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Singularities of polar curves

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0. Introduction

Let ξ be a plane algebroid curve defined over the field $\mathbb{C}$ of the complex numbers. To determine the equisingularity (or topological) type of the polar curves of ξ has been an open problem since the time of M. Noether. Some wrong answers to it may be found in the classical literature. It was not until 1971 that Pham [6] gave examples showing that the equisingularity type of the generic polars of ξ cannot be determined from the equisingularity type of ξ, because it depends on the analytical type of ξ and not only on its equisingularity type. In this paper we give a complete determination of the equisingularitay type of the generic (and most of the non generic) polar curves of ξ under the assumption that ξ is generic in its equisingularity class (theorem 3.1). The main tool to this end is the theory of infinitely near imposed singularities already developed in [2] and applied there to the case of ξ irreducible. Main steps in order to get the result are as follows. First (section 1) we use the unloading principle ([2], 4.2) in order to find an explicit inductive description (1.3) of the cluster $\widetilde{\partial_g(\xi)}$ which will be claimed to describe in turn the singularities of polar curves. Then after giving a precise statement (3.1) of the main claim in section 3, we proceed to prove it in sections 4 and 5. In section 4 we prove the claim for certain infinitely near points we call *initial points* and which are in some sense close to the origin: as the behaviour of a curve at such points is determined by its Newton polygon, for this part of the proof we determine the Newton polygon of the polar curves. For the second half of the claim, in section 5, we use the transformation obtained by successively blowing-up all points in a sequence of initial points. The main point here is that by transforming the polar of a curve we obtain a polar of the transformed curve (5.1). This allows to end the proof using induction together with some technical results, already obtained in section 2, that relate polars relative to different equations of the same curve.

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1. The cluster $\widetilde{\sigma_g(\xi)}$

We place ourselves under the conventions and general hypothesis of [2]. The notions introduced there about curves and infinitely near points will be used without further reference. We shall denote by $e_q(\xi)$ the multiplicity of an ordinary or infinitely near point q on a curve ξ, and by $[\xi \cdot \zeta]_q$ the intersection multiplicity at q of the curves ξ and ζ. An infinitely near point q on a curve ξ will be said to be *non singular on* ξ if and only if the point q itself and all points infinitely near to q on ξ are simple and free. Otherwise we will say that q is a *singular point of* ξ. Notice that on a reduced curve there are finitely many singular points.

Let p be a smooth point of a complex algebraic surface S, O_p the complete local ring of S at p, and assume that ξ is a reduced algebroid curve on S with origin at p. Fix $g \in \mathcal{O}_p$ defining a smooth curve ζ with origin at p. We know from [2] section 8 that the g-polar curves of ξ go through a cluster $\partial_g(\xi)$ which is defined by taking all singular points q on ξ, each with virtual multiplicity $e_q(\xi) - 1$ and furthermore, on each branch γ of ξ, the first $[\gamma \cdot \zeta]_p - 1$ points which are non singular for ξ, each virtually counted once. In the sequel, from these points, those which are singular points of ξ will be called *singular points* of $\partial_g(\xi)$ and the remaining ones will be called *simple* points of $\widetilde{\partial_g(\xi)}$.

Excepted for very simple cases, the cluster $\partial_g(\xi)$ does not satisfy the proximity relations and therefore there are no curves going through $\partial_g(\xi)$ with effective multiplicities equal to the virtual ones. We consider ([2] section 6) a second cluster $\widetilde{\partial_g(\xi)}$ which has the same points as $\partial_g(\xi)$ and virtual multiplicities equal to the effective multiplicities of a generic curve through $\partial_g(\xi)$. The cluster $\widetilde{\partial_g(\xi)}$ is obtained from $\partial_g(\xi)$ by finitely many applications of the unloading principle [2] 4.2. Notice that $\widetilde{\partial_g(\xi)}$ is the only cluster equivalent to $\partial_g(\xi)$ which satisfies the proximity relations. We will give in this section an inductive description of $\widetilde{\partial_g(\xi)}$.

Let K be a cluster with the same points as $\partial_g(\xi)$ and for each $q \in K$ denote by v_q the virtual multiplicty of q in K. Assume that there is a singular point p' of $\partial_g(\xi)$ such that for all singular points q of $\partial_g(\xi)$ equal or infinitely near to p', $v_q = e_q(\xi) - 1$. Assume furthermore that $v_q \leqslant 1$ for all simple points q of $\partial_g(\xi)$. For each branch γ of ξ denote by $s(\gamma)$ the number of simple points of $\partial_g(\xi)$ which belong to γ and are virtually counted once in K. We have:

1.1. LEMMA. *If $s(\gamma) \geqslant e_{p'}(\gamma)$ for all branches γ of ξ, we get a cluster K' equivalent to K by taking:*

(a) *The point p' with virtual multiplicity $e_{p'}(\xi)$.*

(b) *For each branch γ of ξ, the first $s(\gamma) - e_{p'}(\gamma)$ simple points of $\partial_g(\xi)$ on γ virtually counted once, and the remaining ones with virtual multiplicity zero.*

(c) *The remaining points in K with the same virtual multiplicities.*

Proof. We use induction on the number t of singular points of $\partial_g(\xi)$ which are infinitely near to p'.

Let q a point on ξ in the first neighbourhood of p' and assume that q is not a singular point of $\partial_g(\xi)$. Then q is a simple and free point of ξ, it lies on a single branch γ of ξ and it is the only point on γ proximate to p'. Since $s(\gamma) \geqslant e_{p'}(\gamma) \geqslant 1$, we may assume (after using unloading on the simple points on γ if it is needed) that $v_q = 1 = e_q(\xi)$.

If $t > 0$, let $q_1, \ldots, q_r$ be the singular points of $\partial_g(\xi)$ which are proximate to p'. For each branch γ of ξ going through one of these points we have $s(\gamma) \geqslant e_{p'}(\gamma) = \Sigma_i e_{q_i}(\gamma)$, so that the induction hypothesis may be used successively on the q_i in order to get a cluster K'' equivalent to K for which:

(a) Each q_i has virtual multiplicity $e_{q_i}(\xi)$.
(b) On each branch γ of ξ going through one of the points q_i, $s(\gamma) - e_{p'}(\gamma)$ simple points of $\partial_g(\xi)$ are virtually counted once and the remaining ones are taken with virtual multiplicity zero.
(c) The remaining points in K are taken with the same virtual multiplicities.

Put $K'' = K$ if $t = 0$. In any case the point p' has virtual multiplicity $e_{p'}(\xi) - 1$ in K'' whence all points q proximate to p' on ξ have virtual multiplicity $e_q(\xi)$. Since $e_{p'}(\xi) = \Sigma_q e_q(\xi)$, the unloading principle may be applied to K'' at p' giving the cluster K' as calimed. □

In the sequel we call *initial points* of ξ the points on ξ infinitely near to p (p itself excluded) which belong to ζ and also the satellite points which are not preceded by a free point lying outside of ζ. Obviously this notion depends on ζ.

