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Crystal bases of Verma modules for quantum affine Lie algebras

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1. Introduction

The quantized universal enveloping algebra (or quantum group) $U_q(g)$ associated with a symmetrizable Kac-Moody Lie algebra $g$ has been introduced independently by Drinfeld [D] and Jimbo [J] in their study of two dimensional solvable lattice models. The parameter $q$ corresponds to the temperature in the lattice model. In particular, $q = 0$ corresponds to the absolute temperature zero and hence one can expect some simplifications in the zero limit case. Motivated by this observation, one of the authors has introduced the notion of crystal base [K1]. The existence of such bases for any integrable representation of $U_q(g)$ is already known ([K1], [K2], [MM]). In [K2], the global base has been introduced. These bases coincide with Lusztig’s canonical bases for the finite dimensional Lie algebras of type ADE ([Lu1], [Lu2]). More recently, Lusztig and Grojnowski [LG] have proved that the canonical base and the global base coincide for all Kac-Moody Lie algebras with symmetric generalized Cartan matrices.

The crystal base which can be thought of roughly as a base at $q = 0$ provides a powerful combinatorial tool to study the quantum group $U_q(g)$ and its integrable representations. For example, the crystal base theory is very useful in decomposing the tensor products. Of course, for this one needs explicit description of the crystal bases. In [KN], the description of the crystal base for any irreducible highest weight modules for $U_q(g)$ for $g = A_n, B_n, C_n, D_n$ is given. In [N], Nakashima has used these descriptions coupled with the properties of crystal bases to obtain Littlewood-Richardson type rules for decomposing the tensor products. The descriptions of crystal bases for irreducible highest weight $U_q(G_2)$-modules and the tensor product decomposition rules are given in [KM1]. Another description of crystal bases for $U_q(g)$-modules, $g = A_n, B_n, C_n, D_n, E_6, G_2$, using Lakshmibai-Seshadri monomial theory is given in [Li].

Using the Fock space representation, the crystal base for any integrable

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highest weight $U_q(\hat{sl}(n))$-module has been given in [MM] and [JMMO]. It has been observed that the crystal base for these $U_q(\hat{sl}(n))$-modules can be parametrized by certain paths which arise naturally in the context of lattice models. In order to generalize this result to other quantum affine Lie algebras the theory of affine crystals has been introduced in [KMN1] and [KMN2]. Crucial to this theory is the existence of certain crystals known as perfect crystals. The existence of at least one such perfect crystal for each integrable highest weight $U_q(\hat{g})$-module for $g = A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n},$ or $D^{(2)}_{n+1}$ has been given in [KMN3]. Each such perfect crystal gives a path realization of the corresponding crystal base (see [KMN2], [KMN3]).

In this paper, we develop the theory of crystals further and give a path realization for the crystal base of $U_q^-(\hat{g}), g = A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n},$ or $D^{(2)}_{n+1}$. This, in particular, gives a path realization for the crystal bases of Verma modules for the corresponding quantum affine Lie algebras. Also note that the crystal base of any integrable highest weight $U_q(\hat{g})$-module is a surjective image of that of $U_q^-(\hat{g})$. One of the key ingredients in this realization is the energy function which is necessary to determine the weight of a path. One important result in this paper is the explicit description of the energy function as the maximum of certain linear functions in each case. This result was only known for $g = A^{(1)}_n$ (e.g., see [JMMO]) from the corresponding result in the lattice models. It is hoped that these explicit descriptions of the energy functions may provide further information about the corresponding lattice models.

The paper is arranged as follows. In Section 2, we recall basic definitions and develop the theory of crystals further for any symmetrizable Kac-Moody Lie algebra $g$. It is worth pointing out that in this paper we have axiomatized the crystals as purely combinatorial objects. This might be of independent interest to some researchers. In Section 3, we recall certain results from [KMN2] and [KMN3] which are used in the sequel. The core of the paper is Section 4 where we prove the main results like Theorems 4.7, 4.8, and 4.9. These theorems give the desired path realizations provided we have a suitable classical crystal $B_\infty$ (satisfying (4.1)–(4.5)) and the energy function $H : B_\infty \otimes B_\infty \to \mathbb{Z}$ (see definitions in Section 4). In Section 5, we give the descriptions of these crystals $B_\infty$ which satisfy the required conditions. We also determine the corresponding $H$-functions for each quantum affine Lie algebra $U_q(\hat{g}), g = A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_{2n-1}, A^{(2)}_{2n},$ or $D^{(2)}_{n+1}$. Thus we have an explicit path realization of the crystal base for $U_q^-(\hat{g})$ for these quantum affine Lie algebras.

2. The algebra $U_q(\hat{g})$ and crystals

Let $g$ be any symmetrizable Kac-Moody Lie algebra generated by $e_i, f_i$ ($i \in I = \{0, 1, \ldots, n\}$) and the Cartan subalgebra $\mathfrak{h}$ over $\mathbb{Q}$. Let $\{x_i | i \in I\} \subset \mathfrak{h}^*$
and \{h_i \mid i \in I\} \subset \mathfrak{h}\) denote the simple roots and simple coroots, respectively. We normalize the nondegenerate symmetric invariant bilinear form \(\langle , \rangle\) on \(\mathfrak{h}^*\) so that \(\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}_{\geq 0}\). Let \(P\) denote the weight lattice and \(P^*\) denote the dual lattice. Then \(\alpha_i \in P\) and \(h_i \in P^*\) for all \(i \in I\).

The quantized universal enveloping algebra \(U_q(\mathfrak{g})\) is then the \(\mathbb{Q}(q)\)-algebra generated by the symbols \(e_i, f_i (i \in I)\) and \(q^h (h \in P^*)\) with the following defining relations:

\[
q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for all } h, h' \in P^*, \quad (2.1)
\]

\[
q_i^h e_i q_i^{-h} = q_i^{\langle h, \alpha_i \rangle} e_i, \quad q_i^h f_i q_i^{-h} = q_i^{-\langle h, \alpha_i \rangle} f_i \quad \text{for all } i \in I, h \in P^*, \quad (2.2)
\]

\[
[e_i, f_j] = \delta_{ij} t_i - t_i^{-1}, \quad \text{where } q_i = q^{(\alpha_i, \alpha_i)} \text{ and } t_i = q^{(\alpha_i, \alpha_i) h_i} = q_i^{h_i}, \quad (2.3)
\]

\[
\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0 \quad \text{for } i \neq j \text{ and } b = 1 - \langle h_i, \alpha_j \rangle. \quad (2.4)
\]

Here we use the following notations:

\[
[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, \quad [k]_i! = \prod_{m=1}^{k} [m]_i \quad \text{and } e_i^{(k)} = e_i^k/[k]_i!, \quad f_i^{(k)} = f_i^k/[k]_i!.
\]

We understand \(e_i^{(k)} = f_i^{(k)} = 0\) for \(k < 0\).

It is well known that \(U_q(\mathfrak{g})\) has a Hopf algebra structure with a comultiplication \(\Delta\) defined by

\[
\Delta(e_i) = e_i \otimes t_i - t_i^{-1} + 1 \otimes e_i, \\
\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \\
\Delta(q^h) = q^h \otimes q^h
\]

for all \(i \in I, h \in P^*\). The tensor product of two \(U_q(\mathfrak{g})\)-modules has a structure of \(U_q(\mathfrak{g})\)-module via this comultiplication. For \(i \in I\), let \(U_q(\mathfrak{g}_i)\) denote the subalgebra of \(U_q(\mathfrak{g})\) generated by \(e_i, f_i, t_i\) and \(t_i^{-1}\).

For a \(U_q(\mathfrak{g})\)-module \(M\) and \(\lambda \in P\), the \(\lambda\)-weight space of \(M\) is defined by \(M_\lambda = \{ u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \text{ for all } h \in P^* \}\). We say that \(M\) is integrable if \(M = \bigoplus_{\lambda \in \mathfrak{p}} M_\lambda\) and \(M\) is a union of finite-dimensional \(U_q(\mathfrak{g}_i)\)-modules for any \(i \in I\). By the representation theory of \(U_q(\mathfrak{sl}(2))\), any element \(u \in M_\lambda\) can be uniquely written as

\[
u = \sum_{k \geq 0} f_i^{(k)} u_k, \quad (2.5)
\]
where \( u_k \in \text{Ker} \ e_i \cap M_{\lambda + k\alpha_i} \). We define the endomorphisms \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( M \) by
\[
\tilde{e}_i u = \sum f_i^{k-1} u_k, \quad (2.6)
\]
\[
\tilde{f}_i u = \sum f_i^{k+1} u_k. \quad (2.7)
\]

Let \( A \) be the subring of \( \mathbb{Q}(q) \) consisting of the rational functions regular at \( q = 0 \). A crystal lattice of an integrable \( U_q(g) \)-module \( M \) is a free \( A \)-submodule of \( M \) such that \( M \cong \mathbb{Q}(q) \otimes_A L \), \( L = \bigoplus_{\lambda \in P} L_{\lambda} \) where \( L_{\lambda} = L \cap M_{\lambda} \), and \( \tilde{e}_i L \subseteq L, \ \tilde{f}_i L \subseteq L \). A crystal base of the integrable \( U_q(g) \)-module \( M \) is a pair \((L, B)\) such that (i) \( L \) is a crystal lattice of \( M \), (ii) \( B \) is a \( \mathbb{Q} \)-basis of \( L/qL \), (iii) \( B = \bigcup_{\lambda \in P} B_{\lambda} \) where \( B_{\lambda} = B \cap (L_{\lambda}/qL_{\lambda}) \), (iv) \( \tilde{e}_i B \subseteq B \cup \{0\}, \ \tilde{f}_i B \subseteq \{0\} \), and (v) for \( b, b' \in B, \ b' = \tilde{f}_i b \) if and only if \( b = \tilde{e}_i b' \) for \( i \in I \). We sometimes replace (ii) by: \( B_{ps} = B \cup (-B') \) where \( B' \) is a \( \mathbb{Q} \)-basis of \( L/qL \). We call \((L, B_{ps})\) a crystal pseudo-base and \( B_{ps}/\{ \pm 1 \} \) the associated crystal of \((L, B_{ps})\).

