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# Linear forms in $p$-adic logarithms III

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## 0. Introduction and results

**0.1.** The present paper is a continuation of the studies in Yu [Y1], [Y2] and [Y3]. ([Y3] will be referred as [*] in the sequel.) In [*], we have established the $p$-adic analogue of a celebrated theorem of Baker on linear forms in logarithms of algebraic numbers (i.e. Theorem 2 of [B2]) and the $p$-adic analogue of Baker's well-known Sharpening II (i.e. [B1])—except for the replacement of $p$ by $p^2$ in the case $d = 1$, we have established and strengthened the main assertions (Theorems 1, 3 and 4) given in [V].

In the present paper we refine upon our results given in [*] with respect to the dependence on $p$ and to the constants appearing in the inequalities given there; we also give, in Section 6, results of the type of Theorem 2 in [Y2], removing the Kummer condition imposed there. Thus all claims (Theorems 1, 2, 3, 4) of [V] are now established and strengthened.

Let $\alpha_1, \ldots, \alpha_n$ be $n$ ($\geqslant 2$) non-zero algebraic numbers and let $K$ be the field of degree $d$ generated by $\alpha_1, \ldots, \alpha_n$ over the rationals $\mathbb{Q}$. Denote by $p$ a prime number and by $\mathfrak{p}$ any prime ideal of the ring of integers in $K$, lying above $p$, by $e_{\mathfrak{p}}$ the ramification index of $\mathfrak{p}$, and by $f_{\mathfrak{p}}$ the residue class degree of $\mathfrak{p}$. We shall estimate

$$\Xi = \mathrm{ord}_{\mathfrak{p}}(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1),$$

where $b_1, \ldots, b_n$ are rational integers, not all zero, and $\mathrm{ord}_{\mathfrak{p}}$ denotes the exponent to which $\mathfrak{p}$ divides the principal fractional ideal generated by the expression (assumed non-zero) in parentheses. Our results will be in terms of real numbers $h_1, \ldots, h_n$ satisfying $h_1 \leqslant \cdots \leqslant h_n$ and

$$h_j \geqslant \max(h(\alpha_j), |\log \alpha_j|/(10d), \log p) \qquad (1 \leqslant j \leqslant n),$$

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where $h(\alpha)$ denotes the logarithmic absolute height of an algebraic number $\alpha$ (see, for example, Lang [L], Chapter IV) and $\log \alpha_j$ has its imaginary part in $(-\pi, \pi]$. Then as a simple consequence of our main result (see Section 0.2), we have

$$\Xi < \Phi \log(dB),$$

where $B = \max(|b_1|, \ldots, |b_n|, 3)$ and

$$\Phi = 22000(9.5(n+1)d/\sqrt{\log p})^{2(n+1)}(p^{f_\mathfrak{p}} - 1)h_1 \cdots h_n \log(10ndh')$$

with $h' = \max(h_n, 1)$. When $\operatorname{ord}_p b_n = \min \operatorname{ord}_p b_j$, $h'$ can be replaced by $\max(h_{n-1}, 1)$. Suppose that this condition on $\operatorname{ord}_p b_n$ is satisfied and $h'$ is modified accordingly, then for any $\delta$ with $0 < \delta \leqslant 1$, we have

$$\Xi < \max(\Phi \log(\delta^{-1}\Phi|b_n|/h_n), \delta B/|b_n|).$$

Observing $p^{f_\mathfrak{p}} - 1 \leqslant p^d - 1$ (since $f_\mathfrak{p} \leqslant d$), the factor $p^2$ (in the case $d = 1$) in [*] is now replaced by $p - 1$. (See also the remark after the statement of Theorem 2′ in Section 0.2.) Note that this refinement is relevant indeed in several applications. We obtain this refinement with respect to the dependence on $p$ by more careful utilizing the Vahlen-Capelli Theorem quoted in [*], p. 28.

We would like to mention that Dong Pingping [D] recently obtained good estimates for linear forms in $p$-adic logarithms, using an extension of Schneider's method to several variables. However, in his estimates, the algebraic numbers $\alpha_j$'s are subject to the condition that $\operatorname{ord}_\mathfrak{p} \alpha_j = 0$ and $\alpha_j \equiv 1 \pmod{\mathfrak{p}}$ (in our terminology), which becomes $\alpha_j \equiv 1 \pmod{p}$ when $\alpha_j$'s are rational integers. Note that in applications $\alpha_j$'s do not always satisfy such a condition; while our estimates are applicable to any non-zero algebraic $\alpha_j$'s with the only assumption that $\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \neq 0$.

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## 0.2 Detailed statements of the main results

For convenience, throughout this paper, we shall refer to formulae, theorems and so on from [*] by adjoining an asterisk, e.g. (0.1)*, Theorem 2.1*; formulae replacing those from [*] will then be denoted by the same number adjoined with o, so (0.3)° replaces (0.3)*. Let $\alpha_1, \ldots, \alpha_n$ be $n(\geqslant 2)$ non-zero algebraic numbers and

$$K_0 = \mathbb{Q}(\alpha_1, \ldots, \alpha_n),\ D_0 = [K_0 : \mathbb{Q}]. \tag{0.1*}$$

Let $p$ be a prime number. Set

$$q = \begin{cases} 2, & \text{if } p > 2, \\ 3, & \text{if } p = 2. \end{cases} \tag{0.2)*}$$

Let $K$ be an extension field of $K_0$ with degree $D$ over $\mathbb{Q}$. Denote by $\mathfrak{p}$ a prime ideal of the ring of integers in $K$, lying above $p$. For $\alpha \in K \setminus \{0\}$, write $\operatorname{ord}_{\mathfrak{p}} \alpha$ for the exponent of $\mathfrak{p}$ in the prime factorization of the fractional ideal $(\alpha)$; define $\operatorname{ord}_{\mathfrak{p}} 0 = \infty$. Denote by $e_{\mathfrak{p}}$ the ramification index of $\mathfrak{p}$ and by $f_{\mathfrak{p}}$ the residue class degree of $\mathfrak{p}$. We keep the notations $K_{\mathfrak{p}}$ (the completion of $\operatorname{ord}_{\mathfrak{p}}$ is denoted again by $\operatorname{ord}_{\mathfrak{p}}$), $\mathbb{C}_p$ and $\operatorname{ord}_p$ as introduced in [*], p. 17. Hence

$$\operatorname{ord}_{\mathfrak{p}} \beta = e_{\mathfrak{p}} \operatorname{ord}_p \beta \qquad \text{for all } \beta \in K_{\mathfrak{p}}.$$

Write $\zeta_m = e^{2\pi i/m}$ $(m = 1, 2, \ldots)$. We assume that the field $K \supseteq K_0$ satisfies

$$\begin{aligned} &\zeta_3 \in K, && \text{if } p = 2, \\ &\zeta_4 \in K, && \text{if } p \equiv 1 \pmod 4, \\ &\text{either } 2 \nmid f_{\mathfrak{p}} \text{ or } \zeta_4 \in K, && \text{if } p \equiv 3 \pmod 4. \end{aligned} \tag{0.3)°}$$

We shall see that the above amendment to (0.3)* makes the refinement with respect to the dependence on $p$ possible. Let $\mathbb{N}$ be the set of non-negative rational integers and define

$$u := \max\{t \in \mathbb{N} \mid \zeta_{q^t} \in K\}, \tag{0.4)*}$$

$$v := \max\{t \in \mathbb{N} \mid \zeta_{p^t} \in K\}, \tag{0.5)*}$$

$$\alpha_0 := e^{2\pi i/(p^v q^u)}. \tag{0.6)*}$$

Set $\mathbb{L}_K := \{l \in \mathbb{C} \mid e^l \in K\}$. For $l \in \mathbb{L}_K$ define

$$V(l) := \max\left\{h(e^l), \frac{|l|}{2\pi D}, \frac{f_{\mathfrak{p}} \log p}{D}\right\}, \tag{0.7)*}$$

where $h(\alpha)$ denotes the logarithmic absolute height of an algebraic number $\alpha$. Let $V_1, \ldots, V_n$ be real numbers satisfying

$$V_1 \leqslant \cdots \leqslant V_n \tag{0.8)*}$$

and

$$V_j \geqslant V(\log \alpha_j) \quad (1 \leqslant j \leqslant n), \tag{0.9)*}$$

where and in the sequel $\log\alpha_j = \log|\alpha_j| + i\arg\alpha_j$ with $-\pi < \arg\alpha_j \leqslant \pi$ $(1 \leqslant j \leqslant n)$.

Let $b_1, \ldots, b_n \in \mathbb{Z}$, not all zero, and let $B, B_1, \ldots, B_n$ be real numbers such that

$$B \geqslant \max(|b_1|, \ldots, |b_n|, 3), \ \max(1, |b_j|) \leqslant B_j \leqslant B \quad (1 \leqslant j \leqslant n). \tag{0.10°}$$

Set

$$V = \begin{cases} V_{n-1}, & \text{if } \operatorname{ord}_p b_n = \min\limits_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j \text{ or } \log\alpha_n \text{ is linearly dependent} \\ & \text{on } \pi i, \log\alpha_1, \ldots, \log\alpha_{n-1}, \log p \text{ over } \mathbb{Q}, \\ V_n, & \text{otherwise.} \end{cases} \tag{0.11°}$$

(If $\operatorname{ord}_p \alpha_j = 0$ $(1 \leqslant j \leqslant n)$, then "$\log\alpha_n$ is linearly dependent on $\pi i$, $\log\alpha_1, \ldots, \log\alpha_{n-1}, \log p$" is equivalent to "$\log\alpha_n$ is linearly dependent on $\pi i$, $\log\alpha_1, \ldots, \log\alpha_{n-1}$".)

Define

$$\sigma = 1/(p^v q^u f_p \log p), \tag{0.12°}$$

and for primes $p \geqslant 5$

$$\eta_1 = p(p-1)/(p-2)^2 \leqslant \frac{20}{9}, \quad \eta_2 = (p-1)/(p-2) \leqslant \frac{4}{3},$$

$$\eta_3 = p/(p-2) \leqslant \frac{5}{3}, \qquad \eta = \eta_1 \eta_2^n \leqslant \frac{20}{9} \cdot \left(\frac{4}{3}\right)^n. \tag{0.12}$$

THEOREM 1. *Suppose that*

$$\operatorname{ord}_p \alpha_j = 0 \quad (1 \leqslant j \leqslant n) \tag{0.13*}$$

*and*

$$\Theta := (\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) \neq 0. \tag{0.14*}$$

*Then*

$$\operatorname{ord}_p \Theta < C(n+1)^{n+2} n^{n+\sigma} \cdot \frac{p^{f_p} - 1}{q^u (f_p \log p)^{n+2}} \cdot D^{n+2} V_1 \cdots V_n \log(D^2 B)$$
$$\cdot \max(n \log(2^{10} \cdot 3n(n+\sigma) D^2 V), f_p \log p),$$

*where*

$$C = \begin{cases} 210406.38 \cdot 24^n, & \text{if } p = 2, \\ 134627.26 \cdot 20^n, & \text{if } p = 3, \text{ or if } p = 5,\ e_p \geqslant 2, \\ 399171.94 \cdot 20^n, & \text{if } p \geqslant 7,\ e_p \geqslant 2, \\ 67821.82\eta \cdot 10^n, & \text{if } p \geqslant 5,\ e_p = 1. \end{cases}$$

COROLLARY. *Suppose that* (0.14)* *holds and*

$$\alpha_j = p_j/q_j \text{ with } p_j,\ q_j \in \mathbb{Z}\backslash\{0\} \text{ and g.c.d. } (p_j, q_j) = 1 \quad (1 \leqslant j \leqslant n).$$

*Let* $A_1, \ldots, A_n$ *be real numbers such that* $A_1 \leqslant \cdots \leqslant A_n$ *and*

$$A_j \geqslant \max(|p_j|, |q_j|, p) \quad (1 \leqslant j \leqslant n).$$

*Set* $A = A_{n-1}$ *if* $\operatorname{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j$ *or* $\log \alpha_n$ *is linearly dependent on* $\pi i$, $\log \alpha_1, \ldots, \log \alpha_{n-1}, \log p$, *and set* $A = A_n$ *otherwise. Then we have*

$$\operatorname{ord}_p \Theta < C''(n+1)^{2n+4} \frac{p-1}{(\log p)^{n+2}} \log A_1 \cdots \log A_n \log(4B)$$
$$\cdot \max(\log(2^{12} \cdot 3n(n+1) \log A), (\log p)/n),$$

