then one has:

(A) If $\xi > \eta$,

$$\sum_{i=1}^{t} a_{l_i} = a_l \Leftrightarrow \sum_{i=1}^{t} \alpha_j^i < M_j \quad (1 \leqslant j \leqslant s),$$

and in this case must be, $\Sigma_{i=1}^t \alpha_j^i = \alpha_j$ for $1 \leqslant j \leqslant s$ and $\Sigma_{i=1}^t p_i = p$.

(B) If $\xi = \eta$,

$$\sum_{i=1}^{t} a_{l_i} = a_l \Leftrightarrow \sum_{i=1}^{t} \alpha_j^i < M_j \qquad (1 \leqslant j \leqslant s-1),$$

and in this case must be, $\Sigma_{i=1}^t \alpha_j^i = \alpha_j$ for $1 \leqslant j \leqslant s-1$ and $\Sigma_{i=1}^t (p_i + \alpha_s^i) = p + \alpha_s$.

(2.6) PROPOSITION. *Let $g \in k[[X, Y]]$ be a formal series and suppose that for each φ, irreducible component of g, we have*

$$\frac{(\varphi,\ f)}{(\varphi,\ h)} \leqslant \frac{\xi}{(f,\ h)}.$$

Then $(g,\ f) \leqslant a_{(g,h)}$.

Proof. Let $g = \Pi_1^t \varphi_i$ be the factorization of g in irreducible elements and set $l_i = (\varphi_i, h)$. By Lemma (2.4) it suffices to prove that $(\varphi_i, h) \leqslant a_{l_i}$ for any $i = 1, \ldots, t$, that is, we can assume g irreducible.

Denote by $v'_0, \ldots, v'_{s'}, d'_0, \ldots, d'_{s'}, \ldots$ the data defined in (1.3) for g with respect to h and let r be the minimum integer such that

$$(f,\ g) \leqslant \min\{d'_r v_{r+1},\ d_r v'_{r+1}\} = t(r).$$

By (1.7), $\delta_{r+1} = \delta'_{r+1}$ and $d'_0 = (g, h) = d'_r \delta_{r+1}$. Define $\lambda_r, \ldots, \lambda_s$ as $\lambda_r := d'_r$ and

$$\begin{array}{ll}
\lambda_r = \lambda_{r+1} M_{r+1} + \alpha_{r+1}; & \alpha_{r+1} < M_{r+1} \\
\lambda_{r+1} = \lambda_{r+2} M_{r+2} + \alpha_{r+2}; & \alpha_{r+2} < M_{r+2} \\
\quad \cdots & \quad \cdots \\
\lambda_{s-1} = \lambda_s M_s + \alpha_s; & \alpha_s < M_s
\end{array}$$

also define $\alpha_1 = \cdots = \alpha_r = 0$. Now, setting $k = \Sigma_1^s \alpha_i \delta_i < m$, we have

$$(g,\ h) = \lambda_r \delta_{r+1} = \sum_1^s \alpha_i \delta_i + \lambda_s M_s \delta_s = k + \lambda_s m.$$

If $r = s$, then $(g,\ h) = d'_0 = d'_s(f, h) = d'_s m = \lambda_s m$ (in particular $k = 0$) and by the hypothesis

$$(g,\ f) \leqslant \frac{(g,\ h)}{(f,\ h)} \xi = \lambda_s \xi = a_{(g,h)}.$$

Otherwise, $r < s$ and we have

$$\begin{aligned}(g,\ f) \leqslant d'_r v_{r+1} = \lambda_r v_{r+1} &= \alpha_{r+1} v_{r+1} + \lambda_{r+1} M_{r+1} v_{r+1} \\ &\leqslant \alpha_{r+1} v_{r+1} + \lambda_{r+1} v_{r+2} \leqslant \cdots \leqslant \\ &\leqslant \sum \alpha_i v_i + \lambda_s M_s v_s = a_k + \lambda_s \xi = a_{(g,h)}\end{aligned}$$

as we want to prove.

(2.7) REMARK. The Proposition above is known for the Apery basis. Note that, if $(\varphi,\ h) < m$, automatically

$$\frac{(\varphi,\ f)}{(\varphi,\ h)} \leqslant \frac{\eta}{(f,\ h)}.$$

From the proof of Proposition (2.6) and Remark (1.8) we can deduce the conditions in which the equality $(g,\ f) = a_{(g,h)}$ holds. Assume g irreducible and write $(g, h) = pm + k = pm + \Sigma_1^s \alpha_i \delta_i$ with $\alpha_i < M_i$ $(1 \leqslant i \leqslant s)$.

(A) If $(g,\ h) \geqslant m$ and $\xi > \eta$, then:

$$\begin{aligned}(g,\ f) = a_{(g,h)} &\Leftrightarrow \alpha_i = 0 \quad 1 \leqslant i \leqslant s \quad \text{and} \quad (g,\ f) = p\xi \\ &\Leftrightarrow \frac{(f,\ g)}{(g,\ h)} = \frac{\xi}{(f,\ h)}.\end{aligned}$$

(B) If $(g,\ h) \geqslant m$ and $\xi = \eta$, then:

$$\begin{aligned}(f,\ g) = a_{(g,h)} &\Leftrightarrow \alpha_i = 0 \quad 1 \leqslant i \leqslant s-1 \text{ and } (f,\ g) = \alpha_s v_s + p\eta \\ &\Leftrightarrow \frac{(f,\ g)}{(g,\ h)} = \frac{v_s}{\delta_s}.\end{aligned}$$

Note that in this case $(f, g) = d'_{s-1}v_s$ and $d'_{s-1} = pM_s + \alpha_s\delta_s = pM_s + k$.

(C) If $(g, h) < m$, then:

$$(g, f) = a_{(g,h)} \Leftrightarrow \exists r \text{ such that } \alpha_i = 0 \text{ for all } i \neq r \text{ and } (f, g) = \alpha_r v_r$$
$$\Leftrightarrow \frac{(g, f)}{(g, h)} = \frac{v_r}{\delta_r} \quad \text{and} \quad d'_{r-1} < M_r.$$

As consequence, among the numbers $\alpha_1, \ldots, \alpha_s$ there exists at most one not equal to zero, say α_{i_0}, and the condition $\alpha_{i_0}p \neq 0$ implies that $\xi = \eta$ and $i_0 = s$.

(2.8) THEOREM. *Let $f, g \in k[[X, Y]]$; f being irreducible, and suppose*

$$\frac{(\varphi, f)}{(\varphi, h)} \leqslant \frac{\xi}{(f, h)}$$

for each irreducible component φ of g. Let $l = (g, h) = pm + k = pm + \Sigma_1^s \alpha_i\delta_i$ with $\alpha_i < M_i$. If $(g, f) = a_{(g,h)} = p\xi + a_k$, then g can be factorized as $g = A_1A_2 \cdots A_sB$ in such a way that:

(1) *For $i = 1, \ldots, s$ $(i = 1, \ldots, s-1$ if $\xi = \eta)$,*

$$(A_i, h) = \alpha_i\delta_i, \; (A_i, f) = \alpha_i v_i$$

(2) *If $\xi = \eta$ then $B = 1$ and*

$$(A_s, h) = pm + \alpha_s\delta_s, \; (A_s, f) = \rho\eta + \alpha_s v_s$$

(3) *If $\xi > \eta$ then $(B, h) = pm$ and $(B, f) = p\xi$.*

(4) *If $\alpha_i \neq 0$ (resp. if $B \neq 1$), then for any irreducible component, A_{ij} (resp. B_j), of A_i (resp. of B) we have*

$$\frac{(A_{ij}, f)}{(A_{ij}, h)} = \frac{v_i}{\delta_i} \left(\textit{resp. } \frac{(B_j, f)}{(B_j, h)} = \frac{\xi}{(f, h)} \right).$$

Proof. Let $g = \Pi_1^t \varphi_i$ be the factorization of g in irreducible elements. Denote $l_i = (\varphi_i, h) = p_i m + \Sigma_1^s \alpha_j^i \delta_j$ with $\alpha_j^i < M_j$ for $i = 1, \ldots, t$; $j = 1, \ldots, s$. By the results above, we have $(\varphi_i, f) = a_{l_i}$ for any $i = 1, \ldots, t$, and by (2.7) this occurs if and only if:

(1) $\alpha_j^i = 0$ for all j except, at most, for an index $j(i)$.

(2) If $p_i \neq 0$ then $j(i) = s$ and, either $\xi = \eta$ or $\alpha_j^i = 0$ for every j.

(3) $(\varphi_i, f) = p_i\xi + \alpha_{j(i)}^i v_{j(i)}$.

For $k = 1, \ldots, s-1$ (resp. for $k = 1, \ldots, s$ if $\xi > \eta$) consider the set:

$$\Lambda(k) = \{i \in \{1, \ldots, t\} \mid j(i) = k\}.$$

If $\xi > \eta$ define $\Lambda(\xi) = \{1, \ldots, t\} - \bigcup_1^s \Lambda(k)$ and if $\xi = \eta$, $\Lambda(s) = \{1, \ldots, t\} - \bigcup_1^{s-1} \Lambda(k)$. Note that

$$\Lambda(\xi) = \{i \in \{1, \ldots, t\} \mid p_i \neq 0\}$$

$$\Lambda(s) = \{i \in \{1, \ldots, t\} \mid p_i \neq 0 \text{ or } j(i) = s\}.$$

Now, we define

$$A_k = \prod_{i \in \Lambda(k)} \varphi_i \quad \text{for } k = 1, \ldots, s$$

$$B = \prod_{i \in \Lambda(\xi)} \varphi_i \quad \text{if } \xi > \eta.$$

For $k < s$ (resp. $k \leqslant s$ if $\xi > \eta$) we have

$$\sum_{i \in \Lambda(k)} (\varphi_i, f) = \sum_{i \in \Lambda(k)} a_{l_i} \leqslant a_{\Sigma l_i} \tag{$*$}$$

and by (2.5) the inequality in $(*)$ is an equality if and only if $\Sigma_{i \in \Lambda(k)} \alpha_k^i < M_k$.
Now assume that $\xi = \eta$. Then

$$\sum_{k=1}^{s-1} \left(\sum_{i \in \Lambda(k)} \alpha_k^i \right) \delta_k + \sum_{i \in \Lambda(s)} (p_i m + \alpha_s^i \delta_s) = pm + \sum_1^s \alpha_i \delta_i$$

and, since $\Sigma_{i \in \Lambda(k)} \alpha_k^i < M_k$, we have $\Sigma_{i \in \Lambda(k)} \alpha_k^i = \alpha_k$ for any $k < s$ and $(\Sigma\, p_i)m + \Sigma\, \alpha_s^i = pm + \alpha_s \delta_s$. As consequence, $(A_k, h) = \Sigma_i\, \alpha_k^i \delta_k = \alpha_k \delta_k$ and also $(A_k, f) = \Sigma_i\, \alpha_k^i v_k = \alpha_k v_k$ for $k < s$. The last statement in the Theorem for the components A_{ij} of A_i is the Remark (2.7) above, part (C) if $i < s$ and part (B) if $i = s$.

