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RAYMOND ROSS

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K_2 of Fermat curves with divisorial support at infinity

RAYMOND ROSS*

Louisiana State University, Baton Rouge, LA 70803-4918

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Introduction

The Beilinson conjectures ([2], [3]) interpret the special values of the L -functions associated to a projective variety X defined over a number field to the K -theory of X , via a regulator which generalizes that of Dirichlet. In this paper, we examine part of these conjectures in the case where X is the Fermat curve $F_N: x^N + y^N = z^N$ of exponent N , where $N \geq 3$. Our investigation is inspired partly by Beilinson's theorem on modular curves ([3], also see [17]), and by a result of Rohrlich on the divisor class group of F_N [12].

Let us recall what Beilinson's theorem says, following the exposition in [17]. Let X/\mathbf{Q} be a modular curve, and let g be the genus of X . For each modulator curve Y/\mathbf{Q} which covers X via a morphism $\theta_Y: Y \rightarrow X$ defined over \mathbf{Q} , there is a homomorphism $\theta_{Y*}: K_2 Y \rightarrow K_2 X$. Let \mathcal{Q}_Y be the subspace $K_2 Y \otimes \mathbf{Q}$ with divisorial support at the cusps of Y (this is made precise in Section 1 below), and let \mathcal{P}_X denote the subgroup of $K_2 X \otimes \mathbf{Q}$ spanned by $\theta_{Y*}(\mathcal{Q}_Y)$ for all such Y . Then $\mathcal{P}_X \subset H_{\mathcal{D}}^2(X, \mathbf{Q}(2))_{\mathbf{Z}}$ and the image of \mathcal{P}_X in $H_{\mathcal{D}}^2(X, \mathbf{R}(2))$ under the regulator homomorphism reg_X is a \mathbf{Q} -structure of $H_{\mathcal{D}}^2(X, \mathbf{R}(2))$, and

$$\wedge^g \text{reg}_X \mathcal{P}_X = c \wedge^g H_{\mathcal{D}}^2(X, \mathbf{Q}(2)),$$

with $c \equiv L^{(g)}(0, X) \pmod{\mathbf{Q}^*}$.

In general, one must look at \mathcal{P}_X and not just \mathcal{Q}_X . For example, if p is prime, then $X_0(p)$ has exactly two cusps, and therefore only one (up to scalar multiple) modular unit f . It then follows that $\mathcal{Q}_{X_0(p)}$ is trivial.

For F_N , an analogue of the group of modular units is provided by those functions whose divisorial support is contained in the *points at infinity*, which are those points P on F_N such that $xyz(P) = 0$. In partial analogy with Beilinson's construction, we investigate the subgroup of $K_2 F_N$ generated by the images under the transfer maps $K_2 F_{dN} \rightarrow K_2 F_N$ of those elements whose divisorial support is contained in the points at infinity. We show that this subgroup is of positive rank, and that, over $\mathbf{Q}(\mu_{2N})$, it is a cyclic $\text{Aut } F_N$ -module. Using this, we descend to \mathbf{Q} and obtain a bound for the rank of the corresponding group over \mathbf{Q} . In contrast to the modular situation, this

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subgroup is usually of rank smaller than the rank of K_2F_N as predicted by the conjecture, namely, $(N - 1)(N - 2)/2$, the genus of F_N . The rank of this subgroup, and a set of generators for the vector space obtained by tensoring with \mathbf{Q} , will be determined in [14].

We now briefly indicate the organization of this paper. We begin by summarizing what we need from K -theory. In Section 2, we exhibit an explicit element in K_2F_N which we show is of infinite order. In Section 3, we describe the subgroup of K_2F_N which is generated by the K -theoretic transfers of those elements in K_2F_{dN} arising from functions with divisorial support on the points at infinity. We will see that this is simply the group of such elements in K_2F_N . Section 4 is an interlude, where we look at the example of F_4 . In the final section, we show that the subgroup of K_2F_N under investigation is a principal $\text{Aut } F_N$ -module, and use this to determine an upper bound on its rank as an abelian group in case N is an odd prime.

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1. The regulator, Bloch's trick, and symbols with support

We begin by recalling what the regulator of Beilinson and Bloch is in the special case that X is a smooth projective curve defined over \mathbf{Q} . We fix an algebraic closure $\bar{\mathbf{Q}} \subset \mathbf{C}$ of \mathbf{Q} . All algebraic extensions of \mathbf{Q} which arise are tacitly understood to lie in this fixed choice of $\bar{\mathbf{Q}}$.

Let F be a field. By Matsumoto's theorem [10], $K_2F \cong (F^* \otimes F^*)/R$, where R is the subgroup of $F^* \otimes F^*$ generated by the tensors of the form $f \otimes (1 - f)$, with $f \neq 0, 1$. The image of $f \otimes h$ in K_2F is denoted $\{f, h\}$, and is referred to as a *symbol*.

Let X be a smooth projective curve of genus g defined over \mathbf{Q} . The localization sequence provides us with the following exact sequence:

$$\coprod_{P \in X(\bar{\mathbf{Q}})} K_2\mathbf{Q}(P) \rightarrow K_2X \rightarrow K_2\mathbf{Q}(X) \xrightarrow{\tau} \coprod_{P \in X(\mathbf{Q})} \mathbf{Q}(P)^*$$

where

$$\tau = \coprod_{P \in X(\mathbf{Q})} \tau_P,$$

with τ_P being the tame symbol at P :

$$\tau_P(\{f, h\}) = (-1)^{(\text{ord}_P f)(\text{ord}_P h)} \frac{f^{\text{ord}_P h}}{h^{\text{ord}_P f}}(P).$$

By Garland's theorem [9], $K_2\mathbf{Q}(P)$ is torsion, whence K_2X and $\ker \tau$ agree, up to torsion. If k is any finite extension of \mathbf{Q} , then by viewing X as a curve over k , these remarks, with \mathbf{Q} replaced with k , remain valid.

In [2], Beilinson defines a regulator for K_2X , which agrees with that of Bloch [6]. This may be viewed as a homomorphism

$$\text{reg}_X: K_2X \rightarrow H^1(X(\mathbf{C}), \mathbf{R}(1))^-,$$

which is functorial in X . The superscript denotes the -1 -eigenspace of $H^1(X(\mathbf{C}), \mathbf{R}(1))$ under the action of complex conjugation on both $X(\mathbf{C})$ and $\mathbf{R}(1) = 2\pi i\mathbf{R}$. We remark that $H^1(X(\mathbf{C}), \mathbf{R}(1))^-$ is isomorphic to the Deligne cohomology group $H^2_{\mathcal{D}}(X, \mathbf{R}(2))$ mentioned above in the Introduction [18].