*where*

$$C'' = \begin{cases} 11145 \cdot 24^n, & \text{if } p = 2, \\ 7816 \cdot 20^n, & \text{if } p = 3, \\ 2863\eta \cdot 10^n, & \text{if } p \equiv 3 \pmod 4 \text{ and } p \geqslant 7, \\ 4861\eta \cdot 20^n, & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

THEOREM 2. *Suppose that* (0.13)* *and* (0.14)* *hold and*

$$\operatorname{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j. \tag{0.15*}$$

*Let*

$$\Phi = 1.046C(n+1)^{n+1} n^{n+\sigma} \cdot \frac{p^{f_p} - 1}{q^u (f_p \log p)^{n+2}} \cdot D^{n+2} V_1 \cdots V_n$$
$$\cdot \max(n \log(2^{10} \cdot 3n^2 D^2 V_{n-1}), f_p \log p), \tag{0.16°}$$

*where $C$ is given in Theorem* 1. *Let $Z_j = \omega\Phi/V_j$ with*

$$\omega = \begin{cases} \frac{15}{7}, & \text{if } f < n \text{ and } \pi i, \log\alpha_1, \ldots, \log\alpha_n, \log p \text{ are linearly independent over } \mathbb{Q}, \\ 1, & \text{otherwise,} \end{cases} \tag{0.17}^{\circ}$$

$$Q = p^3(10nD)^{2(n+1)}(DV_{n-1})^n. \tag{0.18}^{\circ}$$

*Then for any $j$ with $1 \leqslant j \leqslant n$ and any $\delta$ with $0 < \delta \leqslant Z_j f_{\mathfrak{p}}(\log p)/D$, we have*

$$\operatorname{ord}_{\mathfrak{p}} \Theta < \max(\omega\Phi \log(\delta^{-1} Z_j B_j Q), \delta B/B_j). \tag{0.19}^{\circ}$$

When $\alpha_1, \ldots, \alpha_n$ are non-zero rational numbers, hypothesis (0.13)* in Theorems 1 and 2 may be omitted, as we have seen from the statement of the Corollary to Theorem 1 (for more details, see [*], pp. 95–97.). In the general case, we can also do so. The following Theorems 1′ and 2′ are the version in terms of the additive valuation on $K_0 = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ and without assuming (0.13)*. Denote by $\mathfrak{p}_0$ any prime ideal of the ring of integers in $K_0$, lying above $p$. Let $\operatorname{ord}_{\mathfrak{p}_0}$ be defined with respect to the field $K_0$ and $e_0$, $f_0$ be the ramification index and residue class degree of $\mathfrak{p}_0$, respectively. Let $V_1, \ldots, V_n$ be real numbers satisfying $V_1 \leqslant \cdots \leqslant V_n$ and

$$V_j \geqslant \max\left(h(\alpha_j), |\log\alpha_j|/(10D_0), \tfrac{1}{2}(f_0/D_0)^2 \log p\right) \quad (1 \leqslant j \leqslant n), \tag{0.21}^{*}$$

and let $B, B_1, \ldots, B_n$ and $V$ be defined by $(0.10)^0$ and $(0.11)^{\circ}$.

THEOREM 1′. *Suppose that* (0.14)* *holds. Then*

$$\operatorname{ord}_{\mathfrak{p}_0} \Theta < C'(n+1)^{2n+4} \frac{p^{f_0} - 1}{(\log p)^{n+2}} \cdot (D_0/f_0)^{2n+2} V_1 \cdots V_n$$
$$\cdot \log(4D_0^2 B) \max(\log(2^{13} \cdot 3n(n+1)D_0^3 V), f_0(\log p)/n),$$

*where*

$$C' = \begin{cases} 14860 \cdot 96^n, & \text{if } p = 2, \\ 9649 \cdot 80^n, & \text{if } p = 3, \text{ or if } p = 5,\ e_0 \geqslant 2, \\ 26065 \cdot 80^n, & \text{if } p \geqslant 7,\ e_0 \geqslant 2, \\ 4861\eta \cdot 40^n, & \text{if } p \geqslant 5,\ e_0 = 1. \end{cases}$$

THEOREM 2′. *Suppose that* (0.14)* *and* (0.15)* *hold. Let*

$$\Phi = 1.046C'(n+1)^{2n+3}\frac{p^{f_0}-1}{(\log p)^{n+2}}\cdot(D_0/f_0)^{2n+2}V_1\cdots V_n$$
$$\cdot\max(\log(2^{13}\cdot 3n^2D_0^3V_{n-1}),\ f_0(\log p)/n),$$

*where $C'$ is given in Theorem* 1′. *Let* $Z_j = \dfrac{f_0}{2D_0}\cdot\dfrac{\omega\Phi}{V_j}$, *where $\omega$ is given by* (0.17)° *and*

$$Q = p^3(20nD_0)^{2(n+1)}(4D_0^2V_{n-1})^n.$$

*Then for any $j$ with $1 \leqslant j \leqslant n$ and any $\delta$ with $0 < \delta \leqslant Z_jf_0(\log p)/(2D_0)$, we have*

$$\mathrm{ord}_{\mathfrak{p}_0}\Theta < \max(\omega\Phi\ \log(\delta^{-1}Z_jB_jQ),\ \delta B/B_j).$$

REMARK. We emphasize that $f_0$ is the residue class degree of the prime ideal $\mathfrak{p}_0$ of the ring of integers in $K_0 = \mathbb{Q}(\alpha_1,\ldots,\alpha_n)$, while $f_0$ defined by (0.20)* is, in our notation,

$$\begin{array}{ll} f_0, & \text{if } p \equiv 1 \pmod 4, \\ \max(f_0,\ 2), & \text{otherwise.} \end{array}$$

Thus in current Theorems 1′ and 2′, the main dependence on $p$ is $p^{f_0}-1(\leqslant p^{D_0})$ in general; and it is $p-1$, replacing $p^2$ in Theorems 1′* and 2′* in the case when $p \equiv 3 \pmod 4$ and the residue class degree of $\mathfrak{p}_0$ is 1.

## 1. Preliminaries

We shall improve some lemmas and keep remaining ones in Section 1*.

LEMMA 1.1. *Let $\kappa \geqslant 0$ be the rational integer satisfying*

$$\phi(p^\kappa) \leqslant 2e_{\mathfrak{p}} < \phi(p^{\kappa+1}), \tag{1.1°}$$

*where $\phi$ is the Euler's $\phi$-function. If $\beta \in \mathbb{C}_p$ satisfies*

$$\mathrm{ord}_p(\beta-1) \geqslant 1/e_{\mathfrak{p}},$$

*then*

$$\operatorname{ord}_p(\beta^{p^\kappa}-1) \geqslant \frac{p^\kappa}{2e_p}+\frac{1}{p-1}.$$

*Proof.* By $\gamma^p = \sum_{l=0}^{p} \binom{p}{l}(\gamma-1)^l$, it is easy to verify that for integral $\gamma \in \mathbb{C}_p$ we have

$$\operatorname{ord}_p(\gamma^p-1) \geqslant \min(p \operatorname{ord}_p(\gamma-1), 1+\operatorname{ord}_p(\gamma-1)). \tag{1.2}$$

The lemma is obviously true if $\kappa = 0$. If $\kappa \geqslant 1$, then we obtain by inductive use of (1.2) that

$$\operatorname{ord}_p(\beta^{p^j}-1) \geqslant p^j/e_p, \qquad j = 0, 1, \ldots, \kappa-1. \tag{1.3}$$

[In the proof of (1.3), using (1.2), we may assume $\kappa \geqslant 2$, and note that for $j = 0, 1, \ldots, \kappa-2$ we have $p^{j+1}-p^j \leqslant p^{\kappa-2}(p-1) = \phi(p^\kappa)/p \leqslant 2e_p/p \leqslant e_p$.] On combining (1.3) for $j = \kappa-1$ with (1.2) we obtain, by (1.1)°,

$$\operatorname{ord}_p(\beta^{p^\kappa}-1) \geqslant \min\left(\frac{p^\kappa}{e_p}, 1+\frac{p^{\kappa-1}}{e_p}\right) \geqslant \frac{p^\kappa}{2e_p}+\frac{1}{p-1}.$$

The following lemma will replace Lemma 1.5*.

LEMMA 1.2. *Let $\beta_1, \ldots, \beta_r$ be in a number field $K$ of degree $D$. Suppose*

$$P_{ij} \in \mathbb{Z}[x_1, \ldots, x_r] \qquad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$$

*are such that for each $i = 1, \ldots, m$, $P_{i1}, \ldots, P_{in}$ are not all zero and*

$$\deg_{x_k} P_{ij} \leqslant N_{ik} \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,\ 1 \leqslant k \leqslant r).$$

*Write*

$$\gamma_{ij} = P_{ij}(\beta_1, \ldots, \beta_r) \qquad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$$

*and*

$$X = \left\{\prod_{i=1}^{m}\left(\sum_{j=1}^{n} L(P_{ij})^2\right)^{1/2}\right\}\cdot \exp\left(\sum_{k=1}^{r} h(\beta_k)\sum_{i=1}^{m} N_{ik}\right),$$

*where $L(P_{ij})$ denotes the length of $P_{ij}$, i.e., the sum of the absolute values of its coefficients. If $n > mD$, then there exist $y_1, \ldots, y_n \in \mathbb{Z}$ with*

$$0 < \max_{1 \leqslant j \leqslant n} |y_j| \leqslant X^{D/(n-mD)}$$

*such that*

$$\sum_{j=1}^{n} \gamma_{ij} y_j = 0 \qquad (1 \leqslant i \leqslant m).$$

*Proof.* Apply [B-V] Theorem 12 and the first four lines of the last paragraph of the text of [B-V], p. 31.

Define for $z \in \mathbb{C}$

$$\Delta(z; k) = (z+1)\cdots(z+k)/k! \quad (k \in \mathbb{Z}, k \geqslant 1) \quad \text{and} \quad \Delta(z; 0) = 1,$$

and for $l, m \in \mathbb{N}$

$$\Delta(z; k, l, m) = \frac{1}{m!} \left\{ \frac{d^m}{dy^m} (\Delta(y; k))^l \right\}_{y=z}.$$

For every positive integer $k$, let $v(k) = \text{l.c.m.}\{1, 2, \ldots, k\}$. The following lemma will replace (1.12)* in Lemma 1.6*.

LEMMA 1.3. *Let $q$ be a prime number, $J$ a positive integer, $x$ a rational number such that $q^J x$ is a positive integer. Then*

$$A := q^{Jkl + l\,\mathrm{ord}_q(k!)} (v(k))^m \Delta(x; k, l, m) \in \mathbb{Z}.$$

*Proof.* Note that $A \in \mathbb{Q}$. So it suffices to show $\mathrm{ord}_p A \geqslant 0$ for all primes $p$. We may assume $m \leqslant kl$. Thus

$$\Delta := \Delta(x; k, l, m) = (\Delta(x; k))^l \Sigma((x+j_1)\cdots(x+j_m))^{-1},$$

where the summation is over all selections $j_1, \ldots, j_m$ of $m$ integers from the set $1, \ldots, k$ repeated $l$ times. So

$$q^{Jkl}(k!)^l \Delta \in \mathbb{Z},$$

whence $\operatorname{ord}_q A \geqslant 0$. Further for any $p \neq q$, we have, (writing $y := q^J x$),

$$\begin{aligned}
&\operatorname{ord}_p\{q^{J(kl-m)}(k!)^l(\Delta(x;k))^l((x+j_1)\cdots(x+j_m))^{-1}\} \\
&\quad = l \operatorname{ord}_p\{(y+q^J\cdot 1)\cdots(y+q^J k)\} - \operatorname{ord}_p\{(y+q^J j_1)\cdots(y+q^J j_m)\} \\
&\quad \geqslant l \operatorname{ord}_p(k!) - m\left[\frac{\log k}{\log p}\right] \\
&\quad = \operatorname{ord}_p(k!)^l - \operatorname{ord}_p(v(k))^m.
\end{aligned}$$

Thus

$$\operatorname{ord}_p(q^{Jkl}(v(k))^m\Delta) \geqslant 0,$$

whence $\operatorname{ord}_p A \geqslant 0$. This proves the lemma.

