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SHUN-ICHI KIMURA

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On the characterization of Alexander schemes

SHUN-ICHI KIMURA
Massachusetts Institute of Technology

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Introduction

Let $\bar{\mathscr{M}}_g$ be the moduli space of stable curves of genus g. In [7], Mumford showed that the Chow group of $\bar{\mathscr{M}}_g$ (with rational coefficients) has a ring structure by intersection products when the characteristic is 0, using the fact that $\bar{\mathscr{M}}_g$ is étale locally a quotient of a smooth variety by a finite group, and globally a quotient of a Cohen-Macaulay scheme.

Later, Vistoli gave a more elegant explanation, introducing the notion of Alexander schemes. In [9], Vistoli says "the class of Alexander schemes is a reasonable answer to the question of what is the most natural general class of schemes that behave like smooth schemes from the point of view of intersection theory with rational coefficients." For example, the Chow group of Alexander schemes (tensored with $\mathbb{Q}$) has a ring structure by intersection products. In [10], he showed that if a scheme X is étale locally a quoient of a smooth scheme by a finite group, then X is Alexander (in characteristic 0). Hence $\bar{\mathscr{M}}_g$ is an Alexander scheme, its Chow group has intersection products with rational coefficients, and we do not need a Cohen-Macaulay covering.

In this paper, we will investigate when a scheme is Alexander. The goals are as follows.

(1) Define Alexander schemes in such a way that Vistoli's claim above is clear from the definition, and prove that our definition is (almost) equivalent to Vistoli's definition.

(2) To give a practical criterion to check whether or not a given variety is Alexander.

(3) To define a notion of Alexander morphisms so that a scheme is Alexander if and only if the structure map $X \to \operatorname{Spec} \kappa$ is an Alexander morphism.

Let us look at (1)–(3) more precisely.

(1) When X is a smooth variety, the diagonal map $\Delta_X: X \to X \times X$ is a regular imbedding, so we have a pull-back $\Delta_X^*: A_*(X \times X) \to A_*X$. In [2], Fulton explains all the good properties of smooth varieties only using this

pull-back. For example, the intersection product $[V]\cdot[W]$ is defined to be $\Delta_X^*([V \times W]) \in A_*X$.

In this paper, we will define an Alexander scheme to be a variety that has a "pull-back" $\Delta_X^*: A_*(X \times X)_{\mathbb{Q}} \to A_*X_{\mathbb{Q}}$ (see Definition 1.1 and Remark 1.2). So once we tensor all the Chow groups with $\mathbb{Q}$, then the results in [2] for smooth varieties are valid for Alexander schemes. It is a consequence of our main theorem that this definition is (almost) equivalent to Vistoli's definition (Theorem 4.2, Corollary 4.5).

(2) When $\pi: \tilde{X} \to X$ is a resolution of singularities, then we will show that X is Alexander if and only if there is a "pull-back" $\pi^*: A_*X_{\mathbb{Q}} \to A_*\tilde{X}_{\mathbb{Q}}$ (Definition 4.1). The pull-back π^* lives in $A(\tilde{X} \to X)_{\mathbb{Q}}$ which is easier to calculate than $A(X \to X \times X)_{\mathbb{Q}}$ in which Δ_X^* lives, so it is more practical to use this criterion.

(3) We will define the notion of Alexander morphisms (Definition 2.1). It satisfies the following properties (i)–(iv).

(i) A scheme X is Alexander if and only if the structure morphism $X \to \operatorname{Spec} \kappa$ is an Alexander morphism (Cor. 2.5).
(ii) A composition of Alexander morphisms is an Alexander morphism (Prop. 2.3).
(iii) The property to be an Alexander morphism is stable under base extensions (Prop. 2.3).
(iv) Smooth morphisms are Alexander (Cor. 2.2).

In particular from these properties, we can see that

(a) products of Alexander schemes are Alexander,
(b) open subschemes of Alexander schemes are Alexander.

As an application of our results, we will find a criterion for a cone X over a smooth projective variety S to be Alexander. In particular, we will prove that the cone X over a surface S is Alexander if and only if $A^*S_{\mathbb{Q}} \cong \mathbb{Q}[c_1(\mathcal{O}(1))]$. One exciting implication is that when S is Mumford's fake projective plane [6], then the cone is Alexander if and only if Bloch's conjecture holds for S, namely $A_0S \cong \mathbb{Z}$.

