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The Yang-Baxter and Pentagon equation

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1. Introduction

Let A be a finite-dimensional Hopf algebra and let A° be the dual Hopf algebra with opposite comultiplication (see e.g. [1]). The algebraic tensor product $A \otimes A^\circ$ can be made into a quasi-triangular Hopf algebra (see e.g. [3]). The comultiplication is essentially the tensor product of the comultiplications on A and A°, but the multiplication is in general different from the usual tensor product multiplication. However, when A and A° have a unit 1, the mappings $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ are Hopf algebra embeddings and $a \otimes b = (a \otimes 1)(1 \otimes b)$ in $A \otimes A^\circ$. We will identify A and A° with their images in $A \otimes A^\circ$ and so we will use AA° to denote this quasi-triangular Hopf algebra. Remark that in general A and A° will not commute with each other.

Consider also the tensor product $AA^\circ \otimes AA^\circ$. It is clear that $A^\circ \otimes A$ is a subspace of $AA^\circ \otimes AA^\circ$. The canonical element W (i.e. the identity map when $A^\circ \otimes A$ is identified with the space $L(A, A)$ of linear maps from A to A) is an invertible element in $AA^\circ \otimes AA^\circ$, where now we do take the usual tensor product structure. This W intertwines the comultiplication Δ on AA° with the opposite comultiplication Δ' in the sense that $\Delta(a) = W\Delta'(a)W^{-1}$ when $a \in AA^\circ$. Moreover W satisfies the *Yang-Baxter equation* in $AA^\circ \otimes AA^\circ \otimes AA^\circ$, that is

$$W_{12}W_{13}W_{23} = W_{23}W_{13}W_{12}$$

when $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ and W_{13} is the obvious image of W with 1 in the middle (see e.g. [3] and [14]).

In the infinite-dimensional case, this construction breaks down for several reasons. First the dual space A' of A is no longer a Hopf algebra in the sense that the obvious candidate Δ for the comultiplication will not map A' into $A' \otimes A'$ (but only in $(A \otimes A)'$, which is strictly larger). In many of the well-known examples however, there are enough elements $b \in A'$ such that $\Delta(b) \in A' \otimes A'$, and since these elements form a Hopf algebra (see [12, page 109]), this first difficulty can easily be overcome in many cases.

The second problem is that in the infinite-dimensional case $L(A, A)$ is bigger than $A^\circ \otimes A$ and that the canonical element W is not in $A^\circ \otimes A$. So, strictly speaking, the intertwining property and the Yang-Baxter equation have only a formal meaning. This problem can be overcome by considering finite-dimensional representations π of A so that $(1 \otimes \pi)(W)$ is an element of $A^\circ \otimes \pi(A)$. These elements satisfy the right properties.

In the infinite-dimensional case there is a need for a topological approach using topological tensor products and allowing the comultiplication to go outside the algebraic tensor product. This seems to be very difficult. The C*-algebra approach of Woronowicz to quantum groups ([15]) is not yet completely satisfactory, but has the advantage that much is known about topological tensor products here. The approach of Baaj and Skandalis ([2 and 11]) is close to the C*-algebra approach of Woronowicz. They work with the *Pentagon equation*

$$W_{12}W_{13}W_{23} = W_{23}W_{12},$$

which is similar to the Yang-Baxter equation and is obtained in the finite-dimensional case above if we make $A \otimes A^\circ$ into an algebra in a different way. Moreover, the C*-algebra approach seems to be impossible in some cases (e.g. for the Hopf *-algebra generated by two self-adjoint elements a and b such that a is invertible and $ab = \lambda ba$ with $|\lambda| = 1$), see also [16].

In this paper we make an attempt to get a precise interpretation of the formal construction of Drinfel'd. This has also been done by others. Woronowicz gave solutions of the Yang-Baxter equation (in fact the Braid equation) in the space of linear maps on $A \otimes A \otimes A$ [17]. There is also an attempt to give a precise meaning to the Yang-Baxter equation in the algebraic dual $(AA^\circ \otimes AA^\circ \otimes AA^\circ)'$ by Koornwinder [5]. Here again, some extra conditions are necessary (like the existence of the Hopf subalgebra A° in the dual space A'). We need no extra conditions on A. And we treat the Pentagon equation, as well as the Yang-Baxter equation. We also work in the framework of Hopf*-algebras (so that the W becomes a unitary element).

Our approach is as follows. Let A be any Hopf*-algebra (over $\mathbb{C}$). For any *-algebra D one can make the space $L(A, D)$ of linear maps from A to D into a *-algebra. If $D = \mathbb{C}$ we get of course the space A' of linear functionals on A with the usual *-algebra structure. In section 2 of this paper we introduce the notion of a twisted tensor product of two *-algebras A and B. Our construction is a generalisation of similar constructions in literature. In [13], given two Hopf algebras A and B, an action of A on B and a coaction of B on A satisfying certain compatibility conditions, Takeuchi constructs a Hopf algebra structure on the tensor product $A \otimes B$. S. Majid has elaborated further on this work in [6, 7, 8 and 9]. We work with a pair of *-algebras A and B together with a linear map $R: B \otimes A \to A \otimes B$ satisfying certain conditions and we construct a *-algebra structure on $A \otimes B$.

If we apply our construction to A and A' and the right map R, we get the algebra AA' from above. If we apply it once more to A and $L(A, AA')$, we obtain an algebra that we will denote by $AA' \bar{\otimes} AA'$ because it is $AA' \otimes AA'$ in the finite-dimensional case and because $AA' \otimes AA'$ is a dense subalgebra of $AA' \bar{\otimes} AA'$ in general (if we consider the appropriate topology). The identity map W in $L(A, AA')$ is a unitary in $AA' \bar{\otimes} AA'$. One more application of the above construction yields a *-algebra $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$. The algebra $AA' \bar{\otimes} AA'$ has three obvious embeddings in this algebra. The first one is $x \to 1 \otimes x$. The two others come from the two embeddings $x \to x \otimes 1$ and $x \to 1 \otimes x$ of AA' into $AA' \bar{\otimes} AA'$ that naturally extend to embeddings $L(A, AA') \to L(A, AA' \bar{\otimes} AA')$ and further to $AA' \bar{\otimes} AA' \to AA' \bar{\otimes} (AA' \bar{\otimes} AA')$. The three images W_{23}, W_{12} and W_{13} of W under these maps satisfy the Yang-Baxter equation $W_{23}W_{13}W_{12} = W_{12}W_{13}W_{23}$.

If we start with a different twisting R we obtain the Pentagon equation.

The formulas that we use in the process are well-known, but very often only rigourous in the finite-dimensional case. In the general case, it turns out that the algebras $AA' \bar{\otimes} AA'$ and $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$ are well suited for these formulas.

