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## Degree of local zeta functions and monodromy

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### 1. Introduction

(1.1) Let  $K$  be a  $p$ -adic field, i.e.  $[K:Q_p] < \infty$ . Let  $R$  be the valuation ring of  $K$ ,  $P$  the maximal ideal of  $R$ , and  $\bar{K} = R/P$  the residue field of  $K$ . The cardinality of  $\bar{K}$  is denoted by  $q$ , thus  $\bar{K} = F_q$ . Let  $f(x) \in K[x]$ ,  $x = (x_1, \dots, x_n)$ ,  $f \notin K$ . Igusa's local zeta function of  $f$  with respect to a character  $\chi: R^\times \rightarrow \mathbb{C}^\times$  and a Schwartz–Bruhat function  $\Phi: K^n \rightarrow \mathbb{C}$  is denoted by

$$Z_\Phi(s, \chi) = Z_\Phi(s, \chi, K, f),$$

see e.g. [D3, §1.1], [D2]. When  $\Phi$  is the characteristic function of the residue class  $a \in \bar{K}^n$ , we will write  $Z_a(s, \chi)$  instead of  $Z_\Phi(s, \chi)$ . In this note we will always assume that  $\chi$  is induced by a character  $\chi: \bar{K}^\times \rightarrow \mathbb{C}^\times$ .

In case of good reduction, we showed in [D1] (see also [D3, §4.1]) that  $\deg Z_a(s, \chi) \leq 0$  and  $\deg Z_a(s, \chi_{\text{triv}}) = 0$ , where  $\deg$  means the degree as rational function in  $q^{-s}$  and  $\chi_{\text{triv}}$  is the trivial character. (We put  $\deg 0 = -\infty$ .) In the present note we will prove the following theorem:

(1.2) **THEOREM.** *If  $f$  is defined over a number field  $F \subset \mathbb{C}$ , then for almost all completions  $K$  of  $F$  we have the following:*

*If  $f(0) = 0$  and no eigenvalue of the (complex) local monodromy of  $f$  at 0 has the same order as  $\chi$ , then  $\deg Z_0(s, \chi) < 0$ .*

With an eigenvalue of the (complex) local monodromy of  $f$  at  $a \in f^{-1}(0)$  we mean an eigenvalue of the action of the counter clockwise generator of the fundamental group of  $\mathbb{C} \setminus \{0\}$  on the cohomology (in some dimension) of the Milnor fiber of  $f$  at  $a$  (see e.g. [A] or [D3, §2.1]). It is well known that such an eigenvalue is a root of unity so that we can talk about its order. Theorem 1.2 is a direct consequence of Theorem 1.4 below, whose statement requires some more notation.

(1.3) From now on we assume that  $f \in R[x]$  and  $\bar{f} \neq 0$ , where  $\bar{f}$  denotes the reduction mod  $P$  of  $f$ . We fix a prime  $\ell \nmid q$  and an embedding of  $\mathbb{C}$  into an algebraic closure  $Q_\ell^a$  of  $Q_\ell$ . Thus we can consider  $\chi$  as a character  $\chi: \bar{K}^\times \rightarrow (Q_\ell^a)^\times$ . This  $\chi$  induces a character also denoted by  $\chi$ , of the geometric monodromy group of  $\mathbb{A}_{F_q}^1$  at 0, see 2.1. Let  $F_0$  be the Milnor fibre of  $\bar{f}$  at 0, in the sense of étale topology. We denote by  $H^i(F_0, Q_\ell^a)^\chi$  the component of the  $\ell$ -adic cohomology  $H^i(F_0, Q_\ell^a)$  on which the local geometric monodromy group acts like  $\chi$  times a unipotent action, see 2.3.1.

(1.4) **THEOREM.** *Assume that  $f^{-1}(0)$  has a resolution with tame good reduction mod  $P$  (see 2.2.3 or [D3, 3.2]), and that  $f(0) = 0$ . Then*

$$\lim_{s \rightarrow -\infty} Z_0(s, \chi) = (1 - q)q^{-n} \sum_i (-1)^i \text{Tr}(\sigma_1, H^i(F_0, Q_\ell^a)^\chi),$$

where  $\sigma_1$  is a suitable lifting of the geometric Frobenius (see 3.2).