Assume that a system of local coordinates x, y at p is fixed, in such a way that $g = x$ and hence ζ is the y-axis. Let γ a branch of ξ and assume that a Puiseux series of γ has the form:

$$S(x^{1/n}) = ax^{m/n} + \cdots$$

where $S(T) \in \mathbb{C}[[T]]$, $m/n \leqslant 1$ and $a \neq 0$ if $m/n < 1$. We say that m/n is the initial exponent of γ.

Notice that if $m/n = 1$, i.e., γ is not tangent to ζ, then there are no initial points on γ.

In the case $m/n < 1$ let us recall the description of [2] section 9. We perform the Euclidean algorithm for g.c.d. (m, n)

$$\begin{aligned} m &= 0 \cdot n + n_1 \\ n &= h_1 n_1 + n_2 \\ &\vdots \\ n_{s-2} &= h_{s-1} n_{s-1} + n_s \\ n_{s-1} &= h_s n_s. \end{aligned}$$

Then the initial points on γ are, in successive neighbourhoods of $p = p_{1,1}$:

$$\begin{array}{l} p_{1,2},\ldots,p_{1,h_1},\\ p_{2,1},\ldots,p_{2,h_2},\\ \vdots\\ p_{s,1},\ldots,p_{s,h_s}\end{array}$$

each $p_{i,j}$ being n_i-fold on γ. From them, $p_{1,2},\ldots,p_{2,1}$ are free and lie on ζ and the remaining points are satellite: more precisely, for $i > 1$, $p_{i,1},\ldots,p_{i,h_i}$ and also $p_{i+1,1}$ if $i < s$, are proximate to $p_{i-1,h_{i-1}}$. We say in the sequel that the points $p_{i,j}$ with even (resp. odd) i are even (resp. odd) initial points on γ.

Notice that, once the coordinates are fixed, the points $p_{i,j}$ and their multiplicities n_i depend only on n and m. Then, for any pair of positive integers n, m with $n > m$, it will be useful to denote by $K(n,m)$ the cluster of the points $p_{i,j}$, $i = 1,\ldots,s$, $j = 1,\ldots,h_i$ defined above, each $p_{i,j}$ with virtual multiplicity n_i.

Let p' be the point on γ in the first neighbourhood of p_{s,h_s}, or in the first neighbourhood of p if $m/n = 1$. Denote by E the (germ of the) exceptional divisor on which p' lies: p' being free, E is smooth.

1.2. LEMMA. *We have*

$$\sum_q e_q(\gamma) = [\gamma\cdot\zeta]_p - [\gamma'\cdot E]_{p'}$$

where γ' is the strict transform of γ with origin at p', and the summation runs on the initial points $q = p_{i,j}$ of ξ on γ.

Proof. Follows from an easy computation using that $[\gamma\cdot\zeta]_p = n$ and that $[\gamma'\cdot E]_{p'}$ equals the multiplicity of the last point on γ before p', i.e., n_s if $m/n < 1$ and n if $m/n = 1$. □

Denote by $p_1,\ldots,p_r$ the points on ξ which are free, do not belong to ζ and are not preceded by other free points lying outside of ζ. Let us write E_i for the germ at p_i of the exceptional divisor, g_i for a local equation of E_i and ξ_i for the strict transform of ξ with origin at p_i.

1.3. Theorem. (1) *Assume that there is some branch of ξ non tangent to ζ. Then the cluster $\widetilde{\partial_g(\xi)}$ consists of:*

(a) *The point p with virtual multiplicity $e_p(\xi) - 1$.*

(b) *The initial points q of ξ with virtual multiplicities $e_q(\xi)$.*

(c) *For $i = 1,\ldots,r$, the points of $\widetilde{\partial_{g_i}(\xi_i)}$ with the same virtual multiplicities.*

(2) *If all branches of ξ are tangen to ζ, let $\bar\gamma$ be a branch of ξ with maximal initial exponent. In this case the cluster $\widetilde{\partial_g(\xi)}$ consists of:*

(a) *The point p with virtual multiplicity $e_p(\xi)$.*

(b) *The initial points q on $\bar{\gamma}$ with virtual multiplicities $e_q(\xi)$ for the odd points and $e_q(\xi) - 1$ for the even points, except for the last initial point on $\bar{\gamma}$ which has virtual multiplicity $e_q(\xi) - 1$ in all cases.*

(c) *The remaining initial points q of ξ with virtual multiplicities $e_q(\xi)$.*

(d) *For $i = 1, \ldots, r$, the points of $\widetilde{\partial_{g_i}(\xi_i)}$ with the same virtual multiplicities.*

Proof. Since for any branch γ of ξ the cluster $\partial_g(\xi)$ *has* $[\gamma \cdot \zeta]_p - 1$ simple points on γ, by 1.2, we may iteratively use 1.1 until we reach a cluster K_1 equivalent to $\partial_g(\xi)$ in which the initial points q have multiplicity $e_q(\xi)$. The other singular points q of $\partial_g(\xi)$ still have in K_1 virtual multiplicity $e_q(\xi) - 1$ and there remain in K_1 just $[\gamma' \cdot E']_{p'} - 1$ simple points virtually counted once on each branch γ, if we use for γ the notations of 1.2. Fix one of the points p_i and consider the branches γ going through p_i: for such branches we have $p' = p_i$ and $E' = E_i$ so that the part of K_1 consisting of p_i and the points infinitely near to it agrees, virtual multiplicities included, with $\partial_{g_i}(\xi_i)$. Then, starting from K_1, we use the unloading principle on the same way that lead us from $\partial_{g_i}(\xi_i)$ to $\widetilde{\partial_{g_i}(\xi_i)}$ successively for $i = 1, \ldots, r$. This gives a cluster K_2 equivalent to K_1 just as described in (1)(a), (1)(b) and (1)(c) of the claim.

Now we check the priximity relations at the points of K_2. It is clear that the proximity relations are satisfied at the points q which are equal or infinitely near to one of the p_i. On the other hand, by induction, a such q has virtual multiplicity either $e_q(\xi)$ or $e_q(\xi) - 1$ in K_2 so that the proximity relations at the initial points are also satisfied.

Assume now that there is some branch of ξ non tangent to ζ. Then there is at least one of the p_i, say p_1, in the first neighbourhood of p. Denote by η the curve composed of the branches of ξ going through p_1 and by η' that composed of the remaining ones. From the induction hypothesis applied to $\widetilde{\partial_{g_1}(\xi_1)}$ one can easily see that the sum of virtual multiplicities in K_2 (or in $\widetilde{\partial_{g_1}(\xi_1)}$) of the points on η proximate to p is

$$\sum_q e_q(\xi) - 1 = \sum_q e_q(\eta) - 1 = e_p(\eta) - 1$$

both summations being for q on η and proximate to p. On the other hand, all points q proximate to p on η' have virtual multiplicities in K_2 not bigger than $e_q(\xi) = e_q(\eta')$ so that the sum of these multiplicities cannot be greater than $e_p(\eta')$. Thus, since $e_p(\eta) + e_p(\eta') = e_p(\xi)$ the proximity relation at p is satisfied, K_2 satisfies the proximity relations and then the first half of the claim has been proved.

