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Homology Hopf surfaces

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Introduction

In his fundamental work on compact complex surfaces K. Kodaira classified all complex structures on the topological 4-manifold $S^1 \times S^3$: Every complex surface homeomorphic to $S^1 \times S^3$ is a (primary) Hopf surface [Ko2]. A Hopf surface is by definition a compact complex surface X whose universal covering $\tilde{X}$ is analytically isomorphic to $\mathbb{C}^2 \setminus \{(0, 0)\}$.

From a topological point of view it is very natural to replace S^3 by an arbitrary homology 3-sphere Σ^3 and to ask the following question: When does $S^1 \times \Sigma^3$ admit a complex structure and which complex structures do occur? There exist examples of complex structures on $S^1 \times \Sigma^3$ for many Σ^3 not homeomorphic to the standard 3-sphere. First examples appeared implicitly in a paper by E. Brieskorn and A. Van de Ven [BV] in which they constructed complex structures on $S^1 \times \Sigma^{2n-1}$ for homotopy spheres Σ^{2n-1} of dimension $2n - 1 > 3$. (The case $n = 2$ was excluded in this discussion.)

Beside being a natural classification problem there is at least one further reason for studying complex structures on $S^1 \times \Sigma^3$, namely possible applications to instantons and monopoles. For any integral homology 3-sphere A . Floer has defined certain instanton homology groups [Fl], and there is also a relation between monopoles on Σ^3 and periodic instantons on $S^1 \times \Sigma^3$ [BH]. It should be possible to use complex analytic techniques if $S^1 \times \Sigma^3$ admits a complex structure.

A complex structure on $S^1 \times \Sigma^3$ is an example of a homology Hopf surface. More generally, we call a compact complex surface X a rational (or integral) homology Hopf surface if it has the same rational (or integral) homology as $S^1 \times S^3$. Homology Hopf surfaces are the homologically simplest surfaces with non-zero first Betti number.

Building on Kodaira's results we derive a classification of these surfaces. The algebraic dimension $a(X)$ of a rational homology Hopf surface X can be zero or one. If $a(X) = 0$ then X is a Hopf surface if there exists at least one curve on X ; otherwise it is an Inoue surface. If $a(X) = 1$, then X is an elliptic surface over $\mathbb{P}^1$, obtained from a product $\mathbb{P}^1 \times E$, E an elliptic curve, by means of a finite number of logarithmic transformations:

$$X = L_{x_r}(m_r, \zeta_r) \cdots L_{x_1}(m_1, \zeta_1)(\mathbb{P}^1 \times E),$$

where $x_1, \dots, x_r \in \mathbb{P}^1$ are distinct points, m_i positive integers and $\zeta_i \in \mathbb{C}$ complex numbers with $\sum_{i=1}^r \zeta_i \neq 0$ defining points $[\zeta_i]$ on E of order m_i for $i = 1, \dots, r$.

We determine the fundamental groups of these surfaces and use this result to describe the integral homology Hopf surfaces among them.

In order to construct examples of homology Hopf surfaces we generalize Brieskorn's and Van de Ven's construction of complex structures on $S^1 \times \Sigma^{2n-1}$ from Brieskorn hypersurfaces in $\mathbb{C}^n$. We consider $\mathbb{Z}$ -quotients of general Seifert $\mathbb{C}^*$ -bundles over $\mathbb{P}^1$. If the Euler number $e(\eta)$ of such a bundle $\eta: V_0 \rightarrow \mathbb{P}^1$ is different from zero, then η comes from a normal surface singularity with a $\mathbb{C}^*$ -action. The link of this singularity is a Seifert fibred rational homology 3-sphere Σ^3 with the same Seifert invariants as η . The corresponding $\mathbb{Z}$ -quotient $V_0/\mathbb{Z}$ is an example of a rational homology Hopf surface which is diffeomorphic to $S^1 \times \Sigma^3$. In order to describe which rational homology Hopf surfaces are $\mathbb{Z}$ -quotients of Seifert $\mathbb{C}^*$ -bundles over $\mathbb{P}^1$ we investigate the relation between logarithmic transformations and Dehn twists.

We finally arrive at the following classification of complex structures on $S^1 \times \Sigma^3$, where Σ^3 is a rational homology sphere: If X is a compact complex surface homeomorphic to $S^1 \times \Sigma^3$, then X must be a Hopf surface or an elliptic surface. If such an X exists and if Σ^3 is irreducible with infinite fundamental group, then Σ^3 must be Seifert fibred. Let Σ^3 be Seifert fibred. Then $S^1 \times \Sigma^3$ always has a complex structure coming from a $\mathbb{C}^*$ -singularity whose link is Σ^3 . Suppose that Σ^3 is Seifert fibred with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$; if

$$-2 + \sum_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right) \geq 0,$$

then we can classify the surfaces X homeomorphic to $S^1 \times \Sigma^3$. We state the result for an integral homology 3-sphere Σ^3 ; for the rational case we refer to Theorem 4.2 for the precise description. Let Σ^3 be in addition an integral homology 3-sphere. Then any complex surface X homeomorphic to $S^1 \times \Sigma^3$ is of the form

$$X = L_{x_r}(\alpha_r, \zeta_r) \cdots L_{x_1}(\alpha_1, \zeta_1)(\mathbb{P}^1 \times E)$$

where the complex numbers ζ_i satisfy the following condition: Let $E = \mathbb{C}/\Gamma$ and let $\{1, \omega\}$ be a basis of the lattice Γ , $\omega \in \mathbb{H}$. Then

$$\zeta_i = \frac{1}{\alpha_i} (u_i + v_i \omega), \quad u_i, v_i \in \mathbb{Z}, \quad \text{with}$$

$$\gcd \left(\alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{u_i}{\alpha_i}, \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{v_i}{\alpha_i} \right) = 1.$$

The paper is organized as follows. In Section 1 we study and classify homology Hopf surfaces of algebraic dimension 0. Section 2 is devoted to the case when the algebraic dimension is equal to 1: this is the elliptic surface case. In Section 3 we discuss Seifert $\mathbb{C}^*$ -bundles over $\mathbb{P}^1$ and $\mathbb{Z}$ -quotients of such bundles. Here we describe the relation between such quotients and elliptic homology Hopf surfaces. Section 4 contains the main results on the classification of complex structures on $S^1 \times \Sigma^3$.

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1. Homology Hopf surfaces of algebraic dimension zero

The simplest examples of surfaces with non-zero first Betti numbers are the Hopf surfaces. A *Hopf surface* is a compact complex surface X whose universal covering $\tilde{X}$ is analytically isomorphic to $\mathbb{C}^2 \setminus \{(0, 0)\}$. Special examples are the primary Hopf surfaces. A primary Hopf surface is defined as follows. We define an automorphism g of $\mathbb{C}^2 \setminus \{(0, 0)\}$ by

$$g(z_1, z_2) = (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2),$$

where $m \in \mathbb{N}$, $\alpha_1, \alpha_2, \lambda \in \mathbb{C}$, $0 < |\alpha_1| \leq |\alpha_2| < 1$, $(\alpha_1 - \alpha_2^m)\lambda = 0$. Then the infinite cyclic group $G = \{g^n | n \in \mathbb{Z}\}$ generated by g operates on $\mathbb{C}^2 \setminus \{(0, 0)\}$ properly discontinuously and without fixed points. The quotient X of $\mathbb{C}^2 \setminus \{(0, 0)\}$ by G is a compact complex surface which is called a *primary Hopf surface*. Hopf's original example [Hop] is the case $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $\lambda = 0$. A primary Hopf surface is diffeomorphic to $S^1 \times S^3$. We make the following definition:

DEFINITION. A *rational homology Hopf surface* is a compact complex surface X with $H_*(X, \mathbb{Q}) \cong H_*(S^1 \times S^3, \mathbb{Q})$. An *integral homology Hopf surface* is a compact complex surface X with $H_*(X, \mathbb{Z}) \cong H_*(S^1 \times S^3, \mathbb{Z})$.

It follows from Poincaré duality that a rational (or integral) homology Hopf surface can also be characterized as a surface X with $b_2(X) = 0$ and $H_1(X, \mathbb{Q}) \cong \mathbb{Q}$ (or $H_1(X, \mathbb{Z}) \cong \mathbb{Z}$ respectively).

Examples for integral homology Hopf surfaces are the primary Hopf

surfaces. In fact, Kodaira [Ko2] has shown that a compact complex surface which is homeomorphic to $S^1 \times S^3$ is a primary Hopf surface. Hopf surfaces in general are examples of rational homology Hopf surfaces.

We list some properties of rational homology Hopf surfaces X (cf. [BPV]). Since $b_2(X) = 0$, they are minimal, and since $b_1(X) = 1$, they are non-Kählerian. By [BPV, IV, §2] the essential Hodge numbers are

$$p_g(X) = h^{0,2}(X) = h^2(\mathcal{O}_X) = 0,$$

$$q(X) = h^{0,1}(X) = h^1(\mathcal{O}_X) = 1,$$

$$h^0(\Omega_X^1) = h^{1,0}(X) = 0,$$

$$h^1(\Omega_X^1) = h^{1,1}(X) = 0.$$

From this we get by Noether's formula and the Hirzebruch index theorem

$$c_1^2(X) = 0, \quad c_2(X) = 0.$$

We now want to classify the homology Hopf surfaces. The algebraic dimension $a(X)$ is equal to 0 or 1. For the rest of this section we assume that $a(X) = 0$. The Kodaira dimension $\text{kod}(X)$ of X is then equal to $-\infty$ [BPV, p. 200]. We have to distinguish between two subcases.

