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The Yang-Baxter and Pentagon equation

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1. Introduction

Let $A$ be a finite-dimensional Hopf algebra and let $A^\circ$ be the dual Hopf algebra with opposite comultiplication (see e.g. [1]). The algebraic tensor product $A \otimes A^\circ$ can be made into a quasi-triangular Hopf algebra (see e.g. [3]). The comultiplication is essentially the tensor product of the comultiplications on $A$ and $A^\circ$, but the multiplication is in general different from the usual tensor product multiplication. However, when $A$ and $A^\circ$ have a unit 1, the mappings $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ are Hopf algebra embeddings and $a \otimes b = (a \otimes 1)(1 \otimes b)$ in $A \otimes A^\circ$. We will identify $A$ and $A^\circ$ with their images in $A \otimes A^\circ$ and so we will use $AA^\circ$ to denote this quasi-triangular Hopf algebra. Remark that in general $A$ and $A^\circ$ will not commute with each other.

Consider also the tensor product $AA^\circ \otimes AA^\circ$. It is clear that $A^\circ \otimes A$ is a subspace of $AA^\circ \otimes AA^\circ$. The canonical element $W$ (i.e. the identity map when $A^\circ \otimes A$ is identified with the space $L(A, A)$ of linear maps from $A$ to $A$) is an invertible element in $AA^\circ \otimes AA^\circ$, where now we do take the usual tensor product structure. This $W$ intertwines the comultiplication $\Delta$ on $AA^\circ$ with the opposite comultiplication $\Delta'$ in the sense that $\Delta(a) = W\Delta'(a)W^{-1}$ when $a \in AA^\circ$. Moreover $W$ satisfies the Yang-Baxter equation in $AA^\circ \otimes AA^\circ \otimes AA^\circ$, that is

$$W_{12}W_{13}W_{23} = W_{23}W_{13}W_{12}$$

when $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ and $W_{13}$ is the obvious image of $W$ with 1 in the middle (see e.g. [3] and [14]).

In the infinite-dimensional case, this construction breaks down for several reasons. First the dual space $A'$ of $A$ is no longer a Hopf algebra in the sense that the obvious candidate $\Delta$ for the comultiplication will not map $A'$ into $A' \otimes A'$ (but only in $(A \otimes A)'$, which is strictly larger). In many of the well-known examples however, there are enough elements $b \in A'$ such that $\Delta(b) \in A' \otimes A'$, and since these elements form a Hopf algebra (see [12, page 109]), this first difficulty can easily be overcome in many cases.
The second problem is that in the infinite-dimensional case $L(A, A)$ is bigger than $A^\circ \otimes A$ and that the canonical element $W$ is not in $A^\circ \otimes A$. So, strictly speaking, the intertwining property and the Yang-Baxter equation have only a formal meaning. This problem can be overcome by considering finite-dimensional representations $\pi$ of $A$ so that $(1 \otimes \pi)(W)$ is an element of $A^\circ \otimes \pi(A)$. These elements satisfy the right properties.

In the infinite-dimensional case there is a need for a topological approach using topological tensor products and allowing the comultiplication to go outside the algebraic tensor product. This seems to be very difficult. The C*-algebra approach of Woronowicz to quantum groups ([15]) is not yet completely satisfactory, but has the advantage that much is known about topological tensor products here. The approach of Baaj and Skandalis ([2 and 11]) is close to the C*-algebra approach of Woronowicz. They work with the Pentagon equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12},$$

which is similar to the Yang-Baxter equation and is obtained in the finite-dimensional case above if we make $A \otimes A^\circ$ into an algebra in a different way. Moreover, the C*-algebra approach seems to be impossible in some cases (e.g. for the Hopf *-algebra generated by two self-adjoint elements $a$ and $b$ such that $a$ is invertible and $ab = \lambda ba$ with $|\lambda| = 1$), see also [16].

In this paper we make an attempt to get a precise interpretation of the formal construction of Drinfel'd. This has also been done by others. Woronowicz gave solutions of the Yang-Baxter equation (in fact the Braid equation) in the space of linear maps on $A \otimes A \otimes A$ [17]. There is also an attempt to give a precise meaning to the Yang-Baxter equation in the algebraic dual $(AA^\circ \otimes AA^\circ \otimes AA^\circ)$ by Koornwinder [5]. Here again, some extra conditions are necessary (like the existence of the Hopf subalgebra $A^\circ$ in the dual space $A'$). We need no extra conditions on $A$. And we treat the Pentagon equation, as well as the Yang-Baxter equation. We also work in the framework of Hopf*-algebras (so that the $W$ becomes a unitary element).

Our approach is as follows. Let $A$ be any Hopf*-algebra (over $\mathbb{C}$). For any *-algebra $D$ one can make the space $L(A, D)$ of linear maps from $A$ to $D$ into a *-algebra. If $D = \mathbb{C}$ we get of course the space $A'$ of linear functionals on $A$ with the usual *-algebra structure. In section 2 of this paper we introduce the notion of a twisted tensor product of two *-algebras $A$ and $B$. Our construction is a generalisation of similar constructions in literature. In [13], given two Hopf algebras $A$ and $B$, an action of $A$ on $B$ and a coaction of $B$ on $A$ satisfying certain compatibility conditions, Takeuchi constructs a Hopf algebra structure on the tensor product $A \otimes B$. S. Majid has elaborated further on this work in [6, 7, 8 and 9]. We work with a pair of *-algebras $A$ and $B$ together with a linear map $R: B \otimes A \rightarrow A \otimes B$ satisfying certain conditions and we construct a *-algebra structure on $A \otimes B$. 

If we apply our construction to $A$ and $A'$ and the right map $R$, we get the algebra $AA'$ from above. If we apply it once more to $A$ and $L(A, AA')$, we obtain an algebra that we will denote by $AA' \hat{\otimes} AA'$ because it is $AA' \hat{\otimes} AA'$ in the finite-dimensional case and because $AA' \hat{\otimes} AA'$ is a dense subalgebra of $AA' \otimes AA'$ in general (if we consider the appropriate topology). The identity map $W$ in $L(A, AA')$ is a unitary in $AA' \hat{\otimes} AA'$. One more application of the above construction yields a $\ast$-algebra $AA' \hat{\otimes} (AA' \hat{\otimes} AA')$. The algebra $AA' \hat{\otimes} AA'$ has three obvious embeddings in this algebra. The first one is $x \mapsto x \otimes 1$. The two others come from the two embeddings $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$ of $AA'$ into $AA' \hat{\otimes} AA'$ that naturally extend to embeddings $L(A, AA') \to L(A, AA' \hat{\otimes} AA')$ and further to $AA' \hat{\otimes} AA' \to AA' \hat{\otimes} (AA' \hat{\otimes} AA')$. The three images $W_{23}$, $W_{12}$ and $W_{13}$ of $W$ under these maps satisfy the Yang-Baxter equation $W_{23} W_{13} W_{12} = W_{12} W_{13} W_{23}$.