We refer to [1] and [12] for the terminology and notations in Hopf algebra theory. We use e.g. the standard notations

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

$$\Delta^{(2)}(a) = (\Delta \otimes \iota)\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

(with Δ for the comultiplication and ι for the identity map). There seems to be no standard reference for Hopf *-algebras. So let us recall some of the definitions here (see e.g. [14]). A Hopf *-algebra is a Hopf algebra A over $\mathbb{C}$ with an involution such that the comultiplication Δ and the counit ε are *-homomorphisms and such that $S(S(a)^*)^* = a$ for all $a \in A$, where S is the antipode. The dual space A' is made into a *-algebra by $f^*(a) = f(S(a)^*)^-$, when $a \in A$ and $f \in A'$. In the finite-dimensional case, A' is again a Hopf *-algebra.

Acknowledgements

We like to thank the referee for pointing out to us various articles treating similar twisted tensor product algebras.

2. The twisted tensor product of *-algebras

Let A and B be two algebras and suppose that we have given a linear map $R: B \otimes A \to A \otimes B$ such that

$$R(m \otimes \iota) = (\iota \otimes m)(R \otimes \iota)(\iota \otimes R)$$

$$R(\iota \otimes m) = (m \otimes \iota)(\iota \otimes R)(R \otimes \iota). \tag{1}$$

Here m denotes the product in A as well as B, considered as a linear map $m: A \otimes A \to A$ and similarly for B. As before ι denotes the identity map. Then we will construct an algebra $A \otimes_R B$. As a vector space $A \otimes_R B$ is $A \otimes B$. The product in $A \otimes_R B$ however is not the usual product on $A \otimes B$ but is some twisted product determined by R.

2.1 DEFINITION. We define the product in $A \otimes_R B$ by

$$xy = (m \otimes m)(\iota \otimes R \otimes \iota)(x \otimes y),$$

for $x, y \in A \otimes_R B$.

The conditions (1) on R are necessary for the associativity of the product. They also appear elsewhere in literature (see e.g. [10] and [11]). They are quite natural and generalise the notions of action, coaction and their compatibility in the constructions of Takeuchi and Majid.

If we denote $a \otimes b$ by ab and $R(b \otimes a)$ by ba whenever $a \in A$ and $b \in B$, we can rewrite the conditions on R and the above product in a compact way. The first condition on R becomes

$$(bb')a = b(b'a),$$

provided we define $b(a_1 b_1) = (ba_1)b_1$ and $(a_2 b_2)b_1 = a_2(b_2 b_1)$. The second condition on R becomes

$$b(aa') = (ba)a'$$

if we let $(a_1 b_1)a' = a_1(b_1 a')$ and $a_1(a_2 b_2) = (a_1 a_2)b_2$. It is obvious that all these definitions are compatible with the linear structure of $A \otimes B$ and $B \otimes A$. Moreover it is easy to show that the obvious associativity rules are valid and that indeed the conditions on R take care of the missing ones.

The product is given by $(ab)(a'b') = a(ba')b'$, when $a(a_1 b_1)b'$ is defined as $(aa_1)(b_1 b')$. Also this product is associative but this is not completely obvious. We will prove it in the next proposition.

We will also show that, if A and B are algebras with an identity 1, and if R satisfies certain extra conditions, the maps $a \to a \cdot 1$ and $b \to 1 \cdot b$ are injective homomorphisms of A and B in $A \otimes_R B$. If we identify a and b with their image we get precisely that ab is the product of a with b and that ba is the product of b with a. This should explain why we work with this compact notation and why this compact notation works. If no confusion about the mapping R is possible we will denote $A \otimes_R B$ by AB.

2.2 PROPOSITION. *The product in AB is associative.*

Proof. First notice that the associativity of the multiplication in A and B yields that $(cb)b' = c(bb')$ and $a(a'c) = (aa')c$ for all $a, a' \in A$, $b, b' \in B$ and $c \in AB$.

Now let $a, a', a'' \in A$ and $b, b', b'' \in B$. Then $((ab)(a'b'))(a''b'') = (a(ba')b')(a''b'')$. We have that

$$\begin{aligned}(a(a_1b_1)b')(a''b'') &= ((aa_1)(b_1b'))(a''b'')\\ &= (aa_1)((b_1b')a'')b''\\ &= (aa_1)(b_1(b'a''))b''\end{aligned}$$

for all $a_1 \in A$ and $b_1 \in B$ and since

$$\begin{aligned}(aa_1)(b_1(a_2b_2))b'' &= (aa_1)((b_1a_2)b_2)b''\\ &= (aa_1)(b_1a_2)(b_2b'')\\ &= ((aa_1)b_1)(a_2(b_2b'')\\ &= (a(a_1b_1))((a_2b_2)b'')\end{aligned}$$

for all $a_2 \in A$ and $b_2 \in B$, we get

$$(a(a_1b_1)b')(a''b'') = (a(a_1b_1))((b'a'')b''),$$

and so

$$((ab)(a'b'))(a''b'') = (a(ba'))((b'a'')b'').$$

A similar argument gives us that

$$(ab)((a'b')(a''b'')) = (a(ba'))((b'a'')b''). \qquad \square$$

If A and B have an identity 1 and if R satisfies some natural conditions we can embed A and B in AB in such a way that ab is indeed the product of a and b:

2.3 PROPOSITION. *If R satisfies*

$$R(1 \otimes a) = a \otimes 1 \quad \textit{and} \quad R(b \otimes 1) = 1 \otimes b, \quad \forall a \in A,\ b \in B, \tag{2}$$

then the mappings

$$i_A\colon A \to AB\colon a \mapsto a.1$$

$$i_B\colon B \to AB\colon b \mapsto 1.b$$

are homomorphisms.

The formulas (2) can be rewritten in AB as $1 \cdot a = a \cdot 1$ and $b \cdot 1 = 1 \cdot b$ for $a \in A$ and $b \in B$. Moreover we have that $\forall a,\ a' \in A,\ \forall b,\ b' \in B$:

$$i_A(a)(a'b) = a(a'b); \quad (ab)i_B(b') = (ab)b';$$

$$i_B(b)(ab') = b(ab'); \quad (ab)i_A(a') = (ab)a'.$$

Hence we can identify $i_A(a)$ with a and $i_B(b)$ with b.

Now let A and B be *-algebras. In view of the previous remarks, in the case were A and B have identities, it would be natural to define an involution on AB by $(ab)^* = b^*a^*$. This can only be an involution if $(b^*a^*)^* = ab$. It turns out that this condition on R is sufficient to make AB into an involutive algebra. Remark that this condition in tensor product form is written as $(R(J \otimes J)\sigma)^2 = \iota \otimes \iota$ if we denote the involution on A and B by J and the flip on $A \otimes B$ by σ.