For \( \lambda \in P^+ = \{ \lambda \in P | \langle h_i, \lambda \rangle \geq 0 \) for all \( i \} \), let \( V(\lambda) \) denote the irreducible integrable \( U_q(g) \)-module with highest weight \( \lambda \). Let \( u_\lambda \) be the highest weight vector of \( V(\lambda) \), and let \( L(\lambda) \) be the smallest \( A \)-submodule of \( V(\lambda) \) containing \( u_\lambda \) stable under \( \tilde{f}_i \)'s. Set \( B(\lambda) = \{ b \in L(\lambda)/qL(\lambda) \ | \ b = \tilde{f}_i \cdots \tilde{f}_{i_1} u_\lambda \mod qL(\lambda) \} \setminus \{0\} \). Then \((L(\lambda), B(\lambda))\) is the crystal base of \( V(\lambda) \) (see [K1]). The associated crystal graph is the oriented colored (by \( I \)) graph with \( B(\lambda) \) as the set of vertices and \( b \rightarrow b' \) if and only if \( b' = \tilde{f}_i b \) (hence \( \tilde{e}_i b' = b \)). This graph completely describes the actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( B(\lambda) \). For \( b \in B(\lambda) \), we set
\[
\varepsilon_i (b) = \max\{ k \geq 0 \mid \tilde{e}_i b \neq 0 \}, \quad (2.8)
\]
\[
\varphi_i (b) = \max\{ k \geq 0 \mid \tilde{f}_i b \neq 0 \}. \quad (2.9)
\]

Note that \( B(\lambda) = \bigcup_{\mu \in P} B(\lambda)_{\mu} \) and for \( b \in B(\lambda)_{\mu} \), we have \( \langle h_i, \text{wt}(b) \rangle = \varphi_i (b) - \varepsilon_i (b) \), where \( \text{wt}(b) = \mu \) denotes the weight of \( b \).

Motivated by the nice properties of these crystal graphs, we define the notion of a crystal as follows.

DEFINITION 2.1. A crystal \( B \) is a set \( B = \bigcup_{\lambda \in P} B_{\lambda} \) (wt \( (b) = \lambda \) if \( b \in B_{\lambda} \)) equipped with maps \( \tilde{e}_i : B_{\lambda} \rightarrow B_{\lambda + \alpha_i} \cup \{0\}, \tilde{f}_i : B_{\lambda} \rightarrow B_{\lambda - \alpha_i} \cup \{0\}, \varepsilon_i : B \rightarrow \mathbb{Z} \cup \{ -\infty \}, \varphi_i : B \rightarrow \mathbb{Z} \cup \{ -\infty \} \) for all \( i \in I \) such that
\[
\text{for } b \in B_{\lambda}, \varphi_i (b) = \langle h_i, \lambda \rangle + \varepsilon_i (b), \quad (2.10)
\]
\[
\text{for } b \in B, \text{we have} \quad \varepsilon_i (b) = \varepsilon_i (\tilde{e}_i b) + 1 \quad \text{if } \tilde{e}_i b \neq 0, \quad (2.11)
\]
\[
= \varepsilon_i (\tilde{f}_i b) - 1 \quad \text{if } \tilde{f}_i b \neq 0,
\]
and

\[ \varphi_i(b) = \varphi_i(\tilde{\varepsilon}_i b) - 1 \quad \text{if} \quad \tilde{\varepsilon}_i b \neq 0, \]
\[ = \varphi_i(\tilde{f}_i b) + 1 \quad \text{if} \quad \tilde{f}_i b \neq 0, \]

for \( b, b' \in \mathcal{B}, \tilde{\varepsilon}_i b = b \) if and only if \( b' = \tilde{f}_i b, \)

(2.12)

for \( b \in \mathcal{B}, \varepsilon_i(b) = \varphi_i(b) = -\infty \) implies \( \tilde{\varepsilon}_i b = \tilde{f}_i b = 0. \)

(2.13)

Sometimes, for emphasis we will say that \( \mathcal{B} \) is a \( P \)-weighted crystal.

**DEFINITION 2.2.** For two crystals \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), a morphism of crystals from \( \mathcal{B}_1 \) to \( \mathcal{B}_2 \) is a map \( \psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\} \) such that

\[ \psi(0) = 0, \]
\[ \psi(\tilde{\varepsilon}_i b) = \tilde{\varepsilon}_i \psi(b) \quad \text{for} \quad b, \tilde{\varepsilon}_i b \in \mathcal{B}_1, \quad \text{and} \quad \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for} \quad b, \tilde{f}_i b \in \mathcal{B}_1, \]
\[ \text{for} \quad b \in \mathcal{B}_1, \varepsilon_i(b) = \varepsilon_i(\psi(b)), \quad \varphi_i(b) = \varphi_i(\psi(b)) \quad \text{if} \quad \psi(b) \in \mathcal{B}_2, \]
\[ \text{for} \quad b \in \mathcal{B}_1, \quad \text{wt}(b) = \text{wt}(\psi(b)) \quad \text{if} \quad \psi(b) \in \mathcal{B}_2. \]

(2.14)

(2.15)

(2.16)

(2.17)

A morphism of crystals \( \psi : \mathcal{B}_1 \to \mathcal{B}_2 \) is called an embedding if \( \psi \) is injective.

The crystals and their morphisms form a category. For two crystals \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) we define their tensor product as follows. The underlying set is \( \mathcal{B}_1 \times \mathcal{B}_2 \).

For \( b_1 \in \mathcal{B}_1, b_2 \in \mathcal{B}_2 \), we write \( b_1 \otimes b_2 \) for \( (b_1, b_2) \). We understand \( b_1 \otimes 0 = 0 \otimes b_2 = 0 \). We define the maps \( \varepsilon_i, \varphi_i, \tilde{\varepsilon}_i, \tilde{f}_i \) for \( i \in I \) as follows:

\[ \varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2)) - \langle h_i, \text{wt}(b_1) \rangle, \]
\[ \varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_1), \varphi_i(b_2)) + \langle h_i, \text{wt}(b_2) \rangle, \]
\[ \tilde{\varepsilon}_i(b_1 \otimes b_2) = \tilde{\varepsilon}_i b_1 \otimes b_2 \quad \text{if} \quad \varphi_i(b_1) \geq \varepsilon_i(b_2), \]
\[ = b_1 \otimes \tilde{\varepsilon}_i b_2 \quad \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2), \]
\[ \tilde{f}_i(b_1 \otimes b_2) = \tilde{f}_i b_1 \otimes b_2 \quad \text{if} \quad \varphi_i(b_1) \geq \varepsilon_i(b_2), \]
\[ = b_1 \otimes \tilde{f}_i b_2 \quad \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2), \]
\[ \text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2). \]

(2.18)

(2.19)

(2.20)

(2.21)

(2.22)

Then the following proposition is immediate.

**PROPOSITION 2.3.** If \( \mathcal{B}_1, \mathcal{B}_2, \) and \( \mathcal{B}_3 \) are crystals, then

(a) \( \mathcal{B}_1 \otimes \mathcal{B}_2 \) as defined above is a crystal.

(b) The map \( (\mathcal{B}_1 \otimes \mathcal{B}_2) \otimes \mathcal{B}_3 \to \mathcal{B}_1 \otimes (\mathcal{B}_2 \otimes \mathcal{B}_3) \) given by \( (b_1 \otimes b_2) \otimes b_3 \mapsto \)
$b_1 \otimes (b_2 \otimes b_3)$ is an isomorphism in the category of crystals.

Now we give several examples of crystals which will be of interest to us in this paper.

**EXAMPLE 2.4.** For $\lambda \in P_+$, let $V(\lambda)$ denote the irreducible integrable $U_q(g)$-module with highest weight $\lambda$. Let $(L(\lambda), B(\lambda))$ be the crystal base for $V(\lambda)$. Then the set $B(\lambda)$ is a crystal with the maps $\tilde{e}_i, \tilde{f}_i, e_i, f_i$ defined by (2.6)–(2.9).

**EXAMPLE 2.5.** Let $U_q^-(g)$ be the subalgebra of $U_q(g)$ generated by the $f_i$'s. Then $U_q^-(g)$ has the unique endomorphisms $e_i'$ and $f_i'$ such that

$$[e_i, u] = \frac{t_i e_i'(u) - t_i^{-1} e_i'(u)}{q_i - q_i^{-1}}$$

(2.23)

for any $u \in U_q^-(g)$. Then $e_i'$ and $f_i'$ satisfy the following commutation relations:

$$e_i' f_j = q_i^{-\langle h_i, \omega_j \rangle} f_j e_i' + \delta_{ij}.$$  

(2.24)

Here $f_j$ is considered as the left multiplication operator. Then any element in $u \in U_q^-(g)$ can be uniquely written as

$$u = \sum_{k \geq 0} f_i'^{(k)} u_k,$$

where $e_i'u_k = 0$. Now we define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^-(g)$ by

$$\tilde{e}_i u = \sum f_i'^{(k-1)} u_k,$$

$$\tilde{f}_i u = \sum f_i'^{(k+1)} u_k.$$  

(2.25)

(2.26)

Then we have $\tilde{e}_i \tilde{f}_i = 1$. Let $L(\infty)$ be the smallest $A$-submodule of $U_q^-(g)$ containing 1 that is stable by $f_i$'s. Let $B(\infty)$ be the subset of $L(\infty)/qL(\infty)$ consisting of the nonzero vectors of the form $\tilde{f}_i \cdots \tilde{f}_k \cdot 1 \mod qL(\infty)$. Then $(L(\infty), B(\infty))$ is the crystal base of $U_q^-(g)$ (see [K2]). For $b = \tilde{f}_i \cdots \tilde{f}_k \cdot 1 \in B(\infty)$, define

$$e_i(b) = \max \{k \geq 0 \mid \tilde{e}_i^k b \neq 0\},$$

$$\varphi_i(b) = e_i(b) + \langle h_i, \text{wt}(b) \rangle,$$

(2.27)

(2.28)

where $\text{wt}(b) = -\alpha_i, \cdots -\alpha_i$. Then the set $B(\infty)$ with the maps $\tilde{e}_i, \tilde{f}_i, e_i, f_i$ defined in (2.25)–(2.28) is a crystal. We denote by $u_\infty$ the element in $B(\infty)$ that corresponds to $1 \in U_q^-(g)$. The goal of this paper is to give path realizations of the crystals $B(\infty)$ for the quantum affine Lie algebra of type $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)}, A_{2n-1}^{(2)},$ and $D_{n+1}^{(2)}$. 