To describe reg_X , we assume for simplicity that $\{f, h\} \in \ker \tau$. For details, we refer the reader to [8] or [11]. We view $H^1(X(\mathbf{C}), \mathbf{C})$ as the space of \mathbf{C} -valued functionals on $H_1(X(\mathbf{C}), \mathbf{Z})$. Let γ be a closed path on $X(\mathbf{C})$, and fix a basepoint $P_0 \in \gamma$; assume that f and h are both regular and nonzero on γ . Then

$$\text{reg}_X(\{f, h\})(\gamma) = i \text{Im} \left(\int_{\gamma} \log f d \log h - \log |h(P_0)| \int_{\gamma} d \log f \right). \tag{1}$$

Here, we choose fixed branches of $\log f$ and $\log h$ on some neighborhood of γ , and the integrals are taken starting at P_0 . One may show that this definition depends only upon the class of $\gamma \in H_1(X(\mathbf{C}), \mathbf{Z})$.

Let Ω_X^{1+} denote the subspace of $H^0(X(\mathbf{C}), \Omega_{X/\mathbf{C}}^1)$ which is invariant under the action of complex conjugation on both $X(\mathbf{C})$ and $\Omega_{X/\mathbf{C}}^1$. Under the identification $H^1(X(\mathbf{C}), \mathbf{R}(1))^-$ with $\text{Hom}_{\mathbf{R}}(\Omega_X^{1+}, \mathbf{R})$, we view $\text{reg}_X(\{f, h\})$ as a functional on real 1-forms. One then obtains the following formula for the regulator:

$$\text{reg}_X(\{f, h\})(\omega) = \frac{1}{2\pi i} \int_{X(\mathbf{C})} \log |f| \overline{\log h} \wedge \omega. \tag{2}$$

One can check that this integral converges for any pair of functions on X , and we thereby obtain a homomorphism

$$\text{reg}_X: K_2\mathbf{C}(X) \rightarrow H^1(X(\mathbf{C}), \mathbf{C})$$

which is functorial in X and extends the regulator as defined above. As an immediate consequence of this functoriality, we have the following:

LEMMA 1. *Let $\phi: X \rightarrow \mathbf{P}^1$ be any morphism. Then $\text{reg}_X(\phi^*(\alpha)) = 0$ for all $\alpha \in K_2\mathbf{C}(\mathbf{P}^1)$.*

It follows from this, or directly from (2), that the regulator vanishes on those symbols of which one entry is constant.

We now turn to the transfer map in K -theory, and describe those facts which we will need in the sequel.

Let F/L be a finite Galois extension of fields with Galois group G , and denote by ϕ the inclusion $L \hookrightarrow F$. The two maps

$$\phi_*: K_n F \rightarrow K_n L$$

and

$$\phi^*: K_n L \rightarrow K_n F$$

are then related to the action of Galois on $K_n F$ in the following manner:

$$(\phi^* \circ \phi_*)(\alpha) = \prod_{\sigma \in G} \alpha^\sigma, \quad \alpha \in K_n F \tag{3}$$

$$(\phi_* \circ \phi^*)(\beta) = \beta^{[F:L]}, \quad \beta \in K_n L. \tag{4}$$

This follows from the fact that ϕ^* is induced from the extension of scalars functor $\underline{P}(L) \rightarrow \underline{P}(F)$, and ϕ_* is induced from the restriction of scalars functor $\underline{P}(F) \rightarrow \underline{P}(L)$, where $\underline{P}(A)$ is the category of finitely generated projective A -modules for any commutative ring A .

We will also need to know how the transfer behaves with respect to the regulator map, as follows. Let k/\mathbf{Q} be a finite Galois extension, and let $\psi: \mathbf{Q}(X) \hookrightarrow k(X)$ be the field inclusion.

LEMMA 2. *Let $\alpha \in K_2 k(X)$. Then*

$$\text{reg}_X(\psi_* \alpha) = \text{reg}_X(\psi^* \psi_* \alpha) = \text{reg}_X(\alpha^\Sigma)$$

where

$$\Sigma = \sum_{\sigma \in \text{Gal}(k/\mathbf{Q})} \sigma.$$

Proof. Since the regulator is defined over \mathbf{C} and ψ^* is induced by the field inclusion, we have $\text{reg}_X(\beta) = \text{reg}_X(\psi^* \beta)$ for all $\beta \in K_2 \mathbf{Q}(X)$. Letting $\beta = \psi_* \alpha$, the lemma then follows from (3). □

In particular, if $\text{reg}_X(\alpha) = 0$, then $\text{reg}_X(\psi_*\alpha) = 0$.

We now recall Bloch's trick [4]. Let $S = \{P_0, \dots, P_n\} \subset X(\bar{\mathbf{Q}})$. Choose an embedding $X \hookrightarrow \text{Jac } X$ such that P_0 corresponds to the identity element of $\text{Jac } X$. Assume that the images of the P_i under this embedding are torsion points.

Let f and g be functions in $\bar{\mathbf{Q}}(X)$ whose divisors consist entirely of points in S . One version of Bloch's trick asserts that there is an integer N , functions $\phi_i \in \bar{\mathbf{Q}}(X)^*$, and constants $c_i \in \bar{\mathbf{Q}}^*$ such that $\alpha = \{f, g\}^N \prod_i \{\phi_i, c_i\}$ is in the kernel of the tame symbol. We will refer to such an α as a *normalization* of $\{f, g\}$. Note that $\alpha \in K_2L(X)$ for some finite extension L of \mathbf{Q} ; by taking the K -theoretic transfer from $L(X)$ to $\mathbf{Q}(X)$ we obtain an element in $K_2\mathbf{Q}(X) \cap \ker \tau$. We remark that if k is a finite extension of \mathbf{Q} , $S \subset X(k)$, and $f, g \in k(X)^*$, then ϕ_i and c_i can be chosen to be defined over k .

Utilizing this version of Bloch's trick, we now construct a subgroup of $K_2X \otimes \mathbf{Q}$ which is associated to S .

