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Lin Hong
Wenhuai Shen


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LIN HONG and SHEN WENHUAI*
Department of Mathematics, SCNU, Guangzhou, P.R. China

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1. Introduction

Recall that \( f: X \to Y \in \text{HCW}^* \), the homotopy category of pointed path-connected CW-spaces, is a homotopy epimorphism (monomorphism) if given \( u, v: Y \to Z \in \text{HCW}^* \) (\( u, v: Z \to X \in \text{HCW}^* \)), \( u \circ f = v \circ f \) implies \( u = v \) (\( f \circ u = f \circ v \) implies \( u = v \)) [3].

The purpose of this note is to study the effect of \( p \)-localizing homotopy epimorphisms and homotopy monomorphisms. The following problems are due to Hilton and Roitberg [4].

PROBLEM A. If \( f: X \to Y \) is a homotopy epimorphism (monomorphism) of nilpotent spaces, then is any \( p \)-localized map \( f_p: X_p \to Y_p \) a homotopy epimorphism (monomorphism)?

PROBLEM B. If each \( p \)-localized map \( f_p: X_p \to Y_p \) is a homotopy epimorphism (monomorphism), then is \( f: X \to Y \) a homotopy epimorphism (monomorphism)?

In [4], Hilton and Roitberg obtained some partial information [4, Theorem 4.4, 4.4', 4.5 and 4.5'] for these problems. In this note we shall prove the following theorems.

THEOREM 1. If \( f: X \to Y \) is a homotopy epimorphism of nilpotent spaces, then the \( p \)-localized map \( f_p: X_p \to Y_p \) is a homotopy epimorphism. Conversely, let \( Y \) be quasifinite, if each \( p \)-localized map \( f_p: X_p \to Y_p \) is a homotopy epimorphism, then \( f: X \to Y \) is a homotopy epimorphism.

THEOREM 2. If \( f: X \to Y \) is a homotopy monomorphism of nilpotent spaces, then the \( p \)-localized map \( f_p: X_p \to Y_p \) is a homotopy monomorphism. Conversely, let each homotopy group of \( X \) be finite, if each \( p \)-localized map \( f_p: X_p \to Y_p \) is a homotopy monomorphism, then \( f: X \to Y \) is a homotopy monomorphism.

This answers Problem A affirmatively.

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2. Proofs

At first, we characterize homotopy epimorphisms and homotopy monomorphisms in terms of homotopy pushouts and homotopy pullbacks.

**THEOREM 3.** Let

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{f} & & \downarrow{j_1} \\
Y & \xrightarrow{j_2} & C
\end{array}
\]

be a homotopy pushout in $HCW^*$. Then $f$ is a homotopy epimorphism if and only if $j_1 = j_2$.

Proof. Suppose $f$ is a homotopy epimorphism. It follows from $j_1 \circ f = j_2 \circ f$ that $j_1 = j_2$. Conversely, given two maps $u, v: Y \to Z$ such that $u \circ f = v \circ f$. Since the square is a homotopy pushout, then there is a map $\varphi: C \to Z$ such that $u = \varphi \circ j_1$ and $v = \varphi \circ j_2$. If $j_1 = j_2$, then $u = v$, and so $f$ is a homotopy epimorphism.

**THEOREM 4.** Let

\[
\begin{array}{ccc}
E & \xrightarrow{i_1} & X \\
\downarrow{i_2} & & \downarrow{f} \\
X & \xrightarrow{f} & Y
\end{array}
\]

be a homotopy pullback in $HCW^*$. Assume that $E$ is path-connected (if not, replacing $E$ by the path-component $E^*$ of its base point). Then $f$ is a homotopy monomorphism if and only if $i_1 = i_2$.

Proof. Suppose $f$ is a homotopy monomorphism. It follows from $f \circ i_1 = f \circ i_2$ that $i_1 = i_2$. Conversely, given two maps $u, v: Z \to X$ such that $f \circ u = f \circ v$. Since the square is a homotopy pullback, then there is a map $\varphi: Z \to E$ such that $i_1 \circ \varphi = u$ and $i_2 \circ \varphi = v$. If $i_1 = i_2$, then $u = v$, and so $f$ is a homotopy monomorphism.

Secondly, we must show the question of when we may infer that $C$ and $E$ in Theorem 3 and 4 are nilpotent if $X$ and $Y$ are nilpotent, since we want to localize them.

**LEMMA 1.** If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then $C$ in Theorem 3 is nilpotent.

Proof. Note that the homotopy epimorphism $f: X \to Y$ induces an epimorphism $f_*: \pi_1 X \to \pi_1 Y$ [3, Proposition 1]. By [6, Theorem 2.1], $C$ in Theorem 3 is nilpotent.
LEMMA 2. If $X$ and $Y$ are nilpotent, then $E$ in Theorem 4 is nilpotent.
Proof. See [2, Corollary II.7.6].