If we start with a different twisting $R$ we obtain the Pentagon equation.

The formulas that we use in the process are well-known, but very often only rigourous in the finite-dimensional case. In the general case, it turns out that the algebras $AA' \hat{\otimes} AA'$ and $AA' \otimes (AA' \hat{\otimes} AA')$ are well suited for these formulas.

We refer to [1] and [12] for the terminology and notations in Hopf algebra theory. We use e.g. the standard notations

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

$$\Delta^{(2)}(a) = (\Delta \otimes 1)\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

(with $\Delta$ for the comultiplication and $i$ for the identity map). There seems to be no standard reference for Hopf $\ast$-algebras. So let us recall some of the definitions here (see e.g. [14]). A Hopf $\ast$-algebra is a Hopf algebra $A$ over $\mathbb{C}$ with an involution such that the comultiplication $\Delta$ and the counit $\epsilon$ are $\ast$-homomorphisms and such that $S(S(a)\ast)^\ast = a$ for all $a \in A$, where $S$ is the antipode. The dual space $A'$ is made into a $\ast$-algebra by $f^*(a) = f(S(a)\ast)^\ast$, when $a \in A$ and $f \in A'$. In the finite-dimensional case, $A'$ is again a Hopf $\ast$-algebra.

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We like to thank the referee for pointing out to us various articles treating similar twisted tensor product algebras.

2. The twisted tensor product of $\ast$-algebras

Let $A$ and $B$ be two algebras and suppose that we have given a linear map $R : B \otimes A \to A \otimes B$ such that
Here $m$ denotes the product in $A$ as well as $B$, considered as a linear map $m: A \otimes A \to A$ and similarly for $B$. As before $i$ denotes the identity map. Then we will construct an algebra $A \otimes_R B$. As a vector space $A \otimes_R B$ is $A \otimes B$. The product in $A \otimes_R B$ however is not the usual product on $A \otimes B$ but is some twisted product determined by $R$.

2.1 DEFINITION. We define the product in $A \otimes_R B$ by

$$xy = (m \otimes m)(i \otimes R \otimes i)(x \otimes y),$$

for $x, y \in A \otimes_R B$.

The conditions (1) on $R$ are necessary for the associativity of the product. They also appear elsewhere in literature (see e.g. [10] and [11]). They are quite natural and generalise the notions of action, coaction and their compatibility in the constructions of Takeuchi and Majid.

If we denote $a \otimes b$ by $ab$ and $R(b \otimes a)$ by $ba$ whenever $a \in A$ and $b \in B$, we can rewrite the conditions on $R$ and the above product in a compact way. The first condition on $R$ becomes

$$(bb')a = b(b'a),$$

provided we define $b(a_1b_1) = (ba_1)b_1$ and $(a_2b_2)b_1 = a_2(b_2b_1)$. The second condition on $R$ becomes

$$b(aa') = (ba)a'$$

if we let $(a_1b_1)a' = a_1(b_1a')$ and $a_1(a_2b_2) = (a_1a_2)b_2$. It is obvious that all these definitions are compatible with the linear structure of $A \otimes B$ and $B \otimes A$. Moreover it is easy to show that the obvious associativity rules are valid and that indeed the conditions on $R$ take care of the missing ones.

The product is given by $(ab)(a'b') = a(ba')b'$, when $a(a_1b_1)b'$ is defined as $(aa_1)(b_1b')$. Also this product is associative but this is not completely obvious. We will prove it in the next proposition.

We will also show that, if $A$ and $B$ are algebras with an identity $1$, and if $R$ satisfies certain extra conditions, the maps $a \to a \cdot 1$ and $b \to 1 \cdot b$ are injective homomorphisms of $A$ and $B$ in $A \otimes_R B$. If we identify $a$ and $b$ with their image we get precisely that $ab$ is the product of $a$ with $b$ and that $ba$ is the product of $b$ with $a$. This should explain why we work with this compact notation and why this compact notation works. If no confusion about the mapping $R$ is
possible we will denote \( A \otimes_R B \) by \( AB \).

2.2 PROPOSITION. The product in \( AB \) is associative.

Proof. First notice that the associativity of the multiplication in \( A \) and \( B \) yields that \((cb)b' = c(bb')\) and \(a(a'c) = (aa')c\) for all \(a, a' \in A, b, b' \in B\) and \(c \in AB\).

Now let \(a, a', a'' \in A\) and \(b, b', b'' \in B\). Then \((ab)(a'b')(a''b'') = (ba')(b')a''b''\).

We have that

\[
(a(a_1b_1)b')(a''b'') = ((aa_1)(b_1b'))(a''b'') \\
= (aa_1)((b_1b')a'')b'' \\
= (aa_1)(b_1(b'a''))b''
\]

for all \(a_1 \in A\) and \(b_1 \in B\) and since

\[
(aa_1)(b_1(a_2b_2))b'' = (aa_1)((b_1a_2)b_2)b'' \\
= (aa_1)(b_1a_2)(b_2b'') \\
= ((aa_1)b_1)(a_2(b_2b'') \\
= (a(a_1b_1))((a_2b_2)b'')
\]

for all \(a_2 \in A\) and \(b_2 \in B\), we get

\[
(a(a_1b_1)b')(a''b'') = (a(a_1b_1))((b'')a'')b''),
\]

and so

\[
((ab)(a'b'))(a''b'') = (a(ba'))((b''a'')b'').
\]

A similar argument gives us that

\[
(ab)((a'b')(a''b'')) = (a(ba'))((b'a'')b'').
\]

\(\square\)

If \(A\) and \(B\) have an identity \(1\) and if \(R\) satisfies some natural conditions we can embed \(A\) and \(B\) in \(AB\) in such a way that \(ab\) is indeed the product of \(a\) and \(b\):

2.3 PROPOSITION. If \(R\) satisfies

\[
R(1 \otimes a) = a \otimes 1 \quad \text{and} \quad R(b \otimes 1) = 1 \otimes b, \quad \forall a \in A, b \in B,
\]

(2)
then the mappings

\[ i_A : A \rightarrow AB : a \mapsto a.1 \]
\[ i_B : B \rightarrow AB : b \mapsto 1.b \]

are homomorphisms.

The formulas (2) can be rewritten in \( AB \) as \( 1 \cdot a = a \cdot 1 \) and \( b \cdot 1 = 1 \cdot b \) for \( a \in A \) and \( b \in B \). Moreover we have that \( \forall a, a' \in A, \forall b, b' \in B : \)

\[ i_A(a)(a'b) = a(a'b); \quad (ab)i_B(b') = (ab)b'; \]
\[ i_B(b)(ab') = b(ab'); \quad (ab)i_A(a') = (ab)a'. \]

Hence we can identify \( i_A(a) \) with \( a \) and \( i_B(b) \) with \( b \).