Finally, write $\Sigma\, \alpha_s^i = p'm + \alpha_s'$ with $\alpha_s' < M_s$. Then $\Sigma\, p_i + p' = p$, $\alpha_s' = \alpha_s$ and

$$(A_s, f) = \sum (p_i M_s + \alpha_s^i) v_s = \left(\sum p_i \right) M_s v_s + p' M_s v_s + \alpha_s v_s = p\eta + \alpha_s v_s$$

as we wanted to prove.

The case $\xi > \eta$ can be proved in the same way using the Remark above and similar computations.

(2.9) REMARK. Theorem (2.8) is well-known in some particular cases: Merle in [M] proves this Theorem for the case in which $m = \bar{\beta}_0$, $(f, g) = a_{m-1}$ and $(g, h) = m - 1$ with the main goal of the determination of the polar quotients of f. Ephraim in [E] proves the same result as Merle for an arbitrary $m = (f, h)$. Finally, the same Theorem has been proved by Granja ([G]) for the Apery basis with respect to m.

In the rest of this Section we shall give some applications of (2.8) to a simple case of curves with several branches.

(2.10) APPLICATION: *Diagonal curves with several branches.* Let $f = \Pi_1^d f_i \in k[[X, Y]]$ be such that f_i is irreducible for any $i \in I := \{1, \ldots, d\}$ and $f_i \neq f_j$ if $i \neq j$. Denote by v_i the normalized valuation corresponding to the branch f_i and let $\mathcal{O} = k[[X, Y]]/(f)$ the local ring of the plane curve defined by f. The semigroup of values of f is the subsemigroup, S, of $\mathbb{N}^d$ given by

$$S := \{\underline{v}(g) := (v_1(g), \ldots, v_d(g)) \mid g \in \mathcal{O},\ g \text{ non-zero divisor}\}.$$

We shall say that f is *diagonal* with respect to h if the following conditions are satisfied:

(1) The *m-sequence* for f_i with respect to h is independent of i $(1 \leqslant i \leqslant d)$. In that case, denote it by $m = v_0, \ldots, v_s$. In particular this fact implies that the f_i's are equisingular.
(2) $(f_i, f_j) = \xi = d_{s-1} v_s + c$ independently of $i, j \in I$.

(2.11) From the definition, if f is diagonal then the elements $V_i = (v_i, \ldots, v_i) \in \mathbb{N}^d$ $(i = 0, \ldots, s)$, belongs to the semigroup of values of f and the same is true for $\Psi = (\xi, \ldots, \xi)$. Note that $V_0, \ldots, V_s, \Psi$ are placed in the positive part of the diagonal of $\mathbb{N}^d$. This set of elements permits the complete computation of S (see [D], [Ga]) and these are the reasons for the name *diagonal.*

Consider $g \in k[[X, Y]]$ such that:

$$\underline{v}(g) = \sum_1^s \alpha_i V_i + p\Psi$$

$$(g, h) = \sum_1^s \alpha_i \delta_i + pm. \qquad (2.11.1)$$

where $p < d$ and $\alpha_i < M_i$ for $i = 1, \ldots, s$.

The following results give a decomposition Theorem for g similar to that one in (2.8).

(2.12) LEMMA. *Let f be diagonal with respect to h, and g as in* (2.11.1). *If φ is an irreducible component of g then*

$$\frac{(\varphi,\ f_i)}{(\varphi,\ h)} \leqslant \frac{\xi}{(f_i,\ h)} \qquad \forall\, i \in I \tag{$*$}$$

and as consequence $(\varphi, f_i) = (\varphi, f_j)\ \forall\, i, j \in I$.

Proof. First of all, if the inequality $(*)$ is true, then, taking into account that $\xi = (f_i, f_j)$ and (1.10) we find that $(f_j, \varphi)(f_i, h) \geqslant (f_i, \varphi)(f_j,\ h)$, or equivalently $(f_j, \varphi) \geqslant (f_i, \varphi)$. But the role of f_i and f_j can be interchanged and so $(f_i, \varphi) = (f_j, \varphi)$ for any $i, j \in I$.

We shall prove the first statement in the Lemma by induction on d. The case $d = 1$ is trivial (see (2.7)), so assume $d > 1$ and, also, that there exists φ such that

$$\frac{(\varphi, f_1)}{(\varphi,\ h)} > \frac{\xi}{(f_1,\ h)} \quad \text{or, equivalently } v_1(\varphi) > \frac{(\varphi,\ h)}{m}\ \xi.$$

Let r be the minimum integer such that $v_1(\varphi) \leqslant d_r(\varphi) v_{r+1}$. Then if $r < s$,

$$v_1(\varphi) \leqslant d_r(\varphi) v_{r+1} = d_0(\varphi)\ \frac{d_r}{d_0}\ v_{r+1} \leqslant \frac{(\varphi,\ h)}{d_0}\ \xi.$$

Thus, there must be $r = s$. As consequence $\delta_{s+1} = \delta_{s+1}(\varphi)$ and, since

$$(\varphi,\ h) = d_s(\varphi)(f_1,\ h) = d_s(\varphi) m \leqslant \sum \alpha_i \delta_i + pm,$$

we have $d_s(\varphi) \leqslant p$.

For $i = 2, \ldots, d$, we have $(\varphi,\ f_1)(f_i,\ h) > (f_1,\ f_i)(\varphi,\ h)$ and then

$$v_i(\varphi) = \frac{(\varphi,\ h)}{(f_1,\ h)}\ \xi = d_s(\varphi)\xi.$$

Now, consider $g' = g/\varphi$ and $p' = p - d_s(\varphi)$. Then

$$v_i(g') = \sum_1^s \alpha_j v_j + p'\xi \qquad 2 \leqslant i \leqslant d$$

$$(g',\ h) = \sum_1^s \alpha_j \delta_j + p'm$$

and we can use the induction hypothesis, so all the irreducible components

ψ of g' satisfy

$$\frac{(\psi, f_i)}{(\psi, h)} \leqslant \frac{\xi}{(f_i, h)} \qquad \forall\, i = 2, \ldots, d.$$

By (1.10) there must be $v_1(\psi) \geqslant v_i(\psi)$ and then

$$v_1(g') \geqslant v_i(g') = \sum \alpha_j v_j + p'\xi.$$

But, in this case,

$$v_1(g) = v_1(g') + v_1(\varphi) > v_1(g') + d_s(\varphi)\xi \geqslant v_1(g)$$

and we reach a contradiction.

(2.13) THEOREM. *Let f be diagonal with respect to h and let g be as in* (2.11.1). *Change in Theorem* (2.8) v_i *by* V_i, ξ *by* Ψ *and* $(f, -)$ *by* $\underline{v}(-)$. *Then the resulting decomposition Theorem for g is true.*

Proof. It is a consequence of (2.12) and (2.8).

3. Factorization of the polar of a curve with two branches

Let $f = \Pi_1^d f_i \in k[[X, Y]]$ be such that f_i is irreducible for any $i \in I := \{1, \ldots, d\}$ and $f_i \neq f_j$ if $i \neq j$, and let $h \in k[[X, Y]]$ defining a non-singular algebroid curve. Denote by v_i the normalized valuation corresponding to the branch f_i and by $\bar{\beta}_0^i, \ldots, \bar{\beta}_{g_i}^i$, $v_0^i, \ldots, v_{s_i}^i$, $d_0^i, \ldots, d_{s_i}^i, \ldots$ the data defined in (1.2) and (1.3) for f_i with respect to h $(1 \leqslant i \leqslant d)$. S stands for the semigroup of values of f and S_i for the one of f_i.

(3.1) POLAR CONTACTS. Assume for a moment that k is of characteristic zero. The polar curve of f with respect to h is the algebroid curve defined by:

$$J(f, h) = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial h}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} & \dfrac{\partial h}{\partial y} \end{vmatrix} = 0,$$

and it is well-known (see [M], [E]) that

$$v_i(J(f_i, h)) = c_i - 1 + v_0^i = \sum_{j=1}^{s_i} (M_j^i - 1)v_j^i$$

where c_i is the conductor of the semigroup of values S_i of f_i. Moreover,

$$J(f, h) = \sum_1^d \frac{f}{f_i} J(f_i, h),$$

and then

$$v_i(J(f, h)) = \sum_{j \neq i} (f_i, f_j) + \sum_{j=1}^{s_i} (M_j^i - 1)v_j^i. \tag{3.1.1}$$

By [D] (2.6) and (2.7) the conductor δ of the semigroup S (that is, the minimum element $\delta \in S$ such that $\delta + \mathbb{N}^d \subseteq S$) is given by $pr_i(\delta) = c_i + \Sigma_{j \neq i}(f_i, f_j)$ and the element $\tau = \delta - (1, \ldots, 1)$ belongs to S. Using this notation we find that

$$\underline{v}(J(f, h)) = \tau + \underline{v}(h) = \tau + V_0. \tag{3.1.2}$$

A straightforward computation shows that

$$(J(f, h), h) = \sum_1^d v_0^i - 1. \tag{3.1.3}$$

(3.2) DIAGONAL CASE. Now assume f diagonal with respect to h; then the elements $V_i = (v_i^1, \ldots, v_i^d) = (v_i, \ldots, v_i)$ belong to S and the formulae above provide:

$$\underline{v}(J(f, h)) = \tau + V_0 = \sum_1^s (M_i - 1)V_i + (d - 1)\Psi$$

$$(J(f, h), h) = dv_0 - 1 = \sum_1^s (M_i - 1)\delta_i + (d - 1)v_0.$$

As consequence, Theorem (2.13) gives a decomposition for the polar curve $J(f, h)$ in the special case in which f is diagonal with respect to h.

(3.2.1) REMARK. In the general case (that is, f not necessarily diagonal), we cannot use directly the results of Section 2 in order to give a decomposition of $J(f, h)$. In fact the elements $(v_i^1, \ldots, v_i^d)$ do not belong, in general, to S and also $M_i^k \neq M_i^l$ if $k \neq l$. So, the first step will be to give an arithmetical decomposition of $\tau + V_0$ in "good" elements of S and then to use the arithmetical properties of the semigroup S for such "good" elements in order to give a geometrical decomposition. In this section first we describe the decomposition of $\tau + V_0$ in the case of two branches (see (3.3) and (3.4) below), but a similar description can be made for the case of $d > 2$ branches (see [D2]). After that, the rest of the section is devoted to prove the decomposition Theorems for $J(f, h)$ in the case $d = 2$.