EXAMPLE 2.6. For $\lambda \in P$, consider the set $T_\lambda = \{ t_\lambda \}$ with one element. Define $\varepsilon_i(t_\lambda) = \phi_i(t_\lambda) = -\infty$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$ for $i \in I$ and $\text{wt}(t_\lambda) = \lambda$. Then $T_\lambda$ is a crystal. For any crystals $B$, $B \otimes T_0 \cong T_0 \otimes B \cong B$.

EXAMPLE 2.7. Given any crystal $B$, we can form the following crystals by taking the tensor product with the crystal $T_\lambda$, $\lambda \in P$:

(a) $B \otimes T_\lambda = \{ b \otimes t_\lambda \mid b \in B \}$,

where

$$\text{wt}(b \otimes t_\lambda) = \lambda + \text{wt}(b),$$

$$\tilde{e}_i(b \otimes t_\lambda) = \tilde{e}_i b \otimes t_\lambda, \quad \tilde{f}_i(b \otimes t_\lambda) = \tilde{f}_i b \otimes t_\lambda,$$

$$\varepsilon_i(b \otimes t_\lambda) = \varepsilon_i(b), \quad \phi_i(b \otimes t_\lambda) = \phi_i(b) + \langle h_i, \lambda \rangle.$$

(b) $T_\lambda \otimes B = \{ t_\lambda \otimes b \mid b \in B \}$,

where

$$\text{wt}(t_\lambda \otimes b) = \lambda + \text{wt}(b),$$

$$\tilde{e}_i(t_\lambda \otimes b) = t_\lambda \otimes \tilde{e}_i b, \quad \tilde{f}_i(t_\lambda \otimes b) = t_\lambda \otimes \tilde{f}_i b,$$

$$\varepsilon_i(t_\lambda \otimes b) = \varepsilon_i(b) - \langle h_i, \lambda \rangle, \quad \phi_i(t_\lambda \otimes b) = \phi_i(b).$$

(c) For $\lambda, \mu \in P$,

$$T_\lambda \otimes B \otimes T_\mu = \{ t_\lambda \otimes b \otimes t_\mu \mid b \in B \},$$

where

$$\text{wt}(t_\lambda \otimes b \otimes t_\mu) = \lambda + \mu + \text{wt}(b),$$

$$\tilde{e}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{e}_i b \otimes t_\mu, \quad \tilde{f}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{f}_i b \otimes t_\mu,$$

$$\varepsilon_i(t_\lambda \otimes b \otimes t_\mu) = \varepsilon_i(b) - \langle h_i, \lambda \rangle, \quad \phi_i(t_\lambda \otimes b \otimes t_\mu) = \phi_i(b) + \langle h_i, \mu \rangle.$$

3. Perfect crystals and paths

Let $\mathfrak{g}$ be an indecomposable affine Kac-Moody Lie algebra defined over $\mathbb{Q}$. Let $\{ \alpha_i \mid i \in I \} \subset \mathfrak{h}^*$ and $\{ h_i \mid i \in I \} \subset \mathfrak{h}$ denote the simple roots and simple coroots, respectively. Thus $\{ \alpha_i \mid i \in I \}$ and $\{ h_i \mid i \in I \}$ are linearly independent and $\dim \mathfrak{h} = \# I + 1$. Let $Q = \sum_i \mathbb{Z} x_i$, $Q_+ = \sum_{i \geq 0} \mathbb{Z} x_i$, and $Q_- = -Q_+$. Let $\delta \in Q_+$ be the generator of null roots and let $c \in \sum_i \mathbb{Z}_{\geq 0} h_i$ be the generator of the center. Set
\( h_{cl} = \bigoplus_i Qh_i \subset h \) and \( h_{cl}^* = (\bigoplus_i Qh_i)^* \). and let \( \text{cl}: h^* \to h_{cl}^* \) be the canonical morphism. Then we have an exact sequence

\[ 0 \to Q\delta \to h^* \xrightarrow{\text{cl}} h_{cl}^* \to 0. \]

Fix \( i_0 \in I \) and take an integer \( d \) such that \( \delta - d\sigma_{i_0} \in \Sigma_{i \neq i_0} Z\alpha_i \). For simplicity, we write 0 for \( i_0 \). Let \( \alpha : h_{cl}^* \to h^* \) be a map satisfying \( \text{cl} \circ \alpha = \text{id} \) and \( \alpha \circ \text{cl}(\alpha_i) = \alpha_i \) for \( i \neq 0 \). Let \( \Lambda_i \) be the element of \( h_{cl}^* \subset h^* \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij} \). Hence we have \( \alpha_i = \Sigma_j \langle h_j, \alpha_i \rangle \alpha_j(\Lambda_j) + \delta_{i,0}d^{-1}\delta \). We take \( P = \Sigma_i Z\alpha(\Lambda_i) + Zd^{-1}\delta \subset h^* \) and \( P_{cl} = \text{cl}(P) \subset h_{cl}^* \). An element of \( P \) is called an affine weight and an element of \( P_{cl} \) is called a classical weight. Let \( U_q(g) \) be the quantized universal enveloping algebra associated with \( P \), and let \( U_q'(g) \) be the quantized universal enveloping algebra associated with \( P_{cl} \). A \( P \)-weighted crystal is called an affine crystal and a \( P_{cl} \)-weighted crystal is called a classical crystal.

Let \( B \) be a classical crystal. For \( b \in B \), we set \( \varepsilon(b) = \Sigma_i \varepsilon_i(b)\Lambda_i \) and \( \varphi(b) = \Sigma_i \varphi_i(b)\Lambda_i \). Note that \( \text{wt}(b) = \varphi(b) - \varepsilon(b) \). Set \( P_{cl}^+ = \{ \lambda \in P_{cl} | \langle h_i, \lambda \rangle > 0 \text{ for all } i \in I \} \) and for \( l \in Z_{\geq 0} \), let \( (P_{cl}^+)_l = \{ \lambda \in P_{cl}^+ | \langle c, \lambda \rangle = l \} \).

**DEFINITION 3.1.** For \( l \in Z_{\geq 0} \), we say that \( B \) is a perfect crystal of level \( l \) if

1. \( B \otimes B \) is connected.
2. There exists \( \lambda_0 \in P_{cl} \) such that \( \text{wt}(B) \subset \lambda_0 + \Sigma_{i \neq 0} Z_{\geq 0} \text{cl}(\alpha_i) \) and that \( \#(B_{\lambda_0}) = 1 \).
3. There is a finite dimensional \( U_q'(g) \)-module with a crystal pseudo-base \( (L, B_{ps}) \) such that \( B \cong B_{ps} / \pm 1 \).
4. For any \( b \in B \), we have \( \langle c, \varepsilon(b) \rangle \geq l \).
5. The maps \( c, \varphi : B^{\min} = \{ b \in B | \langle c, \varepsilon(b) \rangle = l \} \to (P_{cl}^+)_l \) are bijective.

The elements in \( B^{\min} \) are called minimal elements.

A \( Z \)-valued function \( H \) on \( B \otimes B \) is called an energy function on \( B \) if for any \( i \in I \) and \( b \otimes b' \in B \otimes B \) such that \( \varepsilon_i(b \otimes b') \neq 0 \), we have

\[
H(\varepsilon_i(b \otimes b')) = H(b \otimes b') \quad \text{if } i \neq 0,
\]

\[
= H(b \otimes b') + 1 \quad \text{if } i = 0 \text{ and } \varphi_0(b) \geq \varepsilon_0(b'),
\]

\[
= H(b \otimes b') - 1 \quad \text{if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b').
\]

(3.6)

The existence of energy functions is proved for the perfect crystals in [KMN2]. From now on, we assume that \( g \) is of rank \( \geq 3 \). For \( \lambda \in P^+ \), let \( B(\lambda) \)
be the affine crystal with highest weight $\lambda$, and denote by $u_\lambda$ the highest weight element of $B(\lambda)$.

**THEOREM 3.2 ([KMN2]).** Let $B$ be a perfect crystal of level 1, and let $b$ be an element of $B$. Then we have an isomorphism of classical crystals

$$B(af(\varepsilon(b))) \otimes B \cong B(af(b))$$

(3.7)

sending $u_{af(\varepsilon(b))} \otimes b$ to $u_{af(\varphi(b))}$. □

For $\mu \in (P_+)_1$, let $b_\mu$ be the unique element of $B$ such that $\varphi(b_\mu) = \mu$. We define the isomorphism $\sigma$ of $(P_+)_1$ by $\varepsilon(b_\mu) = \sigma \mu$. We lift $\sigma$ to an isomorphism of $af(P_+)_1$. Then for $\lambda \in af(P_+)_1$, by Theorem 3.2, we have

$$B(\lambda) \cong B(\sigma \lambda) \otimes B$$

given by $u_\lambda \mapsto u_{\sigma \lambda} \otimes b_{\varepsilon(\lambda)}$. We set $\lambda_k = \sigma^k \lambda$ and $b_k = b_{\varepsilon(\lambda_{k-1})}$ for $k \geq 1$. Thus by applying Theorem 3.2 repeatedly, we obtain an isomorphism of classical crystals

$$\psi_k : B(\lambda) \cong B(\lambda_k) \otimes B^\otimes k$$

(3.8)

given by

$$u_\lambda \mapsto u_{\lambda_k} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1.$$

The sequence $(b_1, b_2, b_3, \ldots)$ is called the ground-state path of weight $\lambda$. A $\lambda$-path in $B$ is a sequence $p = (p(k))_{k \geq 1}$ in $B$ such that $p(k) = b_k$ for $k \gg 0$. We denote by $\Psi(\lambda, B)$ the set of all $\lambda$-paths. The crystal graph structure on $\Psi(\lambda, B)$ is given by (2.20) and (2.21).