Theorem 1.4 is proved in 3.3 using the method of vanishing cycles which we recall in 2.1 and 2.2. A partial converse of Theorem 1.4 is given in 3.4. In Section 4 we propose a conjecture about the holomorphy of  $Z_\Phi(s, \chi)$ . Finally, Section 5 contains an alternative proof of some material in [D2].

## 2. Preliminaries

### (2.1) Local monodromy

We choose a geometric generic point  $\bar{\eta}$  of  $\mathbb{A}_{F_q}^1$ . In particular this choice determines an algebraic closure  $F_q^a$  of  $F_q$ . Let  $S$ , resp.  $S_o$ , be the Henselization at 0 of  $\mathbb{A}_{F_q^a}^1$  resp.  $\mathbb{A}_{F_q}^1$ , and denote by  $\eta$ , resp.  $\eta_o$ , its generic point.

Put  $G_o = \text{Gal}(\bar{\eta}/\eta_o)$  and  $I_o = \text{Gal}(\bar{\eta}/\eta)$ . The group  $G_o$ , resp.  $I_o$ , is called the arithmetical, resp. geometrical, local monodromy group of  $\mathbb{A}_{F_q}^1$  at 0. Via the cover

$$S_o \setminus \{0\} \rightarrow S_o \setminus \{0\}: X \mapsto X^{q-1},$$

with Galois group  $F_q^\times$ , we consider  $F_q^\times$  as a quotient of  $G_o$ . Hence the character  $\chi: F_q^\times \rightarrow (Q_\ell^a)^\times$  induces a homomorphism  $\tilde{\chi}: G_o \rightarrow (Q_\ell^a)^\times$ . The restriction of this homomorphism  $\tilde{\chi}$  to  $I_o$  will be denoted by  $\chi: I_o \rightarrow (Q_\ell^a)^\times$ .

(2.2) *Nearby cycles on the resolution space*

(2.2.1) Let  $h: Y \rightarrow X = \text{Spec } K[x]$  be an (embedded) resolution (of singularities) for  $f^{-1}(0)$  over  $K$  with good reduction mod  $P$ , see [D3, 1.3.1 and 3.2] or [D1]. Reduction mod  $P$  is denoted by  $\bar{\phantom{x}}$ , e.g.  $\bar{Y}, \bar{E}_i$ .

Let  $E_i, i \in T$ , be the irreducible components of  $(f \circ h)^{-1}(0)$ . Denote by  $N_i$ , resp.  $\nu_i - 1$ , the multiplicity of  $E_i$  in the divisor of  $f \circ h$ , resp.  $h^*(dx_1 \wedge \dots \wedge dx_n)$ . Put  $\bar{E}_i = \bar{E}_i \setminus \cup_{j \neq i} \bar{E}_j$ ,  $\bar{E}_i = \bar{E}_i \setminus \cup_{j \neq i} \bar{E}_j$  and  $\bar{E}_I = \cap_{i \in I} \bar{E}_i$ ,  $\bar{E}_I = \bar{E}_I \setminus \cup_{j \notin I} \bar{E}_j$  for any  $I \subset T$ . When  $I = \emptyset$ , put  $\bar{E}_\emptyset = \bar{Y}$ .

(2.2.2) We denote by  $R\Psi_{\bar{f}}(C)$ , resp.  $R\Psi_{\bar{f} \circ \bar{h}}(C)$ , the complex of nearby cycles on  $\bar{f}^{-1}(0) \otimes F_q^a$ , resp.  $(\bar{f} \circ \bar{h})^{-1}(0) \otimes F_q^a$ , associated to a complex  $C$ , see [SGA 7, XIII]. To simplify notation, put  $\Psi_{\bar{f}}^i = R^i\Psi_{\bar{f}}(Q_\ell^a)$  and  $\Psi_{\bar{f} \circ \bar{h}}^i = R^i\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a)$ . If  $f(0) = 0$  then  $(\Psi_{\bar{f}}^i)_0 = H^i(F_0, Q_\ell^a)$ , where  $F_0$  denotes the Milnor fibre of  $\bar{f}$  at 0. It is well known [SGA 7, XIII 2.1.7.1] that