2.4 PROPOSITION. *If R satisfies*

$$R(J \otimes J)\sigma R(J \otimes J)\sigma = \iota \otimes \iota, \tag{3}$$

then $R(J \otimes J)\sigma$ is an involution on $A \otimes_R B$.

Proof. We still have to check that $((ab)(a'b'))^* = (a'b')^*(ab)^*$ for all $a,\ a' \in A$ and $b,\ b' \in B$. So let $a,\ a' \in A$ and $b,\ b' \in B$. Then we have $((ab)(a'b'))^* = (a(ba')b')^*$. Now for $a_1 \in A$, $b_1 \in B$, we have that

$$\begin{aligned}(a(a_1b_1)b')^* &= ((aa_1)(b_1b'))^* \\ &= (b_1b')^*(aa_1)^* \\ &= (b'^*b_1^*)(a_1^*a^*).\end{aligned}$$

One can easily see that this last expression is equal to $b'^*((b_1^*a_1^*)a^*) = b'^*((a_1b_1)^*a^*)$. So $((ab)(a'b'))^* = b'^*((ba')^*a^*) = b'^*((a'^*b^*)a^*)$.

On the other hand we get $(a'b')^*(ab)^* = (b'^*a'^*)(b^*a^*)$, and one can verify that this is also equal to $b'^*((a'^*b^*)a^*)$. □

One can also verify that for *-algebras A and B with identities, and for R satisfying the above conditions, the embeddings i_A and i_B are *-homomorphisms.

We now give some examples.

2.5 EXAMPLES. (i) Let A and B be *-algebras. If we take the flip σ for R, we can check that σ satisfies the conditions and AB becomes the usual tensor product $A \otimes B$ of the two *-algebras.

(ii) Let A be a *-algebra, and B the group algebra of a finite group G. If we have an action α of G on A, we can define $R: B \otimes A \to A \otimes B$ by $R(\Sigma_s s \otimes a_s) = \Sigma_s \alpha_s(a_s) \otimes s$. We show that R satisfies the conditions. If $s, s' \in G$ and $a \in A$,

$$R(m \otimes \iota)(s \otimes s' \otimes a) = R(ss' \otimes a) = \alpha_{ss'}(a) \otimes ss',$$

while on the other hand

$$\begin{aligned}(\iota \otimes m)(R \otimes \iota)(\iota \otimes R)(s \otimes s' \otimes a) &= (\iota \otimes m)(R \otimes \iota)(s \otimes \alpha_{s'}(a) \otimes s') \\ &= (\iota \otimes m)(\alpha_s(\alpha_{s'}(a)) \otimes s \otimes s') \\ &= \alpha_{ss'}(a) \otimes ss'.\end{aligned}$$

Similarly, if $a, a' \in A$ and $s \in G$,

$$R(\iota \otimes m)(s \otimes a \otimes a') = R(s \otimes aa') = \alpha_s(aa') \otimes s,$$

and

$$\begin{aligned}(m \otimes \iota)(\iota \otimes R)(R \otimes \iota)(s \otimes a \otimes a') &= (m \otimes \iota)(\iota \otimes R)(\alpha_s(a) \otimes s \otimes a') \\ &= (m \otimes \iota)(\alpha_s(a) \otimes \alpha_s(a') \otimes s) \\ &= \alpha_s(a)\alpha_s(a') \otimes s,\end{aligned}$$

so that condition (1) is fulfilled. Remark that the first condition of (1) follows from the fact that α_s is an algebra homomorphism and the second follows from the fact that α is a group action.

One can see that condition (2) is fulfilled if A has a unit. Also condition (3) is satisfied. Since $(R(J \otimes J)\sigma)(a \otimes s) = R(s^{-1} \otimes a^*) = \alpha_{s^{-1}}(a^*) \otimes s^{-1}$, we have that

$$\begin{aligned}(R(J \otimes J)\sigma R(J \otimes J)\sigma)(a \otimes s) &= (R(J \otimes J)\sigma)(\alpha_{s^{-1}}(a^*) \otimes s^{-1}) \\ &= \alpha_s(\alpha_{s^{-1}}(a^*)^*) \otimes s \\ &= \alpha_s(\alpha_{s^{-1}}(a)) \otimes s \\ &= a \otimes s.\end{aligned}$$

In this case, AB is the crossed product of A by the action α of G.

(iii) A combination of the first two examples gives us the following. Let A_1 and A_2 be *-algebras, and let B_1 and B_2 be the group algebras of finite groups G_1 and G_2 respectively. Let α be an action of G_1 on A_1 and β be an action of G_2 on A_2. Let $R_1: B_1 \otimes A_1 \to A_1 \otimes B_1$ be as in example (ii) but let $R_2: A_2 \otimes B_2 \to B_2 \otimes A_2$ be defined by $R_2(a \otimes s) = s \otimes \beta_s(a)$. Put $A = A_1 \otimes B_2$ and $B = B_1 \otimes A_2$, and define $R: B \otimes A \to A \otimes B$ as $R = \sigma_{23}(R_1 \otimes R_2)\sigma_{23}$, where $\sigma_{23} = \iota \otimes \sigma \otimes \iota$. One can check that R satisfies conditions (1) and (3), and hence we get a new algebra AB.

We finish this section by formulating some properties of this twisted tensor product.

2.6 PROPOSITION. *Let A, B be *-algebras and $R: B \otimes A \to A \otimes B$ satisfying conditions* (1), (2), (3). *Let A_1 and B_1 also be *-algebras.*

(i) *Suppose $R_1: B_1 \otimes A_1 \to A_1 \otimes B_1$ also satisfies conditions* (1), (2), (3). *If $\varphi: A \to A_1$ and $\psi: B \to B_1$ are *-homomorphisms satisfying $R_1 \circ (\psi \otimes \varphi) = (\varphi \otimes \psi) \circ R$, then $\varphi \otimes \psi: A \otimes_R B \to A_1 \otimes_{R_1} B_1$ is a *-homomorphism of the twisted tensor products.*

(ii) *If the mappings $\varphi: A \to A_1$ and $\psi: B \to B_1$ are bijective *-homomorphisms, then $R_1 := (\varphi \otimes \psi) \circ R \circ (\psi^{-1} \otimes \varphi^{-1})$ satisfies conditions* (1), (2), (3), *and hence defines a twisted tensor product $A_1 \otimes_{R_1} B_1$, isomorphic with $A \otimes_R B$.*

The proof of these properties is straightforward. It is also easy to check that, if A_1, B_1 are subalgebras of A, B respectively such that $R(B_1 \otimes A_1) \subseteq A_1 \otimes B_1$, then $A_1 B_1$ is a subalgebra of AB.