Assume now that all branches of ξ are tangent to ζ. Let us write $\gamma = \bar{\gamma}$ for a branch of ξ with maximal initial exponent m/n and use for the initial points on γ the same notations as before. We put $p_{1,1} = p$ and $e_{i,j} = e_{p_{i,j}}(\xi)$. Notice that we have $m/n < 1$ and

$$\frac{m}{n} = \cfrac{1}{h_1 + \cfrac{1}{\ddots \, h_{s-1} + \cfrac{1}{h_s}}}.$$

If

$$\frac{m'}{n'} = \cfrac{1}{h'_1 + \cfrac{1}{\ddots \, h'_{s-1} + \cfrac{1}{h'_s}}}$$

is any other initial exponent, we have $h'_1 \geqslant h_1$ so that $p_{1,j}$ is the only point in K_2 proximate to $p_{1,j-1}$, for $j = 2, \ldots, h_1$: then one may unload one unity of multiplicity successively from each $p_{1,j}$ on $p_{1,j-1}$ until reaching an equivalent cluster K_3 where the points $p_{1,j}$ have virtual multiplicities $e_{1,j}$ for $j < h_1$ and $e_{1,h_1} - 1$ for $j = h_1$, the other virtual multiplicities being unchanged. It is clear that K_3 satisfies the proximity relations at the points preceding p_{1,h_1}.

If $s = 1$, then K_3 is the cluster described in part 2 of the claim. On the other hand p_{1,h_1} is the last initial point on γ so that there is at least one of the points p_i in the first neighbourhood of p_{1,h_1}. One can see that K_3 satisfies the proximity relation at p_{1,h_1} just as done for K_2 at p when there are branches not tangent to ζ. As the other proximity relations are still satisfied by K_3, if $s = 1$ the proof is complete.

If $s > 1$, then any initial exponent m'/n' has either $h'_1 > h_1$ or $h'_1 = h_1$ and $h'_2 \leqslant h_2$ so that the points proximate to p_{1,h_1} in K_2 are $p_{2,1}, \ldots, p_{2,h_2}$ and, if $s > 2$, $p_{3,1}$. We apply unloading principle at p_{1,h_1} and obtain an equivalent cluster K_4 in which p_{1,h_1} has virtual multiplicity e_{1,h_1} and its proximate points $p_{i,j}$ have virtual multiplicities $e_{i,j} - 1$. After this the proximity relation at p_{1,h_1} is satisfied and it is clear that the proximity relations are also satisfied at each proximate point whose first neighbourhood contains a $p_{i,j}$ whose virtual multiplicity is $e_{i,j} - 1$.

If $s = 2$, then K_4 is the cluster described in the second half of the claim. In this case only the proximity relation at p_{2,h_2} needs to be verified: this may be done as in the preceding cases using that there is one of the points p_i in the first neighbourhood of p_{2,h_2}.

If $s > 2$ the proof continues on the same way: again one unity of multiplicity is unloaded from each odd initial point $p_{3,j}, j > 1$, on the preceding point, and then, if $s > 3$, one unity of multiplicity is unloaded from the even initial points $p_{4,j}$, and also from $p_{5,1}$ if $s > 4$, on p_{3,h_3}, and so on until the last initial point on γ is reached. Then the proof ends as in the preceding cases. □

2. Different g-polars of ξ

Once the cluster $\widetilde{\partial_g(\xi)}$ has been described, we give an auxiliary lemma about g-polars of ξ relative to different equations.

2.1. LEMMA. *Assume that there is a g-polar of ξ that goes through $\widetilde{\partial_g(\xi)}$ with effective multiplicities equal to the virtual ones and has no singularities outside of $\widetilde{\partial_g(\xi)}$. Then:*

(1) *If there is some branch of ξ not tangent to ζ, all g-polars of ξ go through $\widetilde{\partial_g(\xi)}$ with effective multiplicities equal to the virtual ones and have no singularities outside of $\widetilde{\partial_g(\xi)}$.*

(2) *If all branches of ξ are tangent to ζ let q_0 be the last point on ζ with virtual multiplicity $e_p(\xi)$ in $\widetilde{\partial_g(\xi)}$. Then all g-polars of ξ having multiplicity $e_p(\xi)$ at p go through $\widetilde{\partial_g(\xi)}$ with effective multiplicities equal to the virtual ones and have no singularities outside of $\widetilde{\partial_g(\xi)}$ infinitely near to any $q \in \widetilde{\partial_g(\xi)}$, $q \neq q_0$.*

The proof of 2.1 runs as for unibranched curves ξ ([2] 11.2) and therefore we omit it.

3. Behaviour of polar curves

Let, as before, x, y be local coordinates at p with $x = g$ so that ζ is the y-axis. Let ξ be an algebroid curve with origin at p and γ a branch of ξ. Put $n = [\gamma \cdot \zeta]_p$ and let

$$S(x^{1/n}) = \sum_{i \in I(\mathscr{M})} a_i x^{i/n}$$

be a Puiseux series of γ where $\mathscr{M} = \{m_k/n\}_{k=1,\ldots,r}$ is the system of characteristic exponents and

$$I(\mathscr{M}) = \{i \in \mathbb{N} \mid i \in (n, m_1, \ldots, m_{k-1}) \text{ for } i < m_k,\ k = 1, \ldots, r\}$$

$\mathbb{N}$ being the set of positive integers. As it is well known ([9] for instance), $\mathscr{M}$ determines and is determined by n and the topological (or equisingularity) type of γ.

Let us recall that the different Puiseux series of γ are obtained from any one of them by means of substitutions $x^{1/n} \to \varepsilon x^{1/n}$, $\varepsilon^n = 1$. We will denote by F_γ the equation of γ

$$F_\gamma = \prod_{\varepsilon^n = 1} (y - S(\varepsilon x^{1/n}))$$

whose roots as polynomial in y are just the Puiseux series of γ.

The contact between two branches γ and γ' may be defined as

$$C(\gamma, \gamma') = \mathrm{Max}\{\mathrm{Ord}_x(S - S')\}$$

where S and S' are Puiseux series of γ and γ' respectively. Once the characteristic exponents of the branches are known, $C(\gamma, \gamma')$ determines and is determined by the intersection number $[\gamma \cdot \gamma']_p$ ([4]).

Assume we have chosen a curve ξ^0 with origin at p and branches $\gamma_1^0, \ldots, \gamma_l^0$. Put $n_j = [\gamma_j^0 \cdot \zeta]$, $c_{j,t} = C(\gamma_j^0, \gamma_t^0)$ and denote by $\mathscr{M}_j$ the system of characteristic exponents of (a Puiseux series of) γ_j^0.

In the sequel we shall consider all algebroid curves ξ which are equisingular to ξ^0 and whose branches have with ζ the same intersection multiplicities as their corresponding branches of ξ^0.