Choose k so that $S \subset X(k)$, let $\mathcal{U}_S = \{f \in \bar{\mathbf{Q}}(X)^* : \text{ord}_Q(f) = 0 \text{ for all } Q \notin S\}$, and let $\{f_1, \dots, f_m\}$ be generators for $\mathcal{U}_S/\bar{\mathbf{Q}}^*$ defined over k . Let $\{\mathcal{U}_S, \mathcal{U}_S\}$ denote the subgroup of $K_2\bar{\mathbf{Q}}(X)$ generated by symbols $\{f, g\}$ such that $f, g \in \mathcal{U}_S$. Then $\{\mathcal{U}_S, \mathcal{U}_S\}$ is generated by symbols of the following three types:

$$\{c, d\} \quad c, d \in \bar{\mathbf{Q}}^* \tag{5}$$

$$\{f_i, c\} \quad c \in \bar{\mathbf{Q}}^* \tag{6}$$

$$\{f_i, f_j\} \tag{7}$$

The symbols in (5) are torsion, being in $K_2\bar{\mathbf{Q}} = \varinjlim K_2L$, with the limit being taken over all finite extensions L of \mathbf{Q} in $\bar{\mathbf{Q}}$. The symbols in (6) are pullbacks from $K_2\bar{\mathbf{Q}}(\mathbf{P}^1)$; we may therefore apply Bloch's trick on \mathbf{P}^1 and thereby obtain a symbol in $K_2\bar{\mathbf{Q}}(X)$ which is a pullback from $K_2\mathbf{P}^1/\bar{\mathbf{Q}}$. This all occurs over some finite extension of \mathbf{Q} , whence this symbol is torsion.

Let $\mathcal{N}_{X,S}(k)$ be the group generated by a normalization of each of those symbols in (7). This group depends on the normalizations chosen; see the remarks below for more about this dependence. Observe, however, that if $\mathcal{N}_{X,S}$ denotes the group generated by all the symbols in (5)–(7), then $\mathcal{N}_{X,S}(k) \otimes \mathbf{Q}$ and $\mathcal{N}_{X,S} \otimes \mathbf{Q}$ have the same image under the regulator map.

We now put $\mathcal{F}_{X,S}(\mathbf{Q}) = \text{tr } \mathcal{N}_{X,S}(k)$, where tr is the transfer map from $K_2k(X)$ to $K_2\mathbf{Q}(X)$. By (3), $\mathcal{F}_{X,S}(\mathbf{Q}) \subset \ker \tau$, and we may therefore identify $\mathcal{F}_{X,S}(\mathbf{Q}) \otimes \mathbf{Q}$ with a subspace of $K_2X \otimes \mathbf{Q}$, which we shall refer to as the *subspace of $K_2X \otimes \mathbf{Q}$ with divisorial support on S* .

REMARKS. Bloch's trick, as stated in [4], does not specify how to choose N and c_i , or even that the modifications to $\{f, g\}$ to yield an element of $\ker \tau$ are

of the type that we have selected. Any two normalizations of $\{f, g\}$ would differ by a rational multiple and an element of $\Gamma = \ker \tau \cap \ker \text{reg}_X$. Very little, if anything, seems to be known about Γ ; Beilinson conjectures that $(\Gamma \otimes \mathbf{Q}) \cap H^2_{\text{reg}}(X, \mathbf{Q}(2))_{\mathbf{Z}} = 0$. If X is such that $H^2_{\text{reg}}(X, \mathbf{Q}(2))_{\mathbf{Z}} = H^2_{\text{reg}}(X, \mathbf{Q}(2))$, then *conjecturally* our construction yields the same vector space as any version of Bloch's trick. In particular, the Fermat curves satisfy this condition, since their Jacobians have CM: Letting \mathcal{F}_N denote a regular proper model for F_N over \mathbf{Z} , we have the localization exact sequence in K -theory:

$$\coprod_p K'_2 \mathcal{F}_{N,p} \otimes \mathbf{Q} \rightarrow H^2_{\text{reg}}(F_N, \mathbf{Q}(2))_{\mathbf{Z}} \rightarrow H^2_{\text{reg}}(F_N, \mathbf{Q}(2)) \rightarrow \coprod_p K'_1 \mathcal{F}_{N,p} \otimes \mathbf{Q}.$$

By ([19], Theorem 3) the Euler factor $L_p(F_N, s)$ has no pole at $s = 0$; therefore, the right-most vector space is zero ([11], Proposition 4.7.9). Finally, the left-most vector space is zero ([1]).

2. An element of infinite order

Let $\zeta = e^{2\pi i/N}$, and let $A_{i,j}$ denote the automorphism

$$(x, y) \mapsto (\zeta^i x, \zeta^j y) \tag{8}$$

of F_N . Let $t^{1/N}$ denote the principal branch of the N th root function, and let γ denote the following path from $(1, 0)$ to $(0, 1)$ on $F_N(\mathbf{C})$:

$$\begin{aligned} \gamma: [0, 1] &\rightarrow F_N(\mathbf{C}) \\ \gamma: t &\mapsto ((1-t)^{1/N}, t^{1/N}). \end{aligned}$$

For integers m and n , let $\gamma_{m,n}$ denote the following closed path on $F_N(\mathbf{C})$:

$$\gamma_{m,n} = \gamma - A_{m,0}\gamma + A_{m,n}\gamma - A_{0,n}\gamma.$$

By a slight abuse of notation, we will also denote by $\gamma_{m,n}$ the corresponding element of $H_1(F_N(\mathbf{C}), \mathbf{Z})$.

We will need the beta function $B(u, v)$, which is defined for positive real numbers u and v by

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt.$$

THEOREM 1. *The symbol $\{1 - x, 1 - y\}^{2N}$ belongs $\ker \tau$ and has non-zero image under reg_{F_N} . In particular, $\{1 - x, 1 - y\}^{2N}$ represents an element in $K_2 F_N$ of infinite order.*

Proof. An easy calculation shows that $\{1 - x, 1 - y\}^{2N} \in \ker \tau$. We show that $\{1 - x, 1 - y\}^{2N}$ has nontrivial image under the regulator homomorphism by computing its value on the 1-cycle $\gamma_{1,1} + \gamma_{1,-1}$. Let $\varepsilon > 0$ be small, and choose a representative path $\gamma_{m,n,\varepsilon}$ for $\gamma_{m,n}$ such that the initial point of $\gamma_{m,n,\varepsilon}$ is $((1 - \varepsilon)^{1/N}, \varepsilon^{1/N})$, and such that both $1 - x$ and $1 - y$ are nonzero and regular on $\gamma_{m,n,\varepsilon}$. Let $\sigma = \{1 - x, 1 - y\}^{2N}$. Choosing the branches of $\log(1 - x)$ and $\log(1 - y)$ which are real-valued on γ , we find that $\text{reg}_{F_N}(\sigma)(\gamma_{m,n})$ is equal to:

$$2iN \operatorname{Im} \left(\int_{\gamma_{m,n,\varepsilon}} \log(1 - x) d \log(1 - y) - \log(1 - \varepsilon^{1/N}) \int_{\gamma_{m,n,\varepsilon}} d \log(1 - x) \right).$$