## 2. Results subject to a Kummer condition

Let $p$ be a prime number, $K$ be an algebraic number field of degree $D$ over $\mathbb{Q}$. Denote by $\mathfrak{p}$ a prime ideal of the ring of integers in $K$, lying above $p$. Let $\operatorname{ord}_{\mathfrak{p}}$, $e_{\mathfrak{p}}$, $f_{\mathfrak{p}}$ be defined as in Section 0.2. In this section

**we suppose K satisfies (0.3)°, instead of (2.1)*.**

Let $q$, $u$, $v$, $\alpha_0$ be defined by (0.2)* and (0.4)*–(0.6)*. Let $\alpha_1, \ldots, \alpha_n \in K$ $(n \geqslant 2)$. Suppose $V_1, \ldots, V_n$, $V_{n-1}^*$ are real numbers such that

$$V_j \geqslant \max\left(h(\alpha_j), \frac{f_{\mathfrak{p}}\log p}{D}\right) \qquad (1 \leqslant j \leqslant n), \tag{2.8*}$$

$$V_1 \leqslant \cdots \leqslant V_{n-1}, \tag{2.9*}$$

$$V_{n-1}^* = \max(p^{f_{\mathfrak{p}}}, (2^{11}\cdot 3nD^2V_{n-1})^n). \tag{2.10°}$$

Let $b_1, \ldots, b_n \in \mathbb{Z}$, not all zero, $B$, $B'$, $B_n$, $B_0$, $W$, $W^*$ be positive numbers such that

$$B \geqslant \max_{1\leqslant j\leqslant n}|b_j|,\ B' \geqslant \max_{1\leqslant j<n}|b_j|,\ B_n \geqslant |b_n|,\ B_0 \geqslant \min_{1\leqslant j\leqslant n, b_j\neq 0}|b_j|, \tag{2.11*}$$

$$W \geqslant \max\left\{\log\left(1+\frac{n-1}{\rho n^2}\cdot\frac{f_{\mathfrak{p}}\log p}{D}\left(\frac{B_n}{V_1}+\frac{B'}{V_n}\right)\right),\ \rho''\log B_0,\ 2.95\log p\right\}, \tag{2.12°}$$

where

$$\rho = \begin{cases} \frac{16}{3}, & \text{if } p = 2, \\ 5, & \text{if } p > 2 \end{cases} \quad \text{and} \quad \rho'' = \begin{cases} 1, & \text{if } p \mid b_n, \\ 0, & \text{otherwise,} \end{cases}$$

$$W^* \geqslant \max(W,\ n \log(2^{11} \cdot 3nD)). \tag{2.13}^\circ$$

In this section we shall prove the following Theorems and Corollaries.

THEOREM 2.1. *Suppose that*

$$\mathbb{Q}(\alpha_0, \alpha_1, \ldots, \alpha_n) = K, \tag{2.14}^*$$

$$[K(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] = q^{n+1}, \tag{2.15}^*$$

$$\operatorname{ord}_p \alpha_j = 0 \quad (1 \leqslant j \leqslant n), \tag{2.16}^*$$

$$\operatorname{ord}_p b_n = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j \tag{2.17}^*$$

*and*

$$\Theta := (\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) \neq 0, \tag{2.18}^*$$

*then*

$$\operatorname{ord}_p \Theta < ca^n \cdot \frac{(n+1)^{n+1} n^n}{n!} \cdot \frac{p^{f_p} - 1}{q^u (f_p \log p)^{n+2}} \cdot D^{n+2} V_1 \cdots V_n W^* \log V_{n-1}^*,$$

*where c and a are given in the following table, where $\eta_1$ and $\eta_2$ are defined by* (0.12),

$p = 2$

| $n$ | 2, 3, 4 | 5, 6, 7 | $\geqslant 8$ |
|---|---|---|---|
| $c$ | 12563.565 | 12563.565 | 105203.19 |
| $a$ | 64.8 | 64.8 | 48 |

$p = 3$; or $p = 5$, $e_p \geqslant 2$

| | | | |
|---|---|---|---|
| $c$ | 11109.6114 | 16429.477 | 67313.63 |
| $a$ | 56 | 48 | 40 |

$p \geqslant 7$, $e_p \geqslant 2$

| | | | |
|---|---|---|---|
| $c$ | 30260.584 | 48067.457 | 199585.97 |
| $a$ | 56 | 48 | 40 |

$p \geqslant 5,\ e_p = 1$

| $c/\eta_1$ | 5472.1013 | 8250.8311 | 33910.91 |
|---|---|---|---|
| $a/\eta_2$ | 28 | 24 | 20 |

REMARK. Denoting by $c(n)$ and $a(n)$ for the values of $c$ and $a$ for $n \geqslant 2$ in the above table, we see that

$$c(n)(a(n))^n \leqslant c(8)(a(8))^n, \quad n = 2, 3, \ldots, 7.$$

THEOREM 2.2. *In Theorem* 2.1, (2.14)* *may be omitted.*

COROLLARY 2.3. *Suppose that* (2.15)*–(2.18)* *hold. Then*

$$\operatorname{ord}_p \Theta < ca^n \cdot \frac{(n+1)^{n+1} n^n}{n!} \cdot \frac{p^{f_p} - 1}{q^u (f_p \log p)^{n+2}} \cdot D^{n+2} V_1 \cdots V_n \cdot$$
$$\cdot \max(\log B, n \log(2^{11} \cdot 3nD), 2.95 \log p) \max(n \log(2^{11} \cdot 3nD^2 V_{n-1}), f_p \log p),$$

*where c and a are given in Theorem* 2.1.

COROLLARY 2.4. *Let $Z'$, $Z$, $\delta$, $W'$ be positive numbers satisfying*

$$Z' \geqslant ca^n \cdot \frac{(n+1)^{n+1} n^n}{n!} \cdot \frac{p^{f_p} - 1}{q^u (f_p \log p)^{n+2}} \cdot D^{n+2} V_1 \cdots V_{n-1} \cdot$$
$$\cdot \max(n \log(2^{11} \cdot 3nD^2 V_{n-1}), f_p \log p),$$
$$0 < \delta \leqslant Z f_p (\log p)/D$$
$$W' \geqslant \max\left\{\rho' \log\left(\delta^{-1} \frac{f_p \log p}{D} Z B_n\right), n \log(2^{11} \cdot 3nD), 2.95 \log p\right\},$$

*where c, a are given in Theorem* 2.1 *and*

$$\rho' = \begin{cases} 1.0561, & \text{if } n = 2, \\ 1.046, & \text{if } n \geqslant 3. \end{cases}$$

*Suppose that* (2.15)*–(2.18)* *hold. Then*

$$\operatorname{ord}_p \Theta < \max\left(Z' W' V_n, \frac{\delta B}{B_n} \cdot \frac{Z'}{Z}\right).$$

We recall (2.19)*–(2.23)*, (2.25)* and [*] p. 34, 1.1–1.13 (incidentally, we

correct a misprint in 1.1:$\mathbb{C}$ should be replaced by $\mathbb{C}_p$), where (2.24)* with $\theta$ given by

$$\left(1+\frac{1}{2n}\cdot 10^{-100}\right)\theta=\begin{cases}(p-2)/(p-1), & \text{if } p\geqslant 5 \text{ and } e_{\mathfrak{p}}=1,\\ p^{\kappa}/(2e_{\mathfrak{p}}), & \text{otherwise } (\kappa \text{ is given in (1.1)}^{\circ})\end{cases} \tag{2.24}$$

(cf. (1.2)*), follows from current Lemma 1.1.

## 2.1. The statement of a proposition towards the proof of Theorem 2.1

We define $h_0,\ldots,h_7$, $\varepsilon_1$, $\varepsilon_2$ by the following formulae, which will be referred as (2.26)°.

$$h_0=n\log(2^{11}\cdot 3nD),$$

$$h_1=\begin{cases}c_0c_2^nc_4\cdot\dfrac{(n+1)^{n+1}n^n}{n!}\cdot q^n(q-1), & \text{if } p\geqslant 5 \text{ and } e_{\mathfrak{p}}=1,\\ \dfrac{p-1}{p}\cdot c_0(2c_2)^nc_4\cdot\dfrac{(n+1)^{n+1}n^n}{n!}\cdot q^n(q-1)f_{\mathfrak{p}}, & \text{otherwise,}\end{cases}$$

$$h_2=\begin{cases}h_1(c_2n(n+1)q)^{-1}, & \text{if } p\geqslant 5 \text{ and } e_{\mathfrak{p}}=1,\\ h_1(2c_2n(n+1)q)^{-1}, & \text{otherwise,}\end{cases}\qquad 1+\varepsilon_1=(1-1/h_2)^{-n},$$

$$h_3=(h_1-1)/n^2,\quad 1+\varepsilon_2=e^{1/h_3},$$

$$h_4=\begin{cases}c_0c_2^nc_3\cdot\dfrac{(n+1)^{n+1}n^n}{n!}\cdot q^{n-1}(q-1)^2\cdot\dfrac{h_0}{h_0+1}, & \text{if } p\geqslant 5 \text{ and } e_{\mathfrak{p}}=1,\\ c_0(2c_2)^nc_3\cdot\dfrac{(n+1)^{n+1}n^n}{n!}\cdot q^{n-1}(q-1)^2\cdot\dfrac{h_0}{h_0+1}, & \text{otherwise,}\end{cases}$$

$$h_5=c_0c_1c_2^nc_3c_4\cdot\frac{(n+1)^{n+1}n^{n-1}}{n!}\cdot q^{n-1}(q-1)^2\left(\frac{p^{\kappa}}{e_{\mathfrak{p}}\theta}\right)^n h_0,$$

$$h_6^{-1}=(n+2)\frac{\log(nh_0h_5)}{nh_0h_5},$$

$$h_7=c_2n(q-1)\left(1-\frac{1}{2.95c_3e_{\mathfrak{p}}(n+1)}\right)\frac{p^{\kappa}}{e_{\mathfrak{p}}\theta}. \tag{2.26°}$$

In the above formulae and the remaining part of Section 2, $c_0$, $c_1$, $c_2$, $c_3$, $c_4$ are positive numbers given by the following table, denoted by (2.27)° ($\eta_2$ and $\eta_3$ are given in (0.12)). The upper bounds for $(1+\varepsilon_1)(1+\varepsilon_2)$ can be obtained from the above formulae by direct calculations. Blocks I, II, III and IV are for

cases (I) $p = 2$, (II) $p = 3$; or $p = 5$, $e_p \geqslant 2$, (III) $p \geqslant 7$, $e_p \geqslant 2$ and (IV) $p \geqslant 5$, $e_p = 1$, respectively.

Table (2.27)°

| | $n$ | $c_0$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $(1+\varepsilon_1)(1+\varepsilon_2)\leqslant$ |
|---|---|---|---|---|---|---|---|
| I | $2\leqslant n\leqslant 7$ | 5.2987812 | 1.1513473 | $\frac{18}{5}$ | 8.5427482 | 40.176885 | $1 + 1.25 \times 10^{-5}$ |
| | $\geqslant 8$ | 10.758978 | 1.1241612 | $\frac{8}{3}$ | 16.983491 | 85.359335 | $1 + 8 \times 10^{-21}$ |
| II | 2, 3, 4 | 4.5265245 | 0.7784861 | 7 | 23.383422 | 67.410556 | $1+3.95\times 10^{-5}$ |
| | 5, 6, 7 | 5.3803138 | 0.7625948 | 6 | 28.108368 | 71.22897 | $1+2\times 10^{-12}$ |
| | $\geqslant 8$ | 8.3578862 | 0.7593864 | 5 | 44.81934 | 118.31731 | $1+5\times 10^{-21}$ |
| III | 2, 3, 4 | 4.5611992 | 0.7783945 | 7 | 68.048932 | 62.622861 | $1+3.29\times 10^{-5}$ |
| | 5, 6, 7 | 5.3916178 | 0.7625734 | 6 | 83.060633 | 70.376094 | $1 + 2\times 10^{-12}$ |
| | $\geqslant 8$ | 8.3650025 | 0.7593798 | 5 | 133.04796 | 118.07737 | $1+4\times 10^{-21}$ |
| IV | 2, 3, 4 | 4.5430956 | $0.2695256\eta_3$ | 7 | $34.668864\eta_2$ | 64.447861 | $1+5.551 \times 10^{-5}$ |
| | 5, 6, 7 | 5.3850892 | $0.2584152\eta_3$ | 6 | $41.920113\eta_2$ | 70.718705 | $1+1.6 \times 10^{-11}$ |
| | $\geqslant 8$ | 8.3607945 | $0.2562571\eta_3$ | 5 | $66.957173\eta_2$ | 118.19221 | $1+3.7 \times 10^{-19}$ |

It is not difficult to verify that $c_0$, $c_1$, $c_2$, $c_3$, $c_4$ given in (2.27)° satisfy the following two inequalities

$$\begin{aligned}
&\left(1-\frac{1}{2.95c_3e_p(n+1)}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2\\
&\geqslant\left\{\left(1+\frac{1}{c_0-1}\right)\frac{1}{h_5}+\left(1+\frac{1}{2}\cdot\frac{1}{c_0-1}\right)\frac{1}{h_6}\right\}c_1+\left(1+\frac{1}{2}\cdot\frac{1}{c_0-1}\right)\frac{1}{c_2}\\
&\quad+\left\{\frac{1}{q}\left[\log 3\cdot\left(1+\frac{1}{h_0}\right)+1\right]+\frac{1}{c_0-1}\left[1+\frac{1}{h_0}+\frac{1}{n+1}\left(\log 3+\frac{\log(3/e)}{h_0}\right)\right]\right\}\\
&\quad\times\frac{1}{e_p\theta}\cdot\frac{1}{c_3}+\left(1+\frac{1}{h_4}\right)\\
&\quad\times\left\{\left(1+\frac{1}{c_0-1}\right)\frac{1}{n}+2+\frac{\log q}{(q-1)h_0}+\left(\theta+\frac{1}{p-1}\right)\frac{e_p}{q^{n+1}D}\right\}\frac{1}{c_4},
\end{aligned}\tag{2.28°}$$

$$c_1 \geqslant \left(1+\frac{1}{h_7}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q}+\frac{1}{\theta(p-1)}\right)+\frac{1}{e_p\theta q^{n+1}c_3}$$
$$\times\left\{\frac{\log(h_0+1)}{h_0}+\frac{1}{n+1}\left(1-\frac{1}{q}\right)\left(1+\frac{1}{h_7}+\frac{1}{n}+\frac{(n-2)\log q}{h_0}+\frac{\log h_0}{h_0}\right)\right\}. \tag{2.29°}$$

REMARK. The verification is similar to that in Section 2.6*. We also need the following observations.