Let $\mathscr{L}$ be a linear space whose points have complex coordinates $\mathbf{a} = (a_{j,i})$, $j = 1, \ldots, l$, $i \in I(\mathscr{M}_j)$. Consider the subset L of $\mathscr{L}$ defined by the relations

$$a_{j,i} \neq 0$$

for $i/n_j \in \mathscr{M}_j$, $j = 1, \ldots, l$, and

$$\mathrm{Ord}_x\left(\sum_{i \in I(\mathscr{M}_j)} a_{j,i} x^{i/n_j} - \sum_{i \in I(\mathscr{M}_t)} a_{t,i} x^{i/n_t}\right) = c_{j,t}$$

$$\mathrm{Ord}_x\left(\sum_{i \in I(\mathscr{M}_j)} a_{j,i} \varepsilon^i x^{i/n_j} - \sum_{i \in I(\mathscr{M}_t)} a_{t,i} \bar{\varepsilon}^i x^{i/n_t}\right) \leqslant c_{j,t}$$

for $\varepsilon^{n_j} = 1$, $\bar{\varepsilon} n^t = 1$ and $j \neq t$, $j, t = 1, \ldots, l$.

The set L is a Zariski open set (in fact the complementary of a finite number of hyperplanes) of a linear subspace of $\mathscr{L}$.

If $\mathbf{a} = (a_{j,i}) \in L$, let $\xi_{\mathbf{a}}$ be the curve composed of the branches γ_j, $j = 1, \ldots, l$ with Puiseux series

$$\sum_{i \in I(\mathscr{M}_j)} a_{j,i} x^{i/n_j}.$$

It is clear that $\xi_{\mathbf{a}}$ is equisingular to ξ^0, each γ_j corresponding to γ_j^0, and also that $[\gamma_j \cdot \zeta] = n_j = [\gamma_j^0 \cdot \zeta]$. Conversely, it is not hard to see that any curve ξ in such conditions is $\xi = \xi_{\mathbf{a}}$ for some $\mathbf{a} \in L$.

Notice that, in particular, if we take ξ^0 with no branches tangent to ζ, we are dealing with all curves equisingular to ξ^0 whose branches are non tangent to ζ.

If $\xi = \xi_{\mathbf{a}}$ the components $a_{j,i}$ of $\mathbf{a}$ will be called coefficients of ξ. Furthermore we mean for *a curve ξ with general coefficients* any curve $\xi = \xi_{\mathbf{a}}$ with a system of coefficients $\mathbf{a}$ that does not belong to a certain algebraic hypersurface of L.

We put $F_{\gamma_j} = F_j$ and take $F_\xi = F_1 \dots F_l$ as equation of ξ. Then we have:

3.1. THEOREM. *If the curve ξ has general coefficients the polar of ξ, $P_g(F_\xi)$ goes through $\widetilde{\partial_g(\xi)}$ with effective multiplicities equal to the virtual ones and has no singularities outside of $\widetilde{\partial_g(\xi)}$.*

The proof of 3.1 will be given in sections 4 and 5 below. Notice that 2.1 extends the claim of 3.1 to other g-polars. Next corollary gives the behaviour of the generic polars of generic curves with prescribed topological type and needs no proof.

3.2. COROLLARY. *If ξ^0 is taken with no branch tangent to ζ and the curve ξ has general coefficients, the generic polars of ξ go through $\widetilde{\partial(\xi)}$ with effective multiplicities equal to the virtual ones and have no singularities outside of $\widetilde{\partial(\xi)}$.*

4. Proof of 3.1, part one: the initial points

First of all, let us recall some facts about Newton polygons. Let η be a curve, write

$$\sum_{\alpha,\beta} A_{\alpha,\beta} x^\alpha y^\beta$$

the equation of η and denote by $\mathbb{R}^+$ the set of non negative real numbers. The border line of the convex envelope of the set

$$\{(\alpha, \beta) \mid A_{\alpha,\beta} \neq 0\} + (\mathbb{R}^+)^2,$$

consists of two half-lines parallel to (or lying on) the axis and a broken line joining them which is called the *Newton polygon* of η. In the sequel we will consider the Newton polygons and their sides oriented from top to bottom and from left to right, so that we say they start at the vertex with maximal β and minimal α and they end at that with minimal β and maximal α.

It is clear that the x-axis (resp. the y-axis) is a component of η if and only if the Newton polygon ends (resp. starts) out of the α (resp. β) axis. In fact, we will not consider curves having the y-axis as component, so that all our Newton polygons will start at the β-axis.

As it is well known ([7], appendix B, for instance), one of the branches of η has a Puiseux series with leading term

$$ax^{m/n}$$

if and only if there is in the Newton polygon of η a side Γ with slope $-n/m$ and a is a root of the so called equation associated to Γ:

$$\mathscr{E}_\Gamma(z) = \sum_{(\alpha,\beta)\in\Gamma} A_{\alpha,\beta} z^{\beta-\beta_0} = 0,$$

in which β_0 is the ordinate of the end of Γ.

Put $n' = n/\text{g.c.d.}\,(m, n)$ and notice that $\mathscr{E}_\Gamma$ is in fact a polynomial in $\mathbb{C}[z^{n'}]$. Then the group of n'-th roots of unity acts on the roots of $\mathscr{E}_\Gamma(z)$, two conjugate roots giving rise to different Puiseux series of the same branch of η.

Assume that Γ goes from (α_1, β_1) to (α_0, β_0) let $\gamma_1, \ldots, \gamma_k$ be the branches whose Puiseux series has leading term of degree m/n (branches *corresponding* to Γ) and put n_t, $t = 1, \ldots, k$ for the polydromy order of the Puiseux series of γ_t, this is, n_t is the intersection multiplicity of γ_t and the y-axis. Then it follows from the constructive proof of the Puiseux theorem ([7], app. B again) that

$$\beta_1 - \beta_0 = \sum_{t=1}^{k} n_t.$$

We say that $\beta_1 - \beta_0$ is the *height* of Γ and that $\eta_\Gamma = \gamma_1 + \cdots + \gamma_k$ is the component of η associated to Γ. The last equality may be equivalently stated by saying that the height of Γ equals the intersection multiplicity of η_Γ and the y-axis.

The sides with slope bigger than -1 correspond to the branches tangent to the x-axis. We are mainly interested in the other branches, so let us assume in the sequel that $m/n \leqslant 1$. Since all branches γ_t corresponding to Γ have initial exponents $m_t/n_t = m/n$, all of them go through the same set of initial points. In other words, the clusters $K(m_t, n_t)$ (section 1) have the same sets of points. Then an easy computation shows that η_Γ goes through $K(\alpha_0 - \alpha_1, \beta_1 - \beta_0)$ with effective multiplicities equal to the virtual ones and contains no other initial point. Thus we state for future reference:

4.1. LEMMA. *The initial points η_Γ is going through as well as the multiplicities of η_Γ at p and the initial points, depend only on the height and the slope of Γ:*

they are the points and multiplicities in $K(M,N)$, N *being the height and* $-N/M$ the slope of Γ.