Now let \( A \) and \( B \) be *-algebras. In view of the previous remarks, in the case where \( A \) and \( B \) have identities, it would be natural to define an involution on \( AB \) by \( (ab)^* = b^*a^* \). This can only be an involution if \( (b^*a^*)^* = ab \). It turns out that this condition on \( R \) is sufficient to make \( AB \) into an involutive algebra.

Remark that this condition in tensor product form is written as \( (R(J \otimes J) \sigma)^2 = 1 \otimes 1 \) if we denote the involution on \( A \) and \( B \) by \( J \) and the flip on \( A \otimes B \) by \( \sigma \).

2.4 PROPOSITION. If \( R \) satisfies

\[ R(J \otimes J) \sigma R(J \otimes J) \sigma = 1 \otimes 1, \]  \[(3)\]

then \( R(J \otimes J) \sigma \) is an involution on \( A \otimes_R B \).

Proof. We still have to check that \( ((ab)(a'b'))^* = (a'b')^*(ab)^* \) for all \( a, a' \in A \) and \( b, b' \in B \). So let \( a, a' \in A \) and \( b, b' \in B \). Then we have \( ((ab)(a'b'))^* = (a(ba')b')^* \). Now for \( a_1 \in A, b_1 \in B \), we have that

\[ (a(a_1b_1)b')^* = ((aa_1)(b_1b')^* \\
= (b_1b')^*(aa_1)^* \\
= (b^*b_1^*)(a_1^*a^*). \]

One can easily see that this last expression is equal to \( b^*((b_1^*a_1^*)a^*) = b^*((a_1b_1)^*a^*) \). So \( ((ab)(a'b'))^* = b^*((ba')^*a^*) = b^*((a^*b^*)a^*) \).
On the other hand we get \((a'b')^*(ab)^* = (b'^*a'^*)(b^*a^*)\), and one can verify that this is also equal to \(b'^*((a'^*b'^*)a'^*)\).

One can also verify that for \(*\)-algebras \(A\) and \(B\) with identities, and for \(R\) satisfying the above conditions, the embeddings \(i_A\) and \(i_B\) are \(*\)-homomorphisms.

We now give some examples.

2.5 EXAMPLES. (i) Let \(A\) and \(B\) be \(*\)-algebras. If we take the flip \(\sigma\) for \(R\), we can check that \(\sigma\) satisfies the conditions and \(AB\) becomes the usual tensor product \(A \otimes B\) of the two \(*\)-algebras.

(ii) Let \(A\) be a \(*\)-algebra, and \(B\) the group algebra of a finite group \(G\). If we have an action \(\alpha\) of \(G\) on \(A\), we can define \(R: B \otimes A \to A \otimes B\) by \(R(\Sigma_s a_s) = \Sigma_s \alpha_s(a_s) \otimes s\). We show that \(R\) satisfies the conditions. If \(s, s' \in G\) and \(a \in A\),

\[
R(m \otimes i)(s \otimes s' \otimes a) = R(ss' \otimes a) = \alpha_{ss'}(a) \otimes ss',
\]

while on the other hand

\[
(i \otimes m)(R \otimes i)(i \otimes R)(s \otimes s' \otimes a) = (i \otimes m)(R \otimes i)(s \otimes \alpha_{s'}(a) \otimes s')
\]

\[
= (i \otimes m)(\alpha_s(\alpha_{s'}(a)) \otimes s \otimes s')
\]

\[
= \alpha_{ss'}(a) \otimes ss'.
\]

Similarly, if \(a, a' \in A\) and \(s \in G\),

\[
R(i \otimes m)(s \otimes a \otimes a') = R(s \otimes aa') = \alpha_s(aa') \otimes s,
\]

and

\[
(m \otimes i)(i \otimes R)(R \otimes i)(s \otimes a \otimes a') = (m \otimes i)(i \otimes R)(\alpha_s(a) \otimes s \otimes a')
\]

\[
= (m \otimes i)(\alpha_s(a) \otimes \alpha_s(a') \otimes s)
\]

\[
= \alpha_s(a)\alpha_s(a') \otimes s,
\]

so that condition (1) is fulfilled. Remark that the first condition of (1) follows from the fact that \(\alpha_s\) is an algebra homomorphism and the second follows from the fact that \(\alpha\) is a group action.

One can see that condition (2) is fulfilled if \(A\) has a unit. Also condition (3)
is satisfied. Since \((R(J \otimes J)\sigma)(a \otimes s) = R(s^{-1} \otimes a^*) = \alpha_{s^{-1}}(a^*) \otimes s^{-1}\), we have that

\[
(R(J \otimes J)\sigma R(J \otimes J)\sigma)(a \otimes s) = (R(J \otimes J)\sigma)(\alpha_{s^{-1}}(a^*) \otimes s^{-1})
\]
\[
= \alpha_s(\alpha_{s^{-1}}(a^*)) \otimes s
\]
\[
= \alpha_s(\alpha_{s^{-1}}(a)) \otimes s
\]
\[
= a \otimes s.
\]

In this case, \(AB\) is the crossed product of \(A\) by the action \(\alpha\) of \(G\).

(iii) A combination of the first two examples gives us the following. Let \(A_1\) and \(A_2\) be *-algebras, and let \(B_1\) and \(B_2\) be the group algebras of finite groups \(G_1\) and \(G_2\) respectively. Let \(\alpha\) be an action of \(G_1\) on \(A_1\) and \(\beta\) be an action of \(G_2\) on \(A_2\). Let \(R_1: B_1 \otimes A_1 \rightarrow A_1 \otimes B_1\) be as in example (ii) but let \(R_2: A_2 \otimes B_2 \rightarrow B_2 \otimes A_2\) be defined by \(R_2(a \otimes s) = s \otimes \beta(a)\). Put \(A = A_1 \otimes B_2\) and \(B = B_1 \otimes A_2\), and define \(R: B \otimes A \rightarrow A \otimes B\) as \(R = \sigma_{23}(R_1 \otimes R_2)\sigma_{23}\), where \(\sigma_{23} = i \otimes \sigma \otimes i\). One can check that \(R\) satisfies conditions (1) and (3), and hence we get a new algebra \(AB\).

We finish this section by formulating some properties of this twisted tensor product.

2.6 PROPOSITION. Let \(A, B\) be *-algebras and \(R: B \otimes A \rightarrow A \otimes B\) satisfying conditions (1), (2), (3). Let \(A_1\) and \(B_1\) also be *-algebras.