The first case is that there is a curve on X . Then by a result of Kodaira ([Ko1], see also [BPV, V, Theorem (18.7)]) X is a Hopf surface. By [Ko1, II, Theorem 32] X is a quotient space $(\mathbb{C}^2 \setminus \{(0, 0)\})/G$, where G is a group generated by two transformations g and e of $\mathbb{C}^2 \setminus \{(0, 0)\}$

$$g: (z_1, z_2) \mapsto (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2),$$

$$e: (z_1, z_2) \mapsto (\varepsilon_1 z_1, \varepsilon_2 z_2),$$

where m is a positive integer, $\alpha_1, \alpha_2, \lambda \in \mathbb{C}$, $0 < |\alpha_1| \leq |\alpha_2| < 1$, $(\alpha_1 - \alpha_2^m)\lambda = 0$, and $\varepsilon_1, \varepsilon_2$ are primitive l th roots of unity with $(\varepsilon_1 - \varepsilon_2^m)\lambda = 0$. We have $\pi_1(X) \cong G \cong \mathbb{Z} \oplus (\mathbb{Z}/l\mathbb{Z})$, and X is a primary Hopf surface if and only if $l = 1$. Hence X is an integral homology Hopf surface if and only if X is a primary Hopf surface.

The second case is when there are no curves on X . Then by [In1], [Bo1], [Bo2], [LYZ] one has the following theorem:

THEOREM 1.1 (Inoue; Bogomolov; Li, Yau, Zheng). *The rational homology Hopf surfaces with no curves are precisely the following Inoue surfaces:*

- (i) S_M^+ , where $M \in SL(3, \mathbb{Z})$ is a matrix with eigenvalues $\alpha, \beta, \bar{\beta}$ such that $\alpha > 1$ and $\beta \neq \bar{\beta}$.

- (ii) $S_{N,p,q,r,t}^+$, where $N \in SL(2, \mathbb{Z})$ is a matrix with two real eigenvalues $\alpha, 1/\alpha$, $\alpha > 1$, p, q, r are integers, and t is a complex number.
- (iii) $S_{N,p,q,r}^-$, where $N \in GL(2, \mathbb{Z})$ is a matrix with $\det N = -1$ having two real eigenvalues $\alpha, -1/\alpha$ such that $\alpha > 1$, and p, q, r are integers.

Here we used Inoue's notation and we refer for the precise definition of these surfaces to Inoue's paper [In1]. Examining the condition $H_1(X, \mathbb{Z}) \cong \mathbb{Z}$ in each case, we obtain the following proposition.

PROPOSITION 1.1. *Let X be a rational homology Hopf surface without curves. Then X is an integral homology Hopf surface if and only if the following conditions are satisfied:*

- (i) If $X = S_M^+$, then $\det(M - id) = \pm 1$.
- (ii) If $X = S_{N,p,q,r,t}^+$, then $r = \pm 1$, $\det(N - id) = \pm 1$.
- (iii) If $X = S_{N,p,q,r}^-$, then $r \equiv 1 \pmod{2}$, $\det(N - id) = \pm 1$.

REMARK 1.1. Inoue ([In2], see also [Na, §2]) has shown: The surfaces S_M^+ , S_M^- are diffeomorphic to each other, but analytically not isomorphic. They are not deformations of each other. If X is a rational homology Hopf surface and $\pi_1(X)$ is isomorphic to $\pi_1(S_M^+)$ as an abstract group, then X is S_M^+ or S_M^- .

2. Elliptic homology Hopf surfaces

We now consider the case that $a(X) = 1$. Then X admits an elliptic fibration over $\mathbb{P}^1$. All fibres are smooth elliptic curves, possibly multiple for a finite number of fibres. For the Euler characteristic $\chi(X)$ is the sum of the Euler characteristics of the singular fibres. Since the Euler characteristic of a fibre which is not of type mI_0 in Kodaira's notation is greater than 0, it follows that $\chi(X) \geq 0$ and $\chi(X) = 0$ if and only if X has only fibres of type mI_0 , i.e., X has only smooth fibres. In our case $\chi(X) = c_2(X) = 0$.

By a result of Kodaira [Ko1, II, Theorem 27] such a surface is obtained from the cartesian product $\mathbb{P}^1 \times E$ of $\mathbb{P}^1$ with an elliptic curve by means of a finite number of logarithmic transformations. A *logarithmic transformation* is defined as follows: Let X be $\mathbb{P}^1 \times E$. We represent E as $\mathbb{C}/\Gamma$ for a lattice Γ in $\mathbb{C}$. For $\zeta \in \mathbb{C}$ we denote by $[\zeta]$ the corresponding element of $E = \mathbb{C}/\Gamma$. Choose a point $x \in \mathbb{P}^1$, a positive integer m , and a complex number ζ such that $[\zeta]$ is a point of order m in E . Let $D \subset \mathbb{C} \subset \mathbb{P}^1$ be a small disc around 0. Let the cyclic group $\mathbb{Z}_m$ of order m act on $D \times E = D \times \mathbb{C}/\Gamma$ by complex analytic automorphisms in such a way that a generator $\rho = e^{(2\pi\sqrt{-1})/m}$ operates by

$$(z, [y]) \mapsto (\rho z, [y - \zeta]).$$

Form the quotient space $(D \times E)/\mathbb{Z}_m$. Let $B^* = \mathbb{P}^1 \setminus \{x\}$. Define

$$Y := (D \times E)/\mathbb{Z}_m \bigcup_{\varphi} B^* \times E,$$

where $\varphi: ((D \setminus \{0\}) \times E)/\mathbb{Z}_m \rightarrow B^* \times E$ is the identification map defined as follows. If we denote the class of $(z, [y]) \in D \times E$ in $(D \times E)/\mathbb{Z}_m$ by $[z, [y]]$, then

$$\varphi([z, [y]]) = \left(x + z^m, \left[y + \frac{1}{2\pi\sqrt{-1}} m\zeta \log z \right] \right).$$

We denote Y by $L_x(m, \zeta)(X)$ and call Y the space obtained by the logarithmic transformation $L_x(m, \zeta)$ from X . The space Y is again an elliptic surface, but the fibre of Y over x is now a multiple fibre of multiplicity m .

Since this is a local construction, we can repeat it with a point $x_2 \in \mathbb{P}^1$ different from x and possibly different choices of m_2 and ζ_2 , and so on.

By the above mentioned result of Kodaira we get

THEOREM 2.1 (Kodaira). *Any rational homology Hopf surface X with $a(X) = 1$ can be written in the form*

$$X = L_{x_r}(m_r, \zeta_r) \cdots L_{x_1}(m_1, \zeta_1)(\mathbb{P}^1 \times E), \quad (*)$$

where E is an elliptic curve, $x_1, \dots, x_r \in \mathbb{P}^1$ are distinct points, m_i is a positive integer and $\zeta_i \in \mathbb{C}$ such that $[\zeta_i]$ is a point of E of order m_i , for $i = 1, \dots, r$, and

$$\sum_{i=1}^r \zeta_i \neq 0.$$

The representation $(*)$ can be normalized in the following way. Since the logarithmic transformations are commutative, we may assume that $m_1 = \dots = m_{v-1} = 1$ and $m_i \geq 2$ for $i \geq v$. Moreover, it is shown in [Ko1, II, p. 687] that

$$L_{x_{v-1}}(1, \zeta_{v-1}) \cdots L_{x_1}(1, \zeta_1)(\mathbb{P}^1 \times E) = L_x(1, \zeta)(\mathbb{P}^1 \times E),$$

where x is an arbitrary point of $\mathbb{P}^1$. Consider the case in which $r \geq v$. We may assume that $x = x_v$. Then

$$L_{x_v}(m_v, \zeta_v) L_{x_v}(1, \zeta)(\mathbb{P}^1 \times E) = L_{x_v}(m_v, \zeta_v + \zeta)(\mathbb{P}^1 \times E).$$

Therefore the representation $(*)$ can be *normalized* as follows:

- (a) If X has multiple fibres, then r is equal to the number of multiple fibres and $m_i \geq 2$ for $i = 1, 2, \dots, r$.

(b) If X has no multiple fibres, then $r = 1$ and $m_1 = 1$.

Now let X be a surface of type (*). By the canonical bundle formula for elliptic fibrations [BPV, V, Corollary (12.3)] one has

$$\mathcal{K}_X = \mathcal{O}_X \left(-2F + \sum_{i=1}^r (m_i - 1)F_i \right),$$

where F is a regular simple fibre and F_i denotes the reduction of the fibre over x_i . It follows that

$$p_g(X) = \dim H^0(X, \Omega_X^2) = \dim |\mathcal{K}_X| + 1 = 0.$$

The Noether formula

$$1 + p_g(X) - q(X) = 0$$

yields $q(X) = 1$.