$$R\bar{h}_* \circ R\Psi_{\bar{f} \circ \bar{h}} = R\Psi_{\bar{f}},$$

since  $\bar{h}$  is proper and birational. Thus, when  $f(0) = 0$ ,

$$H^i(F_0, Q_\ell^a) = \mathbb{H}^i(\bar{h}^{-1}(0) \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a)), \tag{2.2.2.1}$$

and we have a spectral sequence

$$H^i(\bar{h}^{-1}(0) \otimes F_q^a, \Psi_{\bar{f} \circ \bar{h}}^j) \Rightarrow H^{i+j}(F_0, Q_\ell^a). \tag{2.2.2.2.}$$

Note that  $G_0$  acts on all terms of this spectral sequence, by transport of structure (choice of  $\bar{\eta}$ ), and the spectral sequence is  $G_0$ -equivariant. We recall from [SGA 7, Exp. I Thm 3.3] the following basic facts:

For any  $I \subset T$  with  $I \neq \emptyset$  and any closed point  $s \in \bar{E}_I \otimes F_q^a$ , there is a canonical isomorphism

$$(\Psi_{\bar{f} \circ \bar{h}}^j)_s^{\text{tame}} \cong (\Psi_{\bar{f} \circ \bar{h}}^0)_s^{\text{tame}} \otimes \bigwedge^j (M_I(-1)), \tag{2.2.2.3}$$

where  $M_I$  is the dual of the kernel of the linear map  $(Q_\ell^a)^I \rightarrow Q_\ell^a: (z_i)_{i \in I} \mapsto \sum_{i \in I} N_i z_i$ ,  $M_I(-1)$  is a Tate twist of  $M_I$ , and the superscript *tame* denotes the tame part. Moreover

$$(\Psi_{\bar{f} \circ \bar{h}}^0)_s^{\text{tame}} \cong (Q_\ell^a)^{C_I}, \tag{2.2.2.4}$$

with  $C_I$  a finite set on which  $I_0$  acts transitively, and  $|C_I|$  equal to the largest common divisor of the  $N_i$ ,  $i \in I$ , which is prime to  $q$ .

(2.2.3) Till the end of 2.2.3 we will assume now that the resolution  $h$  has *tame* good reduction, i.e. it has good reduction and  $N_i$  is prime to  $q$  for each  $i \in T$ . Then it easily follows from [K, p. 180] that the action of  $I_0$  on  $\Psi_{\bar{f} \circ \bar{h}}^j$  is tame.

A local calculation shows that the  $\Psi_{\bar{f} \circ \bar{h}}^j$  are lisse on  $\overset{\circ}{E}_I \otimes F_q^a$  and that locally on  $\overset{\circ}{E}_I \otimes F_q^a$  the isomorphisms 2.2.2.3 on the stalks are induced by an isomorphism of the sheaves. Since these isomorphisms are canonical they glue together to a *canonical* isomorphism

$$\Psi_{\bar{f} \circ \bar{h}}^j \cong \Psi_{\bar{f} \circ \bar{h}}^0 \otimes \wedge^j (M_I(-1)) \quad \text{on } \overset{\circ}{E}_I \otimes F_q^a \tag{2.2.3.1}$$

which is compatible with the action of  $G_0$ .

(2.3) *Isotopic components*

(2.3.1) For any constructible  $Q_\ell^a$ -sheaf  $\mathcal{F}$  (or vector space) on which  $I_0$  acts, we denote by  $\mathcal{F}^\chi$  the  $\chi$ -unipotent part of  $\mathcal{F}$ , i.e. the largest subsheaf on which  $I_0$  acts like  $\chi$  times a unipotent action.