(3.3) CASE OF TWO BRANCHES. In the sequel we shall suppose $d = 2$, that is, $f = f_1 f_2 \in k[[X, Y]]$ with f_1 and f_2 irreducibles, $f_1 \neq f_2$ and $h \in k[[X, Y]]$ defining a non-singular algebroid curve. Set $m^1 = v_0^1 = (f_1, h)$, $m^2 = v_0^2 = (f_2, h)$ and denote by $\bar{\beta}_0^1, \ldots, \bar{\beta}_g^1$, $v_0^1, \ldots, v_s^1$, $d_0^1, \ldots$ (resp. $\bar{\beta}_0^2, \ldots, \bar{\beta}_{g'}^2$, $v_0^2, \ldots, v_t^2$, $d_2^0, \ldots$) the data defined in (1.2) and (1.3) for f_1 (resp. f_2) and h. Denote by S the semigroup of values of the algebroid curve given by f, and let $\xi = (f_1, f_2)$ be the intersection multiplicity between f_1 and f_2.

Denote by $(f_1 \mid f_2)_h = (q, c)$ the contact pair of f_1 and f_2 relative to h and assume, without loss of generality, that $d_q^2 v_{q+1}^1 \leqslant d_q^1 v_{q+1}^2$. Thus by (1.7),

$$\text{either} \quad \xi = d_{q-1}^1 v_q^2 + c d_q^1 d_q^2 = d_{q-1}^2 v_q^1 + c d_q^1 d_q^2 \quad \text{or} \quad \xi = d_q^2 v_{q+1}^1$$

depending if $c < \infty$ or $c = \infty$. We shall keep these assumptions in the sequel.

(3.3.1) Let $i \in \mathbb{N}$ such that $0 \leqslant i \leqslant q$, and $\varphi \in k[[X, Y]]$. We have

$$v_1(\varphi) = v_i^1 \Leftrightarrow v_2(\varphi) = v_i^2.$$

Then the elements

$$V_i = (v_i^1, v_i^2) \qquad 0 \leqslant i \leqslant q$$

belong to S. Note that $V_1, \ldots, V_q$ are in the line joining the origin with V_0 and as consequence we can denote $M_i := M_i^1 = M_i^2$ $(1 \leqslant i \leqslant q)$ and $\delta_i = \delta_i^1 = \delta_i^2$ $(1 \leqslant i \leqslant q + 1)$.

(3.3.2) If $c < \infty$ and we take $i > q$, pick $\varphi \in k[[X, Y]]$ in such a way that $v_1(\varphi) = v_i^1$. By (1.8) we have

$$\frac{(f_1, \varphi)}{(\varphi, h)} = \frac{v_i^1}{d_0(\varphi)} = \frac{v_i^1}{\delta_i^1} \geqslant \frac{v_{q+1}^1}{\delta_{q+1}^1} > \frac{\xi}{(f_2, h)},$$

and then, using (1.10), $\xi/(f_1, h) = (\varphi, f_2)/(\varphi, h)$, that is

$$v_2(\varphi) = \frac{(\varphi, h)}{(f_1, h)} \xi = \frac{\xi}{d_{i-1}^1}.$$

In the same way, if $v_2(\varphi) = v_i^2$ then $v_1(\varphi) = \dfrac{\xi}{d_{i-1}^2}$ and thus the following elements belongs to S:

$$V_i^1 = \left(v_i^1, \frac{\xi}{d_{i-1}^1}\right) \quad q+1 \leqslant i \leqslant s$$

$$V_i^2 = \left(\frac{\xi}{d_{i-1}^2}, v_i^2\right) \quad q+1 \leqslant i \leqslant t.$$

(3.3.3) If $\xi = d_q^2 v_{q+1}^1 < d_q^1 v_{q+1}^2$, the argument above remains valid to prove that the following elements belong to S:

$$V_i^1 = \left(v_i^1, \frac{\xi}{d_{i-1}^1}\right) \quad q+2 \leqslant i \leqslant s$$

$$V_i^2 = \left(\frac{\xi}{d_{i-1}^2}, v_i^2\right) \quad q+1 \leqslant i \leqslant t.$$

(3.3.4) Assume, as above, $\xi = d_q^2 v_{q+1}^1 < d_q^1 v_{q+1}^2$ and consider $\varphi\ k[[X, Y]]$ in such a way that $d_{q+1}(\varphi) = 1$ and $v_1(\varphi) = d_q(\varphi) v_{q+1}^1 = d_q^1 v_{q+1}(\varphi)$. Note that:

$$d_q^2 v_{q+1}(\varphi) = d_q^2 \frac{v_{q+1}^1}{d_{q+1}^1} < d_q^1 \frac{v_{q+1}^2}{d_{q+1}^1} = d_q(\varphi) v_{q+1}^2$$

and also (by (1.10)):

$$\frac{(f_1, \varphi)}{(\varphi, h)} = \frac{(f_1, f_2)}{(f_2, h)} \Rightarrow \frac{(f_2, \varphi)}{(\varphi, h)} \geqslant \frac{(f_1, f_2)}{(f_1, h)} \Rightarrow v_2(\varphi) \geqslant d_q^2 v_{q+1}(\varphi).$$

So, as a consequence, $v_2(\varphi) = d_q^2 v_{q+1}(\varphi) = \dfrac{\xi}{d_{q+1}^1}$ and

$$N := \underline{v}(\varphi) = \left(d_q(\varphi) v_{q+1}^1, \frac{\xi}{d_{q+1}^1}\right) = M_{q+1}^1 \left(v_{q+1}^1, \frac{\xi}{d_q^1}\right) \in S.$$

Note that the element $V_{q+1}^1 := \left(v_{q+1}^1, \dfrac{\xi}{d_q^1}\right)$ can be seen as the element missing in (3.3.3), but $V_{q+1}^1 \notin S$, so, the similarity is only formal.

(3.3.5) If $c < \infty$, taking $\varphi \in k[[X, Y]]$ with $d_q(\varphi) = 1$ and $(\varphi \mid f_1)_h =$

$(\varphi \mid f_2)_h = (q, c)$ we find

$$N := \left(\frac{\xi}{d_q^2}, \frac{\xi}{d_q^1}\right) \in S.$$

(3.3.6) REMARK. With notations as above, the elements

$$V_0, \ldots, V_q, V_{l+1}^1, \ldots, V_s^1, V_{q+1}^2, \ldots, V_t^2$$

where $l = q$ if $c < \infty$ or $l = q + 1$ if $c = \infty$, are exactly the values of the maximal contact of f if h is transversal to f (see [D]) or, in another way, the "minimal set of generators of S" in the sense of [D]. In the general case we can take this set as the "$\underline{v}(h)$-*sequence*" of S. The element N defined above has also interesting properies in S and appears sometimes as an additional element to the set of maximal contact values.

(3.4) DECOMPOSITION OF $\tau + V_0$

Let $l = q$ if $c < \infty$ and $l = q + 1$ if $c = \infty$. Then,

$$\tau + V_0 = \sum_1^q (M_i - 1)V_i + \sum_{l+1}^s (M_i^1 - 1)V_i^1 + \sum_{q+1}^t (M_i^2 - 1)V_i^2 + N. \tag{3.4.1}$$

This is obvious using the previous constructions and (3.1.1). Note that, if $c = \infty$

$$N = (M_{q+1}^1 - 1)\left(v_{q+1}^1, \frac{\xi}{d_q^1}\right) + \left(\frac{\xi}{d_q^2}, \frac{\xi}{d_q^1}\right)$$

and then in any case

$$\tau + V_0 = \sum_1^q (M_i - 1)V_i + \sum_{q+1}^s (M_i^1 - 1)V_i^1 + \sum_{q+1}^t (M_i^2 - 1)V_i^2 + \left(\frac{\xi}{d_q^2}, \frac{\xi}{d_q^1}\right) \tag{3.4.2}$$

but in this decomposition we must take into account that not all the summands belong to S if $c = \infty$.

(3.5) NOTATIONS. In the sequel we suppose that g is an element of $k[[X, Y]]$ such that:

$$\underline{v}(g) = \tau + V_0 = \sum_1^q (M_i - 1)V_i + \sum_{l+1}^s (M_i^1 - 1)V_i^1 + \sum_{q+1}^t (M_i^2 - 1)V_i^2 + N$$

$$(g, h) = v_0^1 + v_0^2 - 1 = M_1^1 \cdots M_s^1 + M_1^2 \cdots M_t^2 - 1.$$

The results below give a decomposition for g similar to that one in (2.13). Because of the length, we are going to present the results in three theorems, following the different cases that may appear.

(3.6) FIRST DECOMPOSITION THEOREM. *With conditions and notations as above, g can be factorized as*

$$g = g' A_{l+1} \cdots A_s B_{q+1} \cdots B_t$$

with the following conditions:

(1) *The components A_i, for $l + 1 \leqslant i \leqslant s$, are such that*

$$(A_i, h) = (M_i^1 - 1)\delta_i^1 \quad \textit{and} \quad \underline{v}(A_i) = (M_i^1 - 1)V_i^1.$$

(2) *The components B_j for $q + 1 \leqslant j \leqslant t$, are such that*

$$(B_j, h) = (M_i^2 - 1)\delta_j^2 \quad \textit{and} \quad \underline{v}(B_j) = (M_j^2 - 1)V_j^2.$$

(3) *For any A_{ik} (resp. B_{jk}) irreducible component of A_i (resp. B_j),*

$$\frac{\underline{v}(A_{ik})}{(A_{ik}, h)} = \frac{V_i^1}{\delta_i^1} \quad \left(\textit{resp.}\ \frac{\underline{v}(B_{jk})}{(B_{jk}, h)} = \frac{V_j^2}{\delta_j^2} \right).$$

Proof. Let $r \geqslant l$, $p \geqslant q$ be the smallest integers for which there exists $A_{r+1}, \ldots, A_s$, $B_{p+1}, \ldots, B_t$ satisfying the requirements in the theorem. We are going to prove that $r = l$ and $p = q$. Removing from g the components $A_{r+1}, \ldots, B_t$ we obtain $g^* \in k[[X, Y]]$ such that:

$$\underline{v}(g^*) = \sum_1^q (M_i - 1)V_i + \sum_{l+1}^r (M_i^1 - 1)V_i^1 + \sum_{q+1}^p (M_i^2 - 1)V_i^2 + N$$

$$(g^*, h) = M_1^1 \cdots M_r^1 + M_1^2 \cdots M_p^2 - 1 = \delta_{r+1}^1 + \delta_{p+1}^2 - 1. \tag{3.6.1}$$

The strategy to prove the result will be the following: First, assume $r > l$ (part (A) of the proof). Lemma (3.7) below proves that there exist irreducible components φ of g^* such that

$$\frac{(\varphi, f_1)}{(\varphi, h)} = \frac{v_r^1}{\delta_r^1}.$$

Then we define A_r as the product of all these components. By construction A_r satisfies the conditions in the theorem if $(A_r, h) = (M_r^1 - 1)\delta_r^1$ and $\underline{v}(A_r) = (M_r^1 - 1)V_r^1$. These facts are proved in Lemma (3.8) below.