The author is grateful to the referee and A. Vistoli for useful and thoughtful comments and advices. Theorem 5.3 emerged from a conversation with S. Bloch.

Notations and conventions.

A_*X is the Chow group of X tensored with $\mathbb{Q}$, and $A(X \to Y)$ is the group of the bivariant classes (see [2, Chap. 17]), tensored with $\mathbb{Q}$.

All schemes are algebraic schemes over a fixed field κ. We assume the existence of resolutions of singularities (e.g., char $\kappa = 0$). A variety is an integral scheme.

The notation as in Fig. 1 means that there is a morphism $X \to Y$ and α is an element of $A(X \to Y)$.

Orientation $[f] \in A(X \to Y)$ is the bivarient class determined by the pull-back f^*, for example, when f is flat, a regular imbedding or a locally complete intersection [see 2, §17.4].

1. Definition of Alexander schemes

DEFINITION 1.1. Let X be a scheme. Consider the situation as in Fig. 2, where $f: X \to \operatorname{Spec} \kappa$ is the structure morphism, Δ_X is the diagonal morphism and π_1 and π_2 are the projections. If each connected component of X is pure-dimensional (we need this condition to have the flat pull-back, see [2, Ex. 1.7.1]), then there exist the orientation (i.e., the class defined by the pull-backs) $[f] \in A(X \to \operatorname{Spec} \kappa)$ and its pull-backs $[\pi_i] \in A(X \times X \xrightarrow{\pi_i} X)$, $i = 1, 2$.

We say that X is *Alexander* when X satisfies the following conditions (1) and (2):

(1) Each connected component of X is pure-dimensional (so that $[\pi_i]$'s are well defined);

(2) for the diagonal morphism Δ_X as in Fig. 2, there exists a class $c \in A(X \to X \times X)$ such that $c \cdot [\pi_1] = c \cdot [\pi_2] = 1$ in $A(X \to X) = A^*X$.

REMARK 1.2. We will show that the class $c \in A(X \to X \times X)$ in 2) is unique if it exists (Cor. 3.5). So we can regard c as the orientation of the diagonal morphism (for example, if X is smooth, then c is the orientation of the regular imbedding Δ_X). This c determines the pull-back

$$\Delta_X^* := c \cap \_ : A_*(X \times X) \to A_*X.$$

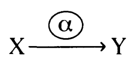

Fig. 1.

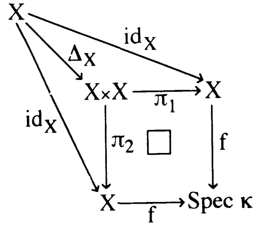

Fig. 2.

REMARK 1.3. All the nice properties of the Chow groups of smooth varieties are deduced from the existence of the pull-back Δ_X^*, or $[\Delta_X] \in A(X \to X \times X)$ (at least, the properties that are explained in [2]). Therefore all the statements in [2] for smooth varieties are valid for Alexander schemes once we tensor all the Chow groups with $\mathbb{Q}$.

REMARK 1.4. We will show that when X is an Alexander scheme, each connected component of X is not only pure dimensional, but also irreducible (Cor. 4.5).

2. Alexander morphisms

DEFINITION 2.1. A morphism $f: X \to Y$ together with a class $c \in A(X \to Y)$ is called an *Alexander morphism* if for any morphism $\varphi: S \to Y$, the following condition (*) is satisfied for the base extension $\tilde{f}: S \times_Y X \to S$ together with $\varphi^* c \in A(S \times_Y X \to S)$.

(*) For any morphism $T \to S \times_Y X$, the homomorphism defined by taking the product with $\varphi^* c$

$$\_ \cdot \varphi^* c: A(T \to S_Y \times X) \to A(T \to S)$$

is bijective (see Fig. 3).

COROLLARY 2.2. (i) *A smooth morphism f together with its orientation* $[f]$ *is an Alexander morphism.*

(ii) *A universal homeomorphism f together with its orientation* $[f]$ (*see* [4, Lemma 3.8]) *is an Alexander morphism.*

Proof. Both classes are stable under base extensions, so we have only to show (*) when $\varphi = \text{id}$. The bijectivities are proved in [2, Prop. 17.4.2] for (i), and [4, Lemma 3.8] for (ii). □

PROPOSITION 2.3. (i) *The composition of Alexander morphisms is an Alexander morphism.*

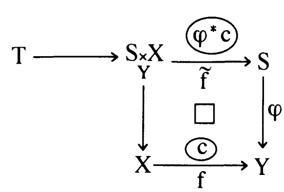

Fig. 3.