We compute the fundamental group $\pi_1(X)$. Represent E as $\mathbb{C}/\Gamma$ and choose a basis of Γ of the form $\{1, \omega\}$ where ω lies in the upper half plane $\mathbb{H}$. Let u_i and v_i be integers with $\gcd(u_i, v_i, m_i) = 1$ such that

$$\zeta_i = \frac{1}{m_i} (u_i + v_i \omega), \quad i = 1, \dots, r.$$

For $i = 1, \dots, r$ choose small open discs D_i in $\mathbb{P}^1$ around the points x_i with $D_i \cap D_j = \emptyset$ for $i \neq j$. Choose a point p_i on the boundary ∂D_i of each disc. Let τ_i be the loop with basepoint p_i going once around x_i on ∂D_i in the counterclockwise direction. Denote by $\pi: X \rightarrow \mathbb{P}^1$ the projection of the elliptic fibration. Let $q_i \in \pi^{-1}(p_i)$ be a basepoint in the fibre $E_i = \pi^{-1}(p_i)$, which is topologically a 2-torus, and let t_i be a lift of τ_i in X with basepoint q_i . Choose loops σ_i, δ_i with basepoint q_i generating $\pi_1(E_i)$. Then t_i, σ_i, δ_i generate $\pi_1(\pi^{-1}(\partial D^2))$ and satisfy the relations $[t_i, \sigma_i] = 1$, $[t_i, \delta_i] = 1$ and $[\sigma_i, \delta_i] = 1$ since $\pi^{-1}(\partial D^2)$ is topologically a 3-torus. (To simplify notation we do not distinguish between loops and their homotopy classes.) Finally let c_i denote the loop $\tau_i \times \{q_i\}$ which is homotopic to zero in $D_i \times E_i$. If we apply first the quotient map $(D_i \setminus \{0\}) \times E_i \rightarrow ((D_i \setminus \{0\}) \times E_i)/\mathbb{Z}_{m_i}$ and then the identification map φ in the definition of the logarithmic transformation $L_{x_i}(m_i, \zeta_i)$, then c_i is mapped to a curve in $\partial(X \setminus \pi^{-1}(D_i))$ which is homotopic to $t_i^{m_i} \sigma_i^{u_i} \delta_i^{v_i}$ in X (cf. also [M, p. 40]). If we choose a basepoint $*$ in $\mathbb{P}^1 \setminus (\bigcup_{i=1}^r D_i)$ and join the basepoints p_i with $*$, then we see that $\tau_1 \tau_2 \cdots \tau_r$ is homotopic to zero in $\mathbb{P}^1$. By

applying Van Kampen's theorem, we find the following presentation of $\pi_1(X)$ (cf. also [GS], [Li, III], [Z]):

$$\pi_1(X) = \left\langle \sigma, \delta, t_1, \dots, t_r \mid \begin{array}{l} [\sigma, \delta] = [t_i, \sigma] = [t_i, \delta] = 1, \\ t_i^{m_i} \sigma^{u_i} \delta^{v_i} = 1, \quad t_1 \cdots t_r = 1 \end{array} \right\rangle.$$

By abelianizing this presentation, we obtain the first homology group with integral coefficients:

$$H_1(X) = \langle \bar{\sigma}, \bar{\delta}, \bar{t}_1, \dots, \bar{t}_r \mid m_i \bar{t}_i + u_i \bar{\sigma} + v_i \bar{\delta} = 0, \bar{t}_1 + \cdots + \bar{t}_r = 0 \rangle.$$

In other words, $H_1(X)$ is the cokernel of the matrix A^t (' means transpose), where

$$A = \begin{pmatrix} m_1 & 0 & \cdots & 0 & u_1 & v_1 \\ 0 & m_2 & \cdots & 0 & u_2 & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & m_r & u_r & v_r \\ 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix}.$$

This matrix is equivalent over $\mathbb{Q}$ to the following matrix

$$\begin{pmatrix} m_1 & 0 & \cdots & 0 & u_1 & v_1 \\ 0 & m_2 & \cdots & 0 & u_2 & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & m_r & u_r & v_r \\ 0 & 0 & \cdots & 0 & \sum \frac{u_i}{m_i} & \sum \frac{v_i}{m_i} \end{pmatrix}.$$

Now $b_1(X) = 1$ if and only if $\text{rank } A = r + 1$. This is in turn equivalent to the condition

$$\sum_{i=1}^r \zeta_i \neq 0.$$

So if X is a surface of type $(*)$ satisfying this condition, then $b_1(X) = 1$. From

$$\chi(X) = 2 - 2b_1(X) + b_2(X) = 0$$

we also derive $b_2(X) = 0$, i.e., X is a rational homology Hopf surface.

We now have seen that the elliptic rational homology Hopf surfaces are precisely the surfaces of type $(*)$ with $\sum_{i=1}^r \zeta_i \neq 0$. Next we examine under which conditions such a surface X is an integral homology Hopf surface. Clearly X is a $\mathbb{Z}$ homology Hopf surface if and only if $\text{coker } A' = \mathbb{Z}$. This means that the $r + 1$ elementary divisors of the matrix A all have to be equal to ± 1 . An equivalent condition is that the greatest common divisor of the $(r + 1) \times (r + 1)$ principal minors of the matrix A is equal to 1. Write A_i for the $(r + 1) \times (r + 1)$ submatrix of A which is obtained by omitting the i th column of A . Then one has

$$\det A_{r+1} = m_1 \cdots m_r \cdot \left(\sum_{i=1}^r \frac{v_i}{m_i} \right),$$

$$\det A_{r+2} = m_1 \cdots m_r \cdot \left(\sum_{i=1}^r \frac{u_i}{m_i} \right),$$

$$\det A_j = \frac{u_j}{m_j} \det A_{r+1} - \frac{v_j}{m_j} \det A_{r+2}, \quad j = 1, \dots, r.$$

PROPOSITION 2.1. *The rational homology Hopf surfaces X with $a(X) = 1$ are precisely the elliptic surfaces of type $(*)$ with $\sum_{i=1}^r \zeta_i \neq 0$. Such a surface is an integral homology Hopf surface if and only if*

$$\gcd(\det A_1, \dots, \det A_{r+2}) = 1.$$

REMARK 2.1. An elliptic surface X of type $(*)$ with $\sum_{i=1}^r \zeta_i = 0$ has $b_1(X) = 2$ and is therefore Kählerian. Examples are bielliptic surfaces [BPV, p. 148].

Now let X be a rational homology Hopf surface with $a(X) = 1$, hence a surface of type $(*)$ with $\sum_{i=1}^r \zeta_i \neq 0$, in normalized representation. We define

$$\kappa(X) := -2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right).$$

We denote by D the unit disc $|z| < 1$ in $\mathbb{C}$. Kodaira has shown in [Ko1, II, Theorem 28]:

THEOREM 2.2 (Kodaira). *The universal covering $\tilde{X}$ of X is given by*

$$\tilde{X} = \begin{cases} \mathbb{C}^2 \setminus \{(0, 0)\}, & \text{if } \kappa(X) < 0, \\ \mathbb{C}^2, & \text{if } \kappa(X) = 0, \\ D \times \mathbb{C}, & \text{if } \kappa(X) > 0. \end{cases}$$

Hence X is an ordinary Hopf surface for $\kappa(X) < 0$ and aspherical (i.e., $\tilde{X}$ is contractible) for $\kappa(X) \geq 0$.

The Kodaira dimension $\text{kod}(X)$ of X is also determined by $\kappa(X)$ [BPV, p. 162–163]:

$$\text{kod}(X) = \begin{cases} -\infty, & \text{if } \kappa(X) < 0, \\ 0, & \text{if } \kappa(X) = 0, \\ 1, & \text{if } \kappa(X) > 0. \end{cases}$$

We determine the solutions of $\kappa \leq 0$. If $r \leq 2$, then $\kappa(X) < 0$. If $r \geq 3$, then $m_i \geq 2$, hence $1 - 1/m_i \geq \frac{1}{2}$, so $r \leq 4$. If $r = 4$, then $m_i = 2$ for $i = 1, \dots, 4$ and $\kappa(X) = 0$. If $r = 3$ then

$$\kappa(X) = 1 - \sum_{i=1}^3 \frac{1}{m_i} \leq 0 \Leftrightarrow \sum_{i=1}^3 \frac{1}{m_i} \geq 1.$$

The solutions are

$$\begin{aligned} \kappa(X) < 0 &\Leftrightarrow (m_1, m_2, m_3) = \begin{cases} (2, 2, m_3), & m_3 \geq 2; \\ (2, 3, m_3), & 3 \leq m_3 \leq 5; \end{cases} \\ \kappa(X) = 0 &\Leftrightarrow (m_1, m_2, m_3) = (2, 3, 6), (2, 4, 4), (3, 3, 3). \end{aligned}$$

If $r \leq 2$, then X is diffeomorphic to $S^1 \times L(\alpha, \beta)$ where $L(\alpha, \beta)$ is a lens space of type (α, β) for appropriate relatively prime integers α and β (see [M, p. 41]).