Thus we may assume $r = l$. Note that if $l = q$ (that is, if $c < \infty$) the role of the branches f_1 and f_2 could be interchanged and (A) would also prove $p = q$. So, we can restrict our attention to the case $l = q + 1$, $\xi = d_q^2 v_{q+1}^1$ and assume $p > q$ (part (B) of the proof). As above, we will prove first (Lemma (3.9)) that there exist irreducible components φ of g^* such that

$$\frac{(\varphi,\ f_2)}{(\varphi,\ h)} = \frac{v_p^2}{\delta_p^2}.$$

Defining B_p as the product of all these components, we shall prove that B_p satisfies the statements in the theorem (Lemma (3.10)).

(A) ASSUME $r > l$.

(3.7) LEMMA. *There exists φ, irreducible component of g^*, such that*

$$\frac{(\varphi, f_1)}{(\varphi, h)} = \frac{v_r^1}{\delta_r^1}.$$

Proof. Recall that d_{r-1}^1 divides v_i^1 for any $i \leqslant r - 1$ and in particular, since $r - 1 > q$, d_{r-1}^1 divides $\frac{\xi}{d_p^2}$ if $p \geqslant q$. As a consequence, d_{r-1}^1 divides $\sum_1^{r-1} (M_i^1 - 1)v_i^1 + \frac{\xi}{d_p^2}$. However, d_{r-1}^1 does not divide $(M_r^1 - 1)v_r^1$.

Taking into account that

$$v_1(g^*) = \sum_1^r (M_i^1 - 1)v_i^1 + \frac{\xi}{d_p^2},$$

the preceding comments prove that there exists φ, irreducible component of g^*, such that d_{r-1}^1 does not divide $v_1(\varphi)$. By (1.8), φ must satisfy

$$\frac{v_1(\varphi)}{(\varphi, h)} \geqslant \frac{v_r^1}{\delta_r^1}. \tag{$*$}$$

Assume that there exists φ, irreducible component of g^*, such that $(*)$ is a strict inequality. By (1.8), $(\varphi, h) = d_r(\varphi)\delta_{r+1}^1 \geqslant \delta_{r+1}^1$.

Now we claim the following, that will be proved later:

With the conditions above, $p = q$.

The inequality

$$\begin{aligned}(\varphi, h) &= d_r(\varphi)\delta^1_{r+1} \leqslant \delta^1_{r+1} + \delta^2_{q+1} - 1 \\ &= \delta_{q+1}(M^1_{q+1}\cdots M^1_r + 1) - 1 = (g^*, h)\end{aligned}$$

implies that $d_r(\varphi) = 1$. Using (1.10) we obtain $v_2(\varphi) = d_r(\varphi)\dfrac{\xi}{d^1_r} = \dfrac{\xi}{d^1_r}$ and as a consequence

$$v_2(g^*/\varphi) = \sum_1^q (M^2_i - 1)v^2_i$$

$$(g^*/\varphi, h) = \delta^2_{q+1} - 1 = \sum_1^q (M^2_i - 1)\delta^2_i.$$

By (2.8), we must have $g^* = \varphi B_1 \cdots B_q$ in such a way that $(B_i, h) = (M^2_i - 1)\delta^2_i$, $v_2(B_i) = (M^2_i - 1)v^2_i$ $(1 \leqslant i \leqslant q)$ and for each B_{ij} irreducible component of B_i:

$$\frac{v_2(B_{ij})}{(B_{ij}, h)} = \frac{v^2_i}{\delta^2_i}.$$

Again by (1.10), $\underline{v}(B_i) = (M_i - 1)V_i$ $(1 \leqslant i \leqslant q)$ and then,

$$v_1(\varphi) = v_1(g^*) - \sum v_1(B_i) = \sum_{q+1}^r (M^1_i - 1)v^1_i + \frac{\xi}{d^2_q}.$$

But, by hypothesis

$$\begin{aligned}v_1(\varphi) > d_{r-1}(\varphi)v^1_r &= M^1_r v^1_r = (M^1_r - 1)v^1_r + v^1_r \\ &> (M^1_r - 1)v^1_r + M^1_{r-1}v^1_{r-1} > \cdots \\ &> \sum_{q+1}^r (M^1_i - 1)v^1_i + v^1_{q+1} \geqslant \sum_{q+1}^r (M^1_i - 1)v^1_i + \frac{\xi}{d^2_q}.\end{aligned}$$

Só, we have a contradiction and the lemma will be proved if we prove the claim. This is a consequence of the following:

(3.7.1) LEMMA. *In the conditions of* (3.6.1), *assume that there exists*

$T \in k[[X, Y]]$ such that T divides g^* and

$$\frac{(\varphi, f_1)}{(\varphi, h)} \geqslant \frac{v_r^1}{\delta_r^1}$$

for each irreducible component φ of T. If $(T, h) \geqslant \delta_{r+1}^1$ then $p = q$.

Proof. Assume that $p > q$, the same argument of (3.7) proves that there exists ψ, irreducible component of g^*, such that

$$\frac{v_2(\psi)}{(\psi, h)} \geqslant \frac{v_p^2}{\delta_p^2}.$$

Note that ψ cannot be an irreducible component of T. If the inequality is a strict inequality, then $(\psi, h) = d_p(\psi)\delta_{p+1}^2$ and so

$$(g^*, h) \geqslant (T, h) + (\psi, h) \geqslant \delta_{r+1}^1 + \delta_{p+1}^2 > (g^*, h).$$

As a consequence, $v_2(\psi) = d_{p-1}(\psi)v_p^2$ and $(\psi, h) = d_{p-1}(\psi)\delta_p^2$. Denote by B_p the product of all the components ψ with the properties above. Then

$$v_2(B_p) = (\textstyle\sum d_{p-1}(\psi))v_p^2 \quad \text{and} \quad (B_p, h) = (\textstyle\sum d_{p-1}(\psi))\delta_p^2.$$

By the construction of ψ it is evident that $\Sigma\, d_{p-1}(\psi) \geqslant M_p^2 - 1$, so we have

$$\delta_{r+1}^1 + M_p^2\delta_p^2 - 1 = (g^*, h) \geqslant \delta_{r+1}^1 + (\textstyle\sum d_{p-1}(\psi))\delta_p^2.$$

As a consequence $\Sigma\, d_{p-1}(\psi) = M_p^2 - 1$ and B_p also satisfies the statements in the theorem. So we have $p = q$ and (3.7.1) is proved.

(3.8) LEMMA. *Let A_r be the product of the irreducible components of g^* such that $v_1(\varphi)\delta_r^1 = (\varphi, h)v_r^1$. Then, A_r satisfies the statements of Theorem (3.6).*

Proof. Let φ be an irreducible component of A_r. Since

$$\frac{v_1(\varphi)}{(\varphi, h)} = \frac{v_r^1}{\delta_r^1} > \frac{v_{q+1}^1}{\delta_{q+1}^1} = \frac{\xi}{(f_2, h)},$$

by (1.10) we have $v_2(\varphi) = d_{r-1}(\varphi)\, \dfrac{\xi}{d_{r-1}^1}$.

Denote $P = \Sigma\, d_{r-1}(\varphi)$, when φ belongs to the set of irreducible components of A_r. Then $\underline{v}(A_r) = PV_r^1$, $(A_r, h) = P\delta_r^1$ and by the construction of φ in (3.7), $P \geqslant (M_r^1 - 1)$. In order to finish the proof it suffices to show that $P = M_r^1 - 1$.

Assume $P \geqslant M_r^1$. Then, by (3.7.1), must be $p = q$. The inequality:

$$(g^*, h) = \delta_{r+1}^1 + \delta_{q+1}^2 - 1 = \delta_{q+1}(M_{q+1}^1 \cdots M_r^1 + 1) - 1 \geqslant (A_r, h) = P\delta_r^1,$$

implies that $P \leqslant M_r^1$, that is, $P = M_r^1$, and as a consequence

$$v_2(A_r) = P\,\frac{\xi}{d_{r-1}^1} = M_r^1\,\frac{\xi}{d_{r-1}^1} = \frac{\xi}{d_r^1}.$$

Now, looking at $g^{**} = g^*/A_r$ we have

$$v_2(g^{**}) = \sum_1^q (M_i^2 - 1)v_i^2 \quad \text{and} \quad (g^{**}, h) = \sum_1^q (M_i^2 - 1)\delta_i^2.$$

Using (2.8) in the same way as in Lemma (3.7) we can prove that

$$v_1(A_r) = \sum_{q+1}^r (M_i^1 - 1)v_i^1 + \frac{\xi}{d_q^2}.$$

And, on the other hand,

$$v_1(A_r) = M_r^1 v_r^1 \underset{(1)}{\geqslant} \sum_{q+1}^r (M_i^1 - 1)v_i^1 + v_{q+1}^1 \underset{(2)}{\geqslant} \sum_{q+1}^r (M_i^1 - 1)v_i^1 + \frac{\xi}{d_q^2}.$$

Note that (1) is an equality if and only if $r = q + 1 > l$. But in this case $l = q$ and $\xi < d_q^2 v_{q+1}^1$, so in (2) we have a strict inequality. In any case we have a contradiction. As a consequence $P = M_r^1 - 1$ and A_r satisfies the statements in the theorem. That is, we have proved that $r = l$.

(B) ASSUME $p > q$.

As we have explained in the sketch of the proof of (3.6) we can restrict our attention to the case $l = q + 1$, $\xi = d_q^2 v_{q+1}^1$ and in this case, after the proof of (A) above, one has:

$$\underline{v}(g^*) = \sum_1^q (M_i - 1)V_i + \left(M_{q+1}^1 v_{q+1}^1, \frac{\xi}{d_{q+1}^1}\right) + \sum_{q+1}^p (M_i^2 - 1)V_i^2$$

$$(g^*, h) = \delta_{q+1}^1 + \delta_{p+1}^2 - 1.$$

(3.9) LEMMA. *There exists φ, irreducible component of g^*, such that*:

$$\frac{v_2(\varphi)}{(\varphi, h)} = \frac{v_p^2}{\delta_p^2}.$$

Proof. As in (3.7), there exists φ such that

$$\frac{v_2(\varphi)}{(\varphi, h)} \geqslant \frac{v_p^2}{\delta_p^2}. \tag{$*$}$$

Assume that in $(*)$ we have a strict inequality; then $\delta_{p+1}(\varphi) = \delta_{p+1}^2$ and by (1.8) and (1.10), $(\varphi, h) = d_p(\varphi)\delta_{p+1}^2$, $v_1(\varphi) = d_p(\varphi)\ \dfrac{\xi}{d^2}$. The inequality $(\varphi, h) \leqslant (g^*, h)$ become:

$$d_p(\varphi)\delta_{p+1}^2 \leqslant \delta_{q+2}^1 + \delta_{p+1}^2 - 1 = \delta_{q+1}(M_{q+1}^1 + M_{q+1}^2 \cdots M_p^2) - 1$$

and then, if α is the natural number

$$\alpha = (M_{q+1}^1 + M_{q+1}^2 \cdots M_p^2) - (M_{q+1}^2 \cdots M_p^2 d_p(\varphi))$$

there must be $0 < \alpha \leqslant M_{q+1}^1$.