(i) Suppose \(R_1: B_1 \otimes A_1 \rightarrow A_1 \otimes B_1\) also satisfies conditions (1), (2), (3). If \(\varphi: A \rightarrow A_1\) and \(\psi: B \rightarrow B_1\) are *-homomorphisms satisfying \(R_1 \circ (\psi \otimes \varphi) = (\varphi \otimes \psi) \circ R\), then \(\varphi \otimes \psi: A \otimes R_1 B \rightarrow A_1 \otimes R_1 B_1\) is a *-homomorphism of the twisted tensor products.

(ii) If the mappings \(\varphi: A \rightarrow A_1\) and \(\psi: B \rightarrow B_1\) are bijective *-homomorphisms, then \(R_1 := (\varphi \otimes \psi) \circ R \circ (\psi^{-1} \otimes \varphi^{-1})\) satisfies conditions (1), (2), (3), and hence defines a twisted tensor product \(A_1 \otimes R_1 B_1\), isomorphic with \(A \otimes R B\).

The proof of these properties is straightforward. It is also easy to check that, if \(A_1, B_1\) are subalgebras of \(A, B\) respectively such that \(R(B_1 \otimes A_1) \subseteq A_1 \otimes B_1\), then \(A_1B_1\) is a subalgebra of \(AB\).

3. The algebras \(AA'\) and \(AA' \otimes AA'\)

Consider a Hopf *-algebra \(A\). For a *-algebra \(D\) we will introduce a *-algebra structure on \(L(A, D)\), the vectorspace of linear \(D\)-valued mappings on \(A\). Then we will define two mappings \(R_1, R_2: L(A, D) \otimes A \rightarrow A \otimes L(A, D)\) satisfying the conditions of section 2, and hence we will get two twisted tensor products \(A \otimes R, L(A, D)\) and \(A \otimes R, L(A, D)\).
The proof of the following proposition is straightforward (see also [1]).

3.1 PROPOSITION. Define multiplication and involution on \( L(A, D) \) by

\[
\begin{align*}
  f_1 \cdot f_2 &= m(f_1 \otimes f_2) \Delta \\
  f^*(a) &= (f(S(a^*))^*)^*,
\end{align*}
\]

where \( f, f_1, f_2 \in L(A, D) \) and \( a \in A \) and where \( m \) denotes multiplication on \( D \). Then \( L(A, D) \) is a \(*\)-algebra.

Remark that we get the algebraic dual \( A' \) of \( A \) with its usual algebra structure if we choose the complex field \( \mathbb{C} \) as algebra \( D \). It will turn out that the algebraic tensor product \( A' \otimes D \) is a \(*\)-subalgebra of \( L(A, D) \).

We now want to define the two mappings \( R_1, R_2 : L(A, D) \otimes A \rightarrow A \otimes L(A, D) \).

For notational convenience we will consider elements of \( A \otimes L(A, D) \) sometimes as linear maps from \( A \) to \( A \otimes D \). So, if \( a \in A \) and \( f \in L(A, D) \) then \((a \otimes f)(x) = a \otimes f(x) \) for all \( x \in A \). Similarly, elements of \( A \otimes L(A, D) \otimes L(A, D) \) will be considered as functions of two variables on \( A \) with values in \( A \otimes D \otimes D \) and other tensor products combining \( A \) and \( L(A, D) \) will be treated in an analogous way. This will make it much easier to write down the proofs in what follows.

3.2 DEFINITION. Let \( A, D \) and \( L(A, D) \) be as above. Define two linear maps \( R_1, R_2 : L(A, D) \otimes A \rightarrow A \otimes L(A, D) \) by

\[
\begin{align*}
  (R_1(f \otimes a))(x) &= \sum_{(a)} a_{(1)} \otimes f(a_{(2)}x) \\
  (R_2(f \otimes a))(x) &= \sum_{(a)} a_{(2)} \otimes f(a_{(3)}xS^{-1}(a_{(1)})).
\end{align*}
\]

It is easy to see that these linear maps are well-defined.

Here we recognise the formulas in [7, page 36] and [13, page 846]. We verify that these mappings satisfy the conditions of section 2.

3.3 PROPOSITION. For \( R = R_1, R_2 \) we have that

(i) \( R(m \otimes i) = (t \otimes m)(R \otimes i)(i \otimes R) \)
(ii) \( R(t \otimes m) = (m \otimes i)(1 \otimes R)(R \otimes i) \)
(iii) \( R(J \otimes J)\sigma R(J \otimes J)\sigma = i \otimes i \).

Proof. We first prove the three relations for \( R_1 \).

(i) Let \( f, g \in L(A, D) \) and \( a, x, y \in A \). Then

\[
((t \otimes R_1)(f \otimes g \otimes a))(x, y) = \sum_{(a)} f(x) \otimes a_{(1)} \otimes g(a_{(2)}y).
\]
So

\[((R_1 \otimes i)(i \otimes R_1)(f \otimes g \otimes a))(x, y) = \sum_{(a)} a_{(1)} \otimes f(a_{(2)}x) \otimes g(a_{(3)}y)\].

Therefore, using the formula for the multiplication in $L(A, D)$, we get

\[((i \otimes m)(R_1 \otimes i)(i \otimes R_1)(f \otimes g \otimes a))(x) = \sum_{(a)(x)} a_{(1)} \otimes f(a_{(2)}x_{(1)})g(a_{(3)}x_{(2)})\]

\[= \sum_{(a)} a_{(1)} \otimes (fg)(a_{(2)}x)\]

\[= (R_1(fg \otimes a))(x)\]

\[= (R_1(m \otimes i)(f \otimes g \otimes a))(x)\].

This proves the first relation.

(iii) Let $f \in L(A, D)$ and $a$, $b$, $x \in A$. Then

\[((R_1 \otimes i)(f \otimes a \otimes b))(x) = \sum_{(a)} a_{(1)} \otimes f(a_{(2)}x) \otimes b.\]

So

\[((i \otimes R_1)(R_1 \otimes i)(f \otimes a \otimes b))(x) = \sum_{(a)(b)} a_{(1)} \otimes b_{(1)} \otimes f(a_{(2)}b_{(2)}x),\]

and

\[((m \otimes i)(i \otimes R_1)(R_1 \otimes i)(f \otimes a \otimes b))(x) = \sum_{(a)(b)} a_{(1)}b_{(1)} \otimes f(a_{(2)}b_{(2)}x)\]

\[= \sum_{(ab)} (ab)_{(1)} \otimes f((ab)_{(2)}x)\]

\[= (R_1(f \otimes ab))(x)\]

\[= (R_1(i \otimes m)(f \otimes a \otimes b))(x).\]

This proves the second relation.