(2.3.2) To the character  $\chi: F_q^\times \rightarrow (Q_\ell^a)^\times$  is associated the lisse rank one  $Q_\ell^a$ -sheaf  $\mathcal{L}_\chi$  on  $\mathbb{A}_{F_q}^1 \setminus \{0\}$ , see [SGA 4 $_{\frac{1}{2}}$ , Sommes Trig.]. The action of the arithmetical monodromy group  $G_0$  at 0 on  $(\mathcal{L}_\chi)_{\bar{\eta}}$  is given by  $\tilde{\chi}^{-1}$ .

Let  $\nu$  be the open immersion  $\nu: \bar{Y} \setminus (\bar{f} \circ \bar{h})^{-1}(0) \hookrightarrow \bar{Y}$  and  $\alpha: \bar{Y} \setminus (\bar{f} \circ \bar{h})^{-1}(0) \rightarrow \mathbb{A}_{F_q}^1 \setminus \{0\}$  the restriction of  $\bar{f} \circ \bar{h}$ . Put  $\mathcal{F}_\chi = \nu_* \alpha^* \mathcal{L}_\chi$ . The cohomology of this sheaf appears in the explicit formula for  $Z_0(s, \chi)$ , see 3.1.

(2.3.3) LEMMA. *There is a canonical isomorphism*

$$\mathcal{F}_\chi|_{(\bar{f} \circ \bar{h})^{-1}(0)} \otimes_{F_q^a} \cong (\Psi_{\bar{f} \circ \bar{h}}^0)^\chi \otimes (\mathcal{L}_\chi)_{\bar{\eta}}.$$

*Proof.* Because the action of  $I_0$  on the stalks of the tame part of  $\Psi_{\bar{f} \circ \bar{h}}^0$  is semi-simple (cf. 2.2.2.4),  $(\Psi_{\bar{f} \circ \bar{h}}^0)^\chi$  equals the largest subsheaf of  $\Psi_{\bar{f} \circ \bar{h}}^0$  on which  $I_0$  acts like  $\chi$ . Moreover there is a canonical isomorphism

$$R\Psi_{\bar{f} \circ \bar{h}}(\alpha^* \mathcal{L}_\chi) \cong R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a) \otimes (\mathcal{L}_\chi)_{\bar{\eta}}. \tag{2.3.3.1}$$

Thus it suffices to prove that there is a canonical isomorphism

$$\mathcal{F}_\chi|_{(\bar{f} \circ \bar{h})^{-1}(0)} \otimes_{F_q^a} \cong (R^0\Psi_{\bar{f} \circ \bar{h}}(\alpha^* \mathcal{L}_\chi))^{I_0}, \tag{2.3.3.2}$$

where the superscript  $I_0$  denotes the largest subsheaf on which  $I_0$  acts trivially. We will denote by an index  $S$  the base change  $S \rightarrow \mathbb{A}_{F_q}^1$ ; for example  $\bar{Y}_S = \bar{Y} \otimes_{\mathbb{A}_{F_q}^1} S$ . Consider the following diagram of natural maps

$$\begin{array}{ccccccc} (\bar{f} \circ \bar{h})^{-1}(0) \otimes F_q^a & \xrightarrow{i} & \bar{Y}_S & \xleftarrow{\nu_S} & (\bar{Y} \setminus (\bar{f} \circ \bar{h})^{-1}(0))_S & \xleftarrow{j} & (\bar{Y}_S)_{\bar{\eta}} \\ \downarrow & & (\bar{f} \circ \bar{h})_S \downarrow & & \alpha_S \downarrow & & \downarrow \\ \{0\} & \longrightarrow & S & \longleftarrow & S \setminus \{0\} = \eta & \xleftarrow{\gamma} & \bar{\eta} \end{array}$$

Consider also the natural map  $\epsilon: S \setminus \{0\} \rightarrow \mathbb{A}_{F_q}^1 \setminus \{0\}$ . By [SGA 4 $\frac{1}{2}$ , Th. finitude 1.9] we have

$$\alpha_S^* \gamma_* (\gamma^* \epsilon^* \mathcal{L}_\chi) \cong j_* (j^* \alpha_S^* \epsilon^* \mathcal{L}_\chi).$$