(ii) *A base extension of an Alexander morphism is an Alexander morphism.*

Proof. Straightforward. For details, see [3, Cor. 3.2.3.9]. □

PROPOSITION 2.4. *Let* $f: X \to Y$ *be a morphism and* $c \in A(X \to Y)$ *an element of the bivariant Chow group. Consider the situation as in Fig.* 4; *then* (1) *and* (2) *are equivalent*:

(1) f *is an Alexander morphism together with* c.

(2) *There exists* $\delta \in A(X \to X_Y \times X)$ *such that* $\delta \cdot f_1^*c = \delta \cdot f_2^*c = 1 \in A^*X$.

Proof. (1) ⇒ (2). Assume that f together with c is Alexander. Then $\pi_1: X \times_Y X \to X$ with f_1^*c is also Alexander by Prop. 2.3(ii). Therefore $f \circ \pi_1: X \times_Y X \to Y$ with f_2^*c c is again Alexander by Prop. 2.3(i). Let C be the class $C = f_2^*c \cdot c = f_1^*c \cdot c \in A(X \times_Y X \to Y)$ (the equality follows from [2, Ex. 17.4.4]). Because $f \circ \pi_1$ with C is Alexander, we have a bijection $\_ \cdot C: A(X \to X \times_Y X) \to A(X \to Y)$ c. So define $\delta \in A(X \to X \times_Y X)$ to be the class that satisfies $\delta \cdot C = c$. Let us show that $\delta \cdot f_1^*c = \delta \cdot f_2^*c = 1$. Because $C = f_1^*c \cdot c = f_2^*c \cdot c$, $\delta \cdot f_1^*c \cdot c = \delta \cdot f_2^*c \cdot c = \delta \cdot C = c$. Because $f: X \to Y$ with c is Alexander, the class $d \in A^*X = A(X \to X)$ that satisfies $d \cdot c = c$ in $A(X \to Y)$ is unique. Each of $\delta \cdot f_1^*c$, $\delta \cdot f_2^*c$ and $1 \in A^*X$ satisfies this condition, so they must coincide.

(2) ⇒ (1). It is easy to check that the condition (2) is stable under base extensions, so, we have only to show that, for any morphism $\psi: T \to X$, the homomorphism $\_ \cdot c$: $A(T \to X) \to A(T \to Y)$ is bijective. One can mimic the proof of [2, Prop. 17.4.2] to show the bijectivity. For details, see [3, Prop. 3.2.4]. □

COROLLARY 2.5. *Assume that each connected component of a scheme* X *is pure dimensional. Then* X *is Alexander if and only if* $f: X \to$ *Spec* κ *together with its orientation* $[f]$ *is an Alexander morphism.*

Proof. Because $[\pi_1]$ and $[\pi_2]$ are pull-backs of $[f]$, the condition (2) of Prop. 2.4 is equivalent to the condition (2) of Definition 1.1. □

REMARK 2.6. In Corollary 2.5, if $f: X \to \operatorname{Spec} \kappa$ is an Alexander morphism

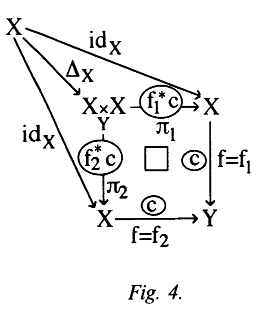

Fig. 4.

with some class $c \in A(X \to \operatorname{Spec} \kappa)$, then one can show that f together with $[f]$ is also Alexander, hence X is Alexander. For details, see [3, Prop. B2]. We do not use this result in this paper.

REMARK 2.7. (i) From Corollary 2.5 and Proposition 2.3, it easily follows that the product of Alexander schemes is also an Alexander scheme.

(ii) From Corollary 2.5 and Corollary 2.2, it follows that when X is Alexander and a morphism $f: Y \to X$ is smooth, then Y is also Alexander. In particular, open subschemes of Alexander schemes are Alexander.

3. Alexander schemes in the sense of Vistoli

DEFINITION 3.1. Let X be an equi-dimensional scheme. We say that X is an *Alexander scheme in the sense of Vistoli* when for any morphism $T \to X$, the evaluation homomorphism $A(T \to X) \to A_* T$ which sends $c \in A(T \to X)$ to $c \cap [X] \in A_* T$ is an isomorphism ([9, Def. 2.1]).

