COMPOSITIO MATHEMATICA

LIN HONG
WENHUAI SHEN

Localization of epimorphisms and monomorphisms in homotopy theory

Compositio Mathematica, tome 91, nº 3 (1994), p. 321-324
<http://www.numdam.org/item?id=CM_1994__91_3_321_0>

Compositio Mathematica **91**: 321–324, 1994.

Localization of epimorphisms and monomorphisms in homotopy theory

LIN HONG and SHEN WENHUAI*
Department of Mathematics, SCNU, Guangzhou, P.R. China

Received 25 January 1993; accepted in final form 14 May 1993

1. Introduction

Recall that $f: X \to Y \in \mathrm{HCW}^*$, the homotopy category of pointed path-connected CW-spaces, is a homotopy epimorphism (monomorphism) if given u, $v: Y \to Z \in \mathrm{HCW}^*$ (u, $v: Z \to X \in \mathrm{HCW}^*$), $u \circ f = v \circ f$ implies $u = v$ ($f \circ u = f \circ v$ implies $u = v$) [3].

The purpose of this note is to study the effect of p-localizing homotopy epimorphisms and homotopy monomorphisms. The following problems are due to Hilton and Roitberg [4].

PROBLEM A. If $f: X \to Y$ is a homotopy epimorphism (monomorphism) of nilpotent spaces, then is any p-localized map $f_p: X_p \to Y_p$ a homotopy epimorphism (monomorphism)?

PROBLEM B. If each p-localized map $f_p: X_p \to Y_p$ is a homotopy epimorphism (monomorphism), then is $f: X \to Y$ a homotopy epimorphism (monomorphism)?

In [4], Hilton and Roitberg obtained some partial information [4, Theorem 4.4, 4.4′, 4.5 and 4.5′] for these problems. In this note we shall prove the following theorems.

THEOREM 1. *If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then the p-localized map f_p: $X_p \to Y_p$ is a homotopy epimorphism. Conversely, let Y be quasifinite, if each p-localized map $f_p: X_p \to Y_p$ is a homotopy epimorphism, then $f: X \to Y$ is a homotopy epimorphism.*

THEOREM 2. *If $f: X \to Y$ is a homotopy monomorphism of nilpotent spaces, then the p-localized map $f_p: X_p \to Y_p$ is a homotopy monomorphism. Conversely, let each homotopy group of X be finite, if each p-localized map $f_p: X_p \to Y_p$ is a homotopy monomorphism, then $f: X \to Y$ is a homotopy monomorphism.*

This answers Problem A affirmatively.

*The second author was partially supported by the National Natural Science Foundation of China.

2. Proofs

At first, we characterize homotopy epimorphisms and homotopy monomorphisms in terms of homotopy pushouts and homotopy pullbacks.

THEOREM 3. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j_1} \\ Y & \xrightarrow{j_2} & C \end{array}$$

be a homotopy pushout in HCW^*. *Then* f *is a homotopy epimorphism if and only if* $j_1 = j_2$.

Proof. Suppose f is a homotopy epimorphism. It follows from $j_1 \circ f = j_2 \circ f$ that $j_1 = j_2$. Conversely, given two maps $u, v\colon Y \to Z$ such that $u \circ f = v \circ f$. Since the square is a homotopy pushout, then there is a map $\varphi\colon C \to Z$ such that $u = \varphi \circ j_1$ and $v = \varphi \circ j_2$. If $j_1 = j_2$, then $u = v$, and so f is a homotopy epimorphism.

THEOREM 4. *Let*

$$\begin{array}{ccc} E & \xrightarrow{i_1} & X \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

be a homotopy pullback in HCW^*. *Assume that* E *is path-connected (if not, replacing* E *by the path-component* E^* *of its base point). Then* f *is a homotopy monomorphism if and only if* $i_1 = i_2$.

Proof. Suppose f is a homotopy monomorphism. It follows from $f \circ i_1 = f \circ i_2$ that $i_1 = i_2$. Conversely, given two maps $u, v\colon Z \to X$ such that $f \circ u = f \circ v$. Since the square is a homotopy pullback, then there is a map $\varphi\colon Z \to E$ such that $i_1 \circ \varphi = u$ and $i_2 \circ \varphi = v$. If $i_1 = i_2$, then $u = v$, and so f is a homotopy monomorphism.

Secondly, we must show the question of when we may infer that C and E in Theorem 3 and 4 are nilpotent if X and Y are nilpotent, since we want to localize them.

LEMMA 1. *If* $f\colon X \to Y$ *is a homotopy epimorphism of nilpotent spaces, then* C *in Theorem* 3 *is nilpotent.*

Proof. Note that the homotopy epimorphism $f\colon X \to Y$ induces an epimorphism $f_*\colon \pi_1 X \to \pi_1 Y$ [3, Proposition 1]. By [6, Theorem 2.1], C in Theorem 3 is nilpotent.

LEMMA 2. *If X and Y are nilpotent, then E in Theorem 4 is nilpotent.*
Proof. See [2, Corollary II.7.6].

Finally, we show p-localization of the square in Theorem 3 (4) is also a homotopy pushout (pullback).