Let as before γ be one of the branches corresponding to Γ and denote by p' the (necessarily free) point on γ in the first neighbourhood of p_{s,h_s}. Put $n' = n/\text{g.c.d.}\,(m,n)$ and $m' = m/\text{g.c.d.}\,(m,n)$ and use the notation $\bar{f}$ to denote the image in $\mathcal{O}_{p'}$ of any $f \in \mathcal{O}_p$. There are local coordinates $\tilde{x}$, $\tilde{y}$ at p' related to $\bar{x}$, $\bar{y}$ by the formulas

$$\bar{x} = \tilde{x}^{n'}$$
$$\bar{y} = \tilde{x}^{m'}(a + \tilde{y})$$

and such that $\tilde{x}$ is an equation of the exceptional divisor E of the composite blowing-up giving rise to p'. The proof of this fact can be found in [2], section 10 for $a = 1$ and the general case follows on the same way. It is worth to remark that former equations have been used since a long time for the classical constructive proof of Puiseux's theorem ([7], App. B, for instance) even if their geometrical significance seems not to have been noticed. The same kind of transformations have been recently used by Oka ([5]). Denote by η' the strict transform of η with origin at p'. We have:

4.2. LEMMA. *The multiplicity of a as a root of* $\mathscr{E}_\Gamma$ *equals the intersection multiplicity of E and* η' *at* p'. *It follows in particular that* p' *and the points infinitely near to* p' *on* γ *are all free and simple on* η *if and only if a is a simple root of the equation associated to* Γ.

Proof. Follows from an easy computation using the coordinates $\tilde{x}, \tilde{y}$. □

Now we will prove part of 3.1. Put $\eta = P_g(F_\xi)$:

4.3. PROPOSITION. *Under the hypothesis of* 3.1, *the polar* η *goes through p and the initial points of* ξ *with effective multiplicities equal to the virtual multiplicities of such points in* $\widetilde{\partial_g(\xi)}$.

Proof. Let us call Δ the Newton polygon of ξ and write $\Gamma_1, \ldots, \Gamma_d$ for the ordered succession of sides of Δ. Let Δ' be the Newton polygon of η. Since the equation of η is just $\partial F_\xi/\partial y$, the first $d-1$ sides of Δ', say $\Gamma'_1, \ldots, \Gamma'_{d-1}$, are the translated of $\Gamma_1, \ldots, \Gamma_{d-1}$ by the vector $(0, -1)$. This is also true for the d-th (and in fact last) side if Δ ends out of the α-axis.

Assume first that ξ has some branch not tangent to the y-axis. An easy computation shows that $e_p(\eta) = e_p(\xi) - 1$. Furthermore, in this case, either Γ_d has slope bigger or equal than -1 or Δ ends outside of the α-axis. Anyway, all sides of Δ' with slope less than -1 are translated of sides of Δ so that the claim for the initial points follows from 4.1.

Then we may assume tht all branches of ξ are tangent to the y-axis and, thus, that all sides Γ_t have slope less than -1 and Δ ends on the α-axis. Denote by $(e-M, N)$ and $(e, 0)$ the vertices of Γ_d (hence $e = e_p(\xi)$) and put $-n'/m'$ $(= -N/M)$ for the slope of Γ_d in irreducible form. By the hypothesis of general coefficients for ξ we may assume $A_{\alpha,\beta} \neq 0$ for $(\alpha, \beta) \in \Gamma_d$. Then, if $N > n'$, there are more than two points with integer coordinates on the side Γ_d and thus Γ_d gives rise to a side Γ'_d of Δ' going from the end of Γ'_{d-1} to $\Omega = (e - m', n' - 1)$. If $N = n'$, then $\Omega = (e - m', n' - 1)$ is just the end of Γ'_{d-1}.

From now on we describe the sides of Δ' between Ω and the α-axis. It is clear that these sides should lie above the line L that goes through Ω and has slope $-n'/m'$. We use the fact, which will be proved below as part of lemma 4.5 that for all points (α, β) above L with $0 \leqslant \beta < n' - 1$, the monomial of degrees (α, β) in $\partial F_\xi/\partial y$ has nonzero coefficient. Write m'/n' as a continued fraction

$$\frac{m'}{n'} = \cfrac{1}{h_1 + \cfrac{1}{\ddots\, \cfrac{}{\dfrac{1}{h_s}}}}$$

and put u_t/v_t for the reduced form of the corresponding reduced fractions, $t = 1, \ldots, s$:

$$\frac{u_t}{v_t} = \cfrac{1}{h_1 + \cfrac{1}{\ddots\, \cfrac{}{\dfrac{1}{h_t}}}}.$$

Put $-v/u\,(u, v > 0)$ for the slope of the first side, say Γ''_1, of Δ' after Ω. Since the end of this side has non-negative ordinate and lies above L, we have $v < n'$ and $u/v > m'/n'$. On the other hand, from the rational numbers with these properties, u/v is that closest to m'/n' because there are no points with integer coordinates and positive ordinate below Γ''_1 and above L. After this a well known geometrical property of continued fractions (see [8] ch. 7) shows that either $u/v = u_{s-1}/v_{s-1}$ if s is even, or $u/v = (u_s - u_{s-1})/(v_s - v_{s-1})$ if s is odd. One may also use Pick's theorem at this point, see [3].

Let us deal with the case s is even first. Since

$$h_s v_{s-1} < h_s v_{s-1} + v_{s-2} = v_s = n'$$

and $v_{s-2} < v_{s-1}$ because of standard properties of continued fractions, Γ_1' ends at

$$\Omega_1 = (e - m' + h_s u_{s-1}, n' - 1 - h_s v_{s-1}) = (e - u_{s-2}, v_{s-2} - 1).$$

If $s > 2$, we iterate by taking Ω_1 and u_{s-2}/v_{s-2} instead of Ω and $m/n = u_s/v_s$, we find the next side has slope $-v_{s-3}/u_{s-3}$ and ends at $\Omega_2 = (e - u_{s-4}, v_{s-4} - 1)$, and so on, until reaching the last vertex $\Omega_{s/2} = (e, 0)$.

For odd s, one easily checks that the first side Γ_0'' after Ω ends at

$$\Omega_0 = (e - m' + u_s - u_{s-1}, n' - v_s + v_{s-1}) = (e - u_{s-1}, v_{s-1} - 1).$$

After this we may deal as for even s, so that, summarizing, for any s, we find after the vertex Ω:

(0) A side Γ_0'' with slope $-(v_s - v_{s-1})/(u_s - u_{s-1})$ and height $v_s - v_{s-1}$ only in the case s is odd

And then, in all cases, t being successively equal to $1, \ldots, [s/2]$,

(t) A side Γ_t'' with slope $-v_{2t-1}/u_{2t-1}$ and height $v_{2t-1}h_{2t}$.