1. If $p \geqslant 5$ and $e_p = 1$, we apply (2.24) and have

$$\left(1-\frac{1}{q}\right)\left(1-\frac{1}{q}+\frac{1}{(p-1)\theta}\right) \leqslant \frac{1}{4}\cdot\frac{p}{p-2}(1+10^{-100}).$$

Otherwise, we have, by (2.24),

$$\theta < \frac{p^\kappa}{2e_p} \leqslant \frac{p}{p-1}, \qquad \frac{1}{e_p\theta} < \frac{2}{p^\kappa}(1+10^{-100}) \quad \text{and}$$
$$\frac{1}{(p-1)\theta} = \left(1+\frac{1}{2n}\cdot 10^{-100}\right)\cdot\frac{2e_p}{p^\kappa(p-1)} < 1+10^{-100}.$$

2. In the case $p = 2$, we have $f_p \geqslant 2$, by (0.3)° and Lemma 1.12*.

3. When $p \geqslant 7$ and $e_p \geqslant 2$, we have $q = 2$ and $D \geqslant 2$, and on writing the term containing $1/c_3$ in inequality (2.28)° as $A/(e_p\theta c_3)$, we have

$$\frac{A}{e_p\theta}\cdot\frac{1}{c_3}+\left(1+\frac{1}{h_4}\right)\frac{e_p\theta}{q^{n+1}D}\cdot\frac{1}{c_4} < (1+10^{-100})\frac{2A}{p^\kappa}\cdot\frac{1}{c_3}+\left(1+\frac{1}{h_4}\right)\frac{p^\kappa}{2^{n+2}D}\cdot\frac{1}{c_4}$$
$$\leqslant (1+10^{-100})2A\cdot\frac{1}{c_3}+\left(1+\frac{1}{h_4}\right)\frac{1}{2^{n+2}D}\cdot\frac{1}{c_4},$$

where the last inequality follows from the fact that

$$\left(1+\frac{1}{h_4}\right)\cdot\frac{7}{6}\cdot\frac{1}{2^{n+1}}\cdot\frac{1}{c_4} \leqslant \frac{2A}{c_3},$$

which can be verified directly.

Set

$$U = (1+\varepsilon_1)(1+\varepsilon_2)c_0c_1c_2^nc_3c_4\cdot\frac{(n+1)^{n+1}n^n}{n!}\cdot q^{2n+1}(q-1)\frac{p^{f_p}-1}{q^u}$$
$$\cdot\frac{(p^\kappa/(e_p\theta))^n}{e_p(f_p\log p)^{n+2}}\cdot D^{n+2}V_1\cdots V_nW^*\log V_{n-1}^*. \tag{2.30°}$$

PROPOSITION 2.1. *Suppose that* (2.14)*–(2.18)* *hold. Then*

$$\operatorname{ord}_p \Theta < U.$$

## 2.2. Notations

The following formulae will be referred as (2.31)°.

$$Y = \frac{e_p f_p \log p}{q^{n+1} D} \cdot U \ (U \text{ is given in } (2.30)^\circ), \qquad S = q\left[\frac{c_3(n+1)DW^*}{f_p \log p}\right],$$

$$T = \left[\frac{Y}{c_1 c_3 e_p \theta W^*}\right], \qquad L_{-1} = [W^*],$$

$$L_0 = \left[\frac{Y}{c_1 c_4 (L_{-1}+1)\log V^*_{n-1}}\right], \qquad L_j = \left[\frac{Y}{c_1 c_2 n p^\kappa S V_j}\right] (1 \leqslant j \leqslant n),$$

$$X_0 = \left\{D \prod_{j=-1}^{n} (L_j+1)\right\}^{1/2} (3^{(L_{-1}+1)} e^{-1})^{T/(n+1)} e^T \left(1 + \frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T$$
$$\cdot \left(e\left(2 + \frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)} \cdot \exp\left(\frac{1}{2} p^\kappa S \sum_{j=1}^{n} L_j V_j + nD \max_{1 \leqslant j \leqslant n} V_j\right). \tag{2.31°}$$

For later convenience we need the following inequalities (2.32)°–(2.47)°, where $(2.j)^\circ = (2.j)^*$ for $j = 32, 34, 38, 46, 47$, which can be easily verified in the current setting.

$$\frac{1}{n+1} q^n S T \theta > \left(1 - \frac{1}{2.95 c_3 e_p (n+1)}\right)\left(1 - \frac{1}{h_1}\right) \frac{1}{c_1} U, \tag{2.33°}$$

$$T(L_{-1}+1) \leqslant \left(1 + \frac{1}{h_0}\right) \frac{1}{e_p \theta} \cdot \frac{1}{c_1 c_3} Y, \tag{2.35°}$$

$$T \log\left(1 + \frac{(n-1)q(B_n L_1 + B' L_n)}{T}\right) \leqslant \frac{1}{e_p \theta} \cdot \frac{1}{c_1 c_3} Y, \tag{2.36°}$$

$$(L_{-1}+1)(L_0+1)\left(\theta + \frac{1}{p-1}\right) \leqslant \left(1 + \frac{1}{h_4}\right)\left(\theta + \frac{1}{p-1}\right) \frac{e_p}{q^{n+1} D} \cdot \frac{1}{c_1 c_4} U, \tag{2.37°}$$

$$(L_{-1}+1)(L_0+1)\log(qL_n) \leqslant 2\left(1 + \frac{1}{h_4}\right) \frac{1}{c_1 c_4} Y, \tag{2.39°}$$

$$nD \max_{1 \leqslant j \leqslant n} V_j \leqslant \frac{1}{h_5} Y, \tag{2.40°}$$

$$\log\left(D \prod_{j=-1}^{n} (L_j + 1)\right) \leqslant \frac{1}{h_6} Y, \tag{2.41°}$$

$$\frac{T \log(L_{-1} + 1)}{\log p} \leqslant \frac{\log(h_0 + 1)}{h_0} \cdot \frac{1}{e_p \theta} \cdot \frac{1}{q^{n+1}} \cdot \frac{1}{c_1 c_3} U. \tag{2.42°}$$

In (2.43)°–(2.45)°, $J$, $k$ are rational integers with $0 \leqslant J \leqslant [\log L_n / \log q]$, $0 \leqslant k \leqslant n$.

$$\left(\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T + 1\right) \operatorname{ord}_p b_n < \left(1 + \frac{1}{h_7}\right)\left(1 - \frac{1}{q}\right) \frac{1}{(n+1) e_p \theta q^{n+1}} \cdot \frac{1}{c_1 c_3} U, \tag{2.43°}$$

$$\left(\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T + 1\right) q^{J+k} S \left(\left(1 - \frac{1}{q}\right) \theta + \frac{1}{p-1}\right)$$
$$< \left(1 + \frac{1}{h_7}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{q} + \frac{1}{\theta(p-1)}\right) \frac{1}{c_1} U, \tag{2.44°}$$

$$\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T \cdot \frac{\log(q^{J+k} S)}{\log p}$$
$$< \left(1 - \frac{1}{q}\right)\left(\frac{1}{n} + \frac{(n-2)\log q}{h_0} + \frac{\log h_0}{h_0}\right) \frac{1}{(n+1) q^{n+1} e_p \theta} \cdot \frac{1}{c_1 c_3} U. \tag{2.45°}$$

Among the above inequalities, we prove $(2.j)°$ for $j = 36, 39, 41, 43, 45$. The remaining ones can be obtained in a way similar to that in [*], using (2.24) instead of (1.3)* and (1.4)*, and using $W \geqslant 2.95 \log p$ instead of $DW \geqslant f_p \log p$ in [*].

*Proof of* (2.36)°. It suffices to prove that

$$\log\left(1 + \frac{(n-1) q (B_n L_1 + B' L_n)}{T}\right) \leqslant \log\left(1 + \frac{n-1}{\rho n^2} \cdot \frac{f_p \log p}{D} \left(\frac{B_n}{V_1} + \frac{B'}{V_n}\right)\right) \leqslant W^*. \tag{2.44}$$

On noting that $h_1 > 2n + 1$ (see (2.26)°) and $c_3 > 8$ we get from (2.33)° that

$$ST > \frac{U}{c_1 q^n \theta}\left(n + 1 - \frac{1}{2}\right)\left(1 - \frac{1}{2n+1}\right) = \frac{nU}{c_1 q^n \theta}.$$

Thus, by (2.31)°, we have for $1 \leqslant j \leqslant n$

$$\frac{L_j}{T} \leqslant \frac{Y}{c_1 c_2 n p^\kappa V_j S T} \leqslant \frac{1}{c_2 q n^2} \cdot \frac{e_p \theta}{p^k} \cdot \frac{f_p \log p}{D V_j}. \tag{2.45}$$

Now (2.44) follows from (2.45), (2.24) and the fact that $c_2 \geqslant \frac{8}{3}$ if $p = 2$ and $c_2 \geqslant 5$ if $p > 2$.

*Proof of* (2.39)°. By (2.31)° and $n^n/n! < e^n/\sqrt{2\pi n}$, we have

$$qL_n \leqslant h_8 e^{n+1} \left(\frac{p^\kappa}{e_p \theta}\right)^n (f_p \log p)^{-n} (c_2 q n)^{n-1} D^n V_1 \cdots V_{n-1} (p^{f_p} - 1) \log V_{n-1}^*, \tag{2.46}$$

where

$$h_8 = \frac{(1+\varepsilon_1)(1+\varepsilon_2) c_0 c_4 q(q-1)}{\sqrt{2\pi n}\, p^\kappa q^u \{1 - (2.95 c_3 e_p (n+1))^{-1}\}} \leqslant 304. \tag{2.47}$$

The inequality in (2.47) can be seen by noting that $n \geqslant 2$, $(1+\varepsilon_1)(1+\varepsilon_2) \leqslant 1 + 10^{-4}$ and

if $p = 2$, then $q = 3$, $c_0 < 11$, $c_4 < 86$, $c_3 > 8$, $\kappa \geqslant 2$, $u \geqslant 1$,

if $p > 2$, then $q = 2$, $c_0 < 9$, $c_4 < 119$, $c_3 > 23$, $\kappa \geqslant 0$, $u \geqslant 1$.

Further, using (2.24) and the fact that $f_p \log p \geqslant \log 3$ if $p > 2$ and $f_p \log p \geqslant \log 4$ if $p = 2$, we get

$$\left(e^{n+1} \left(\frac{p^\kappa}{e_p \theta}\right)^n (f_p \log p)^{-n}\right)^{1/(n-1)}$$
$$\leqslant e^{(n+1)/(n-1)} \left(\frac{2}{f_p \log p}\right)^{n/(n-1)} (1 + 10^{-100})^{1/(n-1)} \leqslant 70.$$

Thus, by $c_2 \leqslant 7$ if $p > 2$ and $c_2 \leqslant \frac{18}{5}$ if $p = 2$, we get

$$e^{n+1} \left(\frac{p^\kappa}{e_p \theta}\right)^n (f_p \log p)^{-n} (c_2 q n)^{n-1} D^n V_1 \cdots V_{n-1} \leqslant 6^{1-n} (V_{n-1}^*)^{(n-1)/n}. \tag{2.48}$$

We now prove

$$304 \cdot 6^{1-n} \log V_{n-1}^* \leqslant (V_{n-1}^*)^{1/n}. \tag{2.49}$$

For fixed $n \geqslant 2$, let $x_0 = (2^{12}n)^n$ and

$$f(x) = 6^n x^{1/n} - 1824 \log x.$$

Then $f(x_0) > 0$ and $f'(x) > 0$ for $x \geqslant x_0$, whence $f(x) > 0$ for $x \geqslant x_0$. This inequality and the fact that $V^*_{n-1} \geqslant (2^{11} \cdot 3nD^2 V_{n-1})^n > x_0$ yield (2.49).