Hence

$$i^*(\nu_S)_* \alpha_S^* \gamma_* (\gamma^* \epsilon^* \mathcal{L}_\chi) \cong i^*(\nu_S)_* j_* (j^* \alpha_S^* \epsilon^* \mathcal{L}_\chi) = R^0\Psi_{\bar{f} \circ \bar{h}}(\alpha^* \mathcal{L}_\chi).$$

Taking  $I_0$ -invariants we get

$$i^*(\nu_S)_* \alpha_S^* (\gamma_* \gamma^* \epsilon^* \mathcal{L}_\chi)^{I_0} \cong (R^0\Psi_{\bar{f} \circ \bar{h}}(\alpha^* \mathcal{L}_\chi))^{I_0}.$$

But  $\epsilon^* \mathcal{L}_\chi \cong (\gamma_* \gamma^* \epsilon^* \mathcal{L}_\chi)^{I_0}$ , hence

$$\mathcal{F}_\chi|_{(\bar{f} \circ \bar{h})^{-1}(0)} \otimes_{F_q^a} \cong i^*(\nu_S)_* \alpha_S^* \epsilon^* \mathcal{L}_\chi \cong (R^0\Psi_{\bar{f} \circ \bar{h}}(\alpha^* \mathcal{L}_\chi))^{I_0}. \quad \square$$

### 3. Cohomological interpretation of $\lim_{s \rightarrow -\infty} Z_0(s, \chi)$

(3.1) Let  $F \in \text{Gal}(F_q^a/F_q)$  be the geometric Frobenius. We recall from [D2] that

$$Z_0(s, \chi) = q^{-n} \sum_{I \subset T} c_{I, \chi, 0} \prod_{i \in I} \frac{q-1}{q^{N_{\beta^i + \nu_i} - 1}}, \tag{3.1.1}$$

where

$$c_{I,\chi,0} = \sum_i (-1)^i \operatorname{Tr}(F, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, \mathcal{F}_\chi)). \quad (3.1.2)$$

Hence

$$\lim_{s \rightarrow -\infty} Z_0(s, \chi) = q^{-n} \sum_{I \subset T} c_{I,\chi,0} (1 - q)^{|I|}. \quad (3.1.3)$$

(3.2) With a *suitable lifting* of the geometric Frobenius (mentioned in the statement of Theorem 1.4) we mean any element  $\sigma_1 \in G_0$  which induces the geometric Frobenius on  $F_q^a$  and which acts trivially on  $(\mathcal{L}_\chi)_{\bar{\eta}}$  (see 2.3.2).

(3.3) *Proof of Theorem 1.4.* We will prove that

$$\frac{q^n}{1 - q} \lim_{s \rightarrow -\infty} Z_0(s, \chi) = \sum_i (-1)^i \operatorname{Tr}(\sigma, H^i(F_0, Q_c^a)^X \otimes (\mathcal{L}_\chi)_{\bar{\eta}}), \quad (3.3.1)$$

for any  $\sigma \in G_0$  which induces the geometric Frobenius  $F$  on  $F_q^a$ . This yields the theorem when we take for  $\sigma$  a suitable lifting  $\sigma_1$  as in 3.2. The right-hand-side of (3.3.1) equals

$$\begin{aligned} & \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H^i(\bar{h}^{-1}(0) \otimes F_q^a, \Psi_{\bar{f} \circ \bar{h}}^j)^X \otimes (\mathcal{L}_\chi)_{\bar{\eta}}), \quad \text{by (2.2.2.2),} \\ &= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, (\Psi_{\bar{f} \circ \bar{h}}^j)^X \otimes (\mathcal{L}_\chi)_{\bar{\eta}})), \\ &= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, (\Psi_{\bar{f} \circ \bar{h}}^0)^X \otimes \\ & \quad \otimes (\mathcal{L}_\chi)_{\bar{\eta}} \otimes \wedge^j (M_I(-1))), \quad \text{by (2.2.3.1),} \\ &= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\sigma, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, \mathcal{F}_\chi) \otimes \\ & \quad \otimes \wedge^j (M_I(-1))), \quad \text{by (2.3.3),} \\ &= \sum_I \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(F, H_c^i((\overset{\circ}{E}_I \cap \bar{h}^{-1}(0)) \otimes F_q^a, \mathcal{F}_\chi)) \operatorname{Tr}(F, \wedge^j (M_I(-1))), \\ &= \sum_I c_{I,\chi,0} (1 - q)^{|I|-1}, \quad \text{by (3.1.2).} \end{aligned}$$