**THEOREM 3.3 ([KMN2]).** The crystal $B(\lambda)$ is isomorphic to $\Psi(\lambda, B)$ given by $B(\lambda) \ni b \mapsto p \in \Psi(\lambda, B)$ where $\psi_k(b) = u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$ for $k \gg 0$. □

The affine weight of elements in $\Psi(\lambda, B)$ can be computed by the energy function $H$ as in the following theorem.

**THEOREM 3.4 ([KMN2]).** If $b \in B(\lambda)$ corresponds to the $\lambda$-path $p = (p(k))_{k \geq 1} \in \Psi(\lambda, B)$, then we have

$$w_{tb} = \lambda + \sum_{k=1}^{\infty} (af(w_tp(k)) - af(w_{tb}))$$

$$- \left( \sum_{k=1}^{\infty} k(H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k)) \right) d^{-1} \delta.$$

□
4. Crystal base of $U_q(g)$ and paths

Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals $B_l$ of level $l$ and $B_l^{\text{min}} = \{b \in B_l | \langle c, \varepsilon(b) \rangle = l \}$. We take the index set $J = \{(l, b) | l \in \mathbb{Z}_{>0}, b \in B_l^{\text{min}} \}$.

**DEFINITION 4.1.** A classical crystal $B_x$ with an element $b_x$ is called a limit of $\{B_l\}_{l \geq 1}$ if it satisfies the following conditions:

\[
\begin{align*}
\text{wt}(b_x) &= 0, \varepsilon(b_x) = \varphi(b_x) = 0, \\
\text{for any } (l, b) \in J, \text{ there exists an embedding of crystals} \\
f_{(l, b)}: T_{\varepsilon(b)} \otimes B_l \otimes T_{- \varphi(b)} &\to B_x
\end{align*}
\]

sending $t_{\varepsilon(b)} \otimes b \otimes t_{- \varphi(b)}$ to $b_x$,

\[
B_x = \bigcup_{(l, b) \in J} \text{Im} f_{(l, b)}. 
\]

If a limit exists for the family $\{B_l\}$, we say that $\{B_l\}$ is a coherent family of perfect crystals. Note that for any $N > 0$, there exists $(l, b) \in J$ such that $\varepsilon_i(b) > N$ and $\varphi_i(b) > N$ for all $i \in I$.

Let $B$ be a classical crystal with an element $b_0 \in B$ satisfying the following properties:

\[
\begin{align*}
\text{for any } N > 0, \text{ there exists } (l_N, b_N) \in J \text{ such that } \\
\varepsilon_i(b_N) > N, \varphi_i(b_N) > N \\
\text{for all } i \in I \text{ and that there exists an embedding} \\
f_{N}: T_{\varepsilon(b_N)} \otimes B_{l_N} \otimes T_{- \varphi(b_N)} &\to B
\end{align*}
\]

sending $t_{\varepsilon(b_N)} \otimes b_N \otimes t_{- \varphi(b_N)}$ to $b_0$,

\[
B = \bigcup_{N \geq 1} \text{Im} f_{N}. 
\]

Since each $B_{l_N}$ is connected, $B$ is also connected.

**LEMMA 4.2.** If $(B, b_0)$ satisfies (4.4) and (4.5), then

(a) for any finite subset $F$ of $B$, there exists $N > 0$ such that $F \subseteq \text{Im} f_{N}$,
(b) $\bar{\varepsilon}_i B \subseteq B, \bar{\varphi}_i B \subseteq B$ for all $i \in I$.

**Proof.** We may assume that $F$ is a connected subcrystal of $B$ containing $b_0$.

Take $N$ such that $\varepsilon_i(b) \geq -N$ and $\varphi_i(b) \geq -N$ for all $i \in I$ and $b \in F$. In this case, we will show that $F \subseteq \text{Im} f_{N}, \bar{\varepsilon}_i F \subseteq B, \text{ and } \bar{\varphi}_i F \subseteq B$. For this, it suffices to show that
(i) for \( b \in F \cap \text{Im } f_N \), we have \( \tilde{e}_b \in \text{Im } f_N \).

(ii) for \( b \in F \cap \text{Im } f_N \), we have \( \tilde{f}_b \in \text{Im } f_N \).

Let \( b = f_N(t_{\epsilon(b)} \otimes b' \otimes t_{-\varphi(b)}) \) for some \( b' \in B_{1n} \). Then \( \epsilon_i(b') = \epsilon_i(b) + \epsilon_i(b_N) > 0 \), which implies \( \tilde{e}_b \in B_{1n} \). Hence \( \tilde{e}_b \in \text{Im } f_N \). The proof for (ii) is similar.

**Lemma 4.3.** Let \((B_x, b_x)\) be a limit of a coherent family \(\{B_i\}_{i \geq 1}\) and suppose that \((B, b_0)\) satisfies the conditions (4.4) and (4.5). Then there exists a unique isomorphism \( B \to B_x \) of crystals sending \( b_0 \) to \( b_x \).

**Proof.** By Lemma 4.2, any finite connected subcrystal \( F \) of \( B \) containing \( b_0 \) is contained in \( \text{Im } f_N \) for some \( N \gg 0 \). Hence there exists an embedding of crystals

\[
F \xrightarrow{f_N^{-1}} T_{\epsilon(b_0)} \otimes B_{1n} \otimes T_{-\varphi(b_0)} \xrightarrow{f_0, b_0} B_x,
\]

which yields an embedding \( B \to B_x \). This is evidently an isomorphism.

**Corollary 4.4.** A limit \((B_x, b_x)\) of a coherent family \(\{B_i\}_{i \geq 1}\) is unique up to isomorphism.

The following proposition is an immediate consequence of the properties of the perfect crystal \( B_l \) (see Definition 3.1).

**Proposition 4.5.**

(a) \( B_x \otimes B_x \) is connected.

(b) For any \( b \in B_x \), \( \langle c, \epsilon(b) \rangle \geq 0 \).

(c) The maps \( \epsilon \) and \( \varphi \) give bijections from \( B_x^{\min} = \{ b \in B_x | \langle c, \epsilon(b) \rangle = 0 \} \) to \( (P_c)_0 \).

Let \( b_0 \in B_x^{\min} \). Then \( b_0 \in \text{Im } f_{(l,b)} \) for some \( (l,b) \in J \) with \( \epsilon_i(b) \gg 0, \varphi_i(b) \gg 0 \). Define \( b' \in B_l \) by

\[
f_{(l,b)}(t_{\epsilon(b)} \otimes b' \otimes t_{-\varphi(b)}) = b_0.
\]

Then \( \epsilon(b') = \epsilon(b) + \epsilon(b_0) \) and \( \varphi(b') = \varphi(b) + \varphi(b_0) \). Hence \( b' \in B_l^{\min} \) and \( \epsilon(b'), \varphi(b') \gg 0 \). Moreover, the map \( T_{-\epsilon(b_0)} \otimes f_{(l,b')} \otimes T_{\varphi(b_0)} \) gives an embedding of crystals

\[
T_{\epsilon(b')} \otimes B_l \otimes T_{-\varphi(b')} \to T_{\epsilon(b_0)} \otimes B_x \otimes T_{-\varphi(b_0)}
\]

such that

\[
t_{\epsilon(b')} \otimes b' \otimes t_{-\varphi(b')} \mapsto t_{\epsilon(b_0)} \otimes b_0 \otimes t_{-\varphi(b_0)}.
\]

Therefore,
satisfies the conditions \((4.4)\) and \((4.5)\). Hence we obtain:

**Lemma 4.6.** For any \(b \in B_{\infty}^{\min}\), there exists a unique isomorphism

\[
\theta_b : B_\infty \rightarrow T_{\epsilon(b)} \otimes B_\infty \otimes T_{-\varphi(b)}
\]

such that

\[
\theta_b(b_\infty) = T_{\epsilon(b)} \otimes b \otimes t_{-\varphi(b)}.
\]  

For \(b', b'' \in B_{\infty}^{\min}\), let \(b\) be an element of \(B_{\infty}\) such that

\[
\theta_{b'}(b') = T_{\epsilon(b')} \otimes b' \otimes t_{-\varphi(b')},
\]

Then \(\epsilon(b) = \epsilon(b') + \epsilon(b'')\) and \(\varphi(b) = \varphi(b') + \varphi(b'')\). Hence we have a well-defined linear automorphism \(\sigma\) on \((P'_{\infty})_0\) such that \(\sigma \varphi(b) = \epsilon(b)\) for \(b \in B_{\infty}^{\min}\).

As in Section 3, a \(\mathbb{Z}\)-valued function \(H\) on \(B_\infty \otimes B_{\infty}\) is called an energy function if \(H(b_\infty \otimes b_\infty) = 0\) and if for any \(i \in I\) and \(b \otimes b' \in B_\infty \otimes B_{\infty}\) such that \(\delta_i(b \otimes b') \neq 0\), we have

\[
H(\tilde{\delta}_i(b \otimes b')) = H(b \otimes b') - 1 \quad \text{if} \quad i = 0 \quad \text{and} \quad \varphi_0(b) < \epsilon_0(b'),
\]

\[
H(b_\infty \otimes b') = 1 \quad \text{if} \quad i = 0 \quad \text{and} \quad \varphi_0(b) \geq \epsilon_0(b').
\]  

Then the existence and the uniqueness of the energy function on \(B_\infty\) follows immediately from the corresponding result on \(B_I\).