REMARK 3.2. In [9], the definition of Alexander scheme has one more condition, namely the commutativity. In this paper, because we assume the existence of resolutions, the commutativity is automatically satisfied. See [2, Ex. 17.4.4] or [3, Rem. 3.1.1.1 and A.3.3] for details.

REMARK 3.3. Assume that the characteristic of the base field κ is 0. When a variety X has quotient singularities étale locally, then X is an Alexander scheme in the sense of Vistoli [10, Cor. 6.4]. Later, we will show that Vistoli's definition is equivalent to ours for varieties (Corollary 4.5), so such varieties are Alexander schemes in our sense, too.

LEMMA 3.4. *If a pure-dimensional scheme X is an Alexander scheme, then X is Alexander in the sense of Vistoli.*

Proof. If X is an Alexander scheme, then the structure morphism $f: X \to \operatorname{Spec} \kappa$ together with $[f]$ is an Alexander morphism by Corollary 2.5. For any morphism $T \to X$, the evaluation map $\_ \cap [X]: A(T \to X) \to A_* T$ is the same as the map $(\_ \cdot [f]) \cap [\operatorname{Spec} \kappa]$. By definition of Alexander morphisms, $\_ \cdot [f]: A(T \to X) \to A(T \to \operatorname{Spec} \kappa)$ is bijective, and by [2, Prop. 17.3.1], $\_ \cap [\operatorname{Spec} \kappa]: A(T \to \operatorname{Spec} \kappa) \to A_* T$ is also bijective. Therefore the evaluation map $\_ \cap [X]$ is bijective, hence X is an Alexander scheme in the sense of Vistoli. □

COROLLARY 3.5. *If X is an Alexander scheme, then $c \in A(X \to X \times X)$ in Definition* 1.1, 2) *is unique.*

Proof. Because $c \cdot [\pi_i] = 1$, we have $c \cap [X \times X] = c \cap ([\pi_i] \cap [X]) = [X]$. By Remark 2.7(i), $X \times X$ is also Alexander, hence Alexander in the sense of

Vistoli by Lemma 3.4. Therefore, the evaluation map $A(X \to X \times X) \to A_*X$ is bijective and such a class c is unique. $\square$

4. Main result

DEFINITION 4.1. When $\pi: \tilde{X} \to X$ is a proper surjective morphism from a smooth scheme $\tilde{X}$, then a class $c \in A(\tilde{X} \to X)$ is called the *orientation* of π if $c \cap [X] = [\tilde{X}] \in A_*\tilde{X}$. We will see that such c is unique if it exists (Remark 4.6).

Theorem 4.2. *Let X be a scheme. Then* (1)–(5) *are equivalent*:

(1) *X is an Alexander scheme.*

(2) *Each of the connected components of X is pure-dimensional and the structure morphism $X \to Spec\ \kappa$ together with the orientation is an Alexander morphism.*

(3) *Each connected component of X is an Alexander scheme in the sense of Vistoli.*

(4) *For any proper surjective morphism $\pi: \tilde{X} \to X$ from a smooth scheme $\tilde{X}$, π has an orientation: there is a class $c \in A(\tilde{X} \to X)$ such that $c \cap [X] = [\tilde{X}]$ (Definition* 4.1).

(5) *There exists a proper surjective morphism $\pi: \tilde{X} \to X$ from a smooth scheme $\tilde{X}$ such that π has an orientation: there is a class $c \in A(\tilde{X} \to X)$ such that $c \cap [X] = [\tilde{X}]$.*

Proof. (1) and (2) are equivalent by Corollary 2.5. (1) Implies (3) by Lemma 3.4. (3) Implies (4) by definition: if X is Alexander in the sense of Vistoli, then for the morphism $\pi: \tilde{X} \to X$, the evaluation homomorphism $A(\tilde{X} \to X) \to A_*\tilde{X}$ is an isomorphism, therefore, there exists $c \in A(\tilde{X} \to X)$ such that $ev(c) = c \cap [X] = [\tilde{X}]$. Because we assume the existence of resolutions, (4) implies (5).

We have only to show that (5) implies (1). We have a proper surjective morphism $\pi: \tilde{X} \to X$ and a class $c \in A(\tilde{X} \to X)$ such that $c \cap [X] = [\tilde{X}]$.

First, let us show that each connected component of X is pure dimensional. Actually, we will show that it is irreducible.