Combining this with 3.1.3 proves 3.3.1 and finishes the proof of Theorem 1.4. □

We now turn to a partial converse of Theorem 1.4. For any finite extension  $L$  of the field  $K$ , the norm from  $L$  to  $K$  is denoted by  $N_{L/K}$ .

(3.4) PROPOSITION. *If  $f$  is defined over a number field  $F \subset \mathbb{C}$ , then for almost all completions  $K$  of  $F$  we have the following: Assume the order of  $\chi$  equals the order of some eigenvalue of the (complex) local monodromy of  $f$  at some complex point of  $f^{-1}(0)$ . Then there are infinitely many unramified extensions  $L$  of  $K$  such that  $\deg Z_{\bar{a}}(s, \chi \circ N_{L/K}, L, f) = 0$  for some integral  $a \in L^n$  with  $f(a) = 0$ .*

*Proof.* It is well known [B] that  $R\Psi_{\bar{f}}(Q_{\bar{t}}^a)[n-1]$  is a perverse sheaf. Let  $C := (R\Psi_{\bar{f}}(Q_{\bar{t}}^a)[n-1])^\chi$  be the maximal subobject (in the category of perverse sheaves) on which  $I_0$  acts like  $\chi$  times a unipotent action. We have  $C \neq 0$  (for almost all completions  $K$  of  $F$ ). Since  $C$  is perverse, there exists a geometric point  $\bar{a}$  of  $\bar{f}^{-1}(0)$  such that  $(H^i(C))_{\bar{a}} \neq 0$  for exactly one  $i$ . The proposition follows now easily from 1.4. □

(3.5) Example. Let  $f(x_1, x_2) = x_2^2 - x_1^3$ . Then the orders of the eigenvalues of the local monodromy are 1 and 6. Thus, for almost all completions,  $\deg Z_0(s, \chi) < 0$  if  $\chi$  has order 2 and 3. (Compare with Proposition 4.5).

#### 4. Holomorphy of $Z_\Phi(s, \chi)$

(4.1) We call a Schwartz–Bruhat function  $\Phi$  on  $K^n$  residual if  $\Phi$  is zero outside  $R^n$  and  $\Phi(x)$  only depends on  $x \bmod P$ .

(4.2) It is well known (see [I1] or [D3, 1.3.2]) that  $Z_\Phi(s, \chi)$  is holomorphic on  $\mathbb{C}$  when the order of  $\chi$  divides no  $N_i$ . The  $N_i$  are not intrinsic, but the order of any eigenvalue of the local monodromy on  $f^{-1}(0)$  divides some  $N_i$  (this follows from 2.2.2.2, 2.2.2.3 and 2.2.2.4). Being very optimistic, we propose the following conjecture:

(4.3) CONJECTURE. *If  $f$  is defined over a number field  $F \subset \mathbb{C}$ , then for almost all completions  $K$  of  $F$  we have the following: when  $\Phi$  is residual,  $Z_\Phi(s, \chi)$  is holomorphic unless the order of  $\chi$  divides the order of some eigenvalue of the (complex) local monodromy of  $f$  at some complex point of  $f^{-1}(0)$ .*

In fact, this might be true for all  $p$ -adic completions  $K$  of  $F$  and for any  $\Phi$ . Veys [V2] verified this when  $f$  has only two variables. Moreover the author showed that the conjecture is true for the relative invariants of a few pre-



homogeneous vector spaces (using Theorem 2 of [I2] and the orbital decomposition).