3. The algebras AA' and $AA' \otimes AA'$

Consider a Hopf *-algebra A. For a *-algebra D we will introduce a *-algebra structure on $L(A, D)$, the vectorspace of linear D-valued mappings on A. Then we will define two mappings $R_1, R_2: L(A, D) \otimes A \to A \otimes L(A, D)$ satisfying the conditions of section 2, and hence we will get two twisted tensor products $A \otimes_{R_1} L(A, D)$ and $A \otimes_{R_2} L(A, D)$.

The proof of the following proposition is straightforward (see also [1]).

3.1 PROPOSITION. *Define multiplication and involution on* $L(A, D)$ *by*

$$f_1 \cdot f_2 = m(f_1 \otimes f_2)\Delta$$
$$f^*(a) = (f(S(a)^*))^*,$$

where $f, f_1, f_2 \in L(A, D)$ *and* $a \in A$ *and where* m *denotes multiplication on* D. *Then* $L(A, D)$ *is a* $*$*-algebra.*

Remark that we get the algebraic dual A' of A with its usual algebra structure if we choose the complex field $\mathbb{C}$ as algebra D. It will turn out that the algebraic tensor product $A' \otimes D$ is a $*$-subalgebra of $L(A, D)$.

We now want to define the two mappings $R_1, R_2 : L(A, D) \otimes A \to A \otimes L(A, D)$. For notational convenience we will consider elements of $A \otimes L(A, D)$ sometimes as linear maps from A to $A \otimes D$. So, if $a \in A$ and $f \in L(A, D)$ then $(a \otimes f)(x) = a \otimes f(x)$ for all $x \in A$. Similarly, elements of $A \otimes L(A, D) \otimes L(A, D)$ will be considered as functions of two variables on A with values in $A \otimes D \otimes D$ and other tensor products combining A and $L(A, D)$ will be treated in an analogous way. This will make it much easier to write down the proofs in what follows.

3.2 DEFINITION. Let A, D and $L(A, D)$ be as above. Define two linear maps $R_1, R_2 : L(A, D) \otimes A \to A \otimes L(A, D)$ by

$$(R_1(f \otimes a))(x) = \sum_{(a)} a_{(1)} \otimes f(a_{(2)}x)$$
$$(R_2(f \otimes a))(x) = \sum_{(a)} a_{(2)} \otimes f(a_{(3)}xS^{-1}(a_{(1)})).$$

It is easy to see that these linear maps are well-defined.
Here we recognise the formulas in [7, page 36] and [13, page 846].
We verify that these mappings satisfy the conditions of section 2.

3.3 PROPOSITION. *For* $R = R_1, R_2$ *we have that*

(i) $R(m \otimes \iota) = (\iota \otimes m)(R \otimes \iota)(\iota \otimes R)$
(ii) $R(\iota \otimes m) = (m \otimes \iota)(\iota \otimes R)(R \otimes \iota)$
(iii) $R(J \otimes J)\sigma R(J \otimes J)\sigma = \iota \otimes \iota$.

Proof. We first prove the three relations for R_1.
(i) Let $f, g \in L(A, D)$ and $a, x, y \in A$. Then

$$((\iota \otimes R_1)(f \otimes g \otimes a))(x, y) = \sum_{(a)} f(x) \otimes a_{(1)} \otimes g(a_{(2)}y).$$

So

$$((R_1 \otimes \iota)(\iota \otimes R_1)(f \otimes g \otimes a))(x, y) = \sum_{(a)} a_{(1)} \otimes f(a_{(2)}x) \otimes g(a_{(3)}y).$$

Therefore, using the formula for the multiplication in $L(A, D)$, we get

$$\begin{aligned}((\iota \otimes m)(R_1 \otimes \iota)(\iota \otimes R_1)(f \otimes g \otimes a))(x) &= \sum_{(a)(x)} a_{(1)} \otimes f(a_{(2)}x_{(1)})g(a_{(3)}x_{(2)}) \\ &= \sum_{(a)} a_{(1)} \otimes (fg)(a_{(2)}x) \\ &= (R_1(fg \otimes a))(x) \\ &= (R_1(m \otimes \iota)(f \otimes g \otimes a))(x).\end{aligned}$$

This proves the first relation.

(ii) Let $f \in L(A, D)$ and $a, b, x \in A$. Then

$$((R_1 \otimes \iota)(f \otimes a \otimes b))(x) = \sum_{(a)} a_{(1)} \otimes f(a_{(2)}x) \otimes b.$$

So

$$((\iota \otimes R_1)(R_1 \otimes \iota)(f \otimes a \otimes b))(x) = \sum_{(a)(b)} a_{(1)} \otimes b_{(1)} \otimes f(a_{(2)}b_{(2)}x),$$

and

$$\begin{aligned}((m \otimes \iota)(\iota \otimes R_1)(R_1 \otimes \iota)(f \otimes a \otimes b))(x) &= \sum_{(a)(b)} a_{(1)}b_{(1)} \otimes f(a_{(2)}b_{(2)}x) \\ &= \sum_{(ab)} (ab)_{(1)} \otimes f((ab)_{(2)}x) \\ &= (R_1(f \otimes ab))(x) \\ &= (R_1(\iota \otimes m)(f \otimes a \otimes b))(x).\end{aligned}$$

This proves the second relation.

(iii) Let $f \in L(A, D)$ and $a, x \in A$. Then

$$\begin{aligned}(R_1(J \otimes J)\sigma(a \otimes f))(x) &= (R_1(f^* \otimes a^*))(x) \\ &= \sum_{(a)} a^*_{(1)} \otimes f^*(a^*_{(2)}x) \\ &= \sum_{(a)} a^*_{(1)} \otimes f(S(a^*_{(2)})^* S(x)^*)^* \\ &= \sum_{(a).} a^*_{(1)} \otimes f(S^{-1}(a_{(2)})S(x)^*)^*.\end{aligned}$$

Now, if $b \in A$ and if g is defined in $L(A, D)$ by $g(x) = f(S^{-1}(b)S(x)^*)^*$, then $g^*(x) = f(S^{-1}(b)x)$. So, if we apply $R_1(J \otimes J)\sigma$ once more, we obtain

$$(R_1(J \otimes J)\sigma R_1(J \otimes J)\sigma(a \otimes f))(x) = \sum_{(a)} a_{(1)} \otimes f(S^{-1}(a_{(3)})a_{(2)}x).$$

But

$$\begin{aligned}\sum_{(a)} S^{-1}(a_{(2)})a_{(1)} &= (m(S^{-1} \otimes \iota)\sigma\Delta)(a) \\ &= (m(\iota \otimes S)\Delta S^{-1})(a) \\ &= \varepsilon(S^{-1}(a))1 = \varepsilon(a)1.\end{aligned}$$

Therefore

$$\begin{aligned}(R_1(J \otimes J)\sigma R_1(J \otimes J)\sigma(a \otimes f))(x) &= \sum_{(a)} a_1 \otimes f(\varepsilon(a_{(2)})x) \\ &= \sum_{(a)} a_1\varepsilon(a_{(2)}) \otimes f(x) \\ &= a \otimes f(x) \\ &= (a \otimes f)(x).\end{aligned}$$

This proves the third equality.