Note that as ε tends toward zero, the imaginary part of the second integral tends toward zero. Since the value of this integral depends only on the homology class of $\gamma_{m,n}$ and not on the basepoint chosen, we conclude that

$$\text{reg}_{F_N}(\sigma)(\gamma_{m,n}) = 2iN \operatorname{Im} \int_{\gamma_{m,n}} \log(1 - x) d \log(1 - y).$$

For any $a, b \in \mathbb{Z}$, let

$$I_{a,b} = \int_0^1 \log(1 - \zeta^a(1 - t)^{1/N}) d \log(1 - \zeta^b t^{1/N}).$$

Then

$$\begin{aligned} I_{a,b} &= \int_0^1 \frac{1}{N} \sum_{k=1}^{\infty} \frac{1}{k} \zeta^{ak} (1 - t)^{k/N} \sum_{j=1}^{\infty} \zeta^{jb} t^{j/N-1} dt \\ &= \frac{1}{N} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k} \zeta^{ka+jb} \int_0^1 (1 - t)^{k/N} t^{j/N-1} dt. \end{aligned}$$

The interchange of the two sums and the integral may be justified by a double application of the Lebesgue dominated convergence theorem. Moreover, the double sum converges absolutely, which may be seen by considering the case $a = b = 0$. Noting that the last integral above is $B(j/N, k/N + 1)$ and using the identity

$$B(u, v + 1) = \frac{v}{u + v} B(u, v),$$

we obtain

$$I_{a,b} = \frac{1}{N} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k+j} \zeta^{ka+jb} B\left(\frac{j}{N}, \frac{k}{N}\right).$$

Returning to $\gamma_{m,n}$, we find that

$$\begin{aligned} \operatorname{reg}_{F_N}(\sigma)(\gamma_{m,n}) &= 2iN \operatorname{Im}(I_{0,0} - I_{m,0} + I_{m,n} - I_{0,n}) \\ &= 2i \operatorname{Im} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k+j} (1 - \zeta^{km})(1 - \zeta^{jn}) B\left(\frac{j}{N}, \frac{k}{N}\right). \end{aligned}$$

Therefore

$$\operatorname{reg}_{F_N}(\sigma)(\gamma_{1,1} + \gamma_{1,-1}) = -4i \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k+j} \sin \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi j}{N}\right) B\left(\frac{j}{N}, \frac{k}{N}\right).$$

Define b_k by

$$b_k = \sum_{j=1}^{\infty} \frac{1}{k+j} \left(1 - \cos \frac{2\pi j}{N}\right) B\left(\frac{j}{N}, \frac{k}{N}\right),$$

and for $1 \leq K \leq N$, let

$$\beta_K = \sum_{k \equiv K \pmod N} b_k.$$

Note that $b_k > 0$ for all k and that the sequence $\{b_k\}$ is monotonically decreasing, so $\beta_1 > \beta_2 > \dots > \beta_N > 0$. Let N' be defined by

$$N' = \begin{cases} \frac{N-1}{2} & \text{if } N \text{ is odd} \\ \frac{N}{2} - 1 & \text{if } N \text{ is even.} \end{cases}$$

Then

$$\begin{aligned} \operatorname{reg}_{F_N}(\sigma)(\gamma_{1,1} + \gamma_{1,-1}) &= -4i \sum_{K=1}^N \beta_K \sin \frac{2\pi K}{N} \\ &= -4i \sum_{K=1}^{N'} (\beta_K - \beta_{N-K}) \sin \frac{2\pi K}{N} \end{aligned}$$

which is nonzero, since each summand is positive. □

3. Symbols with support at infinity and the Fermat tower

We now turn our attention to the elements in K_2F_N which have divisorial support on the points at infinity. By the *points at infinity* we mean the set $S_N = \{P \in F_N(\bar{\mathbf{Q}}) : xyz(P) = 0\}$. There are $3N$ such points, given in projective coordinates by $[\zeta^j, 0, 1]$, $[0, \zeta^j, 1]$, and $[\zeta\zeta^j, 1, 0]$, for $1 \leq j \leq N$, where ζ is a primitive N th root of unity and ξ is a primitive $2N$ th root of unity, which we shall always take to satisfy $\xi^2 = \zeta$.

Let $k = \mathbf{Q}(\mu_{2N})$, and, to fix matters, choose an embedding $\iota : F_N \hookrightarrow \text{Jac } F_N$ defined over \mathbf{Q} and sending the point $[0, 1, 1]$ to the origin. A theorem of Rohrlich [12] asserts that $\iota(S_N)$ is torsion and, in the notation of Section 1, the group $\mathcal{U}_S/\bar{\mathbf{Q}}^*$ is generated by the following functions, all of which are defined over k :

$$x \quad y \quad x - \zeta^j \quad y - \zeta^j \quad x - \xi\zeta^j y \tag{9}$$

$$\left(\prod_{j=0}^{N-1} ((x - \zeta^j)(y - \zeta^j))^j \right)^{1/N} \tag{10}$$

$$\left(\prod_{j=0}^{N-1} ((x - \zeta^j)(x - \xi\zeta^j y))^j \right)^{1/N} \tag{11}$$

$$\left(\prod_{j=0}^{N-1} ((x - \zeta^j)(y - \zeta^j)(x - \xi\zeta^j y))^{j(j+1)/2} \right)^{1/E(N)} \tag{12}$$

with $E(N) = N$ if N is odd and $E(N) = N/2$ if N is even.

We are interested in the space $\mathcal{T}_{F_N, S_N}(\mathbf{Q}) \otimes \mathbf{Q}$, the space of symbols in $K_2F_N \otimes \mathbf{Q}$ with divisorial support at infinity. Recall that this space is $\psi_*(\mathcal{N}_{F_N, S_N}(k) \otimes \mathbf{Q})$, where $\mathcal{N}_{F_N, S_N}(k)$ is the group generated by normalizations of the symbols obtained from the functions in (9)–(12), and ψ_* is the transfer map induced by the field inclusion $\psi^{-1} : \mathbf{Q}(F_N) \hookrightarrow k(F_N)$.