LEMMA 4.3. *Let $Z \subset X$ be an irreducible component that intersects with another irreducible component Y. Then there is no class $d \in A(Z \to X)$ such that $d \cap [X] = [Z]$.*

Proof. Assume that such a class d exists. Then d decomposes into $d = \Sigma_{i=0}^{\dim(X)} d_i$ where $d_i \in A^i(Z \to X)$ by [2, Ex. 17.3.3], and it is easy to see that $d_0 \cap [X] = d_0 \cap m[Z] = [Z]$ where m is the multiplicity of Z in X. So we may assume that $d \in A^0(Z \to X)$ and $d_{|Z} \cap [Z] = 1/m[Z]$. As Z is connected, $A^0(Z \to Z) \cong \mathbb{Q}$ by [4, Prop. 3.10], and $d_{|Z}$ corresponds to $1/m$. At the same time, the restriction of d to Y is 0 because $d_{|Y} \in A^0(Z \cap Y \to Y) = \{0\}$. Take a point $P \in Z \cap Y$, then in $A^0 P \cong \mathbb{Q}$, $(d_{|Z})_{|P} = 1/m$ and $(d_{|Y})_{|P} = 0$, which contradicts to the functoriality of the pull-backs. □

LEMMA 4.4. *If the condition* (5) *holds, then each connected component of X is irreducible, hence pure-dimensional.*

Proof. Assume that X has an irreducible component Z that intersects with another irreducible component Y. Take $\tilde{Z} = Z \times_X \tilde{X}$, then there exists a cycle $\alpha \in A_* \tilde{Z}$ such that $(\pi_{|\tilde{Z}})_* \alpha = [Z]$ (because $\tilde{Z} \to Z$ is proper surjective, the proper push-forward $A_* \tilde{Z} \to A_* Z$ is surjective, see [4, Prop. 1.3]). Because $\tilde{X}$ is smooth, there is a class $\tilde{d} \in A(\tilde{Z} \to \tilde{X})$ such that $\tilde{d} \cap [\tilde{X}] = [\tilde{Z}]$. Define $d = (\pi_{|\tilde{Z}})_*(\tilde{d} \cdot c)$ in $A(Z \to X)$, then $d \cap [X] = [Z]$, but it cannot happen by Lemma 4.3. □

Now, we have to construct a class $\delta \in A(X \to X \times X)$ that satisfies the condition (2) of Definition 1.1.

Because π is proper surjective, $\pi_* : A_* \tilde{X} \to A_* X$ is surjective by [4, Prop. 1.3]. Let $\gamma \in A_* \tilde{X}$ be a class such that $\pi_* \gamma = [X]$. Let $\tilde{\gamma} \in A^* \tilde{X}$ be the class that satisfies $\tilde{\gamma} \cap [\tilde{X}] = \gamma$, and define $b \in A(\tilde{X} \to X)$ to be $\tilde{\gamma} \cdot c$. Because $b \cap [X] = \gamma$, $\pi_*(b \cap [X]) = [X]$.

In the situation as in Fig. 5, let us define $\delta \in A(X \to X \times X)$ to be $\pi_*([\Delta_{\tilde{X}}] \cdot (c \times b))$ where $[\Delta_{\tilde{X}}]$ is the orientation of the regular imbedding $\Delta_{\tilde{X}}$. We will show that this δ satisfies the condition (2) of Definition 1.1.

We have to show that $\delta \cdot [\pi_1] = \delta \cdot [\pi_2] = 1$ in $A^* X$. In order to show that $\delta \cdot [\pi_1] = 1$ it is enough to show that $(\delta \cdot [\pi_1]) \cap [\tilde{X}] = [\tilde{X}]$ because π^* is injective ([4, Lemma 2.1]), and $1 \in A^* \tilde{X}$ is the unique element that satisfies $1 \cap [\tilde{X}] = [\tilde{X}]$. Consider the diagram in Fig. 6 where π_h and π_V are the same morphism $\pi : \tilde{X} \to X$, but π_h is horizontal and π_V is vertical in the diagram.

Let us calculate $(\delta \cdot [\pi_1]) \cap [\tilde{X}]$.

$$(\delta \cdot [\pi_1]) \cap [\tilde{X}] = \delta \cap [\tilde{X} \times X] \quad (\pi_1^* [\tilde{X}] = [\tilde{X} \times X])$$
$$= p_{2*}(\Delta_{\tilde{X}}^{!}(c \times b) \cap [\tilde{X}] \times [X]) \quad (\text{definition of } \delta)$$

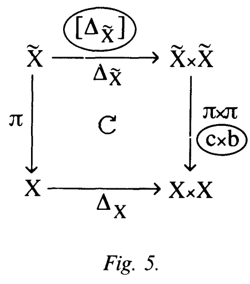

Fig. 5.