Now we prove the relations for R_2.

(*i*) Let $f, g \in L(A, D)$ and $a, x, y \in A$. Then

$$((\iota \otimes R_2)(f \otimes g \otimes a))(x, y) = \sum_{(a)} f(x) \otimes a_{(2)} \otimes g(a_{(3)}yS^{-1}(a_{(1)})).$$

So

$$((R_2 \otimes \iota)(\iota \otimes R_2)(f \otimes g \otimes a))(x, y) = \sum_{(a)} a_{(3)} \otimes f(a_{(4)}xS^{-1}(a_{(2)})) \otimes g(a_{(5)}yS^{-1}(a_{(1)})).$$

Using the formula for the multiplication in $L(A, D)$, we get

$$\begin{aligned}&((\iota \otimes m)(R_2 \otimes \iota)(\iota \otimes R_2)(f \otimes g \otimes a))(x) \\ &\quad = \sum_{(a)(x)} a_{(3)} \otimes f(a_{(4)}x_{(1)}S^{-1}(a_{(2)}))g(a_{(5)}x_{(2)}S^{-1}(a_{(1)})) \\ &\quad = \sum_{(a)} a_{(2)} \otimes (fg)(a_{(3)}xS^{-1}(a_{(1)})) \\ &\quad = R_2(m \otimes \iota)(f \otimes g \otimes a))(x).\end{aligned}$$

This proves the first relation.

(ii) Let $f \in L(A, D)$ and $a, b, x \in A$. Then

$$((R_2 \otimes \iota)(f \otimes a \otimes b))(x) = \sum_{(a)} a_{(2)} \otimes f(a_{(3)} x S^{-1}(a_{(1)})) \otimes b.$$

So

$$((\iota \otimes R_2)(R_2 \otimes \iota)(f \otimes a \otimes b))(x) = \sum_{(a)(b)} a_{(2)} \otimes b_{(2)} \otimes f(a_{(3)} b_{(3)} x S^{-1}(b_{(1)}) S^{-1}(a_{(1)})),$$

and

$$\begin{aligned}((m \otimes \iota)(\iota \otimes R_2)(R_2 \otimes \iota)(f \otimes a \otimes b))(x) &= \sum_{(ab)} (ab)_{(2)} \otimes f((ab)_{(3)} x S^{-1}((ab)_{(1)})) \\ &= (R_2(\iota \otimes m)(f \otimes a \otimes b))(x).\end{aligned}$$

This proves the second relation.

(iii) Let $f \in L(A, D)$ and $a, x \in A$. Then

$$\begin{aligned}(R_2(J \otimes J)\sigma(a \otimes f))(x) &= \sum_{(a)} a^*_{(2)} \otimes f^*(a^*_{(3)} x S^{-1}(a^*_{(1)})) \\ &= \sum_{(a)} a^*_{(2)} \otimes f(S(a^*_{(3)})^* S(x)^* a_{(1)})^*.\end{aligned}$$

Now if $b, c \in A$, and if g is defined in $L(A, D)$ by $g(x) = f(S^{-1}(b)S(x)^*c)^*$, then $g^*(x) = f(S^{-1}(b)xc)$. So, applying $R_2(J \otimes J)\sigma$ once more gives

$$\begin{aligned}(R_2(J \otimes J)\sigma R_2(J \otimes J)\sigma(a \otimes f))(x) &= \sum_{(a)} a_{(3)} \otimes f(S^{-1}(a_{(5)}) a_{(4)} x S^{-1}(a_{(2)}) a_{(1)}) \\ &= \sum_{(a)} a_{(2)} \otimes f(\varepsilon(a_{(3)}) x \varepsilon(a_{(1)})) \\ &= a \otimes f(x) \\ &= (a \otimes f)(x).\end{aligned}$$

This proves the third equality. □

If A and D have a unit, one can easily see that R_1 and R_2 also satisfy the formulas $R(1 \otimes a) = a \otimes 1$ and $R(f \otimes 1) = 1 \otimes f$.

By choosing $\mathbb{C}$ for D, we get two algebras $A \otimes_{R_1} A'$ and $A \otimes_{R_2} A'$, which we will both denote by AA' when no confusion is possible.

For any D we can embed $A' \otimes D$ in $L(A, D)$ by

$$(i(f \otimes d))(a) = f(a)d$$

whenever $a \in A$, $d \in D$ and $f \in A'$. This embedding i is a $*$-homomorphism. In turn i induces an embedding $j = \iota \otimes i : A \otimes A' \otimes D \to A \otimes L(A, D)$. This is also a $*$-homomorphism from $AA' \otimes D$ to $AL(A, D)$.

In the finite-dimensional case these embeddings are also surjective. This is no longer true in the infinite-dimensional case. However, then it is possible to find a suitable vector space topology on the larger space such that the images are dense. We don't want to elaborate further on this, but use this idea as a motivation to denote $L(A, D)$ by $A' \bar{\otimes} D$ and similarly $AL(A, D)$ by $AA' \bar{\otimes} D$. If we want to specify the R, we will also use $AA' \bar{\otimes}_R D$ here as before. It is easily seen that also $A' \bar{\otimes} D$ is a subalgebra of $AA' \bar{\otimes} D$ by the natural embedding $f \mapsto 1f$.

In the future we will omit i and j in our notations and we will consider $A' \otimes D$ as a subalgebra of $A' \bar{\otimes} D$ and $AA' \otimes D$ as a subalgebra of $AA' \bar{\otimes} D$.

Taking AA' for the algebra D gives us an algebra $AA' \bar{\otimes} AA'$. Applying the same construction to this algebra, we get an algebra $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$. This algebra contains the algebra $AA' \bar{\otimes} AA'$ in three different ways. Indeed, we have three embeddings i_{12}, i_{13}, i_{23} of $AA' \bar{\otimes} AA'$ into $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$, by extending the three natural embeddings of $AA' \otimes AA'$ into $AA' \otimes AA' \otimes AA'$.