(iii) Let $f \in L(A, D)$ and $a$, $x \in A$. Then

\[(R_1(J \otimes J)\sigma(a \otimes f))(x) = (R_1(f^* \otimes a^*))(x)\]

\[= \sum_{(a)} a_{(1)}^* \otimes f^*(a_{(2)}^*x)\]

\[= \sum_{(a)} a_{(1)}^* \otimes f(S(a_{(2)}^*)S(x)^*)^*\]

\[= \sum_{(a)} a_{(1)}^* \otimes f(S^{-1}(a_{(2)}S(x)^*))^*.\]
Now, if \( b \in A \) and if \( g \) is defined in \( L(A, D) \) by \( g(x) = f(S^{-1}(b)S(x)^*) \), then \( g^*(x) = f(S^{-1}(b)x) \). So, if we apply \( R_1(J \otimes J)\sigma \) once more, we obtain

\[
(R_1(J \otimes J)\sigma R_1(J \otimes J)\sigma(a \otimes f))(x) = \sum_{(a)} a_{(1)} \otimes f(S^{-1}(a_{(3)})a_{(2)}x).
\]

But

\[
\sum_{(a)} S^{-1}(a_{(2)})a_{(1)} = (mS^{-1} \otimes (i)\sigma\Delta)(a)
\]

\[
= (m(i \otimes S)\Delta S^{-1})(a)
\]

\[
= \varepsilon(S^{-1}(a))1 = \varepsilon(a)1.
\]

Therefore

\[
(R_1(J \otimes J)\sigma R_1(J \otimes J)\sigma(a \otimes f))(x) = \sum_{(a)} a_1 \otimes f(\varepsilon(a_{(2)})x)
\]

\[
= \sum_{(a)} a_1 \varepsilon(a_{(2)}) \otimes f(x)
\]

\[
= a \otimes f(x)
\]

\[
= (a \otimes f)(x).
\]

This proves the third equality.

Now we prove the relations for \( R_2 \).

(i) Let \( f, g \in L(A, D) \) and \( a, x, y \in A \). Then

\[
((i \otimes R_2)(f \otimes g \otimes a))(x, y) = \sum_{(a)} f(x) \otimes a_{(2)} \otimes g(a_{(3)}yS^{-1}(a_{(1)})).
\]

So

\[
((R_2 \otimes i)(i \otimes R_2)(f \otimes g \otimes a))(x, y) = \sum_{(a)} a_{(3)} \otimes f(a_{(4)}xS^{-1}(a_{(2)})) \otimes g(a_{(5)}yS^{-1}(a_{(1)})�
\]

Using the formula for the multiplication in \( L(A, D) \), we get

\[
((i \otimes m)(R_2 \otimes i)(i \otimes R_2)(f \otimes g \otimes a))(x)
\]

\[
= \sum_{(a)(x)} a_{(3)} \otimes f(a_{(4)}x_{(1)}S^{-1}(a_{(2)}))g(a_{(5)}x_{(2)}S^{-1}(a_{(1)})�
\]

\[
= \sum_{(a)} a_{(2)} \otimes (fg)(a_{(3)}xS^{-1}(a_{(1)})�
\]

\[
= R_2(m \otimes i)(f \otimes g \otimes a)(x).
\]
This proves the first relation.

(ii) Let \( f \in L(A, D) \) and \( a, b, x \in A \). Then

\[
((R_2 \otimes i)(f \otimes a \otimes b))(x) = \sum_{(a)} a_{(2)} \otimes f(a_{(3)}xS^{-1}(a_{(1)})) \otimes b.
\]

So

\[
((i \otimes R_2)(R_2 \otimes i)(f \otimes a \otimes b))(x) = \sum_{(a)(b)} a_{(2)} \otimes b_{(2)} \otimes f(a_{(3)}b_{(3)}xS^{-1}(b_{(1)})S^{-1}(a_{(1)})),
\]

and

\[
((m \otimes i)(i \otimes R_2)(R_2 \otimes i)(f \otimes a \otimes b))(x) = \sum_{(ab)} (ab)_{(2)} \otimes f((ab)_{(3)}xS^{-1}((ab)_{(1)})) = (R_2(i \otimes m)(f \otimes a \otimes b))(x).
\]

This proves the second relation.

(iii) Let \( f \in L(A, D) \) and \( a, x \in A \). Then

\[
(R_2(J \otimes J)\sigma(a \otimes f))(x) = \sum_{(a)} a_{(2)}^* \otimes f^*(a_{(3)}^*xS^{-1}(a_{(1)}^*))
\]

\[
= \sum_{(a)} a_{(2)}^* \otimes f(S(a_{(3)}^*)^*S(x)^*q_{(1)}^*)^*.
\]

Now if \( b, c \in A \), and if \( g \) is defined in \( L(A, D) \) by \( g(x) = f(S^{-1}(b)S(x)^*c) \), then \( g^*(x) = f(S^{-1}(b)x) \). So, applying \( R_2(J \otimes J)\sigma \) once more gives

\[
(R_2(J \otimes J)\sigma R_2(J \otimes J)\sigma(a \otimes f))(x) = \sum_{(a)} a_{(3)} \otimes f(S^{-1}(a_{(5)})a_{(4)}xS^{-1}(a_{(2)}q_{(1)}))
\]

\[
= \sum_{(a)} a_{(2)} \otimes f(e(a_{(3)}x)q_{(1)})
\]

\[
= a \otimes f(x)
\]

\[
= (a \otimes f)(x).
\]

This proves the third equality. \( \square \)

If \( A \) and \( D \) have a unit, one can easily see that \( R_1 \) and \( R_2 \) also satisfy the formulas \( R(1 \otimes a) = a \otimes 1 \) and \( R(f \otimes 1) = 1 \otimes f \).

By choosing \( C \) for \( D \), we get two algebras \( A \otimes_{R_1} A' \) and \( A \otimes_{R_2} A' \), which we will both denote by \( AA' \) when no confusion is possible.

For any \( D \) we can embed \( A' \otimes D \) in \( L(A, D) \) by

\[
(i(f \otimes d))(a) = f(a)d
\]
whenever $a \in A$, $d \in D$ and $f \in A'$. This embedding $i$ is a *-homomorphism. In turn $i$ induces an embedding $j = i \otimes i : A \otimes A' \otimes D \to A \otimes L(A, D)$. This is also a *-homomorphism from $AA' \otimes D$ to $AL(A, D)$.

In the finite-dimensional case these embeddings are also surjective. This is no longer true in the infinite-dimensional case. However, then it is possible to find a suitable vector space topology on the larger space such that the images are dense. We don't want to elaborate further on this, but use this idea as a motivation to denote $L(A, D)$ by $A' \otimes D$ and similarly $AL(A, D)$ by $AA' \otimes D$. If we want to specify the $R$, we will also use $AA' \otimes_R D$ here as before. It is easily seen that also $A' \otimes D$ is a subalgebra of $AA' \otimes D$ by the natural embedding $f \mapsto 1f$.