Take \(\lambda \in af(P'_{\infty})\) such that \(\lambda(h_i) \gg 0\) for all \(i\), and let \(B(\lambda)\) be the affine crystal with highest weight \(\lambda\). By Theorem 3.2, we have

\[
B(\lambda) \cong B(\sigma \lambda) \otimes B_I
\]

given by \(u_{\lambda} \mapsto u_{\sigma \lambda} \otimes b_0\), where \(b_0\) is the unique element in \(B_I^{\min}\) such that \(\varphi(b_0) = \lambda\), \(\epsilon(b_0) = \sigma \lambda\). Thus we have an isomorphism of crystals

\[
B(\lambda) \otimes T_{-\lambda} \cong B(\sigma \lambda) \otimes T_{-\sigma \lambda} \otimes T_{\sigma \lambda} \otimes B_I \otimes T_{-\lambda}
\]  

(4.10)
given by
\[ u_\lambda \otimes t_{-\lambda} \mapsto u_{e_\lambda} \otimes t_{-\sigma_\lambda} \otimes b_0 \otimes t_{-\lambda}. \]

Note that \( B(\infty) = \lim_{\lambda} B(\lambda) \otimes T_{-\lambda} \) and \( B_\infty = \lim_{\lambda} T_{\sigma_\lambda} \otimes B_1 \otimes T_{-\lambda} \). Hence by taking the limit of (4.10) we obtain:

**THEOREM 4.7.** There is an isomorphism of crystals

\[ B(\infty) \cong B(\infty) \otimes B_\infty \]
sending \( u_\infty \) to \( u_\infty \otimes b_\infty \).

By applying Theorem 4.7 repeatedly, we obtain an isomorphism of crystals

\[ \psi_k : B(\infty) \cong B(\infty) \otimes B_\infty^\otimes k \]
given by \( u_\infty \mapsto u_\infty \otimes b_\infty \otimes \cdots \otimes b_\infty \otimes b_\infty \).

The sequence \((b_\infty, b_\infty, b_\infty, \ldots)\) is called the ground-state path. A path in \( B_\infty \) is a sequence \( p = (p(k))_{k \geq 1} \) in \( B_\infty \) such that \( p(k) = b_\infty \) for \( k \gg 0 \). We denote by \( \Psi(B_\infty) \) the set of all paths in \( B_\infty \). The actions of \( \tilde{e}_i \) and \( \tilde{f}_i \) \((i \in I)\) are given by (2.20) and (2.21). Then we have

**THEOREM 4.8.** The crystal \( B(\infty) \) is isomorphic to \( \Psi(B_\infty) \) given by \( B(\infty) \ni b \mapsto p \in \Psi(B_\infty) \) where \( \psi_k(b) = u_\infty \otimes p(k) \otimes \cdots \otimes p(1) \) for \( k \gg 0 \).

Now we will compute the weight of a path in \( \Psi(B_\infty) \). Let \( p = (p(k))_{k \geq 1} \in \Psi(B_\infty) \) be a path. By Theorem 4.8, we can find an \( l \gg 0 \) and \( b_0 \in B_1^{\text{min}} \) such that for all \( k \geq 1 \), \( p(k) = f_{l,b}(p_k) \) for \( p_k \in T_{t(b_0)} \otimes T_{-\varphi(b_0)} \). Since \( p(k) = b_\infty \) for \( k \gg 0 \), we also have \( p_k = t_{e(b_0)} \otimes b_0 \otimes t_{-\varphi(b_0)} \) for \( k \gg 0 \). Note that if \( f : B \to B' \) is an embedding of crystals with energy functions \( H \) and \( H' \), respectively, then \( f \otimes f : B \otimes B \to B' \otimes B' \) is also an embedding of crystals and we have \( wtf(b) = wtb \) for \( b \in B \) and \( H'(f(b_1) \otimes f(b_2)) = H(b_1 \otimes b_2) \) for \( b_1, b_2 \in B \). Therefore the following theorem is an immediate consequence of Theorem 3.4.

**THEOREM 4.9.** If \( b \in B(\infty) \) corresponds to the path \( p = (p(k))_{k \geq 1} \in \Psi(B_\infty) \), then we have

\[ wtb = \sum_{k=1}^{\infty} a_k(wtp(k)) - \sum_{k=1}^{\infty} kH(p(k + 1) \otimes p(k))d^{-1} \delta. \]
Here $d$ is an integer such that $\delta - d\alpha_0 \in \sum_{i \neq 0} \mathbb{Z}\alpha_i$.

5. The crystals $B_{\infty}$ and energy function

In this section we will give a description of $B_{\infty}$ and the energy functions by using a coherent family of perfect crystals given in [KMN3] for $g = A_n^{(1)}$, $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$, and $C_n^{(1)}$. We give the proof only in the $A_n^{(1)}$-case. The other cases can be proved similarly. The details of these proofs can be seen in our RIMS preprint [KKM]. In the table below we list the Dynkin diagrams and the corresponding coherent families of perfect crystals without the 0-arrow.

<table>
<thead>
<tr>
<th>Lie algebra</th>
<th>Dynkin diagram</th>
<th>Perfect crystals of level $l$</th>
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<tbody>
<tr>
<td>$A_n^{(1)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(l\Lambda_1)$</td>
</tr>
<tr>
<td>$A_{2n-1}^{(2)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(l\Lambda_1)$</td>
</tr>
<tr>
<td>$B_n^{(1)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(l\Lambda_1)$</td>
</tr>
<tr>
<td>$D_n^{(1)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(l\Lambda_1)$</td>
</tr>
<tr>
<td>$A_{2n}^{(2)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(l\Lambda_1) \oplus B((l-1)\Lambda_1) \oplus \cdots \oplus B(0)$</td>
</tr>
<tr>
<td>$D_{n+1}^{(2)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(l\Lambda_1) \oplus B((l-1)\Lambda_1) \oplus \cdots \oplus B(0)$</td>
</tr>
<tr>
<td>$C_n^{(1)}$</td>
<td>![Dynkin Diagram]</td>
<td>$B(2l\Lambda_1) \oplus B(2(l-1)\Lambda_1) \oplus \cdots \oplus B(0)$</td>
</tr>
</tbody>
</table>

5.1. $A_n^{(1)}$ ($n \geq 2$)

Let $\mathcal{J} = \mathbb{Z}/(n+1)\mathbb{Z}$ and

$$B_{\infty} = \left\{ (v_j)_{j \in \mathcal{J}} \mid \sum_i v_i = 0 \right\}, \quad b_{\infty} = (0, 0, \ldots, 0).$$
Define $\delta_j \in \mathbb{Z}^+$ by $(\delta_j)_i = \delta_{ji}$.

For $b = (v_i)_i$ and $b' = (v'_i)_i$ in $B_\infty$,

$$
\tilde{e}_i b = b + \delta_i - \delta_{i+1} \\
\tilde{f}_i b = b - \delta_i + \delta_{i+1}
$$

(5.1)

$$
wt(b) = \sum_i (v_i - v_{i+1})\Lambda_i, \\
\varphi_i(b) = v_i, \ \varepsilon_i(b) = v_{i+1}.
$$

(5.2)

$$
H(b \otimes b') = \max\{\theta_j(b \otimes b') | 0 \leq j \leq n\}, \text{ where}
$$

$$
\theta_j(b \otimes b') = \sum_{k=1}^j (v'_k - v_k) + v_{j+1}.
$$

5.2. $A_{2n-1}^{(2)}$ $(n \geq 3)$

$$
B_\infty = \left\{ (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1) \in \mathbb{Z}^{2n} \bigg| \sum_{i=1}^n v_i + \sum_{i=1}^n \bar{v}_i = 0 \right\}, \quad b_\infty = (0, 0, \ldots, 0)
$$

For $b = (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1)$ and $b' = (v'_1, \ldots, v'_n, \bar{v}'_n, \ldots, \bar{v}'_1)$ in $B_\infty$,

$$
\tilde{e}_0 b = (v_1, v_2 - 1, \ldots, \bar{v}_2, \bar{v}_1 + 1) \text{ if } v_2 > \bar{v}_2,
$$

$$
= (v_1 - 1, v_2, \ldots, \bar{v}_2 + 1, \bar{v}_1) \text{ if } v_2 \leq \bar{v}_2,
$$

$$
\tilde{e}_n b = (v_1, \ldots, v_n + 1, \bar{v}_n - 1, \ldots, \bar{v}_1),
$$

$$
\tilde{e}_i b = (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, \bar{v}_1) \text{ if } v_{i+1} > \bar{v}_{i+1},
$$

$$
= (v_1, \ldots, \bar{v}_{i+1} + 1, \bar{v}_i - 1, \ldots, \bar{v}_1) \text{ if } v_{i+1} \leq \bar{v}_{i+1},
$$

$$
\tilde{f}_0 b = (v_1, v_2 + 1, \ldots, \bar{v}_2, \bar{v}_1 - 1) \text{ if } v_2 \geq \bar{v}_2,
$$

$$
= (v_1 + 1, v_2, \ldots, \bar{v}_2 - 1, \bar{v}_1) \text{ if } v_2 < \bar{v}_2,
$$

$$
\tilde{f}_n b = (v_1, \ldots, v_n - 1, \bar{v}_n + 1, \ldots, \bar{v}_1),
$$

$$
\tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \text{ if } v_{i+1} \geq \bar{v}_{i+1},
$$

$$
= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1) \text{ if } v_{i+1} < \bar{v}_{i+1},
$$

$$
wt(b) = (\bar{v}_1 - v_1 + \bar{v}_2 - v_2)\Lambda_0
$$

$$
+ \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + (v_n - \bar{v}_n)\Lambda_n,
$$

$$
\varphi_0(b) = \bar{v}_1 + (\bar{v}_2 - v_2)_+, \ \varepsilon_0(b) = v_1 + (v_2 - \bar{v}_2)_+,
$$

$$
\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \text{ for } i = 1, \ldots, n - 1,
$$
\( \varepsilon_i(b) = \tilde{v}_i + (v_{i+1} - \tilde{v}_{i+1}) + \) for \( i = 1, \ldots, n - 1, \)