Denote $g^{**} = g^*/\varphi$. Using the value $\xi = d_q^2 v_{q+1}^1$ we find that

$$\begin{aligned} v_1(g^{**}) &= \sum_1^{q+1} (M_i^1 - 1)v_i^1 + \frac{\xi}{d_p^2} - \frac{\xi}{d_p^2}\, d_p(\varphi) \\ &= \sum_1^{q} (M_i^1 - 1)v_i^1 + v_{q+1}^1(M_{q+1}^1 - 1 + M_{q+1}^2 \cdots M_p^2 - M_{q+1}^2 \cdots M_p^2 d_p(\varphi)) \\ &= \sum_1^{q} (M_i^1 - 1)v_i^1 + v_{q+1}^1(\alpha - 1), \end{aligned}$$

$$(g^{**}, h) = \delta_{q+1}^1\alpha - 1 = \sum_1^q (M_i - 1)\delta_i + (\alpha - 1)\delta_{q+1}^1.$$

Now, we can use (2.8) and then $g^{**} = A_1 \cdots A_q A_{q+1}$ in such a way that $v_1(A_i) = (M_i - 1)v_i^1$, $\forall i \leqslant q$ and $v_1(A_{q+1}) = (\alpha - 1)v_{q+1}^1$. In fact, for the components $A_1, \ldots, A_q$ one has $\underline{v}(A_i) = (M_i - 1)V_i$, $(A_i, h) = (M_i - 1)\delta_i$ and then, for $\phi = \varphi A_{q+1}$ one has:

$$\underline{v}(\phi) = \left(M_{q+1}^1 v_{q+1}^1, \frac{\xi}{d_{q+1}^1}\right) + \sum_{q+1}^{p} (M_i^2 - 1)V_i^2,$$

that is,

$$v_1(\phi) = M^1_{q+1}v^1_{q+1} + \frac{\xi}{d^2_p} - \frac{\xi}{d^2_q} = (M^1_{q+1} - 1 + M^2_{q+1}\cdots M^2_p)v^1_{q+1}$$

$$v_2(\phi) = \sum_{q+1}^{p} (M^2_i - 1)v^2_i + \frac{\xi}{d^1_{q+1}}.$$

Let ψ be an irreducible component of A_{q+1}; then, by (2.8)

$$\frac{v_1(\psi)}{(\psi,\ h)} = \frac{v^1_{q+1}}{\delta^1_{q+1}} = \frac{(f_1, f_2)}{(f_2,\ h)}.$$

So, by (1.10)

$$v_2(\psi) \geqslant \frac{(\psi,\ h)}{d^1_0}\ \xi = d_q(\psi)\ \frac{\xi}{d^1_q}.$$

Adding on the factors ψ of A_{q+1} we obtain

$$v_2(A_{q+1}) \geqslant (\alpha - 1)\ \frac{\xi}{d^1_q}.$$

On the other hand, for φ:

$$\begin{aligned} v_2(\varphi) &> d_{p-1}(\varphi)v^2_p = d_p(\varphi)M^2_p v^2_p = (d_p(\varphi) - 1)M^2_p v^2_p + M^2_p v^2_p \\ &\geqslant (d_p(\varphi) - 1)M^2_p v^2_p + \sum_{q+1}^{p} (M^2_i - 1)v^2_i + v^2_{q+1} \\ &\geqslant (d_p(\varphi) - 1)M^2_p\cdots M^2_{q+1}v^2_{q+1} + v^2_{q+1} + \sum_{q+1}^{p} (M^2_i - 1)v^2_i \\ &= [(d_p(\varphi) - 1)M^2_p\cdots M^2_{q+1} + 1] + \sum (M^2_i - 1)v^2_i \\ &= (M^1_{q+1} - \alpha + 1)v^2_{q+1} + \sum (M^2_i - 1)v^2_i \\ &= \sum (M^2_i - 1)v^2_i + (M^1_{q+1} - \alpha + 1)\ \frac{\xi}{d^1_q}. \end{aligned}$$

Joining both inequalities above

$$v_2(\phi) > \sum_{q+1}^{p} (M^2_i - 1)v^2_i + M^1_{q+1}\ \frac{\xi}{d^1_q} = \sum_{q+1}^{p} (M^2_i - 1)v^2_i + \frac{\xi}{d^1_{q+1}},$$

provides a contradiction. Thus (3.9) is proved.

(3.10) LEMMA. *Let B_p be the product of the irreducible components φ of g^* such that*

$$\frac{v_2(\varphi)}{(\varphi,\ h)} = \frac{v_p^2}{\delta_p^2}.$$

Then B_p satisfies the statements of Theorem (3.6).

Proof. In the same way as in Lemma (3.8), if φ is an irreducible component of B_p, since

$$\frac{\xi}{(f_1,\ h)} < \frac{d_q^1 v_{q+1}^2}{d_0^1} = \frac{v_{q+1}^2}{\delta_{q+1}^2} < \frac{v_p^2}{\delta_p^2} = \frac{v_2(\varphi)}{(\varphi,\ h)}$$

then $v_1(\varphi) = d_{p-1}(\varphi)\ \dfrac{\xi}{d_{p-1}^2}$.

Denote $P = \Sigma\, d_{p-1}(\varphi)$, when φ belongs to the set of irreducible components of B_p. Then $\underline{v}(B_p) = PV_p^2$, $(B_p, h) = P\delta_p^2$ and, by the construction of φ in (3.9), $P \geqslant (M_p^2 - 1)$. In order to finish the proof it suffices to show that $P = M_p^2 - 1$.

Assume $P \geqslant M_p^2$. Since

$$(B_p,\ h) = P\delta_p^2 \leqslant M_1^2 \cdots M_p^2 + M_1^1 \cdots M_{q+1}^1 - 1 = (g^*,\ h)$$

then the natural number λ defined by

$$\lambda = M_{q+1}^1 + M_{q+1}^2 \cdots M_p^2 - M_{q+1}^2 \cdots M_{q+1}^2 P$$

satisfies that $0 < \lambda \leqslant M_{q+1}^1$. Taking into account the computations above, we find for $g^{**} = g^*/B_p$ the following:

$$\begin{aligned}
v_1(g^{**}) &= \sum_1^{q+1} (M_i^1 - 1)v_i^1 + \frac{\xi}{d_p^2} - \frac{\xi}{d_{p-1}^2}\, P \\
&= \sum_1^{q+1} (M_i^1 - 1)v_i^1 + M_{q+1}^2 \cdots M_p^2 v_{q+1}^1 - PM_{q+1}^2 \cdots M_{p-1}^2 v_{q+1}^1 \\
&= \sum_1^{q+1} (M_i^1 - 1)v_i^1 + (\lambda - 1)v_{q+1}^1,
\end{aligned}$$

$$(g^{**}, h) = \delta_{q+1}^1 \lambda - 1 = \sum_1^{q} (M_i^1 - 1)\delta_1^1 + (\lambda - 1)\delta_{q+1}^1.$$

By (2.8), $g^{**} = A_1 \cdots A_q A_{q+1}$ in such a way that $\underline{v}(A_i) = (M_i - 1)V_i$, $\forall i \leqslant q$ and as a consequence, for $\phi = B_p A_{q+1}$,

$$v_2(\phi) = \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + \frac{\xi}{d_{q+1}^1}.$$

If ψ is an irreducible component of A_{q+1}, using (2.8) in the same way as in (3.9) we obtain $v_2(\psi) \geqslant d_q(\psi)\, \frac{\xi}{d_q^1}$ and then $v_2(A_{q+1}) \geqslant (\lambda - 1)\, \frac{\xi}{d_q^1}$. On the other hand, for B_p,

$$\begin{aligned} v_2(B_p) &= Pv_p^2 = (P - M_p^2)v_p^2 + M_p^2 v_p^2 \\ &\geqslant (P - M_p^2)v_p^2 + \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + v_{q+1}^2 \\ &\geqslant (d_{p-1}(B_p) - M_p^2)M_{p-1}^2 \cdots M_{q+1}^2 v_{q+1}^2 + \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + v_{q+1}^2 \\ &= \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + (PM_{p-1}^2 \cdots M_{q+1}^2 - M_p^2 \cdots M_{q+1}^2 + 1)v_{q+1}^2 \\ &= \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + v_{q+1}^2(M_{q+1}^1 - \lambda + 1) \\ &> \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + (M_{q+1}^1 - \lambda + 1)\, \frac{\xi}{d_q^1}. \end{aligned}$$

As a consequence,

$$v_2(\phi) > \sum_{q+1}^{p} (M_i^2 - 1)v_i^2 + M_{q+1}^1 \frac{\xi}{d_q^1}$$

and we get a contradiction. Thus Lemma (3.10) is proved and we have also finished the proof of Theorem (3.6).

(3.11) SECOND DECOMPOSITION THEOREM.
Case: $\xi = d_q^2 v_{q+1}^1 < d_q^1 v_{q+1}^2$.

With conditions and notations as in (3.5), *let* g' *be as in* (3.6) *and assume that* $\xi = d_q^2 v_{q+1}^1 < d_q^1 v_{q+1}^2$. *Then*

$$g' = C_1 \cdots C_q D$$

with the following conditions:

(1) *The components* C_i, *for* $1 \leqslant i \leqslant q$, *are such that*

$$(C_i, h) = (M_i - 1)\delta_i, \quad \underline{v}(C_i) = (M_i - 1)V_i \quad \text{and} \quad \frac{\underline{v}(C_{ij})}{(C_{ij}, h)} = \frac{V_i}{\delta_i}$$

for any C_{ij}, *irreducible component of* C_i.

(2) D *is irreducible*, $(D, h) = \delta^1_{q+2}$ *and*

$$\frac{v_1(D)}{(D, h)} = \frac{v^1_{q+1}}{\delta^1_{q+1}}.$$

(*As a consequence* $\underline{v}(D) = N$.)