M. Kato [Ka, Theorem 10] has shown the following result: Let X and X' be Hopf surfaces and assume that the set of elements of finite order of $\pi_1(X)$ is not a (finite) cyclic group. If $\pi_1(X)$ and $\pi_1(X')$ are isomorphic as abstract groups, then X is diffeomorphic to X' .

If X is a rational homology Hopf surface with $a(X) = 1$ which is not a Hopf surface, then X is aspherical. By a result of P. E. Conner and F. Raymond ([CR1, CR2], see also [M, Corollary 1.17]) two aspherical elliptic homology Hopf surfaces X and X' are diffeomorphic if and only if $\pi_1(X)$ is isomorphic to $\pi_1(X')$.

3. Seifert $\mathbb{C}^*$ -bundles and singularities

We shall now consider examples of rational homology Hopf surfaces related with singularities. For that purpose we recall the basic facts about Seifert $\mathbb{C}^*$ -bundles (see [Ne2]).

A Seifert $\mathbb{C}^*$ -bundle over $\mathbb{P}^1$ is a $\mathbb{C}^*$ -fibration $\eta: V_0 \rightarrow \mathbb{P}^1$ which is locally, near every point $x \in \mathbb{P}^1$, isomorphic to

$$(D \times \mathbb{C}^*)/\mathbb{Z}_\alpha \rightarrow D/\mathbb{Z}_\alpha \cong D$$

for some $\alpha \in \mathbb{Z}$, $\alpha \geq 1$. Here D denotes $\{z \in \mathbb{C} \mid |z| < \varepsilon\}$ for a small $\varepsilon > 0$, and a generator $\rho = e^{(2\pi\sqrt{-1})/\alpha}$ of $\mathbb{Z}_\alpha$ operates on $D \times \mathbb{C}^*$ by

$$\rho(z, w) = (\rho z, \rho^{-\beta} w)$$

for some $\beta \in \mathbb{Z}$ with $\gcd(\alpha, \beta) = 1$ ([Ne2], [Hol]).

Such a Seifert $\mathbb{C}^*$ -bundle is obtained as follows: Consider the product $\mathbb{P}^1 \times \mathbb{C}^*$, and let $x \in \mathbb{P}^1$. Denote $\mathbb{P}^1 \setminus \{x\}$ by B^* . Let α, β be integers with $\alpha \geq 1$ and $\gcd(\alpha, \beta) = 1$. Glue $(D \times \mathbb{C}^*)/\mathbb{Z}_\alpha$ into $B^* \times \mathbb{C}^*$ by the identification map $\psi: ((D \setminus \{0\}) \times \mathbb{C}^*)/\mathbb{Z}_\alpha \rightarrow B^* \times \mathbb{C}^*$ defined by

$$\psi([z, w]) = (x + z^\alpha, z^\beta w)$$

where $[z, w]$ denotes the class of $(z, w) \in (D \setminus \{0\}) \times \mathbb{C}^*$ in $((D \setminus \{0\}) \times \mathbb{C}^*)/\mathbb{Z}_\alpha$. This operation is called a *Dehn twist* $D_x(\alpha, \beta)$. Every Seifert $\mathbb{C}^*$ -bundle $\eta: V_0 \rightarrow \mathbb{P}^1$ is obtained from $\mathbb{P}^1 \times \mathbb{C}^*$ by means of a finite sequence of Dehn twists $D_{x_1}(\alpha_1, \beta_1), \dots, D_{x_r}(\alpha_r, \beta_r)$ where $x_1, \dots, x_r$ are distinct points of $\mathbb{P}^1$ and α_i, β_i integers with $\alpha_i \geq 1$ and $\gcd(\alpha_i, \beta_i) = 1$ for $i = 1, \dots, r$ (see [Ne1, §6] in the case of Seifert manifolds). The corresponding pairs

$$(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$$

are called the *unnormalized Seifert invariants* of $\eta: V_0 \rightarrow \mathbb{P}^1$ (cf. [NR]). The number

$$e(\eta) = - \sum_{i=1}^r \frac{\beta_i}{\alpha_i}$$

is the *Euler number* of the Seifert $\mathbb{C}^*$ -bundle over $\mathbb{P}^1$ with the Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$.

Two Seifert $\mathbb{C}^*$ -bundles over $\mathbb{P}^1$ $\eta: V_0 \rightarrow \mathbb{P}^1$ and $\eta': V'_0 \rightarrow \mathbb{P}^1$ are called *homeomorphic* if there is a fibre preserving homeomorphism of V_0 to V'_0 . It is easy to see that the following moves applied to the Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ of a Seifert $\mathbb{C}^*$ -bundle $\eta: V_0 \rightarrow \mathbb{P}^1$ change the bundle into a homeomorphic one ([Ne1], [NR]):

- (a) permute the indices;
- (b) add or delete a Seifert pair $(1, 0)$;
- (c) replace (α, β) , $(1, b)$ (for any $b \in \mathbb{Z}$) by $(\alpha, \beta + b\alpha)$ and vice versa ([Ne1, Lemma 7.2]).

Let $\eta: V_0 \rightarrow \mathbb{P}^1$ be a Seifert $\mathbb{C}^*$ -bundle. If one adds a point 0 in each fibre, then one obtains a *Seifert line bundle* $\tilde{V} \rightarrow \mathbb{P}^1$ over $\mathbb{P}^1$, i.e., a complex line bundle over $\mathbb{P}^1$ with exceptional fibres of the form $\mathbb{C}/\mathbb{Z}_{\alpha_i}$ over x_i for $i = 1, \dots, r$. Let $S \subset \tilde{V}$ be the zero section of this bundle. Since $\tilde{V}$ is a $\mathbb{Q}$ -homology manifold, one can define an intersection pairing $\mathbb{Q}$ -Poincaré dual to cup product. In this way the Euler number $e(\eta) \in \mathbb{Q}$ can be interpreted as the self-intersection number $S \cdot S$ of S . By Grauert's criterion, S can be blown down to a point if and only if $e(\eta) < 0$. In this case one obtains by blowing down a normal surface singularity (V, p) . The variety $\tilde{V}$ is a partial resolution of (V, p) . It has only cyclic quotient singularities at the points $x_i \in S$; after resolving these singular points one gets a star shaped configuration of exceptional curves.

The singularity (V, p) is a normal surface singularity with a good $\mathbb{C}^*$ -action. "Good" means that p is in the closure of every $\mathbb{C}^*$ -orbit. One has $V_0 = V - \{p\}$, $V_0/\mathbb{C}^* = \mathbb{P}^1$. Conversely, if (V, p) is a normal surface singularity with a good $\mathbb{C}^*$ -action, $V_0 = V - \{p\}$ and $V_0/\mathbb{C}^* = \mathbb{P}^1$, then the $\mathbb{C}^*$ -action defines a Seifert $\mathbb{C}^*$ -bundle $\eta: V_0 \rightarrow \mathbb{P}^1$ over $\mathbb{P}^1$ with $e(\eta) < 0$. Every such bundle is obtained in this way.

We can also compactify the Seifert line bundle $\tilde{V} \rightarrow \mathbb{P}^1$ to a Seifert $\mathbb{P}^1$ -bundle $\mathbb{P}(\tilde{V}) \rightarrow \mathbb{P}^1$ [P]. Let S_∞ be the section at infinity. If we omit the zero section S of $\mathbb{P}(\tilde{V}) \rightarrow \mathbb{P}^1$, then we get a Seifert line bundle $\eta_\infty: V_\infty = \mathbb{P}(\tilde{V}) \setminus S \rightarrow \mathbb{P}^1$ with Seifert invariants $(\alpha_1, -\beta_1), \dots, (\alpha_r, -\beta_r)$. For $1/w$ is a parameter for $\mathbb{P}^1 \setminus \{0\}$ at infinity and

$$\rho \left(z, \frac{1}{w} \right) = \left(\rho z, \frac{1}{\rho^{-\beta} w} \right) = \left(\rho z, \rho^\beta \cdot \frac{1}{w} \right).$$

Therefore we obtain for the self-intersection number $S_\infty \cdot S_\infty$ of the section S_∞

$$S_\infty \cdot S_\infty = e(\eta_\infty) = -e(\eta).$$

This means the following: If $\eta: V_0 \rightarrow \mathbb{P}^1$ is a Seifert $\mathbb{C}^*$ -bundle with $e(\eta) \neq 0$, then we can consider the associated Seifert $\mathbb{P}^1$ -bundle $\mathbb{P}(\tilde{V}) \rightarrow \mathbb{P}^1$ and either blow down the zero section or the section at infinity to get a normal surface singularity.