\( \varphi_n(b) = v_n, \varepsilon_n(b) = \tilde{v}_n, \)

where by definition \( x_+ = \max(x, 0). \)

\[
H(b \otimes b') = \max(\{0_j(b \otimes b'), 0'_j(b \otimes b')|1 \leq j \leq n-1\} \cup \{\eta_j(b \otimes b'), \eta'_j(b \otimes b')|1 \leq j \leq n\}),
\]

where

\[
0_j(b \otimes b') = \sum_{k=1}^{j} (\tilde{v}_k - \tilde{v}_j),
\]

\[
0'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k),
\]

\[
\eta_j(b \otimes b') = \sum_{k=1}^{j} (\tilde{v}_k - \tilde{v}_j') + (\tilde{v}_j' - v_j),
\]

\[
\eta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + (v_j - \tilde{v}_j').
\]

5.3. \( B^{(1)}_a (n \geq 3) \)

\[
B_{\infty} = \left\{(v_1, \ldots, v_n, v_0, \tilde{v}_n, \ldots, \tilde{v}_1) \in \mathbb{Z}^{2n} \times \{0, 1\}|v_0 = 0 \text{ or } 1, \right. \\
\left. \sum_{i=1}^{n} v_i + v_0 + \sum_{i=1}^{n} \tilde{v}_i = 0 \right\}, \quad b_{\infty} = (0, 0, \ldots, 0).
\]

For \( b = (v_1, \ldots, v_n, v_0, \tilde{v}_n, \ldots, \tilde{v}_1) \) and \( b' = (v'_1, \ldots, v'_n, v'_0, v'_n, \ldots, \tilde{v}_1) \) in \( B_{\infty}, \)

\[
\tilde{e}_0 b = (v_1, v_2 - 1, \ldots, \tilde{v}_2, \tilde{v}_1 + 1) \quad \text{if } v_2 > \tilde{v}_2,
\]

\[
= (v_1 - 1, v_2, \ldots, \tilde{v}_2 + 1, \tilde{v}_1) \quad \text{if } v_2 \leq \tilde{v}_2,
\]

\[
\tilde{e}_n b = (v_1, \ldots, v_n, v_0 + 1, \tilde{v}_n - 1, \ldots, \tilde{v}_1) \quad \text{if } v_0 = 0,
\]

\[
= (v_1, \ldots, v_n + 1, v_0 - 1, \tilde{v}_n, \ldots, \tilde{v}_1) \quad \text{if } v_0 = 1,
\]

\[
\tilde{e}_i b = (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, \tilde{v}_1) \quad \text{if } v_{i+1} > \tilde{v}_{i+1},
\]

\[
= (v_1, \ldots, \tilde{v}_{i+1} + 1, \tilde{v}_i - 1, \ldots, \tilde{v}_1) \quad \text{if } v_{i+1} \leq \tilde{v}_{i+1},
\]

\[
\tilde{f}_0 b = (v_1, v_2 + 1, \ldots, \tilde{v}_2, \tilde{v}_1 - 1) \quad \text{if } v_2 \geq \tilde{v}_2,
\]

\[
= (v_1 + 1, v_2, \ldots, \tilde{v}_2 - 1, \tilde{v}_1) \quad \text{if } v_2 < \tilde{v}_2,
\]

\[
= (v_1 + 1, v_2, \ldots, \tilde{v}_2 - 1, \tilde{v}_1) \quad \text{if } v_2 < \tilde{v}_2,
\]
$\bar{f}_n b = (v_1, \ldots, v_n - 1, v_0 + 1, \bar{v}_n, \ldots, \bar{v}_1)$ if $v_0 = 0,$
$= (v_1, \ldots, v_n, v_0 - 1, \bar{v}_n + 1, \ldots, \bar{v}_1)$ if $v_0 = 1,$
$\bar{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1)$ if $v_{i+1} \geq \bar{v}_{i+1},$
$= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1)$ if $v_{i+1} < \bar{v}_{i+1},$

\[
\text{wt}(b) = (\bar{v}_1 - v_1 + \bar{v}_2 - v_2)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i
+ 2(v_n - \bar{v}_n)\Lambda_n,
\]

$\varphi_0(b) = \bar{v}_1 + (\bar{v}_2 - v_2)_+, \varphi_0(b) = v_1 + (v_2 - \bar{v}_2)_+,$
$\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+$ for $i = 1, \ldots, n - 1,$
$\varepsilon_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+$ for $i = 1, \ldots, n - 1,$
$\varphi_n(b) = 2v_n + v_0, \varepsilon_n(b) = 2\bar{v}_n + v_0.$

\[
H(b \otimes b') = \max(\{\varrho_j(b \otimes b'), \varrho'_j(b \otimes b')|1 \leq j \leq n-1\} \cup \{\eta_j(b \otimes b'),
\eta'_j(b \otimes b')|1 \leq j \leq n\}),
\]

where

\[
\varrho_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}_k),
\]

\[
\varrho'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k),
\]

\[
\eta_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}_k) + (\bar{v}_j - v_j),
\]

\[
\eta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + (v_j - \bar{v}_j).
\]

5.4. $D^{(1)}_n \ (n \geq 4)$

\[
B_{\infty} = \left\{(v_1, \ldots, v_n, v_{n-1}, \ldots, \bar{v}_1) \in \mathbb{Z}^{2n-1}_{\geq 0} \left| \sum_{i=1}^{n} v_i + \sum_{i=1}^{n-1} \bar{v}_i = 0 \right\},
\]

\[
b_{\infty} = (0, 0, \ldots, 0).
\]

For $b = (v_1, \ldots, v_n, v_{n-1}, \ldots, \bar{v}_1)$ and $b' = (v'_1, \ldots, v'_n, \bar{v}_{n-1}, \ldots, \bar{v}_1)$ in $B_{\infty},$
\[ \tilde{e}_0 b = (v_1, v_1 - 1, \ldots, v_2, v_1 + 1) \quad \text{if } v_2 > v_1, \]
\[ = (v_1 - 1, v_2, \ldots, v_2 + 1, v_1) \quad \text{if } v_2 \leq v_1, \]
\[ \tilde{e}_{n-1} b = (v_1, \ldots, v_{n-1} + 1, v_n - 1, v_{n-1}, \ldots, v_1), \]
\[ \tilde{e}_n b = (v_1, \ldots, v_n + 1, v_{n-1} - 1, \ldots, v_1), \]
\[ \tilde{e}_i b = (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, v_1) \quad \text{if } v_{i+1} > v_i, \]
\[ = (v_1, \ldots, v_i + 1, v_i - 1, \ldots, v_1) \quad \text{if } v_{i+1} \leq v_i, \]
\[ \tilde{f}_0 b = (v_1, v_2 + 1, \ldots, v_2, v_1 - 1) \quad \text{if } v_2 \geq v_1, \]
\[ = (v_1 + 1, v_2, \ldots, v_2 - 1, v_1) \quad \text{if } v_2 < v_1, \]
\[ \tilde{f}_{n-1} b = (v_1, \ldots, v_{n-1} - 1, v_n + 1, v_{n-1}, \ldots, v_1), \]
\[ \tilde{f}_n b = (v_1, \ldots, v_n - 1, v_{n-1} + 1, \ldots, v_1), \]
\[ \tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, v_1) \quad \text{if } v_{i+1} \geq v_i, \]
\[ = (v_1, \ldots, v_i - 1, v_i + 1, \ldots, v_1) \quad \text{if } v_{i+1} < v_i, \]
\[ \text{wt}(b) = (v_1 - v_1 + v_2 - v_2)\Lambda_0 + \sum_{i=1}^{n-2} (v_i - v_i + v_{i+1} - v_{i+1})\Lambda_i \]
\[ + (v_{n-1} + v_{n-1} - v_n)\Lambda_{n-1} + (v_{n-1} - v_{n-1} + v_n)\Lambda_n, \]
\[ \varphi_0 (b) = \tilde{v}_1 + (\tilde{v}_2 - v_2)_+, \quad e_0 (b) = v_1 + (v_2 - \tilde{v}_2)_+, \]
\[ \varphi_i (b) = v_i + (\tilde{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \ldots, n-2, \]
\[ e_i (b) = \tilde{v}_i + (v_{i+1} - \tilde{v}_{i+1})_+ \quad \text{for } i = 1, \ldots, n-2, \]
\[ \varphi_{n-1} (b) = v_{n-1}, \quad e_{n-1} (b) = v_n + \tilde{v}_{n-1}, \]
\[ \varphi_n (b) = v_{n-1} + v_n, \quad e_n (b) = \tilde{v}_{n-1}, \]
\[ H(b \otimes b') = \max(\{ \varphi_j (b \otimes b'), \varphi'_j (b \otimes b') | 1 \leq j \leq n - 2 \} \cup \{ \eta_j (b \otimes b'), \eta'_j (b \otimes b') | 1 \leq j \leq n \}), \]

where
\[ \varphi_j (b \otimes b') = \sum_{k=1}^{j} (v_k - \tilde{v}_k) \quad \text{for } j = 1, \ldots, n-2, \]
\[ \varphi'_j (b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) \quad \text{for } j = 1, \ldots, n-2, \]
\[ \eta_j(b \otimes b') = \sum_{k=1}^{j} (\tilde{v}_k - \tilde{v}_k') + (\tilde{v}_j - v_j) \quad \text{for } j = 1, \ldots, n - 1, \]
\[ \eta_j'(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + (v_j - \tilde{v}_j) \quad \text{for } j = 1, \ldots, n - 1, \]
\[ \eta_n(b \otimes b') = \sum_{k=1}^{n-1} (\tilde{v}_k - \tilde{v}_k') + v_n, \]
\[ \eta'_n(b \otimes b') = \sum_{k=1}^{n-1} (v'_k - v_k) - v_n. \]