Let X and Y be nilpotent, and the following square (*) be a homotopy pushout, and the following square (**) be a homotopy pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "j_2"] \\ Y \arrow[r, "j_1"] & C \end{tikzcd} \quad \ldots(*) \qquad \begin{tikzcd} E \arrow[r, "i_1"] \arrow[d, "i_2"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \ldots(**)$$

If C and E are nilpotent, then we can localize squares at prime p. Hence we obtain the following commutative squares:

$$\begin{tikzcd} X_p \arrow[r, "f_p"] \arrow[d, "f_p"] & Y_p \arrow[d, "j_{2p}"] \\ Y_p \arrow[r, "j_{1p}"] & C_p \end{tikzcd} \quad \ldots(*)_p \qquad \begin{tikzcd} E_p \arrow[r, "i_{1p}"] \arrow[d, "i_{2p}"] & X_p \arrow[d, "f_p"] \\ X_p \arrow[r, "f_p"] & Y_p \end{tikzcd} \quad \ldots(**)_p$$

LEMMA 3. *If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then the square $(*)_p$ is a homotopy pushout.*
Proof. Let

$$\begin{tikzcd} X_p \arrow[r, "f_p"] \arrow[d, "f_p"] & Y_p \arrow[d, "j_1'"] \\ Y_p \arrow[r, "j_2'"] & C' \end{tikzcd} \quad \ldots(*)'_p$$

be a homotopy pushout. Then there is a map $\varphi: C' \to C_p$ yielding a commutative diagram in HCW*

$$\begin{tikzcd} X_p \arrow[r, "f_p"] \arrow[d, "f_p"] & Y_p \arrow[d, "j_1'"] \arrow[dd, bend left, "j_{1p}"] \\ Y_p \arrow[r, "j_2'"] \arrow[dr, bend right, "j_{2p}"'] & C' \arrow[d, "\varphi"] \\ & C_p \end{tikzcd} ,$$

and hence a map of the Mayer–Vietoris sequence of the square $(*)'_p$ to the p-localization of the Mayer-Vietoris sequence of the square (*). In this map of Mayer-Vietoris sequences all groups except $H_n(C')$ are mapped by the identity.

Thus φ induces an isomorphism of homology groups. Since f is a homotopy epimorphism, $f_*: \pi_1 X \to \pi_1 Y$ is an epimorphism by [3, Proposition 1], and so is $f_{p*}: \pi_1 X_p \to \pi_1 Y_p$. Hence C (so C_p) and C' are nilpotent by [6, Theorem 2.1]. Therefore $\varphi: C' \to C_p$ is a homotopy equivalence by [1].

LEMMA 4. *The square* $(**)_p$ *is a homotopy pullback.*
Proof. See [2, Proposition II.7.9].

Now we can prove Theorem 1 and 2.

Proof of Theorem 1. Let $f: X \to Y$ be a homotopy epimorphism. Then $j_1 = j_2$ in the square (*) by Theorem 3, and C is nilpotent by Lemma 1. So $j_{1p} = j_{2p}$ in the square $(*)_p$. It follows from Lemma 3 and Theorem 3 that $f_p: X_p \to Y_p$ is a homotopy epimorphism. Conversely, let each p-localized map $f_p: X_p \to Y_p$ be a homotopy epimorphism. Then $f_{p*}: \pi_1 X_p \to \pi_1 Y_p$ is an epimorphism [3, Proposition 1]. It follows from [2, Theorem I.3.12] that $f_*: \pi_1 X \to \pi_1 Y$ is an epimorphism, and so C is nilpotent. This implies $j_{1p} = j_{2p}$ in the square $(*)_p$ by Theorem 3. By [2, Theorem II.5.14], we obtain $j_1 = j_2$ in the square (*), and so f is a homotopy epimorphism by Theorem 3.

Proof of Theorem 2. Let $f: X \to Y$ be a homotopy monomorphism. Then $i_1 = i_2$ in the square (**) by Theorem 4, and E is nilpotent by Lemma 2. So $i_{1p} = i_{2p}$ in the square $(**)_p$. It follows from Lemma 4 and Theorem 4 that $f_p: X_p \to Y_p$ is a homotopy monomorphism. Conversely, let each p-localized map $f_p: X_p \to Y_p$ is a homotopy monomorphism. By [4, Theorem 4.5′], $f: X \to Y$ satisfies that $f \circ u' = f \circ v'$ implies $u' = v'$ if given $u', v': W \to X$ and W finite complex. Given $u, v: Z \to X$ such that $f \circ u = f \circ v$. Let $\{Z_\alpha\}$ be the set of finite subcomplex of Z directed by inclusion $i_\alpha: Z_\alpha \to Z$. Then $u \circ i_\alpha = v \circ i_\alpha$ for all α. By [5, Theorem 1], the natural map

$$[Z, X] \to \varprojlim [Z_\alpha, X]$$

is bijective if each homotopy group of X is finite. It follows from $u \circ i_\alpha = v \circ i_\alpha$ that $u = v$, and f is a homotopy monomorphism.

References

1. E. Dror, A generalization of the Whitehead theorem, *Lecture Notes in Math.*, 249 (1971), 13–22.
2. P. Hilton, G. Mislin and J. Roitberg, Localization of nilpotent groups and spaces, North-Holland, *Mathematics studies* 15 (1975).
3. P. Hilton and J. Roitberg, Note on epimorphisms and monomorphisms in homotopy theory, *Proc. Amer. Math. Soc.*, 90 (1984), 316–320.
4. P. Hilton and J. Roitberg, Relative epimorphisms and monomorphisms in homotopy theory, *Compositio Math.*, 61 (1987), 353–367.
5. P. J. Kahn, On inverse limits of homotopy sets, *Proc. Amer. Math. Soc.*, 47 (1975), 487–490.
6. V. Rao, Nilpotency of homotopy pushouts, *Proc. Amer. Math. Soc.*, 87 (1983), 335–341.