$= p_{2*}(\Delta_{\tilde{X}}{}'(c \cap [\tilde{X}]) \times (b \cap [X]))$ ([3, Cor. A.3.2.4])

$= p_{2*}(\Delta_{\tilde{X}}{}'(\pi_V^* c \cap (c \cap [X])) \times \gamma)$ (definition of b, c)

$= p_{2*}(\Delta_{\tilde{X}}{}'(\pi_h^* c \cap (c \cap [X])) \times \gamma)$ ([2, Ex. 17.4.4])

$= p_{2*}(\pi_h^* c \cap (\Delta_{\tilde{X}}^*([\tilde{X}] \times \gamma)))$ ([2, Def. 17.1, C_3])

$= p_{2*}(\pi_h^* c \cap (\Delta_{\tilde{X}}^*(p^*\gamma)))$

($p : \tilde{X} \times \tilde{X} \to \tilde{X}$ is the second projection)

$= p_{2*}(\pi_h^* c \cap \gamma)$ ($\Delta_{\tilde{X}}^* \circ p^* = 1$)

$= c \cap (\pi_{h*}\gamma)$ ([2, Def. 17.1, C_1])

$= c \cap [X]$ (definition of γ)

$= [\tilde{X}]$ (definition of c).

Therefore $\delta \cdot [\pi_1] = 1$.

The proof of $\delta \cdot [\pi_2] = 1$ is similar. For details, see [3, Th. 3.3]. □

COROLLARY 4.5. *A connected scheme X is Alexander if and only if it is Alexander in the sense of Vistoli. In this case, X is irreducible.* □

REMARK 4.6. In Definition 4.1, if an orientation $c \in A(\tilde{X} \to X)$ exists, then such a classs c is unique. Actually, if an orientation c exists, then by Theorem 4.2, each connected component of X is an Alexander scheme in the sense of Vistoli. Therefore, the evaluation map $\_ \cap [X] : A(\tilde{X} \to X) \to A_*\tilde{X}$ is bijective. In particular, the class $c \in A(\tilde{X} \to X)$ that satisfies $c \cap [X] = [\tilde{X}]$ is unique.

REMARK 4.7. The property to be Alexander is étale local, i.e., if $\{\pi_i : U_i \to X\}$ is an étale covering, then X is Alexander if and only if all U_i's are Alexander. For Zariski case, see [3, Th. 5.2]. Etale case will appear elsewhere.

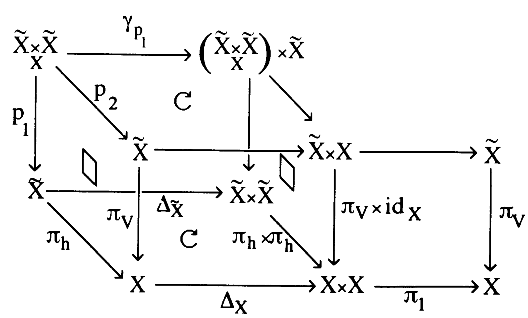

Fig. 6.

5. Cones over smooth varieties

In [9], Vistoli proved that a curve is Alexander if and only if it is geometrically unibranch, and a normal surface over a perfect field is Alexander if and only if the exceptional divisor of its desingularization consists only of rational curves. On the other hand, in the early 60's, Zobel already found that a Weil divisor and a curve in the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ do not have natural intersection number, and hence we see that it cannot be Alexander. In this section, we will find a condition for a cone X over a smooth projective variety S to be Alexander: When X is Alexander, then $A^*S \cong \mathbb{Q}[t]/(t^{n+1})$ where $n = \dim(S)$, and $t = c_1(\mathcal{O}(1))$. Moreover when S is a surface, this is a sufficient condition for X to be Alexander.

Let $S \subset \mathbb{P}^m$ be a smooth projective variety, embed $\mathbb{P}^m \subset \mathbb{P}^{m+1}$ as a hyperplane, take a point $P \in \mathbb{P}^{m+1}$ outside the hyperplane, and let X be the cone over S with the vertex P. Then P is the only singular point of X, and if $\pi: \tilde{X} \to X$ is the blowing-up along P, then π is a desingularization, the exceptional divisor E is canonically isomorphic to S, and $\mathcal{O}(E)_{|E} \cong \mathcal{O}_S(-1)$.