Consider for example the natural embedding $AA' \otimes AA' \to AA' \otimes AA' \otimes 1$. The algebra $AA' \otimes AA'$ is a subalgebra of $AA' \bar{\otimes} AA'$ and $AA' \otimes AA' \otimes 1$ is a subalgebra of $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$. We define i_{12} as the mapping $AA' \bar{\otimes} AA' \to AA' \bar{\otimes} (AA' \bar{\otimes} AA')$ that extends the natural embedding of $AA' \otimes AA'$ into $AA' \otimes AA' \otimes 1$. The two other mappings are given in an analogous way. The exact definition is as follows:

$$i_1 : AA' \to AA' \bar{\otimes} AA' : ab \mapsto j(ab \otimes 1)$$

$$i_2 : AA' \to AA' \bar{\otimes} AA' : ab \mapsto j(1 \otimes ab)$$

$$i_{12} : AA' \bar{\otimes} AA' \to AA' \bar{\otimes} (AA' \bar{\otimes} AA') : af \mapsto a(i_1 \circ f)$$

$$i_{13} : AA' \bar{\otimes} AA' \to AA' \bar{\otimes} (AA' \bar{\otimes} AA') : af \mapsto a(i_2 \circ f)$$

$$i_{23} : AA' \bar{\otimes} AA' \to AA' \bar{\otimes} (AA' \bar{\otimes} AA') : af \mapsto j(1 \otimes af).$$

These mappings are $*$-homomorphisms. Indeed, clearly i_1, i_2 and i_{23} are $*$-homomorphisms, since j is one. The mappings i_{12}, i_{13} can also be checked with straightforward techniques. The injectivity of the mapping i_{12}, i_{13} and i_{23} is clear, and so we really have embeddings of $AA' \bar{\otimes} AA'$ in $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$.

4. The formulas $\Delta(a) = W(a \otimes 1)W^*$ and $\Delta(a) = W\Delta'(a)W^*$ in $AA' \bar{\otimes} AA'$

Again, let A be a Hopf $*$-algebra and let

$$AA' \bar{\otimes}_R AA' = AL(A, AA') = A \otimes_R L(A, AA'),$$

where $R = R_1$ or R_2 as defined in the previous section, and is again omitted in the notation when no confusion is possible. We will consider elements in $AA' \bar{\otimes} AA'$ as functions from A to $A \otimes AA'$ as before.

In this section we will consider the subalgebras $A \otimes A$ and $A' \bar{\otimes} A = L(A, A)$ of $AA' \bar{\otimes} AA'$. We have that

$$(a \otimes b)(x) = a \otimes \varepsilon(x)b$$

$$f(x) = 1 \otimes f(x)$$

for $a, b \in A$ and $f \in L(A, A)$.

We first define W in $L(A, A)$.

4.1 DEFINITION. Let W be the identity map in $L(A, A)$.

Then $W^*(a) = W(S(a)^*)^* = S(a)$ when $a \in A$. Moreover

$$(W^*W)(a) = \sum_{(a)} W^*(a_{(1)})W(a_{(2)}) = \sum_{(a)} S(a_{(1)})a_{(2)} = \varepsilon(a)1.$$

So we get $W^*W = 1$ in the algebra $L(A, A)$. Similarly $WW^* = 1$, so that W is a unitary. When considered as an element in $AA' \bar{\otimes} AA'$, we get $W(x) = 1 \otimes x$ for $x \in A$, and of course also here W is a unitary. Moreover we have the following formulas.

4.2 PROPOSITION. (i) *In* $AA' \bar{\otimes}_{R_1} AA'$ *we have for all* $a \in A$:

$$W(a \otimes 1)W^* = \Delta(a).$$

(ii) *In* $AA' \bar{\otimes}_{R_2} AA'$ *we have for all* $a \in A$:

$$W^*\Delta(a)W = \Delta'(a),$$

where $\Delta' = \sigma\Delta$ *is the opposite comultiplication.*

Proof. In the two cases we have for $a, x \in A$, that

$$(\Delta(a)W)(x) = \left(\sum_{(a)} (a_{(1)} \otimes a_{(2)})W\right)(x) = \sum_{(a)} a_{(1)} \otimes a_{(2)}x.$$

In case (i) we get

$$\begin{aligned}(W(a \otimes 1))(x) &= (R_1(W \otimes a))(x)\\ &= \sum_{(a)} a_{(1)} \otimes W(a_{(2)}x)\\ &= \sum_{(a)} a_{(1)} \otimes a_{(2)}x,\end{aligned}$$

proving the first formula.

In case (ii) we get

$$\begin{aligned}(W\sigma\Delta(a))(x) &= \sum_{(a)} (W(a_{(2)} \otimes a_{(1)}))(x) \\ &= \sum_{(a)} (W(a_{(2)} \otimes 1)(1 \otimes a_{(1)}))(x) \\ &= \sum_{(a)} (W(a_{(2)} \otimes 1))(x)(1 \otimes a_{(1)}) \\ &= \sum_{(a)} a_{(3)} \otimes W(a_{(4)} x S^{-1}(a_{(2)}))a_{(1)} \\ &= \sum_{(a)} a_{(3)} \otimes a_{(4)} x S^{-1}(a_{(2)})a_{(1)}.\end{aligned}$$

We have seen before that $\sum_{(a)} S^{-1}(a_{(2)})a_{(1)} = \varepsilon(a)1$. So we get

$$\begin{aligned}(W\sigma\Delta(a))(x) &= \sum_{(a)} a_{(2)} \otimes a_{(3)} x \varepsilon(a_{(1)}) \\ &= \sum_{(a)} a_{(1)} \otimes a_{(2)} x.\end{aligned}$$

This proves the second formula. □

Remark that, essentially, these formulas determine the commutation rules R_1 and R_2 from $A'A$ to AA'.

We can consider these formulas in some examples.

4.3 EXAMPLE. Consider a finite group G, and let A be the group algebra of G. If we define $\Delta(s) = s \otimes s$, $S(s) = s^{-1}$ and $\varepsilon(s) = 1$ for all $s \in G$, A becomes a Hopf *-algebra. A' is the algebra of linear functions on A, equipped with pointwise multiplication. The element $\sum_{s \in G} \delta_s \otimes s \in A' \otimes A$, considered as a function in $L(A, A)$, is the identical function. So $W = \sum_{s \in G} \delta_s \otimes s$. We then have:

(i) In $AA' \otimes_{R_1} AA'$:

$$\begin{aligned}W(s \otimes 1)W^* &= \sum_u (\delta_u \otimes u)(s \otimes 1) \sum_v (\delta_v \otimes v^{-1}) \\ &= \sum_{u,v} (s \otimes 1)(\delta_{s^{-1}u} \otimes u)(\delta_v \otimes v^{-1}) \\ &= \sum_u (s \otimes 1)(\delta_{s^{-1}u} \otimes u)(\delta_{s^{-1}u} \otimes u^{-1}s) \\ &= \sum_u (s \otimes 1)(\delta_{s^{-1}u} \otimes s)\end{aligned}$$

$$= (s \otimes 1)\left(\sum_u \delta_{s^{-1}u} \otimes s\right)$$

$$= (s \otimes 1)(1 \otimes s) = s \otimes s = \Delta(s).$$

This gives proposition 4.2(i).