We shall now consider a $\mathbb{Z}$ -operation on the total space of a Seifert $\mathbb{C}^*$ -bundle. Let $\eta: V_0 \rightarrow \mathbb{P}^1$ be a Seifert $\mathbb{C}^*$ -bundle over $\mathbb{P}^1$ with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$. Let $\lambda \in \mathbb{C}^*$ be an element of infinite order and denote

by $\langle \lambda \rangle \cong \mathbb{Z}$ the subgroup $\{\lambda^m \mid m \in \mathbb{Z}\}$ of $\mathbb{C}^*$ generated by λ . Multiplication by λ defines a natural action of $\mathbb{Z}$ on V_0 . This action is free if $|\lambda| \neq 1$. For $\lambda^m z = \lambda^l z$, $m \neq l$, $z \in V_0$, implies that $\lambda^{m-l} z = z$ and hence $\lambda^{m-l} = 1$. Without loss of generality we assume that $|\lambda| > 1$. Let X be the quotient

$$X := V_0 / \mathbb{Z}.$$

Then X is a compact complex surface, called a $\mathbb{Z}$ -quotient of η . In the case when V is a Brieskorn hypersurface, such a $\mathbb{Z}$ -quotient was considered by E. Brieskorn and A. Van de Ven [BV]. There is also a remark in [W3, p. 139] concerning this construction. Let Σ be the total space of the unit sphere bundle in V_0 . This space is a Seifert manifold over $\mathbb{P}^1$ with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$.

PROPOSITION 3.1. *The surface $V_0 / \mathbb{Z}$ is diffeomorphic to $S^1 \times \Sigma$.*

Proof. This follows from the fact that the mapping

$$\varphi: \begin{cases} \mathbb{R} \times \Sigma \rightarrow V_0 \\ (t, x) \mapsto \lambda^t x \end{cases}$$

is a $\mathbb{Z}$ -equivariant diffeomorphism: φ is a fibre preserving diffeomorphism and it is $\mathbb{Z}$ -equivariant because for $m \in \mathbb{Z}$

$$\varphi(t + m, x) = \lambda^{t+m} x = \lambda^m \lambda^t x = \lambda^m \varphi(t, x).$$

Since $\mathbb{C}^* / \langle \lambda \rangle$ is an elliptic curve, $V_0 / \mathbb{Z}$ becomes an elliptic surface over $\mathbb{P}^1$. Therefore we have:

COROLLARY 3.1. *Let Σ be any 3-dimensional Seifert manifold over $\mathbb{P}^1$. Then there exists an elliptic surface X over $\mathbb{P}^1$ homeomorphic to $S^1 \times \Sigma$.*

Proof. Consider Σ as a Seifert S^1 -bundle over $\mathbb{P}^1$. A $\mathbb{Z}$ -quotient of the associated Seifert $\mathbb{C}^*$ -bundle has the required property.

If Σ is a rational (integral) homology 3-sphere, then any $\mathbb{Z}$ -quotient $X = V_0 / \mathbb{Z}$ which is homeomorphic to $S^1 \times \Sigma$ is a rational (integral) homology Hopf surface.

REMARK 3.1. A 3-dimensional Seifert manifold Σ over $\mathbb{P}^1$ with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ is a rational homology sphere if and only if

$$e(\Sigma) := - \sum_{i=1}^r \frac{\beta_i}{\alpha_i} \neq 0.$$

It is an integral homology sphere if and only if

$$\alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{\beta_i}{\alpha_i} = \pm 1.$$

This equation implies that the α_i are pairwise relatively prime. To given pairwise relatively prime α_i there is exactly one integral homology 3-sphere Σ as above, up to orientation. The total space V_0 of the corresponding Seifert $\mathbb{C}^*$ -bundle over $\mathbb{P}^1$ is diffeomorphic to a Brieskorn complete intersection [NR].

Any $\mathbb{Z}$ -quotient $V_0/\mathbb{Z}$ is also a surface of type (*).

PROPOSITION 3.2. *The surface $V_0/\mathbb{Z}$ is analytically isomorphic to the surface*

$$L_{x_r} \left(\alpha_r, \frac{\beta_r}{\alpha_r} \right) \cdots L_{x_1} \left(\alpha_1, \frac{\beta_1}{\alpha_1} \right) (\mathbb{P}^1 \times E),$$

where $E = \mathbb{C}^*/\langle \lambda \rangle \cong \mathbb{C}/\Gamma$ with $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\omega$ and $\omega \in \mathbb{H}$ satisfying $\lambda = e^{2\pi\sqrt{-1}\omega}$.

Proof. First we determine ω such that $\mathbb{C}^*/\langle \lambda \rangle$ is isomorphic to $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\omega$. The isomorphism $\mathbb{C}/\Gamma \xrightarrow{\sim} \mathbb{C}^*/\langle \lambda \rangle$ is induced by the mapping $\mathbb{C} \rightarrow \mathbb{C}^*$, $z \mapsto e^{2\pi\sqrt{-1}z}$. Since $|\lambda| > 1$ we have $\lambda = e^{2\pi\sqrt{-1}\omega}$ for some ω with $\text{Im } \omega \neq 0$ determined modulo $2\pi\sqrt{-1}\mathbb{Z}$. Without loss of generality we may choose ω in the upper half plane $\mathbb{H}$. Let Γ be the lattice $\mathbb{Z} \oplus \mathbb{Z}\omega$. We show that Γ is the kernel of the surjective mapping $\mathbb{C} \rightarrow \mathbb{C}^*/\langle \lambda \rangle$. Suppose $e^{2\pi\sqrt{-1}z} \in \langle \lambda \rangle$, i.e. $e^{2\pi\sqrt{-1}z} = \lambda^m$ for $m \in \mathbb{Z}$. Then

$$\begin{aligned} e^{2\pi\sqrt{-1}z} &= e^{2\pi\sqrt{-1}m\omega} \\ \Leftrightarrow e^{2\pi\sqrt{-1}(z-m\omega)} &= 1 \\ \Leftrightarrow z - m\omega &\in \mathbb{Z} \\ \Leftrightarrow z &\in \mathbb{Z} \oplus \mathbb{Z}\omega = \Gamma. \end{aligned}$$

Write for abbreviation $\alpha = \alpha_j$, $\beta = \beta_j$, $\zeta = \beta/\alpha$ for $j \in \{1, \dots, r\}$. Let $\rho = e^{2\pi\sqrt{-1}/\alpha}$ be a generator of $\mathbb{Z}_\alpha$. Since

$$e^{2\pi\sqrt{-1}(y-\zeta)} = e^{2\pi\sqrt{-1}(y-\beta/\alpha)} = \rho^{-\beta} e^{2\pi\sqrt{-1}y},$$

it follows that the induced isomorphism

$$D \times \mathbb{C}/\Gamma \xrightarrow{\sim} D \times \mathbb{C}^*/\langle \lambda \rangle$$

is equivariant with respect to the action of $\mathbb{Z}_\alpha$ on $D \times \mathbb{C}/\Gamma$ (defined in Section 2) and the action of $\mathbb{Z}_\alpha$ on $D \times \mathbb{C}^*/\langle \lambda \rangle$ (defined in this section). This proves Proposition 3.2.

REMARK 3.2. By Proposition 3.2 the invariant $\kappa(V_0/\mathbb{Z})$ is equal to the orbifold Euler characteristic $\chi_{\text{orb}}(V_0/\mathbb{C}^*)$ of $V_0/\mathbb{C}^*$ (cf. [S2]),

$$\chi_{\text{orb}}(V_0/\mathbb{C}^*) = -2 + \sum_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right).$$

PROPOSITION 3.3. *Let*

$$X = L_{x_r}(\alpha_r, \zeta_r) \cdots L_{x_1}(\alpha_1, \zeta_1)(\mathbb{P}^1 \times E)$$

with arbitrary $E = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\omega$, and $\zeta_1, \dots, \zeta_r \in \mathbb{C}$. If X is analytically isomorphic to $V_0/\mathbb{Z}$ (V_0 with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$) by an isomorphism preserving the elliptic fibrations then $\lambda = e^{2\pi\sqrt{-1}\omega}$ and $\zeta_j = \beta_j/\alpha_j + \gamma_j$ for some $\gamma_j \in \Gamma$, $j = 1, \dots, r$.

Proof. By the proof of the previous proposition we have to show (with $\alpha = \alpha_j$, $\beta = \beta_j$, $\zeta = \zeta_j$) that

$$[e^{2\pi\sqrt{-1}(y-\zeta)}] = [\rho^{-\beta} e^{2\pi\sqrt{-1}y}]$$

only if $\zeta = \beta/\alpha + c + d\omega$ for some $c, d \in \mathbb{Z}$. Here the brackets mean the cosets in $\mathbb{C}^*/\langle \lambda \rangle$. But we have

$$[e^{2\pi\sqrt{-1}(y-\zeta)}] = [\rho^{-\beta} e^{2\pi\sqrt{-1}y}]$$

if and only if $e^{2\pi\sqrt{-1}(\zeta-\beta/\alpha)} = \lambda^d$ for some $d \in \mathbb{Z}$. Since $\lambda = e^{2\pi\sqrt{-1}\omega}$, this is equivalent to $\zeta - \beta/\alpha - d\omega = c \in \mathbb{Z}$, i.e. $\zeta = \beta/\alpha + c + d\omega$ for $c, d \in \mathbb{Z}$. This proves Proposition 3.3.

4. Complex structures on $S^1 \times \Sigma^3$

In this section we want to determine the complex structures on $S^1 \times \Sigma^3$, where Σ^3 is a rational or integral homology sphere.

First let X be a rational homology Hopf surface with $a(X) = 1$, hence of the type of Theorem 2.1. We maintain the notation of Section 2.