THEOREM 5.1. *Notations are as above. If X is Alexander, then the Chow ring A^*S is $\mathbb{Q}[t]/(t^{n+1})$ where $n = \dim(S)$, and $t = c_1(\mathcal{O}(1))$.*

Proof. We start by restating our criterion in terms of the Chow group of S.

LEMMA 5.2. *X is Alexander if and only if there is a cycle $\delta \in A_*(S \times S)$ and $\beta \in A_*S$ such that* (i) *and* (ii) *below are satisfied*:

(i) $p_{1*}\delta = 0$, *where $p_1: S \times S \to S$ is the first projection.*

(ii) $c_1(p_2^*\mathcal{O}(1)) \cap \delta + p_1^*\beta = [\Delta_S]$ *where $p_2: S \times S \to S$ is the second projection and $[\Delta_S]$ is the cycle of the diagonal.*

Proof. By Theorem 4.2, X is Alexander if and only if there exists $c \in A(\tilde{X} \to X)$ such that $c \cap [X] = [\tilde{X}]$. We use [4, Th. 3.1] to compute $A(\tilde{X} \to X)$: In Fig. 7, $\pi^*: A(\tilde{X} \to X) \to A(\tilde{X}_X \times \tilde{X} \to \tilde{X})$ is injective, and $\alpha \in A(\tilde{X}_X \times \tilde{X} \to X)$ lies in the image of π^* if and only if $i^*\alpha \in A(E \times E \to E)$ lies

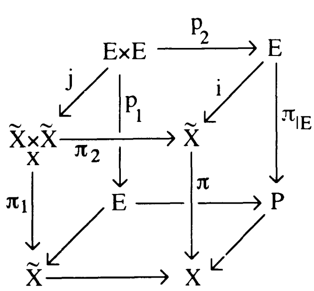

Fig. 7.

in the image of $\pi_{|E}^*: A(E \to P) \to A(E \times E \to E)$.

As P and E are non-singular, the evaluation maps $A(E \to P) \to A_*E$ and $A(E \times E \to E) \to A_*(E \times E)$ are bijective [2, Prop. 17.4.2], and $\pi_{|E}^*$ corresponds to the pull-back p_1^*.

If $c \cap [X] = [\tilde{X}]$, then $(\pi^* c) \cap [\tilde{X}] = [\Delta_{\tilde{X}}] + j_*\delta$ for some $\delta \in A_*(E \times E)$ with $\pi_{1*} j_* \delta = 0$. As $E \to \tilde{X}$ has a section, it implies $p_{1*}\delta = 0$.

Conversely, for some cycle $\delta \in A_*(E \times E)$ with $p_{1*}\delta = 0$, taking the bivariant class $\tilde{c}$ so that $\tilde{c} \cap [\tilde{X}] = [\Delta_{\tilde{X}}] + j_*\delta$, if $\tilde{c}$ comes from some bivariant class $c \in A(\tilde{X} \to X)$, namely $\pi^* c = \tilde{c}$, then $c \cap [X] = [\tilde{X}]$ by the projection formula.

When $\tilde{c} \cap [\tilde{X}] = [\Delta_{\tilde{X}}] + j_*\delta$, $\tilde{c} = \pi^* c$ for some c if and only if $([\Delta_{\tilde{X}}] + j_*\delta) \dot{\tilde{X}} E = p_1^*\beta$ for some $\beta \in A_*E$ by [4, Th. 3.1]. The intersection $([\Delta_{\tilde{X}}] + j_*\delta) \dot{\tilde{X}} E$ is $[\Delta_E] + c_1(p_2^*\mathcal{O}(-1)) \cap \delta$, so the condition is satisfied if and only if $[\Delta_E] = c_1(p_2^*\mathcal{O}(1)) \cap \delta + p_1^*\beta$ for some β. As E is isomorphic to S, Lemma 5.2 is proved. □