We will see below (Theorem 3) that for a suitable choice of normalizations,

$$\mathcal{N}_{F_N, S_N}(k) \otimes \mathbf{Q} = \mathbf{Q}[\Gamma_N] \cdot \{1 - x, 1 - y\}$$

where Γ_N is the automorphism group of F_N . Let $\mathcal{S}_{F_N, S_N}(k)$ denote the \mathbf{Q} -vector space $\mathbf{Q}[\Gamma_N]\{1 - x, 1 - y\}$, and let $\mathcal{L}_{F_N, S_N}(\mathbf{Q}) = \psi_*\mathcal{S}_{F_N, S_N}(k)$. It follows from Theorem 3 that $\mathcal{T}_{F_N, S_N}(\mathbf{Q}) \otimes \mathbf{Q} = \mathcal{L}_{F_N, S_N}(\mathbf{Q})$. We will assume these facts in this section, since the results below do not depend on those discussed here.

For each positive integer d , let

$$\phi_d: F_{dN} \rightarrow F_N$$

be the morphism given by

$$\phi_d: (x, y) \mapsto (x^d, y^d).$$

Let u and v be the standard coordinate functions on F_N , such that $u = x^d$ and $v = y^d$. Put $k_d = \mathbf{Q}(\mu_{2dN})$, so $k_d(F_{dN}) = k_d(F_N)(x, y)$ is an abelian extension of $k_d(F_N)$ with Galois group $G_d \cong \mu_d \times \mu_d$.

For each integer $n \geq 1$, let $A_{i,j}(n)$ (i and j read modulo n), $\sigma(n)$, and $\eta(n)$ denote the following elements of $\Gamma_n = \text{Aut } F_n$:

$$A_{i,j}(n): (x, y) \mapsto (\zeta_n^i x, \zeta_n^j y)$$

$$\sigma(n): (x, y) \mapsto (y, x)$$

$$\eta(n): (x, y) \mapsto \left(\frac{1}{y}, \zeta_n^{-1} \frac{x}{y} \right)$$

where ζ_n is a primitive n th root of unity and $\zeta_n^2 = \zeta$, $\zeta_n^n = -1$. Then Γ_n is generated by $A_{i,j}(n)$, $\sigma(n)$, and $\eta(n)$, and is isomorphic to a semidirect product of $\mu_n \times \mu_n$ and the symmetric group S_3 .

Now fix N , let ζ_N and ξ_N be as above, and for each integer $d \geq 1$, choose ζ_{dN} and ξ_{dN} such that $\zeta_{dN}^d = \zeta_N$ and $\xi_{dN}^d = \xi_N$. Then the map $\Gamma_{dN} \rightarrow \Gamma_N$ given by

$$A_{i,j}(dN) \mapsto A_{i,j}(N)$$

$$\sigma(dN) \mapsto \sigma(N)$$

$$\eta(dN) \mapsto \eta(N)$$

is a homomorphism, the kernel of which is the relative automorphism group $\text{Aut}(F_{dN}/F_N)$. We view $\mathcal{S}_{F_N, S_N}(k)$ as a Γ_{dN} -module via this map. Let $j_d: k_1(F_N) \hookrightarrow k_d(F_N)$ be the field inclusion.

PROPOSITION 1. $\phi_{d*} \mathcal{S}_{F_{dN}, S_{dN}}(k_d) = j_d^* \mathcal{S}_{F_N, S_N}(k)$.

Proof. A direct calculation using (3) shows that

$$\phi_d^* \circ \phi_{d*}: \mathcal{S}_{F_{dN}, S_{dN}}(k_d) \otimes \mathbf{Q} \rightarrow \mathcal{S}_{F_N, S_N}(k) \otimes \mathbf{Q}$$

is Γ_{dN} -equivariant, and that

$$(\phi_d^* \circ \phi_{d*})\{1 - x, 1 - y\} = (\phi_d^* \circ j_d^*)\{1 - u, 1 - v\}.$$

Therefore

$$\begin{aligned}
 (\phi_{d*}^* \circ \phi_{d*}) \mathcal{S}_{F_{dN}, S_{dN}}(k_d) &= (\phi_{d*}^* \circ \phi_{d*})(\mathbf{Q}[\Gamma_{dN}]\{1-x, 1-y\}) \\
 &= \mathbf{Q}[\Gamma_{dN}](\phi_{d*}^* \circ \phi_{d*})\{1-x, 1-y\} \\
 &= \mathbf{Q}[\Gamma_{dN}](\phi_{d*}^* \circ j_d^*)\{1-u, 1-v\} \\
 &= (\phi_{d*}^* \circ j_d^*) \mathbf{Q}[\Gamma_N]\{1-u, 1-v\} \\
 &= (\phi_{d*}^* \circ j_d^*) \mathcal{S}_{F_N, S_N}(k).
 \end{aligned}$$

It follows from (4) that $\ker \phi_{d*}^*$ is trivial (recall that we have tensored with \mathbf{Q}). Therefore $\phi_{d*} \mathcal{S}_{F_{dN}, S_{dN}}(k_d) = j_d^* \mathcal{S}_{F_N, S_N}(k)$. \square

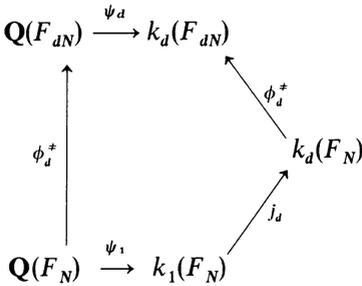
We now descend to \mathbf{Q} . Let $\psi_d: \mathbf{Q}(F_{dN}) \hookrightarrow k_d(F_{dN})$ be the field inclusion.

THEOREM 2. $\phi_{d*} \mathcal{Q}_{F_{dN}, S_{dN}}(\mathbf{Q}) = \mathcal{Q}_{F_N, S_N}(\mathbf{Q})$.