Proof. According to (3.6) we have

$$\underline{v}(g') = \sum_1^q (M_i - 1)V_i + \left(M^1_{q+1}v^1_{q+1}, \frac{\xi}{d^1_{q+1}}\right)$$

$$(g', h) = \delta^1_{q+1} + \delta^2_{q+1} - 1.$$

Assume that there exists φ, irreducible component of g', such that $v_1(\varphi)\delta^1_{q+1} > v^1_{q+1}(\varphi, h)$. Then $\delta_{q+2}(\varphi) = \delta^1_{q+2}$ and since

$$(\varphi, h) = d_{q+1}(\varphi)\delta^1_{q+2} \leqslant \delta^1_{q+2} + \delta_{q+1} - 1 = (g', h)$$

we have $d_{q+1}(\varphi) = 1$. As a consequence, using (1.10),

$$v_2(\varphi) = d_{q+1}(\varphi)\frac{\xi}{d^1_{q+1}} = \frac{\xi}{d^1_{q+1}}$$

$$v_1(\varphi) > \frac{(\varphi, h)}{\delta^1_{q+1}} v^1_{q+1} = d_q(\varphi)v^1_{q+1} = M^1_{q+1}v^1_{q+1}.$$

However,

$$v_2(g'/\varphi) = \sum_1^q (M_i - 1)v^2_i \quad \text{and} \quad (g'/\varphi, h) = \sum_1^q (M_i - 1)\delta_i.$$

So, by (2.8), $g' = \varphi C_1 \cdots C_q$ where $\underline{v}(C_i) = (M_i - 1)V_i$. As a consequence, $v_1(\varphi) = M^1_{q+1}v^1_{q+1}$ and we get a contradiction with the assumption $v_1(\varphi) > M^1_{q+1}v^1_{q+1}$.

We have just proved that, for any irreducible component, φ, of g', one has

$$\frac{v_1(\varphi)}{(\varphi,\ h)} \leqslant \frac{v^1_{q+1}}{\delta^1_{q+1}}.$$

Let $\tilde{f}_1 \in k[[X, Y]]$ be such that the m-sequence of $\tilde{f}_1$ with respect to h is given by $\tilde{v}_i = v^1_i/d^1_{q+1}$ $(0 \leqslant i \leqslant q+1)$ and satisfying $v_1(\tilde{f}_1) = v^2_{q+2}$. For $\tilde{f}_1$ we have $\tilde{d}_i = d^1_i/d^1_{q+1}$ and $\tilde{M}_i = M_i$ $(1 \leqslant i \leqslant q+1)$. For any $\varphi \in k[[X, Y]]$ irreducible, by (1.10), one has the equivalence

$$\frac{(\varphi,\ f_1)}{(\varphi,\ h)} \leqslant \frac{v^1_{q+1}}{\delta^1_{q+1}} \Leftrightarrow \frac{(\varphi,\ \tilde{f}_1)}{(\varphi,\ h)} \leqslant \frac{\tilde{v}^1_{q+1}}{\tilde{\delta}^1_{q+1}} \qquad (*)$$

and, if this condition is satisfied; $(\varphi, \tilde{f}_1)d^1_{q+1} = (\varphi, f_1)$.

Moreover,

$$(g',\ \tilde{f}_1) = \sum_1^q (\tilde{M}_i - 1)\tilde{v}_i + \tilde{M}_{q+1}\tilde{v}_{q+1} = \sum_1^q (\tilde{M}_i - 1)\tilde{v}_i + \tilde{\eta}$$

and

$$(g',\ h) = \tilde{\delta}_{q+2} + \tilde{\delta}_{q+1} - 1 = \sum_1^q (\tilde{M}_i - 1)\tilde{\delta}_i + \tilde{v}_0,$$

where $\tilde{\eta} = \tilde{d}_q\tilde{v}_{q+1} = \tilde{M}_{q+1}\tilde{v}_{q+1}$. Then, using $(*)$, we get

$$\frac{(\varphi, \tilde{f}_1)}{(\varphi,\ h)} \leqslant \frac{\tilde{v}_{q+1}}{\tilde{\delta}_{q+1}} = \frac{\tilde{\eta}}{(\tilde{f}_1,\ h)}$$

for any φ, irreducible component of g'.

Thus we can apply (2.8) and so $g' = C_1 \cdots C_q D$ in such a way that for any $i = 1, \ldots, q$,

$$(C_i,\ h) = (\tilde{M}_i - 1)\tilde{\delta}_1 = (M_i - 1)\delta_i;\ (C_i, \tilde{f}_1) = (\tilde{M}_i - 1)\tilde{v}_i$$

and for any C_{ij} irreducible component of C_i,

$$\frac{(C_{ij}, \tilde{f}_1)}{(C_{ij},\ h)} = \frac{\tilde{v}_i}{\tilde{\delta}_i}.$$

In particular the irreducible components C_{ij} of C_i $(1 \leqslant i \leqslant q)$ satisfy the condition $(*)$ and so

$$v_1(C_i) = (M_i - 1)v_i^1\,; \frac{(C_{ij}, f_1)}{(C_{ij}, h)} = \frac{v_i^1}{\delta_i^1}.$$

Thus, $C_1, \ldots, C_q$ are the components stated in the theorem.

For D, we have $(D, h) = \tilde{v}_0 = \delta_{q+2}^1$, $(D, f_1) = M_{q+1}^1 v_{q+1}^1$ and

$$\frac{(D_i, f_1)}{(D_i, h)} = \frac{v_{q+1}^1}{\delta_{q+1}^1}$$

for any D_i irreducible component of D. It is obvious that $v_2(D) = \dfrac{\xi}{d_{q+1}^1}$ and it only remains to be proven that D is irreducible.

Let φ be an irreducible component of D. We have $v_1(\varphi) = d_q(\varphi)v_{q+1}^1$. If $d_q(\varphi)v_{q+1}^1 = d_q^1 v_{q+1}(\varphi)$, then $(\varphi, h) = d_{q+1}\delta_{q+2}^2$ and so $d_{q+1}(\varphi) = 1$ and $D = \varphi$ as we wanted to prove. Assume that $d_q(\varphi)v_{q+1}^1 < d_q^1 v_{q+1}(\varphi)$ for every component φ of D.

Since $v_1(D) = (\Sigma\, d_q(\varphi))v_{q+1}^1 = M_{q+1}^1 v_{q+1}^1$, then $\Sigma\, d_q(\varphi) = M_{q+1}^1$. Moreover, by (1.8) $v_2(\varphi) \geqslant d_q(\varphi)\, \dfrac{\xi}{d_q^1}$ and, as $v_2(D) = \dfrac{\xi}{d_{q+1}^1} = M_{q+1}^1\, \dfrac{\xi}{d_q^1}$, then $v_2(\varphi) = d_q(\varphi)\, \dfrac{\xi}{d_q^1}$ or, in an equivalent way,

$$\frac{(\varphi, f_2)}{(\varphi, h)} = \frac{\xi}{(f_1, h)}.$$

By the hypothesis,

$$\frac{\xi}{(f_1, h)} < \frac{d_q^1 v_{q+1}^2}{d_0^1} = \frac{v_{q+1}^2}{\delta_{q+1}^1} = \frac{v_{q+1}^2}{\delta_{q+1}^2},$$

and then $v_2(\varphi) < d_q(\varphi)v_{q+1}^2$.

Looking at (1.7) and taking into account that $l_q^2 \geqslant l_q^1$ we deduce that one of the following possibilities must be true

(a) $v_2(\varphi) = d_{q-1}(\varphi)v_q^2 + r d_q^2 d_q(\varphi)$ with $r > l_q^1$. But then,

$$v_2(\varphi) = \frac{d_q(\varphi)}{d_q^1}\,(d_{q-1}^1 v_q^2 + r d_q^1 d_q^2) > \frac{d_q(\varphi)}{d_q^1}\,\xi$$

and we get a contradiction.

(b) $v_2(\varphi) = d_q^2 v_{q+1}(\varphi) < d_q(\varphi) v_{q+1}^2$. But the condition

$$d_q^1 v_{q+1}(\varphi) > d_q(\varphi) v_{q+1}^1$$

implies that

$$v_2(\varphi) = d_q^2 v_{q+1}(\varphi) > d_q^2 \frac{d_q(\varphi) v_{q+1}^1}{d_q^1} = d_q(\varphi) \frac{\xi}{d_q^1}$$

which is also a contradiction.

(3.12) THIRD DECOMPOSITION THEOREM.
Case: $\xi < d_q^2 v_{q+1}^1$.

With conditions and notations as in (3.5), *let* g' *be as in* (3.6) *and assume that* $\xi < d_q^2 v_{q+1}^1 \leqslant d_q^1 v_{q+1}^2$. *Then*

$$g' = C_1 \cdots C_n D$$

with the following conditions:

(1) $n = q$ *if* $\xi > d_{q-1}^2 v_q^1$ *and* $n = q - 1$ *if* $\xi = d_{q-1}^2 v_q^1$.
(2) *The components* C_i, *for* $1 \leqslant i \leqslant n$, *are such that*

$$(C_i, h) = (M_i - 1)\delta_i, \quad \underline{v}(C_i) = (M_i - 1)V_i, \quad \textit{and} \quad \frac{\underline{v}(C_{ij})}{(C_{ij}, h)} = \frac{V_i}{\delta_i}$$

for any C_{ij}, *irreducible component of* C_i.
(3) *If* $\xi > d_{q-1}^2 v_q^1$ *then* D *is irreducible*, $(D, h) = \delta_{q+1}$ *and*

$$\underline{v}(D) = N = \left(\frac{\xi}{d_q^2}, \frac{\xi}{d_q^1}\right).$$

(4) *If* $\xi = d_{q-1}^2 v_q^1$ *then*

$$(D, h) = \delta_{q+1} + (M_q - 1)\delta_q = (2M_q - 1)V_q$$
$$\underline{v}(D) = (M_q - 1)V_q + N = (2M_q - 1)V_q$$

and for any D_i, *irreducible component of* D, *one has*

$$\frac{\underline{v}(D_i)}{(D_i, h)} = \frac{V_q}{\delta_q}.$$

Proof. Consider, for $i = 1, 2$, an algebroid curve given by $\tilde{f}_i \in k[[X, Y]]$ such that $v_i(\tilde{f}_i) = v^i_{q+1}$. Then, the m-sequence for $\tilde{f}_i$ with respect to h is $\tilde{v}_j = v^1_j/d^1_q = v^2_j/d^2_q$ for $j = 0, \ldots, q$. The intersection multiplicity between $\tilde{f}_1$ and $\tilde{f}_2$ is $\tilde{\xi} = \dfrac{\xi}{d^1_q d^2_q}$ and $\tilde{f}_1$ is equisingular to $\tilde{f}_2$. Moreover, as in the proof of (3.11), we obtain

$$\frac{(\varphi, \tilde{f}_i)}{(\varphi, h)} \leqslant \frac{\tilde{\xi}}{(\tilde{f}_i, h)} \Leftrightarrow \frac{(\varphi, f_i)}{(\varphi, h)} \leqslant \frac{\xi}{(f_i, h)}$$

and in this case $\tilde{v}_i(\varphi) = (\varphi, \tilde{f}_i) = \dfrac{v_i(\varphi)}{d^i_q}$.

For g',

$$\underline{\tilde{v}}(g') = (\tilde{v}_1(g'), \tilde{v}_2(g')) = \sum_1^q (M_i - 1)(\tilde{v}_i, \tilde{v}_i) + (\tilde{\xi}, \tilde{\xi})$$

$$(g', h) = 2\delta_{q+1} - 1 = \sum_1^q (M_i - 1)\tilde{\delta}_i + \tilde{v}_0.$$

Thus, we may apply Theorem (2.13) to give the decomposition of g' and it is a straightforward computation (looking to (2.13) and (2.8)) to verify the statements of the theorem for these components.