Once the Newton polygon Δ' has been determined, the claim follows by computing for each side of Δ' the multiplicities of the corresponding component of the polar η at the initial points, using 4.1. □

In particular we have seen:

4.4. COROLLARY (of the proof). *If ξ has general coefficients and all branches of ξ are tangent to the y-axis, then the Newton polygon of $P_x(F_\xi)$ has $(e_p(\xi), 0)$ as last vertex.*

Let, as before, $\gamma_1, \ldots, \gamma_l$ be the branches of ξ put $n_i = [\gamma_i \cdot \zeta]_p$, write m_i/n_i for the initial exponent of γ_i and assume $m_i/n_i < 1$ for all i. We have $e = \Sigma_i m_i$.

Fix a branch γ of ξ corresponding to the last side Γ_d. Assume (after reordering the branches if necessary) that $\gamma = \gamma_1$ and let us drop the index 1 in the sequel, so that we write $m_1 = m$, $n_1 = n$ and $a_{1,i} = a_i$.

Write the equation of η in the form

$$\frac{\partial F_\xi}{\partial y} = \sum_{\alpha,\beta} B_{\alpha,\beta} x^\alpha y^\beta$$

Define

$$T = \{(\alpha, \beta) \in \mathbb{Z}^2 | 0 \leqslant \beta < n' - 1, n\alpha + m\beta > ne - m\}$$

and notice that T contains all points with integer coordinates which are on the Newton polygon Δ' between Ω and the α-axis. Take

$$\delta(\alpha, \beta) = \alpha n' + \beta m' + 2m' - en'$$

for $(\alpha, \beta) \in T$. If the coefficients $a_{j,i}$ are considered as free variables, we have

4.5. LEMMA. *The function δ is injective. For $(\alpha, \beta) \in T$, $B_{\alpha,\beta}$ does non depend on a_i if $i > (n/n')\delta(\alpha, \beta)$ whence $B_{\alpha,\beta}$ is a non-constant linear function of α_i if $i = (n/n')\delta(\alpha, \beta)$.*

Proof. That δ is injective is easy to see: from $\delta(\alpha, \beta) = \delta(\alpha', \beta')$ we obtain $\beta \equiv \beta'$ mod. n', thus $\beta = \beta'$ because of the definition of T, and then $\alpha = \alpha'$.

If $F_\xi = \Sigma_{\alpha,\beta} A_{\alpha,\beta} x^\alpha y^\beta$ we have $B_{\alpha,\beta} = (\beta + 1)A_{\alpha,\beta+1}$ and $\beta + 1 \neq 0$ if $(\alpha, \beta) \in T$, so that we will deal with $A_{\alpha,\beta+1}$ instead of $B_{\alpha,\beta}$.

Let us put $b_j = a_{j,m_j}$ for simplicity and write the equation of ξ in the form

$$\begin{aligned} F_\xi = & \prod_{\varepsilon^n = 1} (y - \varepsilon^m a_m x^{m/n} + \cdots + \varepsilon^i a_i x^{i/n} + \cdots) \\ & \times \prod_{j=2}^{l} \prod_{\varepsilon^{n_j} = 1} (y - \varepsilon^{m_j} b_j x^{m_j/n_j} + \cdots). \end{aligned}$$

Notice that $m/n \geqslant m_j/n_j$ for $j = 2, \ldots, l$ because Γ_d is the last side of Δ. It is clear from the equation F_ξ above that if $A_{\alpha,\beta+1}$ depends on a_i, then

$$\alpha \geqslant (n - \beta - 2)\frac{m}{n} + \frac{i}{n} + \sum_2^l m_j = (n - \beta - 2)\frac{m}{n} + \frac{i}{n} + e - \mathrm{m}$$

the inequality being strict if a_i appears with degree bigger than one. Since the last inequality may be written as

$$i \leqslant n\alpha + m\beta + 2m - ne = \frac{n}{n'}\,\delta(\alpha, \beta)$$

only the effective dependence of $A_{\alpha,\beta+1}$ on a_i, $i = (n/n')\delta(\alpha, \beta)$ remains to be proved.

Then, take $i = (n/n')\delta(\alpha, \beta)$. Notice first that $n/n' = \text{g.c.d.}(m, n)$ divides i and, since $(\alpha, \beta) \in T$,

$$i = n\alpha + m\beta + 2m - ne > m$$

so that $i \in I(\mathscr{M}_1)$, a_i certainly exists and $a_i \neq a_m$. We will show that F_ξ contains a monomial

$$cb_2^{n_2}\cdots b_l^{n_l}a_m^{n-\beta-2}a_i x^\alpha y^{\beta+1}$$

where c is a non zero complex number. For this it is enough to see that

$$\left(\frac{\partial^{n_2}}{\partial b_2^{n_2}}\cdots\frac{\partial^{n_3}}{\partial b_3^{n_3}}\frac{\partial^{n-\beta-2}}{\partial a_m^{n-\beta-2}}\frac{\partial}{a_i}F_\xi\right)_{a_{j,i}}=0$$

contains a non trivial monomial of degrees α, β. Former derivative easily turns out to be equal to

$$(-1)^{n-\beta}n_2!\cdots n_l!\left(\sum \varepsilon_0^i\varepsilon_1^m\cdots\varepsilon_{n-\beta-2}^m\right)x^\alpha y^{\beta+1}$$

where the summation runs on all ordered $(n-\beta-1)$-uples of different n-th roots of unity. All we need to see is

$$\sum \varepsilon_0^i\varepsilon_1^m\cdots\varepsilon_{n-\beta-2}^m\neq 0$$

and this may be easily verified after a computation like that in [1], proof of Prop. 1. □

4.6. REMARK. It follows from 4.5 that the set T may be ordered by the values of δ, and then, each $B_{\alpha,\beta}$, $(\alpha,\beta)\in T$, depends on a coefficient a_i $(i=(n/n')\delta(\alpha,\beta))$ that does not appear in former $B_{\alpha',\beta'}$. Thus, if the coefficients of ξ are general, then the coefficients $B_{\alpha,\beta}$, $(\alpha,\beta)\in T$, and, in particular, those corresponding to points on Δ' beyond Ω, are general too.

We have found no referene for te following elementary result which will be needed later on:

4.7. LEMMA. *Let*

$$\mathscr{F}=z^{\delta_0}(z-a_1)^{\delta_1}\cdots(z-a_s)^{\delta_s}$$

be a polynomial where the a_i *are different non-zero complex numbers and* $\delta_i>0$, $i=0,\ldots,s$. *Write*

$$\frac{d\mathscr{F}}{dz}=z^{\delta_0-1}(z-a_1)^{\delta_1-1}\cdots(z-a_s)^{\delta_s-1}\mathscr{G}.$$

Then, for generic values of the a_i, $\mathscr{G}$ *has no multiple roots.*

Proof. That the (a_i) giving multiple roots for $\mathcal{G}$ describe a Zariski-closed set is clear. Since an easy argument from elementary calculus shows that the roots of $\mathcal{G}$ are always simple if the a_i are real, the claim follows. □

Proposition 4.3 and the next one together prove the part of 3.1 relative to initial points.