THEOREM 4.1. *Let*

$$X = L_{x_r}(m_r, \zeta_r) \cdots L_{x_1}(m_1, \zeta_1)(\mathbb{P}^1 \times E),$$

where $E = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\omega$, $\zeta_i = (1/m_i)(u_i + v_i\omega)$, $\Sigma'_{i=1}\zeta_i \neq 0$, $r > 2$.

(i) *If X is diffeomorphic to a product $S^1 \times \Sigma^3$, Σ^3 a rational (integral) homology sphere, then*

$$(R) \quad \gcd(\det A_{r+1}, \det A_{r+2}) = \gcd(\det A_1, \dots, \det A_{r+2})$$

$$(I) \quad \gcd(\det A_{r+1}, \det A_{r+2}) = 1.)$$

(ii) *If the condition (R) ((I)) is satisfied, then there exists a Seifert 3-manifold Σ' over $\mathbb{P}^1$ with Seifert invariants*

$$(m_1, kv_1 - lu_1), \dots, (m_r, kv_r - lu_r),$$

where $k, l \in \mathbb{Z}$ solve the equation

$$-k \det A_{r+1} + l \det A_{r+2} = \gcd(\det A_{r+1}, \det A_{r+2}), \quad (*)$$

such that X is diffeomorphic to $S^1 \times \Sigma'$. The manifold Σ' is a rational (integral) homology 3-sphere and its topological type is independent of the choice of k and l satisfying the equation (*).

Proof. (i) We first assume that X is diffeomorphic to $S^1 \times \Sigma^3$, Σ^3 a rational homology sphere. Then $\pi_1(X) = \mathbb{Z} \oplus \pi$, where $\pi = \pi_1(\Sigma^3)$. This means that the sequence

$$1 \rightarrow \pi \rightarrow \pi_1(X) \xrightarrow{\varphi} \mathbb{Z} \rightarrow 0$$

splits, i.e. that there exists $\psi: \mathbb{Z} \rightarrow \pi_1(X)$ with $\varphi \circ \psi = \text{id}_{\mathbb{Z}}$, and $\psi(1)$ lies in the center of $\pi_1(X)$. The center of $\pi_1(X)$ is generated by the classes δ and σ , because $G = \pi_1(X)/\langle \delta, \sigma \rangle$ has no center, provided that $r > 2$. This is seen as follows. The group G has a faithful representation as a discrete subgroup of the group of isometries of the sphere S^2 (if $\kappa(X) < 0$), the Euclidean plane E^2 (if $\kappa(X) = 0$), or of the hyperbolic plane H^2 (if $\kappa(X) > 0$). It follows from [S2, Lemma 1.10] that G has no center in the case when $\kappa(X) > 0$, and by similar arguments in the other cases (cf. also [Li, III, p.699f]).

Consider the exact sequence

$$0 \rightarrow \mathbb{Z}^{r+1} \xrightarrow{A^t} \mathbb{Z}^{r+2} \longrightarrow H_1(X) \longrightarrow 0.$$

The mapping $\varphi: \pi_1(X) \rightarrow \mathbb{Z}$ induces a mapping $\bar{\varphi}: H_1(X) \rightarrow \mathbb{Z}$. Since $\psi(1)$ is in the center of $\pi_1(X)$, there exist $k, l \in \mathbb{Z}$ with $\psi(1) = \sigma^k \delta^l$. Let $\rho: \pi_1(X) \rightarrow H_1(X)$ be the abelianization. Then $\rho(\psi(1)) = k\bar{\sigma} + l\bar{\delta} \in H_1(X)$, and hence

$$\bar{\varphi}(k\bar{\sigma} + l\bar{\delta}) = 1.$$

Let T denote the torsion subgroup of $H_1(X)$. Then $\bar{\varphi}: H_1(X) \rightarrow \mathbb{Z}$ splits the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\rho \circ \psi} H_1(X) \longrightarrow T \longrightarrow 0,$$

since $\bar{\varphi} \circ \rho \circ \psi = \varphi \circ \psi = \text{id}_{\mathbb{Z}}$. We can lift $\rho \circ \psi$ to a mapping $\widehat{\rho \circ \psi}: \mathbb{Z} \rightarrow \mathbb{Z}^{r+1}$ with

$$\widehat{\rho \circ \psi}(1) = (0, \dots, 0, k, l)^t.$$

With this mapping we obtain a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathbb{Z}^{r+1} & \xrightarrow{j} & \mathbb{Z}^{r+1} \oplus \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} \longrightarrow 0 \\
 & & \parallel & & \downarrow \Psi & \nearrow \widehat{\rho \circ \psi} & \downarrow \rho \circ \psi \\
 0 & \longrightarrow & \mathbb{Z}^{r+1} & \xrightarrow{A^t} & \mathbb{Z}^{r+2} & \longrightarrow & H_1(X) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & T & = & T \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Let B be the matrix A extended by the row $(0, \dots, 0, k, l)$; B is an $(r+2) \times (r+2)$ matrix. The mapping $\Psi: \mathbb{Z}^{r+1} \oplus \mathbb{Z} \rightarrow \mathbb{Z}^{r+2}$ is given by the matrix B^t . The condition $\bar{\varphi} \circ \rho \circ \psi = \text{id}_{\mathbb{Z}}$ just means: If $\varepsilon_1, \dots, \varepsilon_{r+1}$ are the non-trivial elementary divisors of the matrix A , then $1, \varepsilon_1, \dots, \varepsilon_{r+1}$ are the

elementary divisors of the matrix B . This is equivalent to

$$\det B = \pm \gcd(\det A_1, \dots, \det A_{r+2}).$$

Now we have

$$\det B = -k \det A_{r+1} + l \det A_{r+2}.$$

Therefore there exist $k, l \in \mathbb{Z}$ with $\pm \det B = d := \gcd(\det A_1, \dots, \det A_{r+2})$ if and only if

$$\gcd(\det A_{r+1}, \det A_{r+2}) = \gcd(\det A_1, \dots, \det A_{r+2}).$$

If Σ^3 is an integral homology 3-sphere, then X is an integral homology Hopf surface and it follows from Proposition 2.1 that

$$d = \gcd(\det A_{r+1}, \det A_{r+2}) = 1.$$

In order to prove (ii), assume that condition (R) is satisfied for X . Choose $k, l \in \mathbb{Z}$ with

$$-k \det A_{r+1} + l \det A_{r+2} = \pm d.$$

By replacing k, l by $-k, -l$ if necessary, we may assume that

$$-k \det A_{r+1} + l \det A_{r+2} = d.$$

Define

$$q := -\frac{\det A_{r+1}}{d}, \quad p := -\frac{\det A_{r+2}}{d}.$$

Then $kq - lp = 1$, and hence

$$\begin{pmatrix} k & l \\ p & q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The inverse matrix is

$$\begin{pmatrix} q & -l \\ -p & k \end{pmatrix}.$$

We set

$$\begin{aligned} z &:= \sigma^k \delta^l, \\ h &:= \sigma^p \delta^q. \end{aligned}$$

Then we have

$$\begin{aligned} \sigma &= z^q h^{-l}, \\ \delta &= z^{-p} h^k, \\ t_i^{m_i} \sigma^{u_i} \delta^{v_i} &= t_i^{m_i} z^{qu_i - pv_i} h^{kv_i - lu_i}, \quad i = 1, \dots, r. \end{aligned}$$

Since

$$qu_i - pv_i = -\frac{\det A_{r+1}}{d} u_i + \frac{\det A_{r+2}}{d} v_i = -\frac{1}{d} \det A_i \cdot m_i,$$

we get

$$z^{qu_i - pv_i} = (z^{-(\det A_i)/d})^{m_i}.$$

If we set $\tilde{t}_i := t_i z^{-(\det A_i)/d}$, then we obtain

$$t_i^{m_i} \sigma^{u_i} \delta^{v_i} = \tilde{t}_i^{m_i} h^{kv_i - lu_i}.$$

Therefore we get the following presentation of $\pi_1(X)$:

$$\pi_1(X) = \left\langle z, h, \tilde{t}_1, \dots, \tilde{t}_r \left| \begin{array}{l} z, h \text{ central, } \tilde{t}_1 \cdots \tilde{t}_r = 1, \\ \tilde{t}_i^{m_i} h^{kv_i - lu_i} = 1, \quad i = 1, \dots, r \end{array} \right. \right\rangle.$$

Here we used the fact that $\sum_{i=1}^r \det A_i = 0$. Let

$$\pi := \left\langle h, \tilde{t}_1, \dots, \tilde{t}_r \left| \begin{array}{l} h \text{ central, } \tilde{t}_1 \cdots \tilde{t}_r = 1, \\ \tilde{t}_i^{m_i} h^{kv_i - lu_i} = 1, \quad i = 1, \dots, r \end{array} \right. \right\rangle.$$

Then $\pi_1(X) = \langle z \rangle \times \pi$. The group π is the fundamental group of a Seifert 3-manifold Σ' over $\mathbb{P}^1$ with Seifert invariants

$$(m_1, kv_1 - lu_1), \dots, (m_r, kv_r - lu_r).$$

Changing the sign of k and l means reversing the orientation of this manifold.