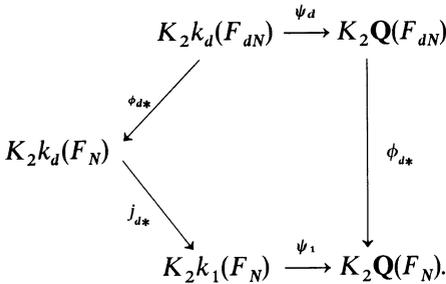
Proof. What we must show is that

$$(\phi_{d*} \circ \psi_{d*}) \mathcal{S}_{F_{dN}, S_{dN}}(k_d) = \psi_{1*} \mathcal{S}_{F_N, S_N}(k).$$

From the following diagram of field inclusions:



we obtain the following commutative diagram in K -theory:



Note that by (4), $(j_{d*} \circ j_d^*)\mathcal{S}_{F_N, S_N}(k) = \mathcal{S}_{F_N, S_N}(k)$. Therefore

$$\begin{aligned} (\phi_{d*} \circ \psi_{d*})\mathcal{S}_{F_{dN}, S_{dN}}(k_d) &= (\psi_{1*} \circ j_{d*} \circ \phi_{d*})\mathcal{S}_{F_{dN}, S_{dN}}(k_d) \\ &= (\psi_{1*} \circ j_{d*} \circ j_d^*)\mathcal{S}_{F_N, S_N}(k) \\ &= \psi_{1*}\mathcal{S}_{F_N, S_N}(k) \end{aligned}$$

as desired. □

4. The Fermat curve of exponent 4

In this section, we examine in some detail the curve F_4 . In particular, we will exhibit a symbol $\{f, g\} \in \mathcal{Q}_{F_4, S_4}(\mathbf{Q})$ such that the divisor of f contains points which are of infinite order under the canonical embedding $F_4 \rightarrow \text{Jac } F_4$; compare with [15]. In some contrast to [15], we can relate this symbol to the L -value $L^{(3)}(F_4, 0)$ in the manner predicted by the Beilinson conjecture.

We begin by showing that the symbols

$$\alpha_1 = \{x^2 + 1, 1 - y\}^8, \quad \alpha_2 = \{1 - x, y^2 + 1\}^8, \quad \alpha_3 = \{1 - x, 1 - y\}^8$$

generate a rank 3 subgroup of $K_2 F_4$. Note that these symbols lie in $\mathcal{Q}_{F_4, S_4}(\mathbf{Q})$.

Let $\gamma_{m,n}$ be as in Section 2. Calculating as in the proof of Theorem 1, we find that

$$\text{reg}_{F_4}(\alpha_1)(\gamma_{m,n}) = \begin{cases} 8i(\beta_1 - \beta_3) & \text{if } n = 1, m \text{ odd} \\ 8i(\beta_3 - \beta_1) & \text{if } n = 3, m \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\beta_j = \sum_{j \equiv J \pmod{4}} \sum_{k=1}^{\infty} \frac{1}{4k + j - 2} B\left(k - \frac{1}{2}, \frac{j}{4}\right).$$

Note that $\beta_1 > \beta_2 > \beta_3 > \beta_4 > 0$; thus in the first two cases the regulator value is non-zero.

A similar calculation for α_2 yields:

$$\text{reg}_{F_4}(\alpha_2)(\gamma_{m,n}) = \begin{cases} 8i(\beta_1 - \beta_3) & \text{if } m = 1, n \text{ odd} \\ 8i(\beta_3 - \beta_1) & \text{if } m = 3, n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Define B_{odd} and B_{even} by

$$B_{\text{odd}} = \sum_{\substack{k \text{ odd} \\ k \geq 1}} \sum_{j=1}^{\infty} \frac{1}{j+k-1} B\left(\frac{2j-1}{4}, \frac{2k-1}{4}\right)$$

$$B_{\text{even}} = \sum_{\substack{k \text{ even} \\ k \geq 2}} \sum_{j=1}^{\infty} \frac{1}{j+k-1} B\left(\frac{2j-1}{4}, \frac{2k-1}{4}\right).$$

Note that $B_{\text{odd}} > B_{\text{even}} > 0$. Calculating as in the proof of Theorem 1, one finds that

$$\text{reg}_{F_4}(\alpha_3)(\gamma_{2,1}) = 2i(B_{\text{odd}} - B_{\text{even}}),$$

which is nonzero. Finally, one may verify that these elements are linearly independent by considering their values on the paths $\gamma_{2,1}$, $\gamma_{1,3}$ and $\gamma_{3,3}$.

It is not difficult to write down an explicit isomorphism $X_0(64) \rightarrow F_4$ which is defined over \mathbf{Q} , and is such that the cusps of $X_0(64)$ correspond precisely with the points at infinity on F_4 . Thus α_1 , α_2 , and α_3 generate the subspace $\mathcal{P}_{X_0(64)}$ of $K_2X_0(64) \otimes \mathbf{Q}$ described in the Introduction. In particular, the images of α_1 , α_2 , and α_3 under reg_{F_4} define a \mathbf{Q} -structure on $H_{\mathcal{D}}^2(F_4, \mathbf{R}(2))$, with volume equal to a rational multiple of $L^{(3)}(F_4, 0)$.

On the other hand, we can approach K_2F_4 “from below”. The Jacobian $J_0(64)$ of $X_0(64)$ is isogenous over \mathbf{Q} to $E \times E' \times E'$, where E and E' are the elliptic curves

$$E: Y^2 = X^3 - 4X$$

$$E': Y^2 = X^3 + 4X,$$

and we have morphisms $\rho: F_4 \rightarrow E$, $\beta: F_4 \rightarrow E'$, and $\beta': F_4 \rightarrow E'$ given by:

$$\rho(x, y) = \left(2 \frac{y^2 + 1}{x^2}, 4 \frac{y(y^2 + 1)}{x^3} \right)$$

$$\beta(x, y) = \left(2 \frac{1 - y^2}{x^2}, 4 \frac{1 - y^2}{x^3} \right)$$

$$\beta'(x, y) = \left(2 \frac{1 - x^2}{y^2}, 4 \frac{1 - x^2}{y^3} \right).$$

One computes that

$$\beta'^* \left\{ 1 - \frac{2X}{Y}, \frac{8X}{Y^2} \right\}^8 = \alpha_1^{-1}$$

$$\beta^* \left\{ 1 - \frac{2X}{Y}, \frac{8X}{Y^2} \right\}^8 = \alpha_2.$$

Noting that the divisors of the pullbacks to F_4 of $8X/Y^2$ and $1 - 2X/Y$ by β and β' consist of points at infinity, we conclude that these functions have torsion divisorial support on E' . A calculation using Rohrlich's explicit version [13] of Bloch's theorem [5] shows that

$$\text{reg}_{E'} \left(\left\{ 1 - \frac{2X}{Y}, \frac{8X}{Y^2} \right\}^8 \right)$$

defines a \mathbf{Q} -structure on $H_{\mathcal{O}}^2(E', \mathbf{R}(2))$ with volume equal to $c'L'(0, E')$, where $c' \in \mathbf{Q}^*$. The conjectures of Beilinson imply that

$$\left\{ 1 - \frac{2X}{Y}, \frac{8X}{Y^2} \right\}$$

is a generator for $K_2 E' \otimes \mathbf{Q}$.