On combining (2.46)–(2.49) and noting that $p^{f_p} - 1 \leqslant V^*_{n-1}$, we obtain $qL_n \leqslant (V^*_{n-1})^2$. Now (2.39)° follows from this inequality and (2.31)°.

*Proof of* (2.41)°. From (2.31)° and (2.26)°, we see that

$$Y \geqslant nh_0h_5 > 4, \qquad D \leqslant Y,$$

$$(L_{-1} + 1)(L_0 + 1) \leqslant \left(1 + \frac{1}{h_4}\right) \frac{Y}{c_1c_4h_0} \leqslant Y,$$

$$L_j + 1 \leqslant \frac{2Y}{c_1c_2np^\kappa SV_j} \leqslant Y (1 \leqslant j \leqslant n).$$

Thus $D \prod_{j=-1}^{n} (L_j + 1) \leqslant Y^{n+2}$, and

$$Y^{-1} \log \left(D \prod_{j=-1}^{n} (L_j + 1)\right) \leqslant (n+2) \frac{\log Y}{Y} \leqslant (n+2) \frac{\log(nh_0h_5)}{nh_0h_5} = h_6^{-1}.$$

*Proof of* (2.43)°. By (2.31)°, we have

$$\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T + 1 \leqslant \left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} \frac{Y}{c_1c_3e_p\theta W^*} + 1.$$

Now

$$\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} \frac{Y}{c_1c_3e_p\theta W^*} \geqslant \left(1 - \frac{1}{q}\right) \frac{1}{n+1} \frac{Y}{c_1c_3e_p\theta W^* L_n}$$

$$\geqslant \left(1 - \frac{1}{q}\right) \frac{1}{n+1} \frac{c_2np^\kappa SV_n}{c_3e_p\theta W^*} \geqslant c_2n(q-1)\left(1 - \frac{1}{2.95c_3e_p(n+1)}\right) \frac{p^\kappa}{e_p\theta} = h_7.$$

Thus

$$\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} T + 1 \leqslant \left(1 + \frac{1}{h_7}\right)\left(1 - \frac{1}{q}\right) \frac{1}{n+1} q^{-J} \frac{Y}{c_1c_3e_p\theta W^*}.$$

Combining this with $\operatorname{ord}_p b_n \leqslant (\log B_0)/\log p \leqslant W^*/\log p$, we obtain (2.43)°.

*Proof of* (2.45)°. We have

$$q^{-J}\log(q^{J+k}S) \leqslant \frac{\log(q^{J+n}S)}{q^{J+n}S}\cdot q^n S \leqslant \log(q^n S)$$

and

$$q^n S \leqslant \frac{q^{n+1}c_3(n+1)D}{f_{\mathfrak{p}}\log p}\cdot W^* \leqslant q^{n-2}\cdot 2^{11}\cdot 3nDW^*.$$

(Here we have used the fact that $c_3 < 17$ if $p = 2$ and $c_3 < 134$ if $p > 2$). Thus, on noting $W^* \geqslant h_0 > 3$, we get

$$(W^*)^{-1}q^{-J}\log(q^{J+k}S) \leqslant (W^*)^{-1}\log(q^n S)$$

$$\leqslant((n-2)\log q + \log(2^{11}\cdot 3nD) + \log W^*)/W^* \leqslant \frac{(n-2)\log q}{h_0} + \frac{1}{n} + \frac{\log h_0}{h_0}.$$

This and the definition of $T$ in (2.31)° yield (2.45)°.

In the sequel, we keep (2.50)*, (2.51)*, (2.52)* for the definitions of $\Lambda_J(z,\tau)$, $D_0$, $D_1$, $\xi_{d_0,d}$. For $\tau = (\tau_0,\ldots,\tau_{n-1})\in\mathbb{N}^n$ write $|\tau| = \tau_0 + \cdots + \tau_{n-1}$.

### 2.3. Construction of the rational integers $p(\lambda, d_0, d)$

We recall that $r_1,\ldots,r_n$ are the rational integers in (2.24)*; $G$, $G_0$, $G_1$ are defined by (2.19)*, (2.20)*; $X_0$ is given in (2.31)°.

LEMMA 2.1. *We adopt the statement of Lemma* 2.1* *for this lemma.*

*Proof.* Below we indicate the modifications in the proof of the lemma. Firstly, we note, by (2.20)* and (2.21)*, that $\zeta^{G_1 s}$ is a root of

$$f(x) := x^{q^{\mu-u}} - (\zeta_{q^u})^s,$$

where $(\zeta_{q^u})^s \notin K^q$ by (0.4)* and the fact that $(s, q) = 1$. We show that $f(x)$ is irreducible over $K$. Note that $K$ satisfies (0.3)°, instead of (2.1)*. If $p = 2$ or $p > 2$ with $\zeta_4 \in K$, then $f(x)$ is irreducible over $K$ by Lemma 1.8*. In the remaining case, where by (0.3)° $p \equiv 3 (\mathrm{mod}\ 4)$ with $f_{\mathfrak{p}}$ odd, we have $p^{f_{\mathfrak{p}}} \equiv 3 (\mathrm{mod}\ 4)$, so $\mu = u = 1$ and $f(x) = x + 1$ is irreducible. Thus

$$\zeta^{G_1 st}, \qquad t = 0, 1, \ldots, q^{\mu-u} - 1$$

are linearly independent over $K$.

Secondly, we apply Lemma 1.2, instead of Lemma 1.5*, to (2.63)* for each $t \in \mathscr{T}$. As noted in [*], p. 43, for each $t \in \mathscr{T}$, in (2.63)* there are

$$M_0 := (1 - 1/q)S\binom{T+n}{n} \tag{2.50}$$

equations and the number of unknowns $p(\lambda, d_0, d)$ is at least

$$\frac{1}{G_0} D(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j + 1 - G_0).$$

So by (2.32)° we may apply Lemma 1.2. Note that we have the following inequalities

$$\deg_{x_j} P_{\lambda,d_0,d;s,\tau} \leqslant p^\kappa L_j s + D \qquad (1 \leqslant j \leqslant n),$$

$$\begin{aligned}
&\sum_{j=1}^{n} h(\alpha_j) \sum_{|\tau|\leqslant T} \sum_{1\leqslant s\leqslant S,(s,q)=1} (p^\kappa L_j s + D) \\
&\quad \leqslant \binom{T+n}{n}\cdot\frac{1}{2}\left(1-\frac{1}{q}\right) S^2 p^\kappa \sum_{j=1}^{n} L_j V_j + M_0 n D \max_{1\leqslant j\leqslant n} V_j \\
&\quad = M_0\left(\frac{1}{2} p^\kappa S \sum_{j=1}^{n} L_j V_j + nD \max_{1\leqslant j\leqslant n} V_j\right).
\end{aligned} \tag{2.51}$$

Now

$$\begin{aligned}
\sum_{|\tau|\leqslant T} \tau_0 &= \sum_{\tau_0=0}^{T} \tau_0 \sum_{\substack{\tau_1+\cdots+\tau_{n-1}\leqslant T-\tau_0 \\ \tau_1\geqslant 0,\ldots,\tau_{n-1}\geqslant 0}} 1 = \sum_{\tau_0=0}^{T} \tau_0 \binom{T-\tau_0+n-1}{n-1} \\
&= \sum_{i=0}^{T} (T-i)\binom{i+n-1}{n-1} = T\sum_{i=0}^{T}\binom{i+n-1}{n-1} - n\sum_{i=1}^{T}\binom{i+n-1}{n} \\
&= T\binom{T+n}{n} - n\binom{T+n}{n+1} = \binom{T+n}{n+1}.
\end{aligned} \tag{2.52}$$

By the estimate of $L(P_{\lambda,d_0,d;s,\tau})$ on p. 42 of [*], we have

$$\begin{aligned}
&\prod_{1\leqslant s\leqslant S,(s,q)=1} \prod_{|\tau|\leqslant T} \left(\sum_{\lambda}\sum_{d_0,d} [L(P_{\lambda,d_0,d;s,\tau})]^2\right)^{1/2} \\
&\quad \leqslant \left\{\left(D\prod_{j=-1}^{n}(L_j+1)\right)^{1/2} e^T\left(1+\frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T \right. \\
&\quad \left. \cdot\left(e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}\right\}^{M_0} \cdot \prod_{1\leqslant s\leqslant S,(s,q)=1}\prod_{|\tau|\leqslant T} (3^{(L_{-1}+1)}e^{-1})^{\tau_0}.
\end{aligned} \tag{2.53}$$

Now the lemma follows at once by applying Lemma 1.2, (2.50)–(2.53), (2.32)° (note that $h(\alpha_0) = 0$).

### 2.4. The main inductive argument

We may adopt the statements of the main inductive argument and Lemmas 2.2*–2.5* for the current main inductive argument and Lemmas 2.2–2.5, indicating the necessary modifications in their proof below.

*Proof of Lemma* 2.2. No change.

*Proof of Lemma* 2.3. We need the fact that

$$\zeta^{G_1 st}, \quad t = 0, 1, \ldots, q^{\mu - u} - 1$$

are linearly independent over $K$, which has been established in the proof of Lemma 2.1.

(2.87)* should be replaced by

$$\begin{aligned}
\operatorname{ord}_p \varphi_{J,t}\left(\frac{s}{q}, \tau\right) &> \left(1 - \frac{1}{q}\right)^2 \frac{1}{n+1} q^k ST0 - (L_{-1} + 1)(L_0 + 1)\left(\theta + \frac{1}{p-1}\right) \\
&> \frac{1}{c_1} U q^{k-n} \left\{\left(1 - \frac{1}{2.95 c_3 e_p (n+1)}\right)\left(1 - \frac{1}{h_1}\right)\left(1 - \frac{1}{q}\right)^2 \right. \\
&\qquad \left. - \left(1 + \frac{1}{h_4}\right)\left(\theta + \frac{1}{p-1}\right) \frac{e_p}{q^{k+1} D} \cdot \frac{1}{c_4}\right\} \\
&\text{for } s \geqslant 1, \quad |\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J} T. \qquad (2.87)^\circ
\end{aligned}$$

For any fixed $t \in \mathscr{T}^{(J)}$ and for

$$1 \leqslant s \leqslant q^{J+k+1} S, \ (s, q) = 1, \ |\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J} T,$$

using Lemma 1.3, we replace (2.88)* by the following

$$\begin{aligned}
&\zeta^{-G_1 st} \zeta^{-sr^{(J)}} q^{J(L_{-1}+1)(L_0+1)+(L_0+1)\operatorname{ord}_q(L_{-1}+1)!} (\nu(L_{-1} + 1))^{\tau \circ} \varphi_{J,t}(s, \tau) \\
&\quad = \sum_{\lambda \in \mathscr{C}_t^{(J)}} \sum_{d_0, d} p^{(J)}(\lambda, d_0, d) q^{J(L_{-1}+1)(L_0+1)+(L_0+1)\operatorname{ord}_q(L_{-1}+1)!} (\nu(L_{-1} + 1))^{\tau \circ} \cdot \\
&\qquad \cdot \Lambda_J(s, \tau) \alpha_0^{h(\lambda_1, \ldots, \lambda_n, d_0, s)} \prod_{j=1}^{n} \alpha_j^{p^k \lambda_j s + k_{jd}} \\
&\quad =: Q_{J,t;s,\tau}(\alpha_0, \alpha_1, \ldots, \alpha_n), \qquad (2.88)^\circ
\end{aligned}$$

with $Q_{J,t;s,\tau}(x_0, x_1, \ldots, x_n) \in \mathbb{Z}[x_0, x_1, \ldots, x_n]$.

In estimating $\log(L(Q_{J,t;s,\tau}))$ we note that

$$(L_0 + 1)\mathrm{ord}_q(L_{-1} + 1)! \cdot \log q \leqslant (L_{-1} + 1)(L_0 + 1) \frac{\log q}{q - 1}$$

$$\leqslant \left(1 + \frac{1}{h_4}\right) \frac{\log q}{(q - 1)h_0} \frac{1}{c_1 c_4} Y,$$

by (2.31)° and $\log V^*_{n-1} \geqslant h_0$.