(3.13) REMARK. Note that in the conditions of (3.5), we have

$$\begin{aligned}(g, h) &= v^1_0 + v^2_0 - 1 = M^1_1 \cdots M^1_s + M^2_1 \cdots M^2_t - 1 \\ &= \sum_1^t (M^2_i - 1)\delta^2_i + M^1_1 \cdots M^1_s \\ &= \sum_1^t (M^2_i - 1)\delta^2_i + \sum_{l+1}^s (M^1_i - 1)\delta^1_1 + \delta^1_{l+1}.\end{aligned}$$

Looking at the methods used in the proofs of (3.6), (3.11) and (3.12) and at the statements in (2.8) and (2.13) we realize that we would have been able to state and prove these theorems in a more general form, namely, we can write in (3.5):

$$\underline{v}(g) = \sum_1^q \alpha_i V_i + \sum_{l+1}^s \alpha^1_i V^1_i + \sum_{q+1}^t \alpha^2_i V^2_i + \varepsilon N$$

$$(g, h) = \sum_1^q \alpha_i \delta_i + \sum_{l+1}^s \alpha^1_i \delta^1_i + \sum_{q+1}^t \alpha^2_i \delta^2_i + \varepsilon \delta^1_{q+1}$$

where $\alpha_i < M_i$, $\alpha_i^1 < M_i^1$, $\alpha_i^2 < M_i^2$ and $\varepsilon = 0, 1$. The statements for the corresponding Theorem (3.6), (3.11) and (3.12) are the adequate generalizations in the same way as (2.8).

4. Geometrical interpretation and the case of more than two branches

(4.1) Let C be the algebroid plane curve given by $f = \Pi_1^d f_i \in k[[X, Y]]$. We shall assume f reduced with d irreducible factors. Let $\pi : X \to \operatorname{Spec}(k[[X, Y]])$ be the canonical minimal resolution of C, $E = \pi^{-1}(0)$ the exceptional divisor of π, $\bar{C}$ the total transform of C and $\tilde{C} = \overline{\pi^{-1}(C - \{0\})}$ the strict transform of C. Recall that $\bar{C}$ is a curve in X with only normal crossings and $\tilde{C}$ has d connected components $\tilde{C}_1, \ldots, \tilde{C}_d$ corresponding to the d branches $C_1, \ldots, C_d$ of C.

Associated with π we can construct the *resolution graph* (also called the *dual graph*), $A(f)$, as the dual figure of E. More precisely:

(i) Each irreducible component of E is represented by a point in $A(f)$.
(ii) Two points in $A(f)$ are joined if and only if their corresponding components in E meet.
(iii) For each irreducible component $\tilde{C}_i$ of $\tilde{C}$ a little arrow is drawn joined to the point corresponding to the only component of E meeting $\tilde{C}_i$.

Denote by $E(P)$ the component of E corresponding to the point P of $A(f)$ and denote by $w(P)$ the number of blow-ups needed to build the divisor $E(P)$. The weighted tree $(A(f), w)$ is an equivalent data to the singularity type of C (see [B-K], [Z]). Frequently, $A(f)$ is also presented as an oriented graph starting in the only point P_0 with weight 1.

Some of the usual terminology for the resolution graph is as follows. Let $P \in A(f)$; $v(P)$ denotes the *valence* of P, that is, the number of points or arrows joined with P in $A(f)$. The *ordinary* points of $A(f)$ are the points $P \in A(f)$ such that $v(P) = 2$ and the *special* (non ordinary) points are divided in *extremal points* (if $v(P) = 1$) and *rupture points* (if $v(P) \geqslant 3$). An *arc* is a geodesic in $A(f)$ joining two special points and with no more special points. In particular, a *dead arc* in $A(f)$ is an arc such that one of its extremes is an extremal point.

(4.2) Let $\varphi \in k[[X, Y]]$ be irreducible. We will say that φ *goes through* $P \in A(f)$ if the strict transform of φ by π meets the component $E(P)$ of E. Fix a non singular algebroid curve given by $h \in k[[X, Y]]$. Then the multi-coefficient of contact of f and φ with respect to h is the element of $\mathbb{Q}^d$:

$$I(f, \varphi, h) = \frac{v(\varphi)}{(\varphi, h)} = \left(\frac{(\varphi, f_1)}{(\varphi, h)}, \ldots, \frac{(\varphi, f_d)}{(\varphi, h)}\right).$$

Note that if h is transversal to φ then $(\varphi, h) = m(\varphi)$ and $\dfrac{(\varphi, f_i)}{(\varphi, h)}$ is the coefficient of contact (in the sense of Hironaka [H]) between f_i and φ, and $\sum_1^d \dfrac{(\varphi, f_i)}{(\varphi, h)}$ is the coefficient of contact between f and φ (following [LMW2]).

Let $P \in A(f)$. A *curvette* at P is a curve germ in X, Θ_P, smooth and transversal to E at a regular point of the component $E(P)$. For such a $P \in A(f)$ and Θ_P we can consider the irreducible curve given by $\pi(\Theta_P)$ and define the *multi-coefficient of insertion* of P with respect to h as:

$$q(P, h) = I(f, \pi(\Theta_P), h).$$

Note that, as above, taking h transversal and adding the coordinates of $q(P, h)$ we obtain the coefficient of insertion $q(P)$ defined in [LMW2].

(4.3) IRREDUCIBLE CASE

Assume that f is irreducible and let $\bar\beta_0, \ldots, \bar\beta_g$, $v_0, \ldots, v_s$ be the numbers defined in (1.2) and (1.3) for f with respect to h. It is well-known that $A(f)$ has $g + 1$ dead arcs and g rupture points. Denote by $P_0, \ldots, P_g$ the extremal points of $A(f)$ ordered by w, that is, $1 = w(P_0) < w(P_1) < \cdots w(P_g)$. Then, $(f, \pi(\Theta_{P_i})) = \bar\beta_i$, i.e., the values of the maximal contact can be obtained by means of curvettes in the extremal points of $A(f)$.

Now consider the resolution graph of fh, $A(fh)$. Following the different possibilities for h with respect to f (see (1.4)) one can also check that s is the number of dead arcs of $A(fh)$ (and therefore the number of extremal points) and $v_1, \ldots, v_s$ can be obtained by means of curvettes at the extremal points of $A(fh)$. Moreover, $v_0 = (f, h)$.

Note that if we take h transversal to f, the extremal point P_0 of $A(f)$ does not appear in $A(fh)$ as an extremal point, because h goes through P_0 and so $v(P_0) = 2$ in $A(fh)$. In the same way, if $(f, h) = \bar\beta_1$ then P_1 is not an extremal point in $A(fh)$. In other case the number of extremal points in $A(fh)$ is $g + 1 = s$.

(4.4) THEOREM. *Let L be a dead arc of $A(fh)$. Then the coefficient of insertion of $P \in L$ with respect to h, $q(P, h)$, does not depend on the point $P \in L$ and thus can be denoted by $q(L)$. The set of the numbers $q(L)$ where L is a dead arc of $A(fh)$ is exactly*

$$\left\{\frac{v_1}{\delta_1}, \ldots, \frac{v_s}{\delta_s}\right\}.$$

Moreover, let $\varphi \in k[[X, Y]]$ be irreducible. Then, φ goes through L if and only if $I(f, \varphi, h) = q(L)$.

The proof when h is transversal to f is well-known ([LMW2], [Z]). The result can be proven for any h using the results in [LMW2] or by direct computation using the Lemma (1.7) and the Noether formulae for the intersection multiplicity in terms of the multiplicities at the infinitely near points.

(4.5) Now assume that C has two branches, that is, $f = f_1 f_2$. Take the conditions and the notations of (3.2) and let l be as at the beginning of (3.4). As consequence of the facts above and the constructions of (3.2), one can easily prove that the elements

$$V_1, \ldots, V_q, V_{l+1}^1, \ldots, V_s^1, V_{q+1}^2, \ldots, V_t^2$$

may be obtained by means of curvettes at the extremal points of $A(fh)$. That is, the set above is exactly the set $\{\underline{v}(\pi(\Theta_P))\}$, when P is an extremal points of $A(fh)$. Moreover, one can prove:

(4.6) THEOREM. *Let L be a dead arc of $A(fh)$. Then the multi-coefficient of insertion of $P \in L$ with respect to h, $q(P, h)$, does not depend of the point $P \in L$ chosen and thus can be denoted by $q(L)$. The set of pairs $q(L)$ where L is a dead arc is exactly the set*

$$\left\{\frac{V_1}{\delta_1}, \ldots, \frac{V_q}{\delta_q}, \frac{V_{l+1}^1}{\delta_{l+1}^1}, \ldots, \frac{V_s^1}{\delta_s^1}, \frac{V_{q+1}^2}{\delta_{q+1}^2}, \ldots, \frac{V_t^2}{\delta_t^2}\right\}.$$

Moreover, let $\varphi \in k[[X, Y]]$ be irreducible. Then, φ goes through L if and only if $I(f, \varphi, h) = q(L)$.

Denote by $L_1, \ldots, L_q, L_{l+1}^1, \ldots, L_s^1, L_{q+1}^2, \ldots, L_t^2$ the dead arcs of $A(fh)$ in such a way that:

$$q(L_i) = \frac{V_i}{\delta_i}, \quad q(L_j^1) = \frac{V_j^1}{\delta_j^1}, \quad q(L_k^2) = \frac{V_k^2}{\delta_k^2},$$

for $1 \leqslant i \leqslant q$, $l + 1 \leqslant j \leqslant s$ and $q + 1 \leqslant k \leqslant t$.

(4.7) Keeping the conditions and notations of (3.2), assume for a moment that h is transversal to f and let R be the *separation point* of f_1 and f_2. That is, let Γ_i be the geodesic in $A(fh)$ joining the only point P_0 with $w(P_0) = 1$ with the arrow corresponding to f_i and let R be the point with the greatest weight in $\Gamma_1 \cap \Gamma_2$. Thus, R is a rupture point in $A(fh)$ and R does not belong to any dead arc except if

$$\xi = (f_1, f_2) = e_q^1 \overline{\beta}_{q+1}^2 = e_q^2 \overline{\beta}_{q+1}^1.$$

In any case, the element N defined in (3.2) can be obtained by means of a curvette in R. Note that if f_1 and f_2 are transversals, R is also a rupture point in $A(fh)$.

Now, take h not necessarily transversal to f. The situation in this case is slightly different. Firstly, the separation point of f_1 and f_2, R, can be an ordinary point in $A(fh)$. Moreover, this point can be different of the rupture point of $A(fh)$ not belonging to any dead arc, as the examples in Remark (4.8) below show. However, similar results to those of the transversal case can be stated:

First of all, the following conditions are equivalent:

(1) No rupture point exists in $A(fh)$ out of a dead arc.
(2) The intersection multiplicity between f_1 and f_2 is $\xi = d_{q-1}^1 v_q^2 = d_{q-1}^2 v_q^1$.
(3) The separation point R of f_1 and f_2 belongs to L_q.