(ii) In $AA' \bar{\otimes}_{R_2} AA'$:

$$W\Delta'(s)W^* = \sum_u (\delta_u \otimes u)(s \otimes s) \sum_v (\delta_v \otimes v^{-1})$$

$$= \sum_{u,v} (s \otimes u)(\delta_{s^{-1}us} \otimes s)(\delta_v \otimes v^{-1})$$

$$= \sum_u (s \otimes u)(\delta_{s^{-1}us} \otimes s)(\delta_{s^{-1}us} \otimes s^{-1}u^{-1}s)$$

$$= \sum_u (s \otimes s)(\delta_{s^{-1}us} \otimes 1) = (s \otimes s) = \Delta(s).$$

This is proposition 4.2(ii). □

4.4 EXAMPLE. Let A be the *-algebra with identity generated by a self-adjoint element h. One can define $\Delta: A \to A \otimes A$ by $\Delta(h) = h \otimes 1 + 1 \otimes h$, $\varepsilon: A \to \mathbb{C}$ by $\varepsilon(h) = 0$, and $S: A \to A$ by $S(h) = -h$. It is easy to verify that A is a Hopf *-algebra. Let B be the *-algebra with identity generated by a self-adjoint element k, with the same Hopf *-algebra structure. Define, for a given $\lambda \in \mathbb{R}$, and for all n, $m \in \mathbb{N}$: $\langle h^n, k^m \rangle = \delta(n, m) n! (i\lambda)^n$, where δ is the Kronecker delta. This is a non-degenerate bilinear mapping $A \times B \to \mathbb{C}$, and since also $\langle \Delta(h^n), k^p \otimes k^q \rangle = \langle h^n, k^p k^q \rangle$ and $\langle h^n, (k^m)^* \rangle = \langle S(h^n)^*, k^m \rangle^-$, we have that $\bar{B} = A'$, when we consider A' with the weak *-topology (see [14]).

For each element $a \in A$, the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\lambda} (k \otimes h)\right)^n (a)$$

reduces to a finite sum, so we can say that the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\lambda} (k \otimes h)\right)^n$$

converges in $L(A, A)$, and we denote it by $\exp\left(\frac{1}{i\lambda}(k \otimes h)\right)$.

Moreover, we have that

$$\exp\left(\frac{1}{i\lambda}(k \otimes h)\right)(h^j) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{(i\lambda)^n}(k^n \otimes h^n)(h^j)\right)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{(i\lambda)^n}\delta(n,j)j!(i\lambda)^j h^n$$
$$= h^j.$$

Thus $\exp\left(\frac{1}{i\lambda}(k \otimes h)\right)$ is the element W.

(i) Since $R_1(k \otimes h)(x) = h \otimes k(x) + 1 \otimes k(hx)$ for all x in A, we have that $R_1(k \otimes h) = h \otimes k + i\lambda(1 \otimes 1)$. So in $A \otimes_{R_1} B$ we have $[k, h] = i\lambda$. Hence in $AA' \bar{\otimes}_{R_1} AA'$ we get

$$W(h \otimes 1)W^* = \exp\left(\frac{1}{i\lambda}(k \otimes h)\right)(h \otimes 1)\exp\left(-\frac{1}{i\lambda}(k \otimes h)\right)$$
$$= h \otimes 1 + \frac{1}{i\lambda}[k \otimes h, h \otimes 1]$$
$$= h \otimes 1 + 1 \otimes h = \Delta(h).$$

This illustrates proposition 4.3(i).

(ii) Since $R_2(k \otimes h)(x) = 1 \otimes k(-xh) + h \otimes k(x) + 1 \otimes k(hx)$ for all x in A, we have that $R_2(k \otimes h) = h \otimes k$. Hence $A \otimes_{R_2} B$ is commutative, and in $AA' \bar{\otimes}_{R_2} AA'$ we get

$$W\Delta'(h)W^* = \exp\left(\frac{1}{i\lambda}(k \otimes h)\right)(h \otimes 1 + 1 \otimes h)\exp\left(-\frac{1}{i\lambda}(k \otimes h)\right)$$
$$= (h \otimes 1 + 1 \otimes h)WW^*$$
$$= h \otimes 1 + 1 \otimes h = \Delta(h).$$

5. The Pentagon and Yang-Baxter equation

The three embeddings i_{12}, i_{13} and i_{23} of $AA' \bar{\otimes} AA'$ in $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$, described in section 3, give rise to the elements $W_{12} = i_{12}(W)$, $W_{13} = i_{13}(W)$ and $W_{23} = i_{23}(W)$. In this section we will show that W satisfies the Pentagon equation in $AA' \bar{\otimes}_{R_1} (AA' \bar{\otimes}_{R_1} AA')$ and the Yang-Baxter equation in $AA' \bar{\otimes}_{R_2} (AA' \bar{\otimes}_{R_2} AA')$.

The *-homomorphism $\iota \otimes \Delta : A' \otimes A \to A' \otimes (A \otimes A)$ can be extended to a *-homomorphism $\iota \otimes \Delta : A' \bar{\otimes} A \to A' \bar{\otimes} (A \otimes A)$, mapping f to $\Delta \circ f$, and later on extended to a *-homomorphism $\iota \otimes \Delta : AA' \bar{\otimes} A \to AA' \bar{\otimes} (AA' \bar{\otimes} AA')$. Then we have the following.

5.1 PROPOSITION. *In $AA' \bar{\otimes} (AA' \bar{\otimes} AA')$ we have that $(\iota \otimes \Delta)(W) = W_{12}W_{13}$.*

Proof. All three elements $(\iota \otimes \Delta)(W)$, W_{12} and W_{13} are in fact in the subalgebra $A' \bar{\otimes} (A \otimes A) = L(A, A \otimes A)$, and it is therefore sufficient to prove the equation in this subalgebra. For all $x \in A$ we have

$$(1 \otimes \Delta)(W)(x) = \Delta(W(x)) = \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

On the other hand

$$\begin{aligned}(W_{12}W_{13})(x) &= \sum_{(x)} W_{12}(x_{(1)})W_{13}(x_{(2)}) \\ &= \sum_{(x)} (x_{(1)} \otimes 1)(1 \otimes x_{(2)}) \\ &= \sum_{(x)} x_{(1)} \otimes x_{(2)}.\end{aligned}$$ □

We now come to the proof of the Pentagon equation.