4.8. PROPOSITION. *Under the hypothesis of* 3.1, *η has no singularities outside of $\widetilde{\partial_g(\xi)}$ infinitely near either to p or to any initial point.*

Proof. First of all we will deal with the point $\bar{p}$ in the first neighbourhood of p and on the x-axis. If $\bar{p}$ does not belong to $\widetilde{\partial_g(\xi)}$, then either ξ has a single smooth branch going through $\bar{p}$, or ξ has no branches through $\bar{p}$ at all. Equivalently, either Δ has a single side with slope bigger than -1 and this side has height one, or all sides of Δ have slope non-bigger than -1. It follows from the description of Δ' given in the proof of 4.3, that in both cases Δ' has no side with slope bigger than -1 and, thus, η does not go through $\bar{p}$. In particular this proves that if $\bar{p}$ is outside of $\widetilde{\partial_g(\xi)}$, then $\bar{p}$ is not a singular point of η.

Now, let γ be any branch of η missing $\bar{p}$. Assume that γ corresponds to a side Γ' of Δ' and that the leading term of a Puiseux series of γ is $ax^{m/n}$ where, necessarily, $m/n \leqslant 1$. Put m'/n' for m/n in irreducible form, denote by p' the first non initial infinitely near point on γ and assume that $p' \notin \widetilde{\partial_g(\xi)}$.

If E is the exceptional divisor of the blowing-up giving rise to p', we need to see that E and the strict transform of γ with origin at p', meet transversally at p', or, equivalently by 4.2, that a is a simple root of $\mathscr{E}_{\Gamma'}$.

Then, using the same notations as in the proof of 4.3, assume first that Γ' is one of the sides Γ''_j between Ω and the end of Δ'. In this case it is clear form 4.5 that $\mathscr{E}_{\Gamma'}$ has no multiple roots, because it is an equation in $z^{n'}$ with general coefficients.

Now, assume that Γ' is one of the sides Γ'_j before Ω. Then there is in Δ a side, say Γ, parallel to Γ'. Let (α, β) be the end of Γ and assume first that $\beta > 0$. It is easy to see that the equations $\mathscr{E}_\Gamma$ and $\mathscr{E}_{\Gamma'}$ are related by

$$\frac{\mathrm{d}}{\mathrm{d}z}(\mathscr{E}_\Gamma z^\beta) = \mathscr{E}_{\Gamma'} z^{\beta-1}.$$

Put $\bar{z} = z^{n'}$, both equations $\mathscr{E}_\Gamma$ and $\mathscr{E}_{\Gamma'}$ being in fact elements of $\mathbb{C}[\bar{z}]$. Last equality easily gives

$$\bar{z}^{\beta-1}\mathscr{E}_{\Gamma'}\mathscr{E}_\Gamma^{n'-1} = \frac{\mathrm{d}}{\mathrm{d}\bar{z}}(\mathscr{E}_\Gamma^{n'}\bar{z}^\beta)$$

Then, if a is a root of $\mathscr{E}_\Gamma$ too, it follows from the last equality that a is a multiple root of $\mathscr{E}_\Gamma$ and p' belongs to $\widetilde{\partial_g(\xi)}$ against the hypothesis. If a is not a root of $\mathscr{E}_\Gamma$, it follows from 4.7 and the former equality that, the coefficients of ξ being general, a is a simple root of $\mathscr{E}_{\Gamma'}$ as wanted.

Lastly, if $\beta = 0$, we have

$$z^{n'-1}\mathscr{E}_{\Gamma'} = \frac{\mathrm{d}}{\mathrm{d}z}\mathscr{E}_\Gamma$$

from which a similar reasoning gets the same result. □

5. Proof of 3.1, part two: induction

Fix p' to be one of the non initial points on ξ which are either in the first neighbourhood of p or in the first neighbourhood of an initial point. The point p' is anyone of the points noted p_i in 1.3. Put ξ' for the strict transform of ξ with origin at p' and E for (the germ of) the exceptional divisor p' is belonging to.

After reordering if necessary, assume that the branches of ξ through p' are $\gamma_1, \ldots, \gamma_{l'}$. If we write, as before, $b_j x^{m_j/n_j}$ for the leading term of the Puiseux series of γ_j, $j = 1, \ldots, l$, because of the way on which we have chosen the Puiseux series of the branches in section 3, all branches through p' have Puiseux series with the same leading term, say,

$$b_j = a,\ m_j/n_j = m'/n' \quad \text{for } j = 1, \ldots, l'$$

where we assume g.c.d.$(m', n') = 1$.

On the other hand, since the other branches do not go through p', we have either

$$m_j/n_j \neq m'/n'$$

or

$$m_j/n_j = m'/n' \quad \text{and} \quad b_j^{n'} \neq a^{n'}$$

for $j = l' + 1, \ldots, l$.

We use the local coordinates at p', $\tilde{x}$, $\tilde{y}$, as introduced in section 4: if $\bar{f}$ denotes the image in $\mathcal{O}_{p'}$ of any $f = \mathcal{O}_p$, we have

$$\bar{x} = \tilde{x}^{n'}$$

$$\bar{y} = \tilde{x}^{m'}(a + \tilde{y})$$

and $\tilde{x}$ is an equation of E.

Write

$$\bar{F}_\xi = \tilde{x}^\delta \tilde{F}_\xi$$

where $\tilde{F}_\xi$ is assumed to have no factor $\tilde{x}$ and, hence, is an equation of ξ'. It is easy to see that the equation of the side of Δ with slope $-n'/m'$ is just $n'\alpha + m'\beta = \delta$, so that in particular we have $m' \leqslant \delta$.

The key fact for the induction is:

5.1. LEMMA. *The strict transform with origin at p' of $P_x(F_\xi)$ is $P_{\tilde{x}}(\tilde{F}_\xi)$.*

Proof. An easy computation is enough: from the definition of $\tilde{F}_\xi$ above,

$$\frac{\partial \bar{F}_\xi}{\partial \tilde{y}} = \tilde{x}^\delta \frac{\partial \tilde{F}_\xi}{\partial \tilde{y}},$$

whence

$$\frac{\partial \bar{F}_\xi}{\partial \tilde{y}} = \tilde{x}^{m'} \overline{\frac{\partial F_\xi}{\partial y}},$$

so that

$$\overline{\frac{\partial F_\xi}{\partial y}} = \tilde{x}^{\delta - m'} \frac{\partial \tilde{F}_\xi}{\partial \tilde{y}}$$

where, like $\tilde{F}_\xi$, $\partial \tilde{F}_\xi / \partial \tilde{y}$ has no factor $\tilde{x}$. □

Put, as before, $F_j = F_{\gamma_j}$ and let us write $I_j = I(\mathcal{M}_j) - \{m_j\}$ for $j = 1, \ldots, l$. If $j \leqslant l'$, take $\bar{n}_j = n_j/n' = m_j/m' = \text{g.c.d.}(m_j, n_j)$ and denote by G_j and G'_j the groups of n_j-th and $\bar{n}_j$-th roots of unity, respectively.