(4.4) *Remark.* Suppose  $f$  is homogeneous. Then for almost all completions  $K$  of  $F$  we have the following: If  $Z(s, \chi)$  is holomorphic, then  $Z(s, \chi) = 0$ , since  $\deg Z(s, \chi) < 0$  (see [D3, 4.1]). For  $s = +\infty$  this yields that

$$S := \sum_{x \in (F_q)^n, \bar{f}(x) \neq 0} \chi(\bar{f}(x))$$

is zero when  $Z(s, \chi)$  is holomorphic. Thus conjecture 4.3 implies a relation between the vanishing of the character sum  $S$  and monodromy. However this relation follows directly from the formula

$$S = (q - 1)q^{n-1} \sum_i (-1)^i \text{Tr}(\sigma_1^{-1}, H^i(F_0, Q_?^{\otimes x^{-1}})),$$

which is easily proved by standard methods.

The following proposition is a partial converse of Conjecture 4.3.

(4.5) **PROPOSITION.** *If  $f$  is defined over a number field  $F \subset \mathbb{C}$ , then for almost all completions  $K$  of  $F$  we have the following: If the order of  $\chi$  divides the order of some eigenvalue of the (complex) local monodromy of  $f$  at some complex point of  $f^{-1}(0)$ , then for infinitely many unramified extensions  $L$  of  $K$ ,  $Z_{\Phi}(s, \chi \circ N_{L/K}, L, f)$  is not holomorphic on  $\mathbb{C}$  for some residual  $\Phi$ .*

I first proved this proposition in the isolated singularity case, see [D3, prop. 4.4.3]. However that proof generalizes directly to the general case, because of Lemma 4.6 below. Indeed by 4.6 and the hypothesis of 4.5 there exists  $a \in f^{-1}(0)$  such that the order  $d$  of  $\chi$  divides the order  $k$  of some reciprocal zero or reciprocal pole of the monodromy zeta function of  $f$  at  $a$ . Hence by A'Campo [A, Thm 3], we have  $\sum_{k|N_i} \chi(\check{E}_i \cap h^{-1}(a)) \neq 0$ . Proceeding now as in my proof of Proposition 4.4.3 of [D3], with  $Z_0$  replaced by  $Z_a$ , we obtain that  $Z_a(s, \chi \circ N_{L/K}, L, f)$  is not holomorphic for infinitely many  $L$ .

(4.6) **LEMMA.** *Let  $f(x) \in \mathbb{C}[x]$ ,  $x = (x_1, \dots, x_n)$ ,  $f \notin \mathbb{C}$ . If  $\lambda$  is an eigenvalue of the (complex) local monodromy of  $f$  at  $b \in f^{-1}(0)$ , then there exists  $a \in f^{-1}(0)$  such that  $\lambda$  is a reciprocal zero or reciprocal pole of the monodromy zeta function of  $f$  at  $a$  (in the sense of [A, p. 233]).*

*Proof.* It is well known [B] that  $R\Psi_f \mathbb{C}[n - 1]$  is a perverse sheaf. Let  $C$  be the maximal subobject (in the category of perverse sheaves) on which the (complex) local monodromy acts like  $\lambda$  times a unipotent endomorphism. The hypothesis of the Lemma implies that  $C \neq 0$ . Since  $C$  is perverse, there

exists  $a \in f^{-1}(0)$  such that  $(H^i(C))_a \neq 0$  for exactly one  $i$ . The lemma follows now easily. □

**5. An alternative proof of some material in [D2]**

In [D2, Thm. 1.1] we proved that certain  $E_i$  do not contribute to poles of  $Z(s, \chi)$ , see also [D3, 4.6]. The proof was based on the following key Lemma 5.1, for which we will now give an alternative proof.