In the future we will omit $i$ and $j$ in our notations and we will consider $A' \otimes D$ as a subalgebra of $A' \otimes D$ and $AA' \otimes D$ as a subalgebra of $A' \otimes D$.

Taking $AA'$ for the algebra $D$ gives us an algebra $AA' \otimes AA'$. Applying the same construction to this algebra, we get an algebra $AA' \otimes (AA' \otimes AA')$. This algebra contains the algebra $AA' \otimes AA'$ in three different ways. Indeed, we have three embeddings $i_{12}$, $i_{13}$, $i_{23}$ of $AA' \otimes AA'$ into $AA' \otimes (AA' \otimes AA')$, by extending the three natural embeddings of $AA' \otimes AA'$ into $AA' \otimes AA' \otimes AA'$.

Consider for example the natural embedding $AA' \otimes AA' \to AA' \otimes AA' \otimes 1$. The algebra $AA' \otimes AA'$ is a subalgebra of $AA' \otimes AA'$ and $AA' \otimes AA' \otimes 1$ is a subalgebra of $AA' \otimes (AA' \otimes AA')$. We define $i_{12}$ as the mapping $AA' \otimes AA' \to AA' \otimes (AA' \otimes AA')$ that extends the natural embedding of $AA' \otimes AA'$ into $AA' \otimes AA' \otimes AA'$. The two other mappings are given in an analogous way. The exact definition is as follows:

$$i_1 : AA' \to AA' \otimes AA' : ab \mapsto j(ab \otimes 1)$$
$$i_2 : AA' \to AA' \otimes AA' : ab \mapsto j(1 \otimes ab)$$
$$i_{12} : AA' \otimes AA' \to AA' \otimes (AA' \otimes AA') : af \mapsto a(i_1 \circ f)$$
$$i_{13} : AA' \otimes AA' \to AA' \otimes (AA' \otimes AA') : af \mapsto a(i_2 \circ f)$$
$$i_{23} : AA' \otimes AA' \to AA' \otimes (AA' \otimes AA') : af \mapsto j(1 \otimes af).$$

These mappings are *-homomorphisms. Indeed, clearly $i_1$, $i_2$ and $i_{23}$ are *-homomorphisms, since $j$ is one. The mappings $i_{12}$, $i_{13}$ can also be checked with straightforward techniques. The injectivity of the mapping $i_{12}$, $i_{13}$ and $i_{23}$ is clear, and so we really have embeddings of $AA' \otimes AA'$ in $AA' \otimes (AA' \otimes AA')$.

**4. The formulas $\Delta(a) = W(a \otimes 1)W^*$ and $\Delta(a) = W\Delta(a)W^*$ in $AA' \otimes AA'$**

Again, let $A$ be a Hopf *-algebra and let

$$AA' \otimes_R AA' = AL(A, AA') = A \otimes_R L(A, AA'),$$
where \( R = R_1 \) or \( R_2 \) as defined in the previous section, and is again omitted in the notation when no confusion is possible. We will consider elements in \( AA' \otimes AA' \) as functions from \( A \) to \( A \otimes AA' \) as before.

In this section we will consider the subalgebras \( A \otimes A \) and \( A' \otimes A = L(A, A) \) of \( AA' \otimes AA' \). We have that

\[
(a \otimes b)(x) = a \otimes \varepsilon(x)b
\]

\[
f(x) = 1 \otimes f(x)
\]

for \( a, b \in A \) and \( f \in L(A, A) \).

We first define \( W \) in \( L(A, A) \).

4.1 DEFINITION. Let \( W \) be the identity map in \( L(A, A) \).

Then \( W^*(a) = W(S(a)^*)^* = S(a) \) when \( a \in A \). Moreover

\[
(W^*W)(a) = \sum_{(a)} W^*(a_{(1)})W(a_{(2)}) = \sum_{(a)} S(a_{(1)})a_{(2)} = \varepsilon(a)1.
\]

So we get \( W^*W = 1 \) in the algebra \( L(A, A) \). Similarly \( WW^* = 1 \), so that \( W \) is a unitary. When considered as an element in \( AA' \otimes AA' \), we get \( W(x) = 1 \otimes x \) for \( x \in A \), and of course also here \( W \) is a unitary. Moreover we have the following formulas.

4.2 PROPOSITION. (i) In \( AA' \otimes_R, AA' \) we have for all \( a \in A \):

\[
W(a \otimes 1)W^* = \Delta(a).
\]

(ii) In \( AA' \otimes_{R_1} AA' \) we have for all \( a \in A \):

\[
W^*\Delta(a)W = \Delta'(a),
\]

where \( \Delta' = \sigma\Delta \) is the opposite comultiplication.

Proof. In the two cases we have for \( a, x \in A \), that

\[
(\Delta(a)W)(x) = \left( \sum_{(a)} (a_{(1)} \otimes a_{(2)})W \right)(x) = \sum_{(a)} a_{(1)} \otimes a_{(2)}x.
\]

In case (i) we get

\[
(W(a \otimes 1))(x) = (R_1(W \otimes a))(x)
\]

\[
= \sum_{(a)} a_{(1)} \otimes W(a_{(2)}x)
\]

\[
= \sum_{(a)} a_{(1)} \otimes a_{(2)}x,
\]
proving the first formula.

In case (ii) we get

\[(W \sigma \Delta(a))(x) = \sum_{(a)} (W(a_{(2)} \otimes a_{(1)}))(x)\]

\[= \sum_{(a)} (W(a_{(2)} \otimes 1)(1 \otimes a_{(1)}))(x)\]

\[= \sum_{(a)} (W(a_{(2)} \otimes 1))(x)(1 \otimes a_{(1)})\]

\[= \sum_{(a)} a_{(3)} \otimes W(a_{(4)} x S^{-1}(a_{(2)}))a_{(1)}\]

\[= \sum_{(a)} a_{(3)} \otimes a_{(4)} x S^{-1}(a_{(2)})a_{(1)}\]

We have seen before that \(\sum_{(a)} S^{-1}(a_{(2)})a_{(1)} = \varepsilon(a)1\). So we get

\[(W \sigma \Delta(a))(x) = \sum_{(a)} a_{(2)} \otimes a_{(3)} x \varepsilon(a_{(1)})\]

\[= \sum_{(a)} a_{(1)} \otimes a_{(2)} x.\]

This proves the second formula. \(\square\)

Remark that, essentially, these formulas determine the commutation rules \(R_1\) and \(R_2\) from \(A'A\) to \(AA'\).

We can consider these formulas in some examples.