5.2. LEMMA. *If $j \leqslant l'$ we have*

$$\bar{F}_j = \tilde{x}^{\delta_j} U_j F_{\tilde{\gamma}_j}$$

where U_j is invertible in $\mathcal{O}_{p'}$ and $\delta_j > 0$. Furthermore

$$\tilde{S}_j = \sum_{i \in I_j} a_{j,i} \tilde{x}^{(i - m_j/\bar{n}_j)}$$

is a Puiseux series for $\tilde{\gamma}_j$, $\bar{n}_j = [E \cdot \tilde{\gamma}_j]_{p'}$ and $U_j(0,0)$ and $(\partial U_j/\partial \tilde{y})(0,0)$ depend only on the coefficient a.

Proof. From

$$F_j = \prod_{\varepsilon \in G_j} \left(y - \varepsilon^{m_j} a x^{m_j/n_j} - \sum_{i \in I_j} a_{j,i} \varepsilon^i x^{i/n_j} \right)$$

we obtain

$$\bar{F}_j = \tilde{x}^{m' y n_j} \prod_{e \in G_j} \left(\tilde{y} + a - \varepsilon^{m_j} a - \sum_{i \in I_j} a_{j,i} \varepsilon^i \tilde{x}^{(i - m_j)/\bar{n}_j} \right).$$

Then the claim follows by taking

$$U_j = \prod_{e \in G_j - G'_j} \left(\tilde{y} + a - \varepsilon^{m_j} a - \sum_{i \in I_j} a_{j,i} \varepsilon^i \tilde{x}^{(i - m_j)/\bar{n}_j} \right).$$

□

Next lemma follows from a similar but easier computation.

5.3. LEMMA. *For $j > l'$ we have*

$$\bar{F}_j = \tilde{x}^{\delta_j} U_j$$

where U_j is invertible in $\mathcal{O}_{p'}$ and $\delta_j > 0$. Furthermore $U_j(0,0)$ and $(\partial U_j/\partial \tilde{y})(0,0)$ depend only on the coefficients a and b_j.

We choose $\tilde{S}_j, j = 1, \ldots, l'$ as Puiseux series of the branches of ξ', the conditions of section 3 being still satisfied by the $\tilde{S}_j$. Then the coefficients of ξ' are coefficients of ξ so that, in particular, if ξ has general coefficients, ξ' has general coefficients too. Thus, in the sequel we will assume, using induction, that ξ' satisfies the claim of 3.1, this is, that $P_{\tilde{x}}(F_{\xi'})$ goes through $\widetilde{\partial_{\tilde{x}}(\xi')}$ with effective multiplicities equal to the virtual ones and has no singularities outside of $\widetilde{\partial_{\tilde{x}}(\xi')}$.

In view of the inductive description of $\widetilde{\partial_g(\xi)}$ given at 3.1, once the behaviour of $P_g(F_\xi)$ at p and the initial points is given by 4.3 and 4.7, the last piece in order to achieve the proof of 3.1 is:

5.4. PROPOSITION. *The strict transform of $P_g(F_\xi)$ with origin at p' goes through $\widetilde{\partial_{\tilde{x}}(\xi')}$ with effective multiplicities equal to the virtual ones and has no singularities outside of $\widetilde{\partial_{\tilde{x}}(\xi')}$.*

Proof. From 5.1 we know the strict transform of $P_g(F_\xi)$ to be a polar of ξ', namely $P_{\tilde{x}}(\tilde{F}_\xi)$. On the other hand, because of the induction hypothesis, $P_{\tilde{x}}(F_{\xi'})$ goes through $\widetilde{\partial_{\tilde{x}}(\xi')}$ with effective multiplicities equal to the virtual ones and has no singularities outside of $\widetilde{\partial_{\tilde{x}}(\xi')}$.

Then, if there is some branch of ξ' not tangent to E, the claim follows directly from 2.1(a).

Assume now that all branches of ξ' are tangent to E. Because of the induction hypothesis we have $e_{p'}(\xi') = e_{p'}(P_{\tilde{x}}(F_{\xi'})) = e'$ (say). Put

$$U = U_1 \cdots U_{l'} U_{l'+1} \cdots U_l$$

so that from 5.2 and 5.3 we have

$$\tilde{F}_\xi = UF_{\xi'}$$

and thus,

$$\frac{\partial \tilde{F}_\xi}{\partial \tilde{y}} = U \frac{\partial F_{\xi'}}{\partial \tilde{y}} + \frac{\partial U}{\partial \tilde{y}} F_{\xi'}. \tag{1}$$

Use the plane $\alpha\beta$ to draw Newton polygons of equations in $\tilde{x}$, $\tilde{y}$, α being the degree in $\tilde{x}$ and β the degree in $\tilde{y}$.

The Newton polygon of $F_{\xi'}$ ends at $(e', 0)$ because all branches of ξ' are tangent to E, and that of $\partial F_{\xi'}/\partial \tilde{y}$ ends at the same point by 4.4. Then, it is clear that the Newton polygon of $\partial F_{\xi'}/\partial \tilde{y}$ lies below that of $F_{\xi'}$ the two polygons meeting just at their common end.

We will see first that the terms of bidegree $(e', 0)$ may not be cancelled in (1) which implies that $\partial \tilde{F}_\xi/\partial \tilde{y}$ has the same Newton polygon as $\partial F_{\xi'}/\partial \tilde{y}$.

Assume that $\tilde{\gamma}_1$ is one of the branches of ξ' with maximal initial exponent, so that it corresponds to the last side of the Newton polygon of ξ'. If $(\bar{m}_1 - m_1)/\bar{n}_1$ is the initial exponent of $\tilde{\gamma}_1$, it follows from 4.5 that the term of bidegree $(e', 0)$ in $\partial F_{\xi'}/\partial \tilde{y}$ depends on $a_{1,2\bar{m}_1 - m_1}$ whence this coefficient does not appear either in the term of bidegree $(e', 0)$ of F_ξ (by an easy computation) or in $U(0,0)$, $(\partial U/\partial \tilde{y})(0,0)$ (by 5.2 and 5.3). Thus it is clear that, under the hypothesis of general coefficients for ξ, the monomials of bidegree $(e', 0)$ in (1) cannot be cancelled.

It follows in particular that $P_{\tilde{x}}(\tilde{F}_\xi)$ has multiplicity e' at p' so that 2.1(b) may be applied to: take h to be the greater integer strictly less than $\bar{n}_1/(\bar{m}_1 - m_1)$ and call q_0 the point on E in the $(h-1)$-th neighbourhood of p', then the polar $P_{\tilde{x}}(\tilde{F}_\xi)$ goes through $\partial_{\tilde{x}}(\xi')$ with effective multiplicities equal to the virtual ones and has no singularities outside of $\partial_{\tilde{x}}(\xi')$ infinitely near to any $q \neq q_0$.

Thus only the non existence of singularities in the first neighbourhood of q_0 needs to be verified. Since $P_{\tilde{x}}(\tilde{F}_\xi)$ has the same Newton polygon as $P_{\tilde{x}}(F_{\xi'})$ as seen before, it is equivalent to see that the equation associated to the last side has no multiple roots, and this in turn follows easily from 4.5 and the analysis of the monomial of bidegree $(e', 0)$ we made above. □

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