(5.1) LEMMA. [D2, 4.1]. *Assume the notation of 2.2.1. and 2.3.2. Let  $\chi$  be a character of  $\bar{K}^\times$  of order  $d$ , and  $i_0 \in T$ . Suppose  $E_{i_0}$  is proper,  $d|N_{i_0}$ , and  $E_{i_0}$  intersects no  $E_j$  with  $d|N_j$ ,  $j \neq i_0$ . Then*

$$H_c^i(\mathring{E}_{i_0} \otimes F_q^a, \mathcal{F}_\chi) = 0 \quad \text{for all } i \neq n - 1.$$

(5.2) *An alternative proof for Lemma 5.1.* A local calculation, using the hypothesis of the Lemma, shows that for every closed point  $s \in \bar{E}_{i_0} \setminus \mathring{E}_{i_0}$  the local monodromy of  $\mathcal{F}_\chi|_{\mathring{E}_{i_0}}$  at  $s$  has no invariants. Hence by [SGA 4 $\frac{1}{2}$ , Sommes Trig. 1.19.1] and tame ramification, we have

$$H_c^i(\mathring{E}_{i_0} \otimes F_q^a, \mathcal{F}_\chi) = H^i(\mathring{E}_{i_0} \otimes F_q^a, \mathcal{F}_\chi), \quad \text{for all } i.$$

Thus by Poincaré duality we only have to prove the Lemma for  $i > n - 1$ . Because  $\bar{E}_{i_0}$  is proper,  $\bar{h}(\bar{E}_{i_0})$  is finite. Hence we may assume that  $\bar{h}(\bar{E}_{i_0}) = \{0\}$ . We claim that

$$H_c^i(\mathring{E}_{i_0} \otimes F_q^a, \mathcal{F}_\chi) \subset (\Psi_{\bar{f}}^i)_0^\chi \otimes (\mathcal{L}_\chi)_{\bar{\eta}}, \quad \text{for all } i. \tag{5.2.1}$$

This claim proves the Lemma since it is well known that  $\Psi_{\bar{f}}^i = 0$  when  $i > n - 1$ , see [SGA 7, Exp. I Th. 4.2].

From 2.2.2.3, 2.2.2.4 and the hypothesis of the Lemma, it follows that

$$(\Psi_{\bar{f} \circ \bar{h}}^i)_s^\chi = 0 \tag{5.2.2}$$

for any closed point  $s \in \bar{E}_{i_0} \setminus \mathring{E}_{i_0}$  and  $i \geq 0$ , and also for any closed point  $s \in \mathring{E}_{i_0}$  and  $i \geq 1$ . (Indeed the  $\chi$ -unipotent part is contained in the tame part.) Thus applying the Mayer–Vietoris sequence for  $\bar{h}^{-1}(0) = \bar{E}_{i_0} \cup (\bar{h}^{-1}(0) \setminus \mathring{E}_{i_0})$  and the spectral sequence of hypercohomology we obtain

$$\mathbb{H}^i(\bar{h}^{-1}(0) \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a))^\chi = \mathbb{H}^i(\bar{E}_{i_0} \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a))^\chi \oplus$$

$$\oplus \mathbb{H}^i((\bar{h}^{-1}(0) \setminus \bar{E}_{i_0}) \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a))^\chi.$$

Together with 2.2.2.1 this yields

$$\mathbb{H}^i(\bar{E}_{i_0} \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a))^\chi \subset (\Psi_{\bar{f}}^i)_0^\chi, \quad \text{for all } i. \quad (5.2.3)$$

Again by 5.2.2 we have

$$\begin{aligned} \mathbb{H}^i(\bar{E}_{i_0} \otimes F_q^a, R\Psi_{\bar{f} \circ \bar{h}}(Q_\ell^a))^\chi &= H^i(\bar{E}_{i_0} \otimes F_q^a, (\Psi_{\bar{f} \circ \bar{h}}^0)^\chi) \\ &= H_c^i(\bar{E}_{i_0} \otimes F_q^a, (\Psi_{\bar{f} \circ \bar{h}}^0)^\chi) \end{aligned} \quad (5.2.4)$$

by degeneration of (the  $\chi$ -unipotent part of) the spectral sequence of hypercohomology. The claim 5.2.1 follows now from 5.2.3, 5.2.4 and Lemma 2.3.3. This terminates the proof of Lemma 5.1.  $\square$

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