4.3 EXAMPLE. Consider a finite group \(G\), and let \(A\) be the group algebra of \(G\). If we define \(\Delta(s) = s \otimes s\), \(S(s) = s^{-1}\) and \(\varepsilon(s) = 1\) for all \(s \in G\), \(A\) becomes a Hopf *-algebra. \(A'\) is the algebra of linear functions on \(A\), equipped with pointwise multiplication. The element \(\sum_{s \in G} \delta_s \otimes s \in A' \otimes A\), considered as a function in \(L(A, A)\), is the identical function. So \(W = \sum_{s \in G} \delta_s \otimes s\). We then have:

(i) In \(AA' \otimes_R AA'\):

\[W(s \otimes 1)W^* = \sum_u (\delta_u \otimes u)(s \otimes 1) \sum_v (\delta_v \otimes v^{-1})\]

\[= \sum_{u,v} (s \otimes 1)(\delta_{s^{-1}u} \otimes u)(\delta_v \otimes v^{-1})\]

\[= \sum_u (s \otimes 1)(\delta_{s^{-1}u} \otimes u)(\delta_{s^{-1}u} \otimes u^{-1} s)\]

\[= \sum_u (s \otimes 1)(\delta_{s^{-1}u} \otimes s)\]
This gives proposition 4.2(i).

(ii) In $AA' \otimes_{\mathbb{R}} AA'$:

$$W\Delta'(s)W^* = \sum_u (\delta_u \otimes u)(s \otimes s) \sum_v (\delta_v \otimes v^{-1})$$

$$= \sum_{u, v} (s \otimes u)(\delta_{s^{-1}u} \otimes s)(\delta_v \otimes v^{-1})$$

$$= \sum_{u} (s \otimes u)(\delta_{s^{-1}u} \otimes s)(\delta_{s^{-1}u} \otimes s^{-1}u^{-1}s)$$

$$= \sum_{u} (s \otimes s)(\delta_{s^{-1}u} \otimes 1) = (s \otimes s) = \Delta(s).$$

This is proposition 4.2(ii). \hfill \Box

4.4 EXAMPLE. Let $A$ be the *-algebra with identity generated by a self-adjoint element $h$. One can define $\Delta: A \to A \otimes A$ by $\Delta(h) = h \otimes 1 + 1 \otimes h$, $\varepsilon: A \to \mathbb{C}$ by $\varepsilon(h) = 0$, and $S: A \to A$ by $S(h) = -h$. It is easy to verify that $A$ is a Hopf *-algebra. Let $B$ be the *-algebra with identity generated by a self-adjoint element $k$, with the same Hopf *-algebra structure. Define, for a given $\lambda \in \mathbb{R}$, and for all $n, m \in \mathbb{N}$:

$$\langle h^n, k^m \rangle = \delta(n, m)n!(i\lambda)^n,$$

where $\delta$ is the Kronecker delta. This is a non-degenerate bilinear mapping $A \times B \to \mathbb{C}$, and since also $\langle \Delta(h^n), k^p \otimes k^q \rangle = \langle h^n, k^p k^q \rangle$ and $\langle h^n, (k^m)^* \rangle = \langle S(h^n)^*, k^m \rangle$, we have that $B = A'$, when we consider $A'$ with the weak *-topology (see [14]).

For each element $a \in A$, the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{i\lambda} (k \otimes h) \right)^n (a)$$

reduces to a finite sum, so we can say that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{i\lambda} (k \otimes h) \right)^n$$

converges in $L(A, A)$, and we denote it by $\exp \left( \frac{1}{i\lambda} (k \otimes h) \right)$. 
Moreover, we have that

\[
\exp\left(\frac{1}{i\lambda} (k \otimes h)\right)(h') = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\lambda}\right)^n (k^n \otimes h^n)(h')
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\lambda}\right)^n \delta(n, j)! (i\lambda)^j h^n
\]

\[
= h^j.
\]

Thus \(\exp\left(\frac{1}{i\lambda} (k \otimes h)\right)\) is the element \(W\).

(i) Since \(R_1(k \otimes h)(x) = h \otimes k(x) + 1 \otimes k(hx)\) for all \(x\) in \(A\), we have that \(R_1(k \otimes h) = h \otimes k + i\lambda(1 \otimes 1)\). So in \(A \otimes_{R_1} B\) we have \([k, h] = i\lambda\). Hence in \(AA' \otimes_{R_1} AA'\) we get

\[
W(h \otimes 1)W^* = \exp\left(\frac{1}{i\lambda} (k \otimes h)\right)(h \otimes 1) \exp\left(-\frac{1}{i\lambda} (k \otimes h)\right)
\]

\[
= h \otimes 1 + \frac{1}{i\lambda} [k \otimes h, h \otimes 1]
\]

\[
= h \otimes 1 + 1 \otimes h = \Delta(h).
\]

This illustrates proposition 4.3(i).

(ii) Since \(R_2(k \otimes h)(x) = 1 \otimes k(-xh) + h \otimes k(x) + 1 \otimes k(hx)\) for all \(x\) in \(A\), we have that \(R_2(k \otimes h) = h \otimes k\). Hence \(A \otimes_{R_2} B\) is commutative, and in \(AA' \otimes_{R_2} AA'\) we get

\[
W\Delta(h)W^* = \exp\left(\frac{1}{i\lambda} (k \otimes h)\right)(h \otimes 1 + 1 \otimes h) \exp\left(-\frac{1}{i\lambda} (k \otimes h)\right)
\]

\[
= (h \otimes 1 + 1 \otimes h)WW^*
\]

\[
= h \otimes 1 + 1 \otimes h = \Delta(h).
\]

5. The Pentagon and Yang-Baxter equation

The three embeddings \(i_{12}, i_{13}\) and \(i_{23}\) of \(AA' \otimes AA'\) in \(AA' \otimes (AA' \otimes AA')\), described in section 3, give rise to the elements \(W_{12} = i_{12}(W), W_{13} = i_{13}(W)\) and \(W_{23} = i_{23}(W)\). In this section we will show that \(W\) satisfies the Pentagon equation in \(AA' \otimes_{R_1} (AA' \otimes AA')\) and the Yang-Baxter equation in \(AA' \otimes_{R_2} (AA' \otimes_{R_2} AA')\).
The \(*\)-homomorphism \(i \otimes \Delta: A' \otimes A \to A' \otimes (A \otimes A)\) can be extended to a \(*\)-homomorphism \(i \otimes \Delta: A' \otimes A \to A' \otimes (A \otimes A)\), mapping \(f\) to \(\Delta \circ f\), and later on extended to a \(*\)-homomorphism \(i \otimes \Delta: AA' \otimes A \to AA' \otimes (AA' \otimes AA')\). Then we have the following.

5.1 \textbf{PROPOSITION.} In \(AA' \otimes (AA' \otimes AA')\) we have that \((i \otimes \Delta)(W) = W_{12}W_{13}\).