Now assuming that there exist $s$, $\tau$ with

$$1 \leqslant s \leqslant q^{J+k+1}S, \ (s, q) = 1, \ |\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J} T$$

such that

$$\varphi_{J,t}(s, \tau) \neq 0,$$

we have, (similarly as in [*]), by Lemma 1.3*,

$$\begin{aligned}
\mathrm{ord}_p \varphi_{J,t}(s, \tau) &\leqslant \mathrm{ord}_p Q_{J,t;s,\tau}(\alpha_0, \alpha_1, \ldots, \alpha_n) \\
&< \frac{1}{c_1} U q^{k-n} \left\{ \left( \frac{1}{q}\left(1 + \frac{1}{c_0 - 1}\right) \frac{1}{h_5} + \frac{1}{q}\left(1 + \frac{1}{2(c_0 - 1)}\right) \frac{1}{h_6} \right) c_1 \right. \\
&\quad + \left(1 + \frac{1}{2q(c_0 - 1)}\right) \frac{1}{c_2} + \frac{1}{q}\left(\log 3 \cdot \left(1 + \frac{1}{h_0}\right) + 1 \right. \\
&\quad \left. + \frac{1}{c_0 - 1}\left(1 + \frac{1}{h_0} + \frac{\log 3}{n+1} + \frac{\log(3/e)}{(n+1)h_0}\right)\right) \frac{1}{e_p \theta} \cdot \frac{1}{c_3} \\
&\quad \left. + \left(1 + \frac{1}{h_4}\right)\left(\frac{1}{n}\left(1 + \frac{1}{q(c_0 - 1)}\right) + \frac{2}{q} + \frac{\log q}{q(q-1)h_0}\right) \frac{1}{c_4} \right\}.
\end{aligned}$$

[Here we have used the fact that

$$\begin{aligned}
\log(3^{(L_{-1}+1)} e^{-1})^{T/(n+1)} &= \frac{T}{n+1}((\log 3)L_{-1} + \log(3/e)) \\
&\leqslant \frac{1}{n+1}\left(\log 3 + \frac{\log(3/e)}{h_0}\right) \frac{Y}{e_p \theta c_1 c_3}.]
\end{aligned}$$

This together with (2.28)° yields

$$\operatorname{ord}_p \varphi_{J,t}(s, \tau) < \frac{1}{c_1} U q^{k-n} \left\{ \left(1 - \frac{1}{2.95 c_3 e_p (n+1)}\right)\left(1 - \frac{1}{h_1}\right)\left(1 - \frac{1}{q}\right)^2 \right.$$
$$\left. - \left(1 + \frac{1}{h_4}\right)\left(2\left(1 - \frac{1}{q}\right) + \left(\theta + \frac{1}{p-1}\right)\frac{e_p}{q^{n+1}D}\right)\frac{1}{c_4}\right\}. \qquad (2.89)^\circ$$

On noting, by definition (2.24) of $\theta$, that

$$2\left(1 - \frac{1}{q}\right) + \left(\theta + \frac{1}{p-1}\right)\frac{e_p}{q^{n+1}D} > \left(\theta + \frac{1}{p-1}\right)\frac{e_p}{q^{k+1}D},$$

we see that (2.89)° contradicts (2.87)°. This contradiction proves that for any fixed $t \in \mathscr{T}^{(J)}$,

$$\varphi_{J,t}(s, \tau) = 0 \quad \text{for} \quad 1 \leqslant s \leqslant q^{J+k+1}S, \ (s, q) = 1$$

and

$$|\tau| \leqslant \left(1 - \frac{k+1}{n+1}\left(1 - \frac{1}{q}\right)\right) q^{-J} T.$$

This fact and (2.78)* imply (2.74)* for $k+1$, and the proof of Lemma 2.3 is complete.

*Proof of Lemma* 2.4. Now (2.91)* is replaced by

$$\operatorname{ord}_p \varphi_{J,t}\left(\frac{s}{q}, \tau\right) > \left(1 - \frac{1}{q}\right)^2 \frac{1}{n+1} q^n S T \theta - (L_{-1} + 1)(L_0 + 1)\left(\theta + \frac{1}{p-1}\right)$$
$$> \frac{1}{c_1} U \left\{ \left(1 - \frac{1}{2.95 c_3 e_p (n+1)}\right)\left(1 - \frac{1}{h_1}\right)\left(1 - \frac{1}{q}\right)^2 \right.$$
$$\left. - \left(1 + \frac{1}{h_4}\right)\left(\theta + \frac{1}{p-1}\right)\frac{e_p}{q^{n+1}D} \cdot \frac{1}{c_4}\right\} \quad \text{for } s \geqslant 1, \ |\tau| \leqslant q^{-(J+1)}T. \qquad (2.91)^\circ$$

For any fixed $t \in \mathscr{T}^{(J)}$ and for $1 \leqslant s \leqslant q^{J+1}S$, $(s, q) = 1$, $|\tau| \leqslant q^{-(J+1)}T$, using Lemma 1.3, we replace (2.95)* by

$$\xi^{-(r^{(J)} + tG_1)s} q^{(J+1)(L_{-1}+1)(L_0+1) + (L_0+1)\operatorname{ord}_q(L_{-1}+1)!} (\nu(L_{-1} + 1))^{\tau_\circ} \varphi_{J,t}\left(\frac{s}{q}, \tau\right)$$
$$= \sum_{\lambda \in \mathscr{C}_t^{(J)}} \sum_{d_0, d} p^{(J)}(\lambda, d_0, d) q^{(J+1)(L_{-1}+1)(L_0+1) + (L_0+1)\operatorname{ord}_q(L_{-1}+1)!} (\nu(L_{-1}+1))^{\tau_\circ}.$$

$$\cdot \Lambda_J\left(\frac{s}{q}, \tau\right)(\alpha_0^{1/q})^{h^*(\lambda_1,\ldots,\lambda_n,d_0,s)} \prod_{j=1}^{n} (\alpha_j^{1/q})^{p^\kappa \lambda_j s + qk_{jd}}$$

$$=: Q^*_{J,t;s,\tau}(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}), \tag{2.95°}$$

with $Q^*_{J,t;s,\tau}(x_0, x_1, \ldots, x_n) \in \mathbb{Z}[x_0, x_1, \ldots, x_n]$. Assuming that there exist $s$, $\tau$ satisfying $1 \leqslant s \leqslant q^{J+1}S$, $(s, q) = 1$, $|\tau| \leqslant q^{-(J+1)}T$, such that

$$\varphi_{J,t}\left(\frac{s}{q}, \tau\right) \neq 0,$$

we should have, by Lemma 1.3* and (2.28)°,

$$\begin{aligned}
\operatorname{ord}_p \varphi_{J,t}\left(\frac{s}{q}, \tau\right) &\leqslant \operatorname{ord}_p Q^*_{J,t;s,\tau}(\alpha_0^{1/q}, \alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) \\
&\leqslant \frac{1}{c_1} U \left\{\left(\left(1 + \frac{1}{c_0 - 1}\right)\frac{1}{h_5} + \left(1 + \frac{1}{2(c_0 - 1)}\right)\frac{1}{h_6}\right) c_1 + \left(1 + \frac{1}{2(c_0 - 1)}\right)\frac{1}{c_2}\right. \\
&\quad + \left[\frac{1}{q}\left(\log 3 \cdot \left(1 + \frac{1}{h_0}\right) + 1\right) + \frac{1}{c_0 - 1}\left(1 + \frac{1}{h_0} + \frac{1}{n+1}\left(\log 3 + \frac{\mathrm{lo}(3/e)}{h_0}\right)\right)\right]\frac{1}{e_p \theta} \\
&\quad \left. \cdot \frac{1}{c_3} + \left(1 + \frac{1}{h_4}\right)\left(\frac{1}{n}\left(1 + \frac{1}{c_0 - 1}\right) + 2 + \frac{\log q}{(q-1)h_0}\right)\frac{1}{c_4}\right\} \\
&\leqslant \frac{1}{c_1} U \left\{\left(1 - \frac{1}{2.95 c_3 e_p (n+1)}\right)\left(1 - \frac{1}{h_1}\right)\left(1 - \frac{1}{q}\right)^2\right. \\
&\quad \left. - \left(1 + \frac{1}{h_4}\right)\left(\theta + \frac{1}{p-1}\right)\frac{e_p}{q^{n+1} D} \cdot \frac{1}{c_4}\right\},
\end{aligned}$$

contradicting (2.91)°. This contradiction proves Lemma 2.4.

*Proof. of Lemma* 2.5. We need only to show that the polynomial in (2.99)*

$$x^{q^{\mu-u+1}} - \zeta_{q^\mu} \quad \text{(recalling (0.4)*, (2.19)* and (2.20)*)}$$

is irreducible in $K'[x]$ with $K' = K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$. By Kummer condition (2.15)*, we have $[K'(\alpha_0^{1/q}) : K'] = q$. So $\alpha_0 \notin (K')^q$, whence

$$\zeta_{q^u} = \alpha_0^{p^v} \notin (K')^q, \tag{2.100*}$$

since $(p, q) = 1$. If $p = 2$ or $p > 2$ with $\zeta_4 \in K$, then the irreducibility follows from (2.100)* by Lemma 1.8*. In the remaining case, where $p \equiv 3 \pmod 4$ with $f_p$ odd (see (0.3)°), we have $q = 2$ and $p^{f_p} \equiv 3 \pmod 4$, whence $\mu = u = 1$ and

$$x^{q^{u-u+1}} - \zeta_{q^u} = x^2 + 1$$

is irreducible in $K'[x]$ by (2.100)*. Now we can adopt the proof of Lemma 2.5*.

Thus we have established the main inductive argument for $J = 0, 1, \ldots, [\log L_n/\log q] + 1$.

### 2.5. Completion of the proof of Proposition 2.1

On noting that $(2.j)^\circ = (2.j)^*$ for $j = 46, 47$, we can adopt Section 2.5*.

### 2.6. Proof of Theorem 2.1

On observing (0.2)*, (2.24), (1.1)° and the fact that

$$\left(1 + \frac{1}{2n} \cdot 10^{-100}\right)^n < \exp\left(\frac{1}{2} \cdot 10^{-100}\right) < 1 + 10^{-100},$$

we see that Theorem 2.1 is a direct consequence of Proposition 2.1.

### 2.7. Proof of Theorem 2.2 and Corollaries 2.3, 2.4

*Proof of Theorem* 2.2. Similar to the proof of Theorem 2.2*. We need only to note the following. Set

$$K' := \mathbb{Q}(\alpha_0, \alpha_1, \ldots, \alpha_n) \subseteq K, \quad \mathfrak{p}' = \mathfrak{p} \cap O_{K'},$$

where $O_{K'}$ is the ring of integers in $K'$. We see that if $p \equiv 3 \pmod 4$ and $2 \mid f_{\mathfrak{p}'}$ then $2 \mid f_{\mathfrak{p}}$ (since $f_{\mathfrak{p}'} \mid f_{\mathfrak{p}}$), whence $\zeta_4 \in K$ (by (0.3)°), that is, $u \geqslant 2$. But $\zeta_{2^u} = \alpha_0^{p^n} \in K'$, whence $\zeta_4 \in K'$. With the above observation, we conclude that $K'$ together with $\mathfrak{p}'$ satisfies condition (0.3)°.

*Proof of Corollary* 2.3. By (2.8)*, (2.11)*, (2.12)°, (2.13)° and $n \geqslant 2$, in Theorems 2.1 and 2.2 we may choose

$$W^* = \max(\log B, n \log(2^{11} \cdot 3nD), 2.95 \log p).$$

Now the corollary follows at once.