Note that

$$f = 1 - x/y \quad \text{and} \quad g = \frac{1 - y^2}{1 - x^2},$$

while functions on F_4 , actually define functions on E , which we also denote by f and g . The divisors of f and g on E consist of torsion points, and computing as above we find that $\text{reg}_E(\{f, g\})$ determines a \mathbf{Q} -structure on $H_{\mathcal{O}}^2(E, \mathbf{R}(2))$ with volume equal to $cL'(0, E)$ with $c \in \mathbf{Q}^*$. The Beilinson conjectures imply that $\{f, g\}$ is a generator for $K_2 E \otimes \mathbf{Q}$. Put $\tilde{\alpha}_3 = \rho^* \{f, g\}^8$.

There is a decomposition of cohomology induced by ρ , β , and β' :

$$H^1(F_4(\mathbf{C}), \mathbf{C}) \cong H^1(E(\mathbf{C}), \mathbf{C}) \oplus H^1(E'(\mathbf{C}), \mathbf{C}) \oplus H^1(E'(\mathbf{C}), \mathbf{C})$$

which is orthogonal with respect to the pairing

$$\langle \omega, \eta \rangle = \frac{1}{2\pi i} \int_{F_4(\mathbf{C})} \omega \wedge \bar{\eta},$$

where we are using DeRham cohomology. Therefore $\alpha_1, \alpha_2,$ and $\tilde{\alpha}_3$ span a 3-dimensional subspace V of $K_2F_4 \otimes \mathbf{Q}$, and the image of V under the regulator is a \mathbf{Q} -structure of $H_{\mathcal{D}}^2(F_4, \mathbf{R}(2))$ with volume equal to a rational multiple of $L^{(3)}(F_4, 0)$.

If the Beilinson conjectures are true, we should have $V = \mathcal{L}_{F_4, S_4}(\mathbf{Q})$ (and both should be equal to $K_2F_4 \otimes \mathbf{Q}$). We now verify that it is indeed the case that $V = \mathcal{L}_{F_4, S_4}(\mathbf{Q})$. Note that it suffices to check that $\tilde{\alpha}_3 \in \mathcal{T}_{F_4, S_4}(\mathbf{Q}) \otimes \mathbf{Q}$.

A computation using (3) shows that, up to torsion,

$$\alpha_3^2 \alpha_1 \alpha_2 = \rho^*(\rho_* \alpha_3).$$

Using the algorithm in [16], we compute that, up to torsion,

$$\rho^*(\rho_* \alpha_3) = \tilde{\alpha}_3.$$

Therefore, $\tilde{\alpha}_3 \in \mathcal{L}_{F_4, S}(\mathbf{Q})$, as desired.

In closing, we note that the divisor of $1 - x/y$ contains points which are not points at infinity on F_4 ; by a result of Coleman [7], these points are of infinite order in $\text{Jac } F_4$.

5. The group of symbols with support at infinity

In this section, we fix our attention on F_N , and let $k = k_1 = \mathbf{Q}(\mu_{2N})$, $\mathcal{N} = \mathcal{N}_{F_N, S_N}(k)$, $\mathcal{S} = \mathcal{S}_{F_N, S_N}(k)$, $\mathcal{T} = \mathcal{T}_{F_N, S_N}(\mathbf{Q})$, $\psi = \psi_1: \mathbf{Q}(F_N) \hookrightarrow k(F_N)$, and $G = G_1 = \text{Gal}(k/\mathbf{Q})$.

In general, $\mathcal{T} \otimes \mathbf{Q}$ cannot be equal to $K_2F_N \otimes \mathbf{Q}$, provided that one accepts the Beilinson conjectures. For the predicted rank of K_2F_N is the genus of F_N , which equals $(N - 1)(N - 2)/2$, and we have

PROPOSITION 2. *Let p be an odd prime. Then $\dim_{\mathbf{Q}} \mathcal{T}_{F_p, S_p}(\mathbf{Q}) \otimes \mathbf{Q} \leq 3(p - 1)$. In particular, for $p > 7$, we have $\dim_{\mathbf{Q}} \mathcal{T}_{F_p, S_p}(\mathbf{Q}) \otimes \mathbf{Q} < \text{genus}(F_p)$.*

Proposition 2 is a consequence of the following:

THEOREM 3. *With an appropriate choice of normalizations, $\mathcal{N} \otimes \mathbf{Q} = \mathcal{S} = \mathbf{Q}[\Gamma_N]\{1 - x, 1 - y\}$.*

Proof. By (9)–(12), it is clear that $\mathcal{N} \otimes \mathbf{Q}$ is generated over \mathbf{Q} by normalizations of the following symbols:

$$\{x - \zeta^j, y - \zeta^k\}, \{x - \zeta^j, x - \xi \zeta^k y\}, \{y - \zeta^j, x - \xi \zeta^k y\} \tag{13}$$

$$\{x - \xi \zeta^j y, x - \xi \zeta^k y\}, \{x - \xi \zeta^j y, x\}, \{x - \xi \zeta^j y, y\} \tag{14}$$

$$\{x - \zeta^j, y\}, \{y - \zeta^j, x\}, \{x - \zeta^j, x - \zeta^k\}, \{y - \zeta^j, y - \zeta^k\} \tag{15}$$

$$\{x - \zeta^j, x\}, \{y - \zeta^j, y\}, \{x, y\} \tag{16}$$

for $0 \leq j, k \leq N - 1$. Note that we do not need to include symbols for which one of the entries is a function listed in (10)–(12), because we have tensored with \mathbf{Q} .

The symbols in (16) are torsion in $K_2k(F_N)$.

Although not torsion, appropriate powers of the symbols in (15) are pullbacks from $K_2k(\mathbf{P}^1)$, and therefore have trivial normalizations.