\textit{Proof.} All three elements \((i \otimes \Delta)(W), W_{12}\) and \(W_{13}\) are in fact in the subalgebra \(A' \otimes (A \otimes A) = L(A, A \otimes A)\), and it is therefore sufficient to prove the equation in this subalgebra. For all \(x \in A\) we have

\[(1 \otimes \Delta)(W)(x) = \Delta(W(x)) = \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.\]

On the other hand

\[(W_{12}W_{13})(x) = \sum_{(x)} W_{12}(x_{(1)})W_{13}(x_{(2)})\]

\[= \sum_{(x)} (x_{(1)} \otimes 1)(1 \otimes x_{(2)})\]

\[= \sum_{(x)} x_{(1)} \otimes x_{(2)}. \quad \square\]

We now come to the proof of the Pentagon equation.

5.2 \textbf{THEOREM.} In \(AA' \otimes_R (AA' \otimes_R AA')\) we have \(W_{12}W_{13}W_{23} = W_{23}W_{12}\).

\textit{Proof.} Because of proposition 5.1 it will be sufficient to prove

\[W_{23}W_{12}W_{23}^* = (i \otimes \Delta)(W).\]

We consider this equation in the subalgebra \(A' \otimes (AA' \otimes_R AA') = L(A, AA' \otimes_R AA').\) So let \(a \in A\). Then, we have

\[W_{12}(a) = a \otimes 1\]

\[W_{23}(a) = \varepsilon(a)W.\]

So

\[(W_{23}W_{12}W_{23}^*)(a) = \sum_{(a)} W_{23}(a_{(1)})W_{12}(a_{(2)})W_{23}^*(a_{(3)})\]

\[= \sum_{(a)} \varepsilon(a_{(1)})W(a_{(2)} \otimes 1)\varepsilon(a_{(3)})W^*\]

\[= W(a \otimes 1)W^* = \Delta(a). \quad \square\]
Similarly we can prove the Yang-Baxter equation.

5.3 THEOREM. In $AA' \otimes_{R_2} (AA' \otimes_{R_2} AA')$ we have $W_{12}W_{13}W_{23} = W_{23}W_{13}W_{12}$.

Proof. In Proposition 5.1 we saw that here $(i \otimes \Delta)(W) = W_{12}W_{13}$. It is not hard to see that $(i \otimes \Delta')(W) = W_{13}W_{12}$. Therefore we must show that

$$W_{23}(i \otimes \Delta')(W) = (i \otimes \Delta)(W)W_{23}.$$ 

We can do this again in $A' \otimes (AA' \otimes_{R_2} AA') = L(A, AA' \otimes_{R_2} AA')$. So let $a \in A$. Then

$$(W_{23}(i \otimes \Delta')(W))(a) = \sum W_{23}(a_{(1)})(i \otimes \Delta')(W)(a_{(2)})$$

$$= \sum \varepsilon(a_{(1)})W\Delta'(a_{(2)})$$

$$= W\Delta'(a),$$

while

$$((i \otimes \Delta)(W)W_{23}(a) = \sum ((i \otimes \Delta)W)(a_{(1)})W_{23}(a_{12})$$

$$= \Delta(a)W.$$ 

This proves the Yang-Baxter equation. □

We now verify these relations in our examples.

5.4 EXAMPLE. Take the example of the group algebra of a finite group $G$ (see example 4.3). Here $W$ is given by $\sum_{s \in G} \delta_s \otimes s$ and so

$$W_{12} = \sum_{s \in G} \delta_s \otimes s \otimes 1$$

$$W_{13} = \sum_{s \in G} \delta_s \otimes 1 \otimes s$$

$$W_{23} = \sum_{s \in G} 1 \otimes \delta_s \otimes s.$$ 

Therefore we get

$$W_{12}W_{13}W_{23} = \sum_{s, t, u} (\delta_s \otimes s \otimes 1)(\delta_t \otimes 1 \otimes t)(1 \otimes \delta_u \otimes u)$$

$$= \sum_{s, u} \delta_s \otimes s\delta_u \otimes su.$$
and this proves the Pentagon equation.

(ii) In $AA' \otimes_{R_s} (AA' \otimes_{R_s} AA')$ we have

$$W_{23}W_{12} = \sum_{s,t,i,u} (1 \otimes \delta_s \otimes t)(1 \otimes \delta_t \otimes u)$$

$$= \sum_{s,t} \delta_i \otimes \delta_s t \otimes s$$

$$= \sum_{s,t} \delta_i \otimes t \delta_{s-1s} \otimes s$$

$$= \sum_{s,u} \delta_i \otimes t \delta_u \otimes tu,$$

and this proves the Yang-Baxter equation.

5.5 EXAMPLE. Now let us consider the case of an algebra generated by a single self-adjoint element (as in example 4.4). Here $W$ is given by the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{i\lambda} (k \otimes h) \right)^n.$$

This is not an element in $B \otimes A$, but in $L(A, A)$ it can be seen as a limit of elements in $B \otimes A$.

(i) In $AA' \otimes_{R_s} (AA' \otimes_{R_s} AA')$ we have

$$W_{23}W_{12}W_{23}^* = (1 \otimes W)(W \otimes 1)(1 \otimes W^*)$$

$$= (1 \otimes W) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \left( \frac{1}{i\lambda} (k \otimes h) \right)^n \otimes 1 \right) (1 \otimes W^*)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{i\lambda} \right)^n k^n \otimes (W(h \otimes 1)^n W^*).$$
We already know from example 4.4 (ii) that $W(h \otimes 1)W^* = h \otimes 1 + 1 \otimes h$, so that $W(h \otimes 1)^nW^* = (h \otimes 1 + 1 \otimes h)^n$. This gives

$$W_{23}W_{12}W_{23}^* = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1}{i\lambda} \right)^n k^n \otimes (h \otimes 1 + 1 \otimes h)^n$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \left( \frac{1}{i\lambda} \right)^n k^n \otimes h^j \otimes h^{n-j}$$

$$= \left( \sum_{n=0}^{\infty} \left( \frac{1}{i\lambda} \right)^n k^n \otimes h^n \otimes 1 \right) \left( \sum_{m=0}^{\infty} \left( \frac{1}{i\lambda} \right)^m k^m \otimes 1 \otimes h^m \right)$$

$$= W_{12}W_{13}.$$

So we get $W_{23}W_{12} = W_{12}W_{13}W_{23}$, and this is the Pentagon equation.

(ii) We already know that $A \otimes_{R_a} B$ is a commutative algebra and one can check in a similar way that also $AA' \otimes_{R_a} AA'$ and $AA' \otimes_{R_a} (AA' \otimes_{R_a} AA')$ are commutative. Hence the Yang-Baxter equation $W_{23}W_{13}W_{12} = W_{12}W_{13}W_{23}$ is trivially satisfied in $AA' \otimes_{R_a} (AA' \otimes_{R_a} AA')$.

6. References