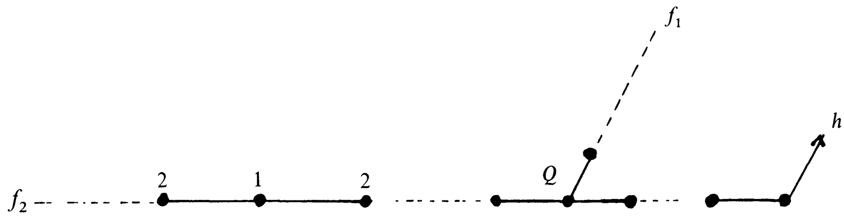
If f satisfies these equivalent conditions, then the element N of S can be obtained by means of a curvette in the separation point R of f_1 and f_2. However, note that in this case $q(L_q) = q(R)$ and so, from the point of view of the multi-coefficient with respect to h, the irreducible curves going through R are indistinguishable from the curves going through L_q.

If f does not satisfy the equivalent conditions above, there exists exactly one point Q such that Q is a rupture point and Q does not belong to any dead arc. In this case, N can be obtained by means of a curvette in Q, $N = \underline{v}(\pi(\Theta_Q))$.

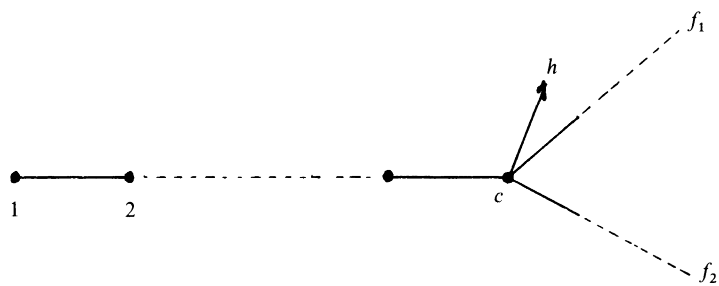
(4.8) REMARK. Note that the point Q stated above is not, in general, the separation point of f_1 and f_2. By example, take f_1 and f_2 transversal and let h be such that $v_1(h) = v_0^1 = \overline{\beta}_1^1$. Then the separation point of f_1 and f_2 is obviously the point P_0 of weight 1. However, the point Q is the rupture point of $A(f_1)$ with minimal weight. Note that this point does not belong to any dead arc in $A(fh)$ because the two dead arcs in $A(f_1)$ going through Q disappear, the first one by the resolution graph of f_2 and the second one by the arrow corresponding to h (see the first figure below).

Note that in this case: $\xi = \overline{\beta}_0^1 \overline{\beta}_0^2 = d_0^2 v_1^1 < d_0^1 v_1^2$. In fact, one can characterize in these terms the cases in which Q is not the separation point.

An example in which the point Q stated above does not exist, is the



following. Take f_1 and f_2 such that $\xi = c\bar{\beta}_0^1\bar{\beta}_0^2$ and let h be such that $(f_1, h) = c\bar{\beta}_0^1$, $(f_2, h) = c\bar{\beta}_0^2$. Then the situation in $A(fh)$ is



and the point Q does not exist. Note that, in this case $v_0^1 = c\bar{\beta}_0^1$, $v_0^2 = c\bar{\beta}_0^2$ and so $\xi = v_0^1 v_1^2 = v_1^1 v_0^2$.

The results proved in Section 3 can be stated, taking into account the comments above, as follows:

(4.9) DECOMPOSITION THEOREM. *Let $g \in k[[X, Y]]$ in the conditions of* (3.5). *Then g can be decomposed as*

$$g = C_1 \cdots C_n D A_{l+1} \cdots A_s B_{q+1} \cdots B_t$$

in such a way that:

(1) *For $j = l+1, \ldots, s$, the irreducible components of A_j go through L_j^1 and*

$$\underline{v}(A_j) = (M_j^1 - 1)V_j^1, \qquad (A_j, h) = (M_j^1 - 1)\delta_j^1 .$$

(2) *For $k = q+1, \ldots, t$, the irreducible components of B_k go through L_k^2 and*

$$\underline{v}(B_k) = (M_k^2 - 1)V_k^2, \qquad (B_k, h) = (M_k^2 - 1)\delta_k^2 .$$

(3) $n = q - 1$ *if and only if* $\xi = d^1_{q-1} v^2_q = d^2_{q-1} v^1_q$. (*That is, if and only if* f *satisfies the equivalent conditions of* (4.7)). *In other case* $n = q$.
(4) *For* $i = 1, \ldots, n$, *the irreducible components of* C_i *go through* L_i *and*

$$\underline{v}(C_i) = (M_i - 1)V_i, \qquad (C_i, h) = (M_i - 1)\delta_i.$$

(5) *If* $n = q - 1$ *the irreducible components of* D *go through* L_q *and*

$$\underline{v}(D) = (2M_q - 1)V_q, \qquad (D, h) = (2M_q - 1)\delta_q.$$

(6) *If* $n = q$, D *is irreducible and goes through* Q *as a smooth curve.*

(4.10) REMARK. In particular, we obtain the Theorem (2.1) in [LMW2] in a slightly more precise form. Note also that the multi-quotient polars, that is, the set

$$\{I(f, \varphi, h) \mid \varphi \text{ is an irreducible component of } J(f, h)\},$$

is exactly

$$\{q(P, h) \mid P \text{ is a rupture point of } A(fh)\}.$$

The number of packages of components in (4.9) is exactly the number of rupture points. We must remark finally the convenience of handle $A(fh)$ (or equivalently, the curve fh) instead of $A(f)$ (or f) for the different statements about polars.

(4.11) REMARK. In the proofs of the results above we have only needed the facts

$$\underline{v}(g) = \tau + V_0, \qquad (g, h) = (f, h) - 1$$

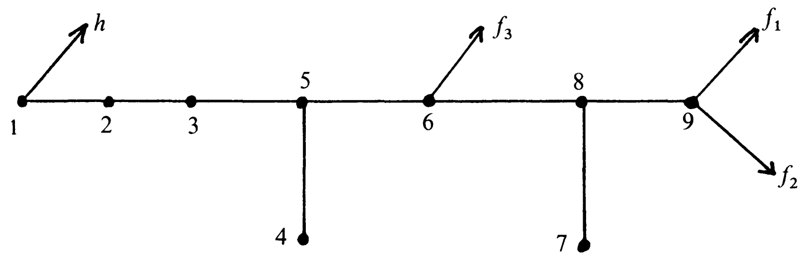
and after that we have used the arithmetic of the semigroup of values of f. So, for the case of two branches (and, of course, for the case of one branch), topological or analytical properties of the polars are not needed. In other words, a curve with the same intersection multiplicities than the polar with f_1, f_2 and h, has a similar behavior (in the sense of (4.9)). However, for $d \geqslant 3$ branches this is not true as the following example shows.

(4.12) EXAMPLE. Consider $f_1, f_2, f_3 \in \mathbb{C}[[X, Y]]$ defined, in terms of its

Hamburger–Noether expansions, by:

$$f_1:\begin{cases} y = x^3 z_1 \\ x = z_1^2 + z_1^3 z_2 \\ z_1 = z_2^2 \end{cases} \qquad f_2:\begin{cases} y = x^3 z_1 \\ x = z_1^2 + z_1^3 z_2 \\ z_1 = z_2^2 + z_2^3 \end{cases} \qquad f_3:\begin{cases} y = x^3 z_1 \\ x = z_1^2 + z_1^3 \end{cases}$$

and take h transversal to $f = f_1 f_2 f_3$. The resolution weighted tree $A(fh)$ is:



and by (2.1) in [LMW2], there are irreducible components of $J(f, h)$ going through the rupture points 6 and 9 and through the two dead arcs (in fact these ones go through 4 and 7).

Note that f_1 is equisingular to f_2. We have $\bar{\beta}_0^1 = \bar{\beta}_0^2 = 4$, $\bar{\beta}_1^1 = \bar{\beta}_1^2 = 14$, $\bar{\beta}_2^1 = \bar{\beta}_2^2 = 31$ and the conductors of the semigroups $S(f_1)$ and $S(f_2)$ are $c_1 = c_2 = 42$. For f_3 we have $\bar{\beta}_0^3 = 2$, $\bar{\beta}_1^3 = 7$ and $c_3 = 6$. The intersection multiplicities are:

$$(f_1, f_2) = 63 \qquad (f_1, f_3) = (f_2, f_3) = 30.$$

As consequence: $\underline{v}(J(f, h)) = \tau + V_0 = (138, 138, 67)$ and $(J(f, h), h) = m(J(f, h)) = m(f) - 1 = 9$.

Let g_1 be the irreducible curve given by:

$$g_1:\begin{cases} y = x^3 z_1 \\ x = z_1^2 + z_1^3 z_2 \\ z_1 = 2z_2^2 + z_2^2 z_3 \\ z_2 = z_3^2 + \cdots \end{cases}.$$

Then, $m(g_1) = (g, h) = 8$ and $\underline{v}(g_1) = (124, 124, 60)$. So, for $g = Yg_1$,

$$\underline{v}\,(g) = (138, 138, 67), \quad (g, h) = m(g) = 9.$$

That is, g has the same intersection multiplicities with f_1, f_2, f_3 and h as $J(f, h)$.

However, it is obvious that g cannot be decomposed in a similar way as $J(f, h)$. In fact g has only two branches, $Y = 0$ goes through the point 4 in $A(fh)$ and g_1 goes through 8 in $A(fh)$ smooth but tangent to the divisor with weight 8.

(4.13) REMARK. The obstruction in order to realize similar factorization results for the case $d \geqslant 3$ is neither the arithmetical decomposition of τ (see [D2]) nor the components corresponding to dead arcs of a single branch (some particular results can be given for such components). The main problem is the behavior on the dead arcs corresponding to more than one branch. To do the factorization, the results at the end of section 2 and the beginning of section 3 should be improved. In the case of three branches f_1, f_2, f_3, in general, one must add two different intersection multiplicities for each branch, say for example $(f_1, f_2) \geqslant e^1_{g-1}\bar{\beta}^1_g$ and $(f_1, f_3) < e^3_{g-1}\bar{\beta}^1_g$; but then the element in the semigroup given by

$$\sum_1^g (N_i^1 - 1)\bar{\beta}_i^1 + (f_1, f_2) + (f_1, f_3),$$

which provides the expression for the contact with the branch f_1, does not permit the decomposition of $J(f, h)$ in a nice form as in Theorem (2.8) because the second number (f_1, f_3) introduces a hard distortion in the arithmetic we need. One can find examples as the one above with a good selection of the numbers (f_1, f_2) and (f_1, f_3).

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