*Proof of Corollary* 2.4. By (2.15)*–(2.18)* we may apply Theorems 2.1 and 2.2 with $V_n$ replaced by

$$V_n' := \max\left(V_n, \frac{\delta B}{B_n Z W'}\right).$$

We may replace $B'$, $B_0$ in (2.12)° by $B$, $B_n$. By (2.8)*, $0 < \delta \leqslant Zf_p(\log p)/D$ and $W' > 1$, we get

$$\frac{B_n}{V_1} + \frac{B}{V'_n} \leqslant \frac{B_n}{V_1} + \delta^{-1} Z B_n W' \leqslant 2\delta^{-1} Z B_n W'.$$

By (2.12)°, (2.13)°, $n \geqslant 2$ and the fact that $2(n-1)/(\rho n^2) \leqslant \frac{1}{10}$, to prove the corollary, it suffices to prove

$$\max\left\{\log\left(1 + \frac{1}{10}\psi W'\right), \log B_n, 2.95 \log p, n \log(2^{11} \cdot 3nD)\right\} \leqslant W',$$

where

$$\psi = \delta^{-1} \frac{f_p \log p}{D} Z B_n.$$

By the hypotheses on $Z$, $\delta$ and $W'$, we need only to show that

$$\log\left(1 + \frac{1}{10}\psi W'\right) \leqslant W'.$$

Again by the hypothesis on $W'$ and the fact that $x - \log(1 + \frac{1}{10}\psi x)$ is increasing for $x \geqslant 1$, it suffices to prove

$$\log\left(1 + \frac{1}{10}\psi y\right) \leqslant y \quad \text{for } y = \max(\rho' \log \psi, n \log(2^{11} \cdot 3nD)). \tag{2.54}$$

Let

$$g(x) = \rho' \log x - \log\left(1 + \frac{\rho'}{10} x \log x\right).$$

It is readily verified that

$$g((2^{11} \cdot 3n)^{n/\rho'}) > 0 \qquad \text{and} \qquad g'(x) > 0 \quad \text{for } x \geqslant (2^{11} \cdot 3n)^{n/\rho'}.$$

Now if $y = n \log(2^{11} \cdot 3nD)$, then $\psi \leqslant (2^{11} \cdot 3nD)^{n/\rho'}$, and hence

$$y - \log\left(1 + \frac{1}{10}\psi y\right) \geqslant y - \log\left(1 + \frac{1}{10} \cdot (2^{11} \cdot 3nD)^{n/\rho'} y\right) = g((2^{11} \cdot 3nD)^{n/\rho'}) > 0.$$

If $y = \rho' \log \psi$, then $\psi \geqslant (2^{11} \cdot 3nD)^{n/\rho'}$, and hence

$$y - \log\left(1 + \frac{1}{10}\psi y\right) = g(\psi) \geqslant g((2^{11} \cdot 3nD)^{n/\rho'}) > 0.$$

This proves (2.54), and the proof of the corollary is thus complete.

## 3. Propositions for Kummer descent

Let $K$, $D$, $p$, $q$, $u$, $v$, $\alpha_0$, $\mathfrak{p}$, $\mathrm{ord}_{\mathfrak{p}}$, $f_{\mathfrak{p}}$ be defined in Section 0.2. Note that (0.3)° replaces (2.1)*. However, on modifying their proof accordingly, it is readily verified that the statements of Propositions 3.1*–3.4* and Lemma 3.1* remain true for $K$ satisfying (0.3)°. (3.1)* should be changed to $q \leqslant p^v q^u \leqslant 3D$, and the inequality $D \geqslant 2$ in Section 3* should be replaced by $D \geqslant 1$. As an example, we list modifications to the proof of Lemma 3.1*. (We leave the other modifications to the reader.) $\pi/q$ in (3.36)* should be replaced by $2\pi/q$. The extreme right-hand side of (3.38)* should be replaced by $\frac{1}{8}(q-1)n(n+1) + \frac{1}{4}(q+7)$. (3.39)* should be replaced by

$$\begin{aligned}\frac{|\lambda_1|}{2\pi D} &\leqslant \frac{1}{q}\left\{V + \frac{q-1}{qD} + (q-1)\cdot\frac{1}{2}(2 + \cdots + n)V_{n-1} + \frac{|t|}{D}\right\}\\ &\leqslant \frac{1}{q}V + \left(1 - \frac{1}{q}\right)V_{n-1}\left\{\frac{1}{4}n(n+1) - \frac{1}{2} + \frac{1}{f_{\mathfrak{p}}\log p}\left(\frac{1}{q} + \frac{1}{8}n(n+1) + \frac{q+7}{4(q-1)}\right)\right\}\\ &\leqslant \frac{1}{q}V + \frac{1}{2}\left(1 - \frac{1}{q}\right)U_0 \leqslant \frac{q+1}{2q}V.\end{aligned}$$

$10^5$ appearing in the last line of p. 76 of [*] and in lines 3, 4, 7 of p. 77 of [*] should be replaced by 3500.

## 4. Proof of Theorem 1, its Corollary and Theorem 1′

This is very similar to Section 4*. We just indicate modifications.

*Proof of Theorem* 1. (4.2)* should be replaced by $q \leqslant p^v q^u \leqslant 3D$. (4.6)* should be replaced by $V_0 \leqslant \sigma V_j \leqslant (1/\log 9)V_j$ $(1 \leqslant j \leqslant n)$, since now (0.12)° gives $\sigma \leqslant 1/(2\log 3)$. We deal with cases (*a*)–(*h*) as in Section 4*.

In case (*a*) we modify the proof in [*] as follows. (4.7)* should be replaced by the inequality in the statement of Theorem 1, the right-hand side of which we denote by $U_1$ (see the remark below the statement of Theorem 2.1), (4.9)*

should be replaced by

$$B > 10^{n+4}(n+1)^{n+2}n^{n-1}\,\frac{p^{f_p}-1}{(f_p\log p)^2}\,D^2V_{n-1}, \tag{4.9°}$$

(4.19)* should be replaced by

$$(n+1)\log(D^2B) \geqslant \max(n\ \log(2^{11}\cdot 3nD),\ 2.95\log p), \tag{4.19°}$$

which need to be verified. To prove (4.19)° it suffices to show

$$(D^2B)^{n+1} \geqslant (D^2B)^3 \geqslant p^{2.95},$$

since $B \geqslant 2^{11}\cdot 3n$ by (4.9)°. Now by the inequality $DV_{n-1} \geqslant f_p\log p$ (see (0.7)* and (0.9)*), (4.9)° yields

$$D^2B > 10^4\,\frac{p^{f_p}-1}{f_p\log p}.$$

Applying the inequality that for $x > 1$

$$(\log x)^3 < 10^5(x-1)^{1/20}\ (\text{since } (\log x)^{60} < 60!(x-1))$$

to $x = p^{f_p}$, we see that

$$(D^2B)^3 > 10^{12}\cdot\frac{(p^{f_p}-1)^3}{(f_p\log p)^3} > 10^7(p^{f_p}-1)^{2.95} \geqslant 10^7\left(\frac{1}{2}\,p^{f_p}\right)^{2.95} > p^{2.95}.$$

This proves (4.19)°. In cases (*b*)–(*h*), the proof in [*] can be modified similarly.

*Proof of the Corollary of Theorem* 1. We choose

$$K = \begin{cases} \mathbb{Q}(\zeta_3), & \text{if } p = 2,\\ \mathbb{Q}(\zeta_4), & \text{if } p \equiv 1 \pmod 4,\\ \mathbb{Q}, & \text{if } p \equiv 3 \pmod 4.\end{cases}$$

Then $K$ satisfies (0.3)°. We may assume

$$\operatorname{ord}_p \alpha_j = 0 \qquad (1 \leqslant j \leqslant n),$$

arguing similarly to the proof of Corollary 2*. Thus we can apply Theorem 1.

In the sequel we verify the case when $p \equiv 3 \pmod 4$ with $p \geqslant 7$. We can choose $C$ in Theorem 1 as

$$C = \begin{cases} 67821.82\eta \cdot 10^n, & \text{if } n \geqslant 5, \\ 10944.2026\eta \cdot 14^n, & \text{if } n = 2,\ 3,\ 4. \end{cases} \tag{4.1}$$

The choice of $C$ in (4.1) for $n = 2, 3, 4$ is justified by Corollary 2.3 with the values $c$ and $a$ for the case $p \geqslant 5$ with $e_{\mathfrak{p}} = 1$ and $n = 2, 3, 4$ (recalling (0.12)), together with the propositions for Kummer descent, as in the proof of Theorem 1.

Similarly to the proof of Corollary 2*, it is easy to verify that we may take

$$V_j = \log A_j \quad (1 \leqslant j \leqslant n), \qquad V = \log A.$$

Next, we show that if $\sigma$ is fixed with $0 < \sigma < 0.46$ then

$$g(x) = \frac{x^{x+1+\sigma}}{(x+1)^{x+2}} \quad \text{is decreasing for } x \geqslant 2. \tag{4.2}$$

By the inequality $\log(1 + y) \geqslant \frac{4}{5}y$ for $0 \leqslant y \leqslant \frac{1}{2}$, we see that

$$\frac{d}{dx}(\log g(x)) = \frac{1+\sigma}{x} - \frac{1}{x+1} - \log\left(1 + \frac{1}{x}\right) < \frac{1.46}{x} - \frac{1}{x+1} - \frac{4}{5}\cdot\frac{1}{x} < 0$$

for $x \geqslant 2$,

whence (4.2) follows.

Now $p \equiv 3 \pmod 4$ with $p \geqslant 7$, $K = \mathbb{Q}$, $\mathfrak{p} = p\mathbb{Z}$, $q = 2$, $D = f_{\mathfrak{p}} = u = 1$, $v \geqslant 0$, and by (0.12)°, we have $0 < \sigma \leqslant 1/(2\log 7) < 0.46$. By (4.2) we see that for $n \geqslant 5$

$$67821.82 \cdot q^{-u}(n+1)^{n+2}n^{n+1+\sigma} \leqslant \tfrac{1}{2} \cdot 67821.82 \cdot g(5)(n+1)^{2n+4} \leqslant 2863(n+1)^{2n+4}.$$

Further, for $n = 2, 3, 4$,

$$\begin{aligned} &10944.2026 \times 14^n \cdot q^{-u} \cdot (n+1)^{n+2}n^{n+1+\sigma} \\ &\quad = 5472.1013 \times 14^n g(n)(n+1)^{2n+4} < 2863 \times 10^n (n+1)^{2n+4}. \end{aligned}$$

Now on applying Theorem 1 with (4.1) and the above two inequalities, we obtain the corollary for the case when $p \equiv 3 \pmod 4$ with $p \geqslant 7$. We omit here the details for the remaining cases.

*Proof of Theorem* 1′. Similar to the proof of Theorem 1′*. We just indicate the modifications. We can keep (4.79)*–(4.84)* and (4.87)*, where $e_{\mathfrak{p}_0}$, $f_{\mathfrak{p}_0}$ are now written as $e_0$, $f_0$; while (4.85)* should be replaced by $|\log \alpha'_j| \leqslant (D_0/f_0)(10D_0 + 1)V_j$ $(1 \leqslant j \leqslant n)$, by (0.21)*. Now we choose

$$K = \begin{cases} K_0(\zeta_3), & \text{if } p = 2, \\ K_0(\zeta_4), & \text{if } p \equiv 1 \pmod 4 \text{ or if } p \equiv 3 \pmod 4 \text{ with } 2 \mid f_0, \\ K_0, & \text{if } p \equiv 3 \pmod 4 \text{ with } 2 \nmid f_0, \end{cases} \tag{4.3}$$

and let $\mathfrak{p}$ be any prime ideal of the ring of integers in $K$ such that $\mathfrak{p} \supseteq \mathfrak{p}_0$. Thus by (4.3), and by Lemma 1.12* and the Lemma in [*], Appendix, we have

$$e_{\mathfrak{p}} = e_0, \quad f_{\mathfrak{p}} = \begin{cases} f_0, & \text{if } p > 2, \\ \max(f_0, 2), & \text{if } p = 2. \end{cases} \tag{4.4}$$

So $K$ satisfies (0.3)°. Further (4.90)* remains true. Note that "$\log \alpha_n$ is linearly dependent on $\pi i$, $\log \alpha_1, \ldots, \log \alpha_{n-1}$, $\log p$" is equivalent to "$\log \alpha'_n$ is linearly dependent on $\pi i$, $\log \alpha'_1, \ldots, \log \alpha'_{n-1}$, $\log p$". Now we can apply Theorem 1 to $\operatorname{ord}_{\mathfrak{p}}((\alpha'_1)^{b_1} \cdots (\alpha'_n)^{b_n} - 1)$, utilizing (4.81)*, (4.87)*, (4.90)* and (4.4). We verify the case when $p \geqslant 7$, $e_0 \geqslant 2$. For the constant $C$ in Theorem 1, we may take

$$C = \begin{cases} 399171.94 \cdot 20^n, & \text{if } n \geqslant 5, \\ 60521.168 \cdot 28^n, & \text{if } n = 2, 3, 4 \end{cases} \tag{4.5}$$

by a similar argument given in the proof of the Corollary of Theorem 1. There are five subcases:

$p \equiv 1 \pmod 4$ with $\zeta_4 \notin K_0$,

or $\zeta_4 \in K_0$,

$p \equiv 3 \pmod 4$ with $2 \nmid f_0$,

or $2 \mid f_0$ and $\zeta_4 \notin K_0$,

or $2 \mid f_0$ and $\zeta_4 \in K_0$.