Finally, we show $p$-localization of the square in Theorem 3 (4) is also a homotopy pushout (pullback).

Let $X$ and $Y$ be nilpotent, and the following square (*) be a homotopy pushout, and the following square (**) be a homotopy pullback

\[
\begin{align*}
X & \xrightarrow{f} Y \\
& \downarrow j_2 \quad \cdots (*) \\
Y & \xrightarrow{j_1} C
\end{align*}
\]

\[
\begin{align*}
E & \xrightarrow{i_1} X \\
& \downarrow i_2 \quad f \quad \cdots (**) \\
X & \xrightarrow{f} Y
\end{align*}
\]

If $C$ and $E$ are nilpotent, then we can localize squares at prime $p$. Hence we obtain the following commutative squares:

\[
\begin{align*}
X_p & \xrightarrow{f_p} Y_p \\
& \downarrow f_p \quad \cdots (*)_p \\
Y_p & \xrightarrow{j_{1p}} C_p
\end{align*}
\]

\[
\begin{align*}
E_p & \xrightarrow{i_{1p}} X_p \\
& \downarrow i_{2p} \quad f_p \quad \cdots (**)_p \\
X_p & \xrightarrow{f_p} Y_p
\end{align*}
\]

LEMMA 3. If $f: X \to Y$ is a homotopy epimorphism of nilpotent spaces, then the square $(*)_p$ is a homotopy pushout.
Proof. Let

\[
\begin{align*}
X_p & \xrightarrow{f_p} Y_p \\
& \downarrow f_p \quad \cdots (*)_p \\
Y_p & \xrightarrow{j_{1p}} C'
\end{align*}
\]

be a homotopy pushout. Then there is a map $\varphi: C' \to C_p$ yielding a commutative diagram in $HCW^*$

and hence a map of the Mayer–Vietoris sequence of the square $(*)_p$ to the $p$-localization of the Mayer-Vietoris sequence of the square (*). In this map of Mayer-Vietoris sequences all groups except $H_n(C')$ are mapped by the identity.
Thus \( \varphi \) induces an isomorphism of homology groups. Since \( f \) is a homotopy epimorphism, \( f_*: \pi_1 X \to \pi_1 Y \) is an epimorphism by [3, Proposition 1], and so is \( f_*^p: \pi_1 X_p \to \pi_1 Y_p \). Hence \( C \) (so \( C_p \)) and \( C' \) are nilpotent by [6, Theorem 2.1]. Therefore \( \varphi: C' \to C_p \) is a homotopy equivalence by [1].

**Lemma 4.** The square (**) \( \to \) is a homotopy pullback.

**Proof.** See [2, Proposition II.7.9].

Now we can prove Theorem 1 and 2.

**Proof of Theorem 1.** Let \( f: X \to Y \) be a homotopy epimorphism. Then \( j_1 = j_2 \) in the square (*) by Theorem 3, and \( C \) is nilpotent by Lemma 1. So \( j_1^p = j_2^p \) in the square (*\( \to \)) \( \to \). It follows from Lemma 3 and Theorem 3 that \( f_*^p: X_p \to Y_p \) is a homotopy epimorphism. Conversely, let each \( p \)-localized map \( f_*^p: X_p \to Y_p \) be a homotopy epimorphism. Then \( f_*^p: \pi_1 X_p \to \pi_1 Y_p \) is an epimorphism [3, Proposition 1]. It follows from [2, Theorem I.3.12] that \( f_*: \pi_1 X \to \pi_1 Y \) is an epimorphism, and so \( C \) is nilpotent. This implies \( j_1^p = j_2^p \) in the square (**) \( \to \). It follows from Lemma 4 and Theorem 4 that \( f_*: X \to Y \) satisfies that \( f \circ u = f \circ v \) implies \( u = v \) if given \( u \), \( v: W \to X \) and \( W \) finite complex. Given \( u, v: Z \to X \) such that \( f \circ u = f \circ v \). Let \( \{Z_\alpha\} \) be the set of finite subcomplex of \( Z \) directed by inclusion \( i_\alpha: Z_\alpha \to Z \). Then \( u \circ i_\alpha = v \circ i_\alpha \) for all \( \alpha \). By [5, Theorem 1], the natural map

\[
[Z, X] \to \lim [Z_\alpha, X]
\]

is bijective if each homotopy group of \( X \) is finite. It follows from \( u \circ i_\alpha = v \circ i_\alpha \) that \( u = v \), and \( f \) is a homotopy monomorphism.

**References**