Note that since $r > 2$, $\pi_1(\Sigma') \cong \pi$ is not finite cyclic. By Corollary 3.1, there is an elliptic surface X' diffeomorphic to $S^1 \times \Sigma'$. If X is a Hopf surface, then $\pi \cong \pi_1(\Sigma')$ is finite, so X' is a Hopf surface, too. Otherwise both X and X' are aspherical. By the results of Kato and Conner and Raymond mentioned at the end of Section 2, X is diffeomorphic to X' .

Note that Σ' is a rational homology 3-sphere. Since condition (I) implies condition (R), the same holds under the assumption (I). Moreover, if condition (I) is satisfied, then X is an integral homology Hopf surface and hence Σ' is an integral homology 3-sphere.

We finally show, using the classification of Seifert manifolds according to [NR, Theorem 1.1], that the topological type of the Seifert manifold Σ' is independent of the choice of k and l satisfying the equation (*).

For the Euler number of Σ^3 we have

$$e(\Sigma^3) = l \sum_{i=1}^r \frac{u_i}{m_i} - k \sum_{i=1}^r \frac{v_i}{m_i} = \frac{d}{m_1 \cdots m_r},$$

so the Euler number is independent of the choice of k and l .

Let k', l' also be integers satisfying

$$-k' \det A_{r+1} + l' \det A_{r+2} = d.$$

By [NR, Theorem 1.1] it remains to show that

$$k'v_i - l'u_i \equiv kv_i - lu_i \pmod{m_i} \quad \text{for } i = 1, \dots, r.$$

Now

$$k \det A_{r+1} - l \det A_{r+2} = k' \det A_{r+1} - l' \det A_{r+2}$$

implies

$$(k - k') \det A_{r+1} = (l - l') \det A_{r+2}.$$

Set $g := \gcd(k - k', l - l')$. Then we have

$$\frac{\det A_{r+1}}{d} = \frac{l - l'}{g}, \quad \frac{\det A_{r+2}}{d} = \frac{k - k'}{g}.$$

Since $d = \gcd(\det A_1, \dots, \det A_{r+2})$, d must also divide

$$\det A_i = \frac{u_i}{m_i} \det A_{r+1} - \frac{v_i}{m_i} \det A_{r+2},$$

hence

$$\frac{\det A_i}{d} = \frac{u_i}{m_i} \frac{\det A_{r+1}}{d} - \frac{v_i}{m_i} \frac{\det A_{r+2}}{d} \in \mathbb{Z}.$$

This implies

$$u_i \frac{\det A_{r+1}}{d} \equiv v_i \frac{\det A_{r+2}}{d} \pmod{m_i},$$

hence

$$u_i \frac{l - l'}{g} \equiv v_i \frac{k - k'}{g} \pmod{m_i},$$

so

$$u_i(l - l') \equiv v_i(k - k') \pmod{m_i}.$$

Since this is true for all $i = 1, \dots, r$, we see that the topological type of Σ' is independent of the choice of k and l . This finishes the proof of (ii) and hence the proof of Theorem 4.1.

REMARK 4.1. We have already remarked that if $r \leq 2$ then X is diffeomorphic to $S^1 \times L(\alpha, \beta)$, where $L(\alpha, \beta)$ is a lens space.

REMARK 4.2. Condition (I) implies that $m_1, \dots, m_r$ are pairwise relatively prime, because a common divisor of m_i and m_j , $i \neq j$, $1 \leq i, j \leq r$, would be a common divisor of $\det A_{r+1}$ and $\det A_{r+2}$. The converse is in general not true, as one can easily see.

COROLLARY 4.1. *Let X be as in Theorem 4.1. If X is homeomorphic to a product $S^1 \times \Sigma^3$, Σ^3 a rational homology sphere, then there exists a Seifert manifold Σ' over $\mathbb{P}^1$ with $\pi_1(\Sigma') \cong \pi_1(\Sigma^3)$.*

Proof. By Theorem 4.1(ii) there exists a Seifert manifold Σ' over $\mathbb{P}^1$ such that X is diffeomorphic to $S^1 \times \Sigma'$. Hence $\pi_1(X) \cong \mathbb{Z} \oplus \pi_1(\Sigma^3) \cong \mathbb{Z} \oplus \pi_1(\Sigma')$. Using the fact that X is a rational homology Hopf surface and hence $H_1(X, \mathbb{Q}) = \mathbb{Q}$, one can easily derive from this that $\pi_1(\Sigma^3) \cong \pi_1(\Sigma')$. This proves Corollary 4.1.

REMARK 4.3. Note that there remains a cancellation problem: If $S^1 \times \Sigma^3$ is diffeomorphic to $S^1 \times \Sigma'$, does this imply that Σ^3 is diffeomorphic to Σ' ? In the sequel we shall consider conditions under which we can conclude from $S^1 \times \Sigma^3$ being homeomorphic to $S^1 \times \Sigma'$ that Σ^3 is homeomorphic to Σ' .

We now classify complex structures on $S^1 \times \Sigma^3$. Recall that a 3-manifold M is *irreducible* if any embedded 2-sphere in M bounds a 3-ball.

THEOREM 4.2. *Let Σ^3 be a rational homology 3-sphere.*

(i) *If X is a compact complex surface homeomorphic to $S^1 \times \Sigma^3$, then X must be a Hopf surface or an elliptic surface.*

(ii) *If there exists a complex structure on $S^1 \times \Sigma^3$ and if Σ^3 is irreducible with infinite fundamental group, then Σ^3 must be Seifert fibred.*

(iii) *If Σ^3 is Seifert fibred, then $S^1 \times \Sigma^3$ admits complex structures. Suppose that Σ^3 is Seifert fibred with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ such that*

$$\chi_{\text{orb}}(\Sigma^3/S^1) = -2 + \sum_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right) \geq 0.$$

Then the complex structures on $S^1 \times \Sigma^3$ are precisely the surfaces

$$X = L_{x_r}(\alpha_r, \zeta_r) \cdots L_{x_1}(\alpha_1, \zeta_1)(\mathbb{P}^1 \times E)$$

where $\zeta_i = (1/\alpha_i)(u_i + v_i\omega)$ with

$$\gcd(\det A_{r+1}, \det A_{r+2}) = \gcd(\det A_1, \dots, \det A_{r+2}) = \pm \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{\beta_i}{\alpha_i},$$

such that there exist $k, l \in \mathbb{Z}$ and $\varepsilon \in \{0, 1\}$ with $kv_i - lu_i \equiv (-1)^\varepsilon \beta_i \pmod{\alpha_i}$ for all $i = 1, \dots, r$ and

$$k \sum_{i=1}^r \frac{v_i}{\alpha_i} - l \sum_{i=1}^r \frac{u_i}{\alpha_i} = (-1)^\varepsilon \sum_{i=1}^r \frac{\beta_i}{\alpha_i}.$$

(iv) *If Σ^3 is in addition an integral homology 3-sphere, then the complex structures on $S^1 \times \Sigma^3$ are precisely the surfaces*

$$X = L_{x_r}(\alpha_r, \zeta_r) \cdots L_{x_1}(\alpha_1, \zeta_1)(\mathbb{P}^1 \times E)$$

where $\zeta_i = (1/\alpha_i)(u_i + v_i\omega)$ with

$$\gcd\left(\alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{u_i}{\alpha_i}, \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{v_i}{\alpha_i}\right) = 1.$$

Proof. (i) Let Σ^3 be a rational homology 3-sphere, and let X be a compact complex surface homeomorphic to $S^1 \times \Sigma^3$. Then X is a rational homology Hopf surface with $\pi_1(X) = \mathbb{Z} \oplus \pi_1(\Sigma^3)$. Assume that X is neither a Hopf surface nor an elliptic surface. By the results of Section 1, X is then an Inoue

surface. We shall show that for such a surface it is impossible to have $\pi_1(X) = \mathbb{Z} \oplus \pi_1(\Sigma^3)$.

First assume that X is of type $S_M^\pm$. Then X is differentiably a 3-torus bundle over a circle [In1], so we have the following diagram for the fundamental group $\pi_1(X)$ with an exact horizontal and vertical sequence:

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & & \downarrow & & & & \\
 & & \pi_1(\Sigma^3) & & & & \\
 & \swarrow \psi & \downarrow & \searrow \varphi & & & \\
 0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \pi_1(X) & \longrightarrow & \mathbb{Z} \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & \mathbb{Z} & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

Since $H_1(\Sigma^3)$ is a torsion group, every element of $\pi_1(\Sigma^3)$ is mapped to zero by the composite homomorphism φ and there exists an injective homomorphism $\psi: \pi_1(\Sigma^3) \rightarrow \mathbb{Z}^3$. But this forces $\pi_1(\Sigma^3)$ to be trivial, and hence $\pi_1(X) = \mathbb{Z}$, which is impossible for an Inoue surface.

If X is of type $S_{N,p,q,r,t}^+$, then X is differentiably a fibre bundle over a circle, where the typical fibre F is a circle bundle over a 2-torus [In1]. So we have the following exact sequences:

$$1 \rightarrow \pi_1(F) \rightarrow \pi_1(X) \rightarrow \mathbb{Z} \rightarrow 0,$$

$$0 \rightarrow \mathbb{Z} \rightarrow \pi_1(F) \rightarrow \mathbb{Z}^2 \rightarrow 0.$$

By applying the above arguments to the first and then to the second sequence, we get the same contradiction.