5.5. $A_{2n}^{(2)} (n \geq 2)$

\[ B_\infty = \{ (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1) | v_i, \bar{v}_i \in \mathbb{Z} \} = \mathbb{Z}^{2n}, \quad b_\infty = (0, 0, \ldots, 0). \]

For $b = (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1)$ and $b' = (v'_1, \ldots, v'_n, \bar{v}'_n, \ldots, \bar{v}'_1)$ in $B_\infty$,

\[ \tilde{e}_0 b = (v_1 - 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 > \bar{v}_1, \]
\[ = (v_1, \ldots, \bar{v}_2, \bar{v}_1 + 1) \quad \text{if } v_1 \leq \bar{v}_1, \]
\[ \tilde{e}_n b = (v_1, \ldots, v_n + 1, \bar{v}_n - 1, \ldots, \bar{v}_1), \]
\[ \tilde{e}_i b = (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1}, \]
\[ = (v_1, \ldots, \bar{v}_{i+1} + 1, \bar{v}_i - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \leq \bar{v}_{i+1}, \]
\[ \tilde{f}_0 b = (v_1 + 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 \geq \bar{v}_1, \]
\[ = (v_1, \ldots, \tilde{v}_2, \tilde{v}_1 - 1) \quad \text{if } v_1 < \bar{v}_1, \]
\[ \tilde{f}_n b = (v_1, \ldots, v_n - 1, \bar{v}_n + 1, \ldots, \bar{v}_1), \]
\[ \tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \geq \bar{v}_{i+1}, \]
\[ = (v_1, \ldots, \tilde{v}_{i+1} - 1, \tilde{v}_i + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1}, \]

\[ \text{wt}(b) = 2(\bar{v}_1 - v_1)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + (v_n - \bar{v}_n)\Lambda_n, \]
\[ \varphi_0(b) = -s(b) + (\bar{v}_1 - v_1), \quad \varphi_0(b) = -s(b) + (v_1 - \bar{v}_1), \]
\[ \varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \ldots, n - 1, \]
\[ \varphi_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1, \ldots, n - 1, \]
\[ \varphi_n(b) = v_n, \quad \varphi_n(b) = \bar{v}_n, \]
where by definition \( s(b) = \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \bar{v}_i \).

\[
H(b \otimes b') = \max\{0, j(b \otimes b'), 0' j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b') \mid 1 \leq j \leq n\},
\]

where

\[
0_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + s(b') - s(b),
\]

\[
0'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v'_k - v_k) + s(b) - s(b'),
\]

\[
\eta_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + 2(\bar{v}_j - v_j) + s(b') - s(b),
\]

\[
\eta'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v'_k - v_k) + 2(v_j - \bar{v}_j) + s(b) - s(b').
\]

5.6. \( D_{n+1}^{(2)} (n \geq 2) \)

\[
B_\infty = \{(v_1, \ldots, v_n, v_0, \bar{v}_n, \ldots, \bar{v}_1) \mid v_0 = 0 \text{ or } 1, v_i, \bar{v}_i \in \mathbb{Z}\} = \mathbb{Z}^n \times \{0, 1\},
\]

\[
b_\infty = (0, 0, \ldots, 0) \in B_\infty.
\]

For \( b = (v_1, \ldots, v_n, v_0, \bar{v}_n, \ldots, \bar{v}_1) \) and \( b' = (v'_1, \ldots, v'_n, v'_0, \bar{v}'_n, \ldots, \bar{v}'_1) \) in \( B_\infty \),

\[
\tilde{e}_0 b = (v_1 - 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 > \bar{v}_1,
\]

\[
= (v_1, \ldots, \bar{v}_2, \bar{v}_1 + 1) \quad \text{if } v_1 \leq \bar{v}_1,
\]

\[
\tilde{e}_n b = (v_1, \ldots, v_n, v_0 + 1, \bar{v}_n - 1, \ldots, \bar{v}_1) \quad \text{if } v_0 = 0,
\]

\[
= (v_1, \ldots, v_n + 1, v_0 - 1, \bar{v}_n, \ldots, \bar{v}_1) \quad \text{if } v_0 = 1,
\]

\[
\tilde{e}_i b = (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1},
\]

\[
= (v_1, \ldots, v_i + 1 + 1, \bar{v}_i - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \leq \bar{v}_{i+1},
\]

\[
\tilde{f}_0 b = (v_1 + 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 \geq \bar{v}_1,
\]

\[
= (v_1, \ldots, \bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_1 < \bar{v}_1,
\]

\[
\tilde{f}_n b = (v_1, \ldots, v_n - 1, v_0 + 1, \ldots, \bar{v}_1) \quad \text{if } v_0 = 0,
\]

\[
= (v_1, \ldots, v_n, v_0 - 1, \bar{v}_n + 1, \bar{v}_n, \ldots, \bar{v}_1) \quad \text{if } v_0 = 1,
\]

\[
\tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \geq \bar{v}_{i+1},
\]

\[
= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},
\]
\[
\text{wt}(b) = 2(\bar{v}_1 - v_1)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + 2(v_n - \bar{v}_n)\Lambda_n,
\]

\[
\varphi_0(b) = -s(b) + 2(\bar{v}_1 - v_1), \quad \varphi_0(b) = -s(b) + 2(\bar{v}_1 - v_1),
\]

\[
\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \ldots, n - 1,
\]

\[
e_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1, \ldots, n - 1,
\]

\[
\varphi_n(b) = 2v_n + v_0, \quad e_n(b) = 2\bar{v}_n + v_0,
\]

where by definition \(s(b) = \sum_{i=1}^{n} v_i + v_0 + \sum_{i=1}^{n} \bar{v}_i\).

\[
H(b \otimes b') = \max\{0_j(b \otimes b'), 0'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b')\} 1 \leq j \leq n,
\]

where

\[
0_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}_k') + s(b') - s(b),
\]

\[
0'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k' - v_k) + s(b) - s(b'),
\]

\[
\eta_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}_k) + 2(\bar{v}_j - v_j) + s(b') - s(b),
\]

\[
\eta'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v_k' - v_k) + 2(v_j - \bar{v}_j) + s(b) - s(b').
\]

5.7. \(C_n^{(1)}(n \geq 2)\)

\[
B_{\infty} = \left\{ (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1) \in \mathbb{Z}^{2n} \middle| \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \bar{v}_i \in 2\mathbb{Z} \right\},
\]

\[
b_{\infty} = (0, 0, \ldots, 0) \in B_{\infty}.
\]

For \(b = (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1)\) and \(b' = (v'_1, \ldots, v'_n, \bar{v}'_n, \ldots, \bar{v}'_1)\) in \(B_{\infty}\), we have

\[
\tilde{e}_0 b = (v_1 - 2, v_2, \ldots, \bar{v}_2, \bar{v}_1) \quad \text{if } v_1 \geq \bar{v}_1 + 1,
\]

\[= (v_1 - 1, v_2, \ldots, \bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_1 = \bar{v}_1 + 1,
\]

\[= (v_1, v_2, \ldots, \bar{v}_2, \bar{v}_1 + 2) \quad \text{if } v_1 \leq \bar{v}_1,
\]

\[
\tilde{e}_n b = (v_1, \ldots, v_n + 1, \bar{v}_n - 1, \ldots, \bar{v}_1),
\]

\[
\tilde{e}_j b = (v_1, \ldots, v_j + 1, v_{j+1} - 1, \ldots, \bar{v}_1) \quad \text{if } v_{j+1} > \bar{v}_{j+1},
\]

\[= (v_1, \ldots, \bar{v}_{j+1} + 1, \bar{v}_j - 1, \ldots, \bar{v}_1) \quad \text{if } v_{j+1} \leq \bar{v}_{j+1},
\]
\[ \tilde{f}_0 b = (v_1 + 2, v_2, \ldots, \tilde{v}_2, \tilde{v}_1) \quad \text{if} \ v_1 \geq \tilde{v}_1, \]
\[ = (v_1 + 1, v_2, \ldots, \tilde{v}_2, \tilde{v}_1 - 1) \quad \text{if} \ v_1 = \tilde{v}_1 - 1, \]
\[ = (v_1, v_2, \ldots, \tilde{v}_2, \tilde{v}_1 - 2) \quad \text{if} \ v_1 \leq \tilde{v}_1 - 2, \]
\[ \tilde{f}_n b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \tilde{v}_1), \]
\[ \tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \tilde{v}_1) \quad \text{if} \ v_{i+1} \geq \tilde{v}_{i+1}, \]
\[ = (v_1, \ldots, \tilde{v}_{i+1} - 1, \tilde{v}_1 + 1, \ldots, \tilde{v}_1) \quad \text{if} \ v_{i+1} < \tilde{v}_{i+1}, \]

\[ \text{wt}(b) = (\tilde{v}_1 - v) \Lambda_0 + \sum_{i=1}^{n-1} (v_i - \tilde{v}_i + \tilde{v}_{i+1} - v_{i+1}) \Lambda_i + (v_n - \tilde{v}_n) \Lambda_n, \]
\[ \varphi_0(b) = -\frac{1}{2}s(b) + (\tilde{v}_1 - v_1)_+, \]
\[ \varepsilon_0(b) = -\frac{1}{2}s(b) + (v_1 - \tilde{v}_1)_+, \]
\[ \varphi_i(b) = v_i + (\tilde{v}_{i+1} - v_{i+1})_+ \quad \text{for} \ i = 1, \ldots, n-1, \]
\[ \varepsilon_i(b) = \tilde{v}_i + (v_{i+1} - \tilde{v}_{i+1})_+ \quad \text{for} \ i = 1, \ldots, n-1, \]
\[ \varphi_n(b) = v_n, \varepsilon_n(b) = \tilde{v}_n, \]

where by definition
\[ s(b) = \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \tilde{v}_i. \]

\[ H(b \otimes b') = \max\{0_j(b \otimes b'), 0_j(b \otimes b'), \eta_j(b \otimes b'), \eta_j(b \otimes b') \mid 1 \leq j \leq n\}, \]

where
\[ 0_j(b \otimes b') = \sum_{k=1}^{j-1} (\tilde{v}_k - \tilde{v}_k') + \frac{1}{2}(s(b') - s(b)), \]
\[ 0_j'(b \otimes b') = \sum_{k=1}^{j-1} (v_k' - v_k) + \frac{1}{2}(s(b') - s(b)), \]
\[ \eta_j(b \otimes b') = \sum_{k=1}^{j-1} (\tilde{v}_k - \tilde{v}_k') + (\tilde{v}_j - v_j) + \frac{1}{2}(s(b) - s(b)), \]
\[ \eta_j'(b \otimes b') = \sum_{k=1}^{j-1} (v_k' - v_k) + (v_j' - \tilde{v}_j) + \frac{1}{2}(s(b) - s(b')). \]