5.2 THEOREM. *In $AA' \bar{\otimes}_{R_1} (AA' \bar{\otimes}_{R_1} AA')$ we have $W_{12}W_{13}W_{23} = W_{23}W_{12}$.*

Proof. Because of proposition 5.1 it will be sufficient to prove

$$W_{23}W_{12}W_{23}^* = (\iota \otimes \Delta)(W).$$

We consider this equation in the subalgebra $A' \bar{\otimes} (AA' \bar{\otimes}_{R_1} AA') = L(A, AA' \bar{\otimes}_{R_1} AA')$. So let $a \in A$. Then, we have

$$W_{12}(a) = a \otimes 1$$
$$W_{23}(a) = \varepsilon(a)W.$$

So

$$\begin{aligned}(W_{23}W_{12}W_{23}^*)(a) &= \sum_{(a)} W_{23}(a_{(1)})W_{12}(a_{(2)})W_{23}^*(a_{(3)}) \\ &= \sum_{(a)} \varepsilon(a_{(1)})W(a_{(2)} \otimes 1)\varepsilon(a_{(3)})W^* \\ &= W(a \otimes 1)W^* = \Delta(a).\end{aligned}$$ □

Similarly we can prove the Yang-Baxter equation.

5.3 THEOREM. *In* $AA' \bar{\otimes}_{R_2} (AA' \bar{\otimes}_{R_2} AA')$ *we have* $W_{12}W_{13}W_{23} = W_{23}W_{13}W_{12}$.

Proof. In Proposition 5.1 we saw that here $(\iota \otimes \Delta)(W) = W_{12}W_{13}$. It is not hard to see that $(\iota \otimes \Delta')(W) = W_{13}W_{12}$. Therefore we must show that

$$W_{23}(\iota \otimes \Delta')(W) = (\iota \otimes \Delta)(W)W_{23}.$$

We can do this again in $A' \bar{\otimes} (AA' \bar{\otimes}_{R_2} AA') = L(A, AA' \bar{\otimes}_{R_2} AA')$. So let $a \in A$. Then

$$\begin{aligned}(W_{23}(\iota \otimes \Delta')(W))(a) &= \sum W_{23}(a_{(1)})(\iota \otimes \Delta')(W)(a_{(2)}) \\ &= \sum \varepsilon(a_{(1)}) W\Delta'(a_{(2)}) \\ &= W\Delta'(a),\end{aligned}$$

while

$$\begin{aligned}((\iota \otimes \Delta)(W))W_{23}(a) &= \sum_{(a)} ((\iota \otimes \Delta)W)(a_{(1)})W_{23}(a_{12}) \\ &= \Delta(a)W.\end{aligned}$$

This proves the Yang-Baxter equation. □

We now verify these relations in our examples.

5.4 EXAMPLE. Take the example of the group algebra of a finite group G (see example 4.3). Here W is given by $\sum_{s\in G} \delta_s \otimes s$ and so

$$W_{12} = \sum_{s\in G} \delta_s \otimes s \otimes 1$$

$$W_{13} = \sum_{s\in G} \delta_s \otimes 1 \otimes s$$

$$W_{23} = \sum_{s\in G} 1 \otimes \delta_s \otimes s.$$

Therefore we get

$$\begin{aligned}W_{12}W_{13}W_{23} &= \sum_{s,t,u} (\delta_s \otimes s \otimes 1)(\delta_t \otimes 1 \otimes t)(1 \otimes \delta_u \otimes u) \\ &= \sum_{s,u} \delta_s \otimes s\delta_u \otimes su.\end{aligned}$$

(i) In $AA' \bar{\otimes}_{R_1} (AA' \bar{\otimes}_{R_1} AA')$ we have

$$\begin{aligned} W_{23}W_{12} &= \sum_{s,t} (1 \otimes \delta_s \otimes s)(\delta_t \otimes t \otimes 1) \\ &= \sum_{s,t} \delta_t \otimes \delta_s t \otimes s \\ &= \sum_{s,t} \delta_t \otimes t\delta_{t^{-1}s} \otimes s \\ &= \sum_{s,u} \delta_t \otimes t\delta_u \otimes tu, \end{aligned}$$

and this proves the Pentagon equation.

(ii) In $AA' \bar{\otimes}_{R_2} (AA' \bar{\otimes}_{R_2} AA')$ we have

$$\begin{aligned} W_{23}W_{13}W_{12} &= \sum_{s,t,u} (1 \otimes \delta_s \otimes s)(\delta_t \otimes 1 \otimes t)(\delta_u \otimes u \otimes 1) \\ &= \sum_{s,t} \delta_t \otimes \delta_s t \otimes st \\ &= \sum_{s,t} \delta_t \otimes t\delta_{t^{-1}st} \otimes st \\ &= \sum_{s,u} \delta_t \otimes t\delta_u \otimes tu, \end{aligned}$$

and this proves the Yang-Baxter equation. □

5.5 EXAMPLE. Now let us consider the case of an algebra generated by a single self-adjoint element (as in example 4.4). Here W is given by the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\lambda} (k \otimes h) \right)^n.$$

This is not an element in $B \otimes A$, but in $L(A, A)$ it can be seen as a limit of elements in $B \otimes A$.

(i) In $AA' \bar{\otimes}_{R_1} (AA' \bar{\otimes}_{R_1} AA')$ we have

$$\begin{aligned} W_{23}W_{12}W_{23}^* &= (1 \otimes W)(W \otimes 1)(1 \otimes W^*) \\ &= (1 \otimes W) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left(\frac{1}{i\lambda} (k \otimes h) \right)^n \otimes 1 \right) (1 \otimes W^*) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{i\lambda} \right)^n k^n \otimes (W(h \otimes 1)^n W^*). \end{aligned}$$

We already know from example 4.4 (ii) that $W(h \otimes 1)W^* = h \otimes 1 + 1 \otimes h$, so that $W(h \otimes 1)^n W^* = (h \otimes 1 + 1 \otimes h)^n$. This gives

$$\begin{aligned}
W_{23}W_{12}W_{23}^* &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{i\lambda}\right)^n k^n \otimes (h \otimes 1 + 1 \otimes h)^n \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{n!} \frac{n!}{j!(n-j)!}\left(\frac{1}{i\lambda}\right)^n k^n \otimes h^j \otimes h^{n-j} \\
&= \left(\sum_{n=0}^{\infty}\left(\frac{1}{i\lambda}\right)^n k^n \otimes h^n \otimes 1\right)\left(\sum_{m=0}^{\infty}\left(\frac{1}{i\lambda}\right)^m k^m \otimes 1 \otimes h^m\right) \\
&= W_{12}W_{13}.
\end{aligned}$$

So we get $W_{23}W_{12} = W_{12}W_{13}W_{23}$, and this is the Pentagon equation.

(ii) We already know that $A \bar{\otimes}_{R_2} B$ is a commutative algebra and one can check in a similar way that also $AA' \bar{\otimes}_{R_2} AA'$ and $AA' \bar{\otimes}_{R_2} (AA' \bar{\otimes}_{R_2} AA')$ are commutative. Hence the Yang-Baxter equation $W_{23}W_{13}W_{12} = W_{12}W_{13}W_{23}$ is trivially satisfied in $AA' \bar{\otimes}_{R_2} (AA' \bar{\otimes}_{R_2} AA')$.

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