The symbols in (14) are also pullbacks from $K_2k(\mathbf{P}^1)$. Indeed, letting $\eta = \eta(N)$, one finds that:

$$\begin{aligned} \eta^*\{x - \zeta^j y, x - \zeta^k y\}^{2N} &= \{1 - \zeta^j x, 1 - \zeta^k x\}^{2N} \left\{1 - x^N, \frac{1 - \zeta^j x}{1 - \zeta^k x}\right\}^2 \\ \eta^*\{x - \zeta^j y, x\}^{2N} &= \{1 - \zeta^j x, 1 - x^N\}^{-2} \\ (\eta^2)^*\{x - \zeta^j y, y\}^{2N} &= \{1 - y^N, y - \zeta^j\}^{-2}. \end{aligned}$$

Thus the symbols in (14)–(16) all have trivial normalizations. We now show that the symbols in (13) have normalizations which lie in \mathcal{S} , and explicitly determine these normalizations. Given a symbol $\{f, g\}$, we will denote a normalization of $\{f, g\}$ by $v(f, g)$.

Since $\{x - \zeta^j, y - \zeta^k\}^{2N} = A_{j,-k}^* \{1 - x, 1 - y\}^{2N}$, we take

$$v(x - \zeta^j, y - \zeta^k) = \{x - \zeta^j, y - \zeta^k\}^{2N} = A_{j,-k}^* \{1 - x, 1 - y\}^{2N}. \tag{17}$$

In examining the other symbols in (13), we will need the following. Since $\{1 - x, y\}^{2N} = \{1 - x, 1 - x^N\}^2 \in K_2k(x)$, we apply Bloch’s trick and select $\delta = \prod_i \{f_i(x), c_i\}$ with $f_i \in k(x)^*$ and $c_i \in k^*$ such that

$$\{1 - x, y\}^{2N} \cdot \delta = \{1 - x, 1 - x^N\}^2 \cdot \delta \in \ker \tau \cap K_2k(x).$$

Therefore $\{1 - x, y\}^{2N} \cdot \delta$ is torsion; let M denote its order. We thus have

$$\{1 - x, y\}^{2MN} = \{1 - x, 1 - x^N\}^{2M} = \delta^{-M}. \tag{18}$$

We note that the image of δ under any automorphism of F_N is a product of symbols in which one entry is constant.

A routine calculation shows that we may select normalizations of the remaining symbols in (13) as follows:

$$\begin{aligned} v(x - \zeta^j, x - \zeta^k y) &= \{x - \zeta^j, x - \zeta^k y\}^{2MN} \cdot [A_{j,-k}^*(\eta^2)^*](\delta^M) \\ &= [A_{j,-k}^*(\eta^2)^*] \{1 - x, 1 - y\}^{-2MN}. \end{aligned} \tag{19}$$

and

$$\begin{aligned} v(y - \zeta^j, x - \xi \zeta^k y) &= \{y - \zeta^j, x - \xi \zeta^k y\}^{2MN} \cdot [\eta^* A_{j,-k}^* \sigma^*](\delta^M) \\ &= [\eta^* A_{j,-k}^* \sigma^*] \{1 - x, 1 - y\}^{-MN}. \end{aligned} \tag{20}$$

From this it is clear that $\mathcal{N} \otimes \mathbf{Q} \subset \mathcal{S}$; a routine verification using (17)–(20) yields the reverse inclusion. \square

Proof of Proposition 2. Note that as \mathbf{Q} -vector spaces, $\mathcal{F} \otimes \mathbf{Q} \cong \psi^* \mathcal{F} \otimes \mathbf{Q} \cong \psi^* \psi_* \mathcal{N} \otimes \mathbf{Q}$. Let $\Sigma = \Sigma_{\sigma \in G} \sigma$. Then by (3) we see that $\mathcal{F} \otimes \mathbf{Q} \cong \mathcal{N}^\Sigma \otimes \mathbf{Q}$; in light of the previous result, we consider \mathcal{S}^Σ .

We first make a convenient choice of spanning set for \mathcal{S} . Since p is odd, $x - \xi \zeta^k y = x + \zeta^{k'} y$, where $k' = (2k + 1 + p)/2$. In what amounts to a re-indexing, let

$$\begin{aligned} T_{j,k} &= \{x - \zeta^j, y - \zeta^k\} \\ U_{j,k} &= \{x - \zeta^j, x + \zeta^k y\} \\ V_{j,k} &= \{y - \zeta^j, x + \zeta^k y\}. \end{aligned}$$

Then \mathcal{S} is spanned by the normalizations $v((T_{j,k}), v(U_{j,k}),$ and $v(V_{j,k})$ of these symbols.

Therefore \mathcal{S}^Σ is generated by $v(T_{j,k})^\Sigma, v(U_{j,k})^\Sigma,$ and $v(V_{j,k})^\Sigma$. One finds, using (18), (19), and (20) that $v(T_{j,k})^\Sigma = (T_{j,k}^\Sigma)^{2N}, v(U_{j,k})^\Sigma = (U_{j,k}^\Sigma)^{2MN},$ and $v(V_{j,k})^\Sigma = (V_{j,k}^\Sigma)^{2MN}.$

Assume that $(j, k) \neq (0, 0)$. Let $B_{j,k}$ denote any one of $T_{j,k}, U_{j,k},$ and $V_{j,k}$. If a is an integer relatively prime to p , then $B_{j,k}^\Sigma = B_{aj,ak}^\Sigma,$ where the subscripts are read modulo p . This follows from the fact that $B_{j,k}^{\sigma_a} = B_{aj,ak},$ where $\sigma_a: \zeta \mapsto \zeta^a$ is the Frobenius automorphism at a .

It follows that \mathcal{S}^Σ is generated by the following $3(p + 2)$ elements:

$$\begin{array}{ccc} T_{0,0}^\Sigma & T_{1,0}^\Sigma & T_{j,1}^\Sigma \\ U_{0,0}^\Sigma & U_{1,0}^\Sigma & U_{j,1}^\Sigma \\ V_{0,0}^\Sigma & V_{1,0}^\Sigma & V_{j,1}^\Sigma \end{array}$$

for $0 \leq j \leq p - 1$. Note that we have the following relations:

$$\begin{aligned} \prod_{j=0}^{p-1} T_{j,1}^{2\Sigma} &= 1 \\ T_{0,0} &= T_{0,1}^{-\Sigma} \\ T_{0,0} &= T_{1,0}^\Sigma \end{aligned}$$

So we may discard the elements $T_{0,0}^{\Sigma}$, $T_{0,1}^{\Sigma}$, and $T_{1,0}^{\Sigma}$. The same remark holds, *mutatis mutandis*, for the $U_{j,k}^{\Sigma}$ and the $V_{j,k}^{\Sigma}$, since they are obtained from the $T_{j,k}^{\Sigma}$ by automorphisms. So \mathcal{F} is generated over \mathbf{Q} by $3(p-1)$ symbols, as claimed. \square

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