Finally let X be of type $S_{N,p,q,r}^-$. Then X has a surface Y of the previous type as unramified double covering [In1], so that there exists an exact sequence

$$1 \rightarrow \pi_1(Y) \rightarrow \pi_1(X) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

Now $\pi_1(X) = \mathbb{Z} \oplus \pi_1(\Sigma^3)$ implies that the following sequence with some group T whose abelianization is a torsion group is exact:

$$1 \rightarrow T \rightarrow \pi_1(Y) \rightarrow \mathbb{Z} \rightarrow 0.$$

But we have already seen that this is impossible.

This shows that the fundamental group of an Inoue surface can never be a direct sum $\mathbb{Z} \oplus \pi_1(\Sigma^3)$, and hence an Inoue surface can never be homeomorphic to $S^1 \times \Sigma^3$. This proves part (i).

In order to prove (ii), assume that Σ^3 is irreducible with infinite fundamental group. Let X be again a compact complex surface homeomorphic to $S^1 \times \Sigma^3$. By (i), X must be a Hopf surface or an elliptic surface. By our assumptions, X must have algebraic dimension $a(X) = 1$. For otherwise, according to Section 1, X would be a Hopf surface with $\pi_1(X) = \mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z}$. This would imply that $\pi_1(\Sigma^3) = \mathbb{Z}/l\mathbb{Z}$, which contradicts our assumption that $\pi_1(\Sigma^3)$ is infinite. Therefore $a(X) = 1$ and X is an elliptic surface.

It follows from Corollary 4.1 that $\pi_1(\Sigma^3)$ is the fundamental group of a Seifert manifold Σ' . Since $\pi_1(\Sigma^3)$ is infinite, we derive from [W1, (10.2) and (7.1)] that Σ' is irreducible. P. Scott [S1], generalizing a result of F. Waldhausen [W2], has shown that if M is a closed, orientable, irreducible Seifert manifold with infinite fundamental group and N a closed, orientable, irreducible 3-manifold and if $\pi_1(M)$ is isomorphic to $\pi_1(N)$, then M is homeomorphic to N . This result implies that Σ^3 is homeomorphic to Σ' , hence Σ^3 is Seifert fibred. This proves (ii).

For the proof of (iii), let Σ^3 be a Seifert fibred rational homology 3-sphere. By Corollary 3.1, $S^1 \times \Sigma^3$ admits a complex structure. Suppose that Σ^3 is Seifert fibred with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ such that $\chi_{\text{orb}}(\Sigma^3/S^1) \geq 0$. Let X be any compact complex surface homeomorphic to $S^1 \times \Sigma^3$. Since $\chi_{\text{orb}}(\Sigma^3/S^1) \geq 0$, $r \geq 3$. Hence $\pi_1(\Sigma^3)$ cannot be cyclic, and therefore $a(X) = 1$. Hence X is an elliptic surface

$$X = L_{x_s}(m_s, \zeta_s) \cdots L_{x_1}(m_1, \zeta_1)(\mathbb{P}^1 \times E)$$

for certain positive integers m_i , $\zeta_i = (1/m_i)(u_i + v_i\omega) \in \mathbb{C}$, $\sum_{i=1}^s \zeta_i \neq 0$ and

$$\gcd(\det A_{s+1}, \det A_{s+2}) = \gcd(\det A_1, \dots, \det A_{s+2}).$$

By Corollary 4.1, $\pi_1(\Sigma^3)$ is the fundamental group of a Seifert manifold Σ' with Seifert invariants

$$(m_1, kv_1 - lu_1), \dots, (m_s, kv_s - lu_s)$$

where $k, l \in \mathbb{Z}$ solve the equation

$$-k \det A_{s+1} + l \det A_{s+2} = \gcd(\det A_{s+1}, \det A_{s+2}).$$

Now P. Orlik, E. Vogt, and H. Zieschang ([OVZ], see also [O, 5.3, Theorem 6]) have proved the following result: If M and M' are large Seifert manifolds with isomorphic fundamental groups, then M and M' are homeomorphic. Here large means that certain Seifert manifolds with small fundamental groups are excluded. In particular, if $\chi_{\text{orb}}(\Sigma^3/S^1) \geq 0$, then Σ^3 and Σ' are large (unless the Seifert invariants are $(2, 1)$, $(2, 1)$, $(2, -1)$, $(2, -1)$; but in this case the Seifert fibration over $\mathbb{P}^1$ is also unique up to homeomorphism). So we can apply this result and find that Σ^3 and Σ' are homeomorphic. Therefore $r = s$ and there exists $\varepsilon \in \{0, 1\}$ such that $m_i = \alpha_i$, $kv_i - lu_i \equiv (-1)^\varepsilon \beta_i \pmod{\alpha_i}$ for $i = 1, \dots, r$, and

$$e(\Sigma') = -k \sum_{i=1}^r \frac{v_i}{\alpha_i} + l \sum_{i=1}^r \frac{u_i}{\alpha_i} = -(-1)^\varepsilon \sum_{i=1}^r \frac{\beta_i}{\alpha_i} = (-1)^\varepsilon e(\Sigma^3).$$

Moreover, since

$$\det A_{r+1} = \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{v_i}{\alpha_i}, \quad \det A_{r+2} = \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{u_i}{\alpha_i},$$

we get

$$\gcd(\det A_{r+1}, \det A_{r+2}) = \gcd(\det A_1, \dots, \det A_{r+2}) = -(-1)^\varepsilon \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{\beta_i}{\alpha_i}.$$

On the other hand, if X is a surface as in claim (iii), then according to Theorem 4.1 there exists a Seifert 3-manifold Σ' over $\mathbb{P}^1$ with Seifert invariants

$$(\alpha_1, kv_1 - lu_1), \dots, (\alpha_r, kv_r - lu_r),$$

where $k, l \in \mathbb{Z}$ solve the equation

$$-k \det A_{r+1} + l \det A_{r+2} = \gcd(\det A_{r+1}, \det A_{r+2}),$$

such that X is diffeomorphic to $S^1 \times \Sigma'$. The topological type of Σ' is independent of the choice of $k, l \in \mathbb{Z}$ satisfying the above equation. By assumption $kv_i - lu_i \equiv (-1)^\varepsilon \beta_i \pmod{\alpha_i}$ for all $i = 1, \dots, r$ and some $\varepsilon \in \{0, 1\}$ and $e(\Sigma') = (-1)^\varepsilon e(\Sigma^3)$. According to [NR, Theorem 1.1] Σ' is therefore homeomorphic to Σ^3 . This proves (iii).

We finally show (iv). Let Σ^3 be a Seifert fibred integral homology sphere with Seifert invariants $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ and $\chi(\Sigma^3/S^1) \geq 0$. According to (iii), the complex structures on $S^1 \times \Sigma^3$ are precisely the surfaces

$$X = L_{x_r}(\alpha_r, \zeta_r) \cdots L_{x_1}(\alpha_1, \zeta_1)(\mathbb{P}^1 \times E)$$

with $\zeta_i = (1/\alpha_i)(u_i + v_i\omega)$, where u_i, v_i satisfy the following conditions:

$$(a) \quad \gcd(\det A_{r+1}, \det A_{r+2}) = \gcd(\det A_1, \dots, \det A_{r+2}) = \pm \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{\beta_i}{\alpha_i},$$

and there exist $k, l \in \mathbb{Z}$ and $\varepsilon \in \{0, 1\}$ with

$$(b) \quad kv_i - lu_i \equiv (-1)^\varepsilon \beta_i \pmod{\alpha_i}, \quad \text{for all } i = 1, \dots, r,$$

$$(c) \quad k \sum_{i=1}^r \frac{v_i}{\alpha_i} - l \sum_{i=1}^r \frac{u_i}{\alpha_i} = (-1)^\varepsilon \sum_{i=1}^r \frac{\beta_i}{\alpha_i}.$$

Since $\alpha_1 \cdots \alpha_r \sum_{i=1}^r \beta_i/\alpha_i = \pm 1$, (a) is equivalent to the condition stated in (iv). It follows from (a) that there exist $k, l \in \mathbb{Z}$ and $\varepsilon \in \{0, 1\}$ with

$$\alpha_1 \cdots \alpha_r \left(k \sum_{i=1}^r \frac{v_i}{\alpha_i} - l \sum_{i=1}^r \frac{u_i}{\alpha_i} \right) = (-1)^\varepsilon \alpha_1 \cdots \alpha_r \sum_{i=1}^r \frac{\beta_i}{\alpha_i}.$$

This implies (c). On the other hand, by reduction modulo α_i we get (b), since $\alpha_1, \dots, \alpha_r$ are pairwise relatively prime. This proves (iv) and finishes the proof of Theorem 4.2.

If Σ^3 is a Seifert fibred rational homology sphere then we can rephrase Theorem 4.2 as follows: Since $e(\Sigma^3) \neq 0$, the Seifert $\mathbb{C}^*$ -bundle $\eta: V_0 \rightarrow \mathbb{P}^1$ with the same Seifert invariants as Σ^3 defines a singularity. So there is always one complex structure coming from a singularity. But there may be more: the precise classification is given in Theorem 4.2.

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