5.8. The proof in the case $A_n^{(1)} \ (n \geq 2)$

We shall give the proof only in the case of $A_n^{(1)} \ (n \geq 2)$. The other cases are similar. For any positive integer $l$, we first recall from [KMN3] the description
of the perfect crystal $B_i$ which is isomorphic to $B(l \Lambda_i)$ as crystals for $U_q(A_n)$:

$$B_i = \left\{ (x_i) \in \mathbb{Z}^l \mid x_i \geq 0, \sum_i x_i = l \right\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are the same as that of Section 5.1 replacing $v$ with $x$, as well as $\varepsilon_i$, $\varphi_i$ and $w_t$. In this case, we have $B_i^{\min} = B_i$. Furthermore, if $\lambda = \sum_i k_i \Lambda_i \in (P_+^i)_{1i}$, then $\sigma \lambda = \sum_i k_i \Lambda_{i-1}$.

Now let $B_\infty$ be the crystal defined in Section 5.1.

For $b_0 = (m_i) \in B_i^{\min}$, let $\lambda = \varphi(b_0)$ and $\sigma \lambda = \varepsilon(b_0)$. Then $\lambda = \sum_i m_i \Lambda_i$ and $\sigma \lambda = \sum_i m_{i+1} \Lambda_i$. For $b = (x_1, x_2, \ldots, x_{n+1}) \in B_i$, we define the map

$$f_{(l,bo)}: T_{\sigma \lambda} \otimes B_i \otimes T_{-\lambda} \rightarrow B_\infty$$

by $f_{(l,bo)}(t_{\sigma \lambda} \otimes b \otimes t_{-\lambda}) = b' = (v_i)_l$,

where $v_i = x_i - m_i$.

Note that $f_{(l,bo)}(t_{\sigma \lambda} \otimes b_0 \otimes t_{-\lambda}) = b_\infty$. It is straightforward to check that $f_{(l,bo)}$ satisfies the condition (2.15). By Example 2.7, we have

$$wt(t_{\sigma \lambda} \otimes b \otimes t_{-\lambda}) = wt(b) + \sigma \lambda - \lambda$$

$$= \sum_i (x_i - x_{i+1}) \Lambda_i - \sum_i (m_i - m_{i+1}) \Lambda_i$$

$$= wt(b'),$$

$$\varphi_i(t_{\sigma \lambda} \otimes b \otimes t_{-\lambda}) = \varphi_i(b) + \langle h_i, -\lambda \rangle = x_i - m_i = \varphi_i(b'),$$

$$\varepsilon_i(t_{\sigma \lambda} \otimes b \otimes t_{-\lambda}) = \varepsilon_i(b) - \langle h_i, \sigma \lambda \rangle = x_{i+1} - m_{i+1} = \varepsilon_i(b').$$

Hence for any $b_0 \in B_i^{\min}$, $f_{(l,bo)}: T_{\sigma \lambda} \otimes B_i \otimes T_{-\lambda} \rightarrow B_\infty$ is a morphism of crystals. By definition, it is clear that $f_{(l,bo)}: T_{\sigma \lambda} \otimes B_i \otimes T_{-\lambda} \rightarrow B_\infty$ is an embedding and that

$$B_\infty = \bigcup_{(l,bo) \in J} \text{Im } f_{(l,bo)}.$$

Therefore $B_\infty$ is the limit of the coherent family of perfect crystals $\{B_i\}_{i \geq 1}$ for $U_q(A_n^{(1)})$. 
THEOREM 5.1. For \( b = (v_1, \ldots, v_{n+1}) \) and \( b' = (v'_1, \ldots, v'_{n+1}) \) in \( B_{n} \),

\[
H(b \otimes b') = \max\{ \theta_j(b \otimes b') \mid 0 \leq j \leq n \},
\]

where

\[
\theta_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + v'_{j+1}.
\]

Proof. Recall that \( \varphi_o(b) = v_{n+1} \) and \( \varepsilon_0(b') = v'_1 \). If \( v_{n+1} \geq v'_1 \), by (2.20) and (5.1), we have

\[
\tilde{\varepsilon}_o(b \otimes b') = \tilde{\varepsilon}_o b \otimes b' = (v_1 - 1, \ldots, v_{n+1} + 1) \otimes (v'_1, \ldots, v'_{n+1}).
\]

Hence

\[
\theta_o(\tilde{\varepsilon}_o b \otimes b') = \theta_o(b \otimes b')
\]

and for \( i = 1, \ldots, n \),

\[
\theta_i(\tilde{\varepsilon}_o b \otimes b') = \theta_i(b \otimes b') + 1.
\]

But since

\[
\theta_n(b \otimes b') - \theta_o(b \otimes b') = v_{n+1} - v'_1 \geq 0,
\]

and \( \theta_n(\tilde{\varepsilon}_o b \otimes b') - \theta_o(\tilde{\varepsilon}_o b \otimes b') = v_{n+1} + 1 - v'_1 > 0 \),

we have \( H(\tilde{\varepsilon}_o b \otimes b') = H(b \otimes b') + 1 \).

If \( v_{n+1} < v'_1 \), then by (2.20) and (5.1), we have

\[
\tilde{\varepsilon}_o(b \otimes b') = b \otimes \tilde{\varepsilon}_o b' = (v_1, \ldots, v_{n+1}) \otimes (v'_1 - 1, \ldots, v'_{n+1} + 1).\]

Hence

\[
\theta_n(b \otimes \tilde{\varepsilon}_o b') = \theta_n(b \otimes b')
\]
and for $i = 0, 1, \ldots, n - 1$,

$$\theta_i(b \otimes \tilde{e}_0 b') = \theta_i(b \otimes b') - 1.$$  

But since

$$\theta_0(b \otimes b') - \theta_n(b \otimes b') = \nu_i - \nu_{n+1} > 0,$$

and

$$\theta_0(b \otimes \tilde{e}_0 b') - \theta_n(b \otimes \tilde{e}_0 b') = \nu'_i - 1 - \nu_{n+1} \geq 0,$$

we have $H(b \otimes \tilde{e}_0 b') = H(b \otimes b') - 1$.

For $i = 1, \ldots, n$, recall that $\varphi_i(b) = \nu_i$ and $\varepsilon_i(b') = \nu'_{i+1}$. If $\nu_i \geq \nu'_{i+1}$, by (2.20) and (5.1), we have

$$\tilde{e}_i(b \otimes b') = \tilde{e}_i b \otimes b' = (\nu_1, \ldots, \nu_i + 1, \nu_{i+1} - 1, \ldots, \nu_{n+1}) \otimes (\nu'_1, \ldots, \nu'_{n+1}).$$

Hence $\theta_i(\tilde{e}_i b \otimes b') = \theta_i(b \otimes b') - 1$

and for $j \neq i$

$$\theta_j(\tilde{e}_i b \otimes b') = \theta_j(b \otimes b').$$

But since

$$\theta_{i-1}(b \otimes b') - \theta_i(b \otimes b') = \nu_i - \nu'_{i+1} \geq 0,$$

and

$$\theta_{i-1}(\tilde{e}_i b \otimes b') - \theta_i(\tilde{e}_i b \otimes b') = \nu_i - \nu'_{i+1} + 1 > 0,$$

we have $H(\tilde{e}_i b \otimes b') = H(b \otimes b')$.

If $\nu_i < \nu'_{i+1}$, then by (2.20) and (5.1), we have

$$\tilde{e}_i(b \otimes b') = b \otimes \tilde{e}_i b' = (\nu_1, \ldots, \nu_{n+1}) \otimes (\nu'_1, \ldots, \nu'_i + 1, \nu'_{i+1} - 1, \ldots, \nu'_{n+1}).$$
Hence

\[ 0_{i-1}(b \otimes \tilde{e}_i b') = 0_{i-1}(b \otimes b') + 1 \]

and for \( j \neq i - 1 \)

\[ 0_j(b \otimes \tilde{e}_i b') = 0_i(b \otimes b'). \]

But since

\[ 0_i(b \otimes b') - 0_{i-1}(b \otimes b') = \nu_{i+1} - \nu_i > 0, \]

and

\[ 0_i(b \otimes \tilde{e}_i b') - 0_{i-1}(b \otimes \tilde{e}_0 b') = \nu_{i+1} - \nu_i - 1 \geq 0, \]

we have \( H(b \otimes \tilde{e}_i b') = H(b \otimes b') \),

which proves the theorem. \( \square \)

References


Crystal bases of Verma modules for quantum affine Lie algebras