Now, if X is Alexander, then we have δ and β as in Lemma 5.2. Consider $[\Delta_S] = c_1(p_2^*\mathcal{O}(1)) \cap \delta + p_1^*\beta$ as a correspondence from S to itself, then its action on the Chow group A_*S is the identity. Let us take a d-cycle $\gamma \in A_d S$, then

$$\gamma = [\Delta_S]_*\gamma = \{c_1(p_2^*\mathcal{O}(1)) \cap \delta\}_*\gamma + (p_1^*\beta)_*\gamma = c_1(\mathcal{O}(1)) \cap (\delta_*\gamma) + (\beta, \gamma) \cdot [S]$$

where we consider $\delta \in A_*(S \times S)$ as a correspondence, and (β, γ) is the intersection number. If $d < n = \dim(S)$, then $(\beta, \gamma) = 0$ and $\gamma = c_1(\mathcal{O}(1)) \cap (\delta_*\gamma)$, so $c_1(\mathcal{O}(1)) \cap \_ : A_{d+1}S \to A_d S$ is surjective for $d = 0, 1, \ldots, n-1$, and $c_1(\mathcal{O}(1))^n \cap \_ : A_n S \to A_0 S$ is bijective, so the Chow ring of S is generated by $c_1(\mathcal{O}(1))$ over $A^0 S \cong \mathbb{Q}$. □

THEOREM 5.3. *If S is a surface over an algebraically closed field, the converse of Theorem* 5.1 *holds: If $A^*S \cong \mathbb{Q}[t]/(t^3)$, then a cone over S is Alexander.*

Proof. If $A^*S \cong \mathbb{Q}[t]/(t^3)$, then $q = \dim(\mathrm{Pic}(S)) = 0$, and $p_g = 0$ by [5], so the cycle map $A_*S \to H_{2*}(S, \mathbb{Q})$ is bijective. Let us consider the Chow group $A_*(S \times S)$. As the exterior product $A_0 S \otimes A_0 S \to A_0(S \times S)$ is surjective (the image contains the generators), $A_0(S \times S) \cong \mathbb{Q}$. By [1], the homologically equivalent to 0 part of $A_2(S \times S)$ is representable by some Abelian variety A, and by [8], A satisfies $2 \dim(A) \leqslant \dim H^3(S \times S, \mathbb{Q}) = 0$, so $A_2(S \times S) \cong H_4(S \times S, \mathbb{Q})$. So by writing $[D] = c_1(\mathcal{O}(1)) \cap [S]$, we have

$$[\Delta_S] = [S \times Pt] + \frac{[D \times D]}{(D, D)} + [Pt \times S],$$

where (D, D) is the intersection number. Hence

$$\delta = \frac{1}{(D, D)}([S \times D] + [D \times S]) \quad \text{and} \quad \beta = [Pt]$$

satisfy the conditions of Lemma 5.2. □

Such a surface S must satisfy $q = p_g = 0$, and $\rho(S) = 1$. So far, the only known examples are $\mathbb{P}^2$ and Mumford's fake projective plane [6]. If Bloch's conjecture holds for the fake projective plane S, namely $A_0 S \cong \mathbb{Q}$, then the cone over S is Alexander.

CONJECTURE 5.4. If S is a smooth variety and $A^*S = \mathbb{Q}[t]/(t^{n+1})$, then the cone over S is Alexander.

References

[1] Bloch, S. and Srinivas, V.: Remarks on correspondences and algebraic cycles. *Am. J. Math.* 105 (1983) 1235–1253.
[2] Fulton, W.: *Intersection Theory*. Springer, 1984.
[3] Kimura, S.: On varieties whose Chow groups have intersection products with $\mathbb{Q}$-coefficients. Thesis, University of Chicago, 1990.
[4] Kimura, S.: Fractional intersection and bivariant theory. *Com. Algebra* 20 (1992) 285–302.
[5] Mumford, D.: Rational equivalences of 0-cycles on surfaces. *J. Math. Kyoto Univ.* 9 (1969) 195–204.
[6] Mumford, D.: An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$. *Am. J. Math.* 101 (1979) 233–244.
[7] Mumford, D.: Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and Geometry* (dedicated to I. Shafarevich), vol. II., Progress in Math. 36, Birkhäuser.
[8] Murre, J. P.: Applications of algebraic K-theory to the theory of algebraic cycles. In *Algebraic Geometry Sitges* (1983), Springer Lecture Note 1124.
[9] Vistoli, A.: Alexander duality in intersection theory. *Compositio Math.* 70 (1989) 199–225.
[10] Vistoli, A.: Intersection theory on algebraic stacks and on their moduli spaces. *Invent. Math.* 97 (1989) 613–670.
[11] Zobel, A.: On the non-specialization of intersections on a singular variety. *Mathematika* 8 (1961) 39–44.