In the first subcase, we have

$$D = 2D_0,\ p \geqslant 13,\ q = 2,\ u \geqslant 2,\ v \geqslant 0,\ f_{\mathfrak{p}} \geqslant 1,$$

and by (0.12)°, we have $\sigma \leqslant 1/(2^2 \log 13) < 0.46$. By (4.2), (4.5) and the fact that

$g(n) \leqslant g(5)$ for $n \geqslant 5$, we see that

$$C(n+1)^{n+2}n^{n+1+\sigma}2^{2n+2}q^{-u} \leqslant C\cdot 4^n g(n)(n+1)^{2n+4} \leqslant 26065\cdot 80^n(n+1)^{2n+4}$$

Now on applying Theorem 1 with (4.5) and the above inequality, we obtain Theorem 1′ for the subcase

$$p \geqslant 7 \text{ with } p \equiv 1 \pmod 4 \text{ and } \zeta_4 \notin K_0.$$

The other subcases for the case $p \geqslant 7$, $e_0 \geqslant 2$ can be verified similarly. Thus Theorem 1′ for $p \geqslant 7$, $e_0 \geqslant 2$ is established. We omit the details for the remaining cases.

## 5. Proof of Theorems 2 and 2′

*Proof of Theorem* 2. Similar to the proof of Theorem 2*. We need

$$\log x \leqslant x^{1/7} \quad \text{for} \quad x \geqslant 10^{10}, \qquad \log x \leqslant x^{1/6} \quad \text{for } x \geqslant 10^9. \tag{5.1°}$$

By Lemma 1.12* and (0.3)°, we see that if $p = 2$ then $D \geqslant 2$ and $f_{\mathfrak{p}} \geqslant 2$. Further, using $DV_j \geqslant f_{\mathfrak{p}} \log p$ $(1 \leqslant j \leqslant n)$, we get

$$\frac{D}{f_{\mathfrak{p}} \log p} Q > \begin{cases} 10^{10}, & \text{if } n \geqslant 3, \\ 10^9, & \text{if } n = 2. \end{cases} \tag{5.2°}$$

Recalling in Corollary 2.4 $\rho' = 1.0561$ if $n = 2$ and $\rho' = 1.046$ if $n \geqslant 3$, we show that

$$\rho' \log\left(\frac{D}{f_{\mathfrak{p}} \log p} Q\right) \geqslant \max(n \log(2^{11} \cdot 3nD),\ 2.95 \log p). \tag{5.3°}$$

We shall verify the case $p > 2$ and leave the case $p = 2$ to the reader. By (0.18)°, $D \geqslant 1$, $DV_{n-1} \geqslant f_{\mathfrak{p}} \log p \geqslant \log 3$, it is easy to see that (5.3)° is true for $n = 2$, and to show (5.3)° for $n \geqslant 3$, it suffices to prove that for $n \geqslant 3$

$$f(n) := \left(\frac{27}{\log 3}\right)^{\rho'} n^{2(n+1)\rho'-n} \cdot 10^{2(n+1)\rho'} \cdot \left(\frac{2^{11}\cdot 3}{(\log 3)^{\rho'}}\right)^{-n} \geqslant 1, \tag{5.4}$$

where $\rho' = 1.046$. By studying the behaviour of the function $\log f(x)$ for $x \geqslant 2$, it is readily seen that

$$f(n) \geqslant \min(f(9), f(10)) = f(10) > 1 \quad \text{for } n = 3, 4, \ldots,$$

whence (5.4) and (5.3)° (for $p > 2$) follow.

In the proof of Theorem 2 we always observe that for any fixed $j$ with $1 \leqslant j \leqslant n$ if (0.19)° holds for $Z_j > 0$ and any $\delta$ with $0 < \delta \leqslant Z_j f_{\mathfrak{p}} (\log p)/D$, then so does (0.19)° for any $Z_j'' \geqslant Z_j$ and any $\delta''$ with $0 < \delta'' \leqslant Z_j'' f_{\mathfrak{p}} (\log p)/D$.

We need to deal with cases $(a)$–$(h)$ as in Section 5*. In the sequel we modify the proof for case $(a)$ in Section 5*. (5.5)* should be replaced by

$$\Psi_1 = \begin{cases} \dfrac{1}{1.046}\Phi, & \text{if } n \geqslant 3, \\ C_2 C^{-1} \dfrac{1}{1.046}\Phi, & \text{if } n = 2, \end{cases} \tag{5.5°}$$

where $C_2 = 2c(\frac{1}{2}a)^n$ and $c$, $a$ are given in Theorem 2.1 for $n = 2$.

We can keep (5.6)*–(5.9)* with $\rho' = 1.0561$ if $n = 2$ and $\rho' = 1.046$ if $n \geqslant 3$, rewriting the first equality of (5.8)* as $Z = Z_n = \Phi/V_n$. On noting (5.7)*, (5.3)° and (0.18)°, it is readily seen that for any $\delta$ with $0 < \delta \leqslant Z_n f_{\mathfrak{p}} (\log p)/D$,

$$\max\left\{\rho' \log\left(\delta^{-1} \frac{f_{\mathfrak{p}} \log p}{D} Z_n B_n''\right), n \log(2^{11} \cdot 3nD), 2.95 \log p\right\}$$
$$\leqslant \rho' \log(\delta^{-1} Z_n B_n Q). \tag{5.10°}$$

Thus (0.19)° for case (a) with $j = n$ follows, similarly to [*], from Corollary 2.4.

Suppose now $1 \leqslant j < n$. We replace the first equality of (5.11)* by

$$Z = Z_j = \begin{cases} \dfrac{15}{7} \cdot \dfrac{\Phi}{V_j} = \dfrac{15}{7} \cdot \dfrac{\rho' \Psi_1}{V_j}, & \text{if } n \geqslant 3 \text{ (see (5.5)°)}, \\ \dfrac{13}{6} \cdot \dfrac{\rho' \Psi_1}{V_j}, & \text{if } n = 2, \end{cases} \tag{5.6}$$

and replace the first part of (5.12)* by

$$\rho' \Psi_1 = \begin{cases} \dfrac{7}{15} Z_j V_j, & \text{if } n \geqslant 3, \\ \dfrac{6}{13} Z_j V_j, & \text{if } n = 2. \end{cases} \tag{5.7}$$

By (5.1)°, (5.2)° and the inequalities $B_j \geqslant 1$ and $\delta \leqslant Z_j f_{\mathfrak{p}} (\log p)/D$, we have

$$\log(\delta^{-1} Z_j B_j Q) \leqslant \begin{cases} (\delta^{-1} Z_j B_j Q)^{1/7}, & \text{if } n \geqslant 3, \\ (\delta^{-1} Z_j B_j Q)^{1/6}, & \text{if } n = 2. \end{cases}$$

By the same argument as in pp. 100–101 of [*], we get for any $\delta$ with $0 < \delta \leqslant Z_j f_{\mathfrak{p}} (\log p)/D$

$$\operatorname{ord}_{\mathfrak{p}} \Theta < \max(Z_j V_j \log(\delta^{-1} Z_j B_j Q), \delta B/B_j). \tag{5.8}$$

Now, by (5.6), we see that $Z_j V_j = \frac{15}{7}\Phi$ for $n \geqslant 3$, and $Z_j V_j = \frac{13}{6}\rho' \Psi_1 \leqslant \frac{15}{7}\Phi$ for $n = 2$ (by (5.5)°). Thus (0.19)° for case (*a*) with $1 \leqslant j < n$ follows from (5.8), and the proof of Theorem 2 for case (*a*) is complete. We omit the details for cases (*b*)–(*h*).

*Proof of Theorem* 2′. By arguments similar to the proof of Theorem 1′, one can easily deduce Theorem 2′ from Theorem 2. We omit the details.

**6.** By similar arguments, we can remove Kummer condition (0.6) (with $q$ satisfying (0.1)) imposed on Theorem 2 of [Y2] and prove the following results, among which Theorem 3 is a result subject to new Kummer condition (2.15)* (with $q = 3$ if $p = 2$ and $q = 2$ if $p > 2$). Their proof is simpler than that in Sections 2, 4, 5 and will be omitted here. In Theorem 3, notations are as those in Section 2. In Theorem 4, its Corollary and Theorem 4′, we use notations as those for Theorem 1, its Corollary and Theorem 1′, respectively.

THEOREM 3. *Suppose that* (2.15)*–(2.18)* *hold. Then*

$$\operatorname{ord}_{\mathfrak{p}} \Theta < c'(a')^n \frac{(n(n+1))^{n+1}}{n!} \cdot \frac{p^{f_{\mathfrak{p}}} - 1}{q^u (f_{\mathfrak{p}} \log p)^{n+2}} \cdot e_{\mathfrak{p}} D^{n+2} V_1 \cdots V_n$$
$$\cdot \{\max(\log B, n \log(2^{11} \cdot 3nD), 2.95 \log p)\}^2,$$

*where $c'$ and $a'$ are given in the following table, where $\eta_1$ and $\eta_2$ are defined by* (0.12),

$p = 2$

| $n$ | 2, 3, 4 | 5, 6, 7 | $\geqslant 8$ |
|---|---|---|---|
| $c'$ | 1286.7449 | 1286.7449 | 10498.48 |
| $a'$ | 54 | 54 | 45 |

$p = 3$; *or* $p = 5$, $e_{\mathfrak{p}} \geqslant 2$

| | | | |
|---|---|---|---|
| $c'$ | 434.6988 | 723.3362 | 2674.6353 |
| $a'$ | 56 | 48 | 40 |

$p \geqslant 7$, $e_{\mathfrak{p}} \geqslant 2$

| | | | |
|---|---|---|---|
| $c'$ | 589.9375 | 1028.0929 | 3857.9104 |
| $a'$ | 56 | 48 | 40 |

$p \geqslant 5,\ e_p = 1$

| | | | |
|---|---|---|---|
| $c'/\eta_1$ | 219.1304 | 360.3486 | 1333.696 |
| $a'/\eta_2$ | 28 | 24 | 20 |

THEOREM 4. *Suppose that* (0.13)* *and* (0.14)* *hold. Then*

$$\operatorname{ord}_p \Theta < C_1(n+1)^{n+3} n^{n+1+\sigma} \cdot \frac{p^{f_p}-1}{q^u (f_p \log p)^{n+2}} \cdot e_p D^{n+2} \cdot V_1 \cdots V_n \log^2(D^2 B),$$

*where*

$$C_1 = \begin{cases} 20996.96 \cdot (\frac{45}{2})^n, & \text{if } p = 2, \\ 5349.2706 \cdot 20^n, & \text{if } p = 3, \text{ or if } p = 5,\ e_p \geqslant 2, \\ 7715.8208 \cdot 20^n, & \text{if } p \geqslant 7,\ e_p \geqslant 2, \\ 2667.392\eta \cdot 10^n, & \text{if } p \geqslant 5,\ e_p = 1. \end{cases}$$

COROLLARY. *Under the hypotheses of the Corollary of Theorem* 1 (*without setting* $A$), *we have*

$$\operatorname{ord}_p \Theta < C_1''(n+1)^{2n+5} \frac{p-1}{(\log p)^{n+2}} \log A_1 \cdots \log A_n \log^2(4B),$$

*where*

$$C_1'' = \begin{cases} 1038 \cdot (\frac{45}{2})^n, & \text{if } p = 2, \\ 311 \cdot 20^n, & \text{if } p = 3, \\ 113\eta \cdot 10^n, & \text{if } p \equiv 3 \pmod 4 \text{ and } p \geqslant 7, \\ 192\eta \cdot 20^n, & \text{if } p \equiv 1 \pmod 4. \end{cases}$$

THEOREM 4′. *Suppose that* (0.14)* *holds. Then*

$$\operatorname{ord}_{p_0} \Theta < C_1'(n+1)^{2n+5} \frac{p^{f_0}-1}{(\log p)^{n+2}} e_0 \left(\frac{D_0}{f_0}\right)^{2n+2} V_1 \cdots V_n \log^2(4D_0^2 B),$$

*where*

$$C_1' = \begin{cases} 1384 \cdot 90^n, & \text{if } p = 2, \\ 384 \cdot 80^n, & \text{if } p = 3, \text{ or if } p = 5,\ e_0 \geqslant 2, \\ 504 \cdot 80^n, & \text{if } p \geqslant 7,\ e_0 \geqslant 2, \\ 192\eta \cdot 40^n, & \text{if } p \geqslant 5,\ e_0 = 1. \end{cases}$$

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