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Curves of genus ten on K3 surfaces

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Introduction

Let $C$ denote a smooth complete algebraic curve and $L$ a line bundle on $C$. There is a natural map, called the Wahl or Gaussian map,

$$\Phi_L: \bigwedge^2 H^0(C, L) \to H^0(C, \Omega_C^1 \otimes L \otimes^2)$$

which sends $s \wedge t$ to $s dt - t ds$. J. Wahl made the striking observation that if $C$ is embeddable in a K3 surface then $\Phi_L$ is not onto for $L = \Omega_C^1$ ([W], Thm. 5.9); this raises the natural problem of studying the stratification of the moduli space of curves $\mathcal{M}_g$ by the rank of the Wahl map $\Phi(C) = \Phi_{\Omega_C^1}$. Roughly speaking, our main theorem says that the closure of the locus of curves of genus 10 which lie on a K3 is equal to the locus where $\Phi(C)$ fails to be surjective.

In order to state the theorem precisely and explain what is special about the case of genus 10, we need to introduce some spaces. Let $\mathcal{P}_g$ be the moduli space of K3 surfaces with a polarization of genus $g$, $\mathcal{P}_g$ the union, over all $S \in \mathcal{P}_g$, of the linear series $|\mathcal{O}_S(1)|$. Let $\mathcal{H}$ be the closure of the image of the natural rational map $\mu: \mathcal{P}_g \to \mathcal{M}_g$. As the dimension of $\mathcal{P}_g$ is $19 + g$ and the dimension of $\mathcal{M}_g$ is $3g - 3$, one might naively expect $\mu$ to be dominant for $g < 10$ and finite onto its image for $g \geq 11$. These expectations hold for $g \leq 9$ ([M], Thm. 6.1) and for odd $g \geq 11$ and even $g \geq 20$ ([M-M], Thm. 1), but for $g = 10$, Mukai showed that $\mu$ is not dominant ([M], Thm. 0.7). This exceptional behavior is due to the fact that the general K3 surface of genus 10 is a codimension 3 plane section of a certain 5-fold, so that when a curve lies on a general K3, it in fact lies on a 3-dimensional family of them. One of our first tasks is to show that $\mathcal{H}$ is a divisor when $g = 10$.

Over the open subset $\mathcal{M}_g^{10}$ of $\mathcal{M}_g$ of curves without automorphisms we have the relative Wahl map; let $\mathcal{W}^o$ denote its degeneracy locus and $\mathcal{W}$ the closure of $\mathcal{W}^o$ in $\mathcal{M}_g$. It is a theorem of Ciliberto-Harris-Miranda [C-H-M] that $\mathcal{W}$ is a

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divisor (i.e. the Wahl map does not degenerate everywhere), and by Wahl's theorem $\mathcal{K} \leq \mathcal{W}$. Our result can then be stated as follows.

**THEOREM.** We have an equality of divisors

$$\mathcal{W} = 4\mathcal{K}.$$

Moreover, for the general curve $C$ of genus 10 which can be embedded in a $K3$ surface, the codimension of the image of the Wahl map $\Phi(C)$ is 4.

It is worth remarking that a priori not every curve of genus 10 on a $K3$ appears in $\mathcal{K}$: the variety $\mathcal{P}$ consists of pairs $(S, C)$ where $\mathcal{O}_S(C)$ is indivisible in $\text{Pic}(S)$. But by Wahl’s theorem, every curve on a $K3$ has a degenerate Wahl map, so by the theorem defines a point of $\mathcal{K}$. It would be interesting to see explicitly a family of curves polarizing $K3$ s of genus 10 degenerating, for instance, to a plane sextic (which polarizes a $K3$ of genus 2).

We also note that Voisin proved ([V] Prop. 3.3) that the corank of $\Phi(C)$ is at most 3 for a genus 10 curve satisfying certain hypothesis (3.1)(i), (ii) and (iii) (loc. cit.). These hypotheses hold for a general curve, and (i) holds for a general curve on a $K3$. It follows that either (ii) or (iii) fails for the general curve of genus 10 on a $K3$; as Voisin pointed out to us, a dimension counting argument suggests that it is (iii) which fails generically.

To prove the theorem we first study the cohomology of a certain 5-fold $X$, which is a homogeneous space for the exceptional Lie group $G_2$, using a theorem of Bott as in [M]. This allows us to show, in Section 2, that $\mathcal{K}$ is a divisor and that for every $C$ which is a smooth codimension 4 plane section of $X$, the corank of $\Phi(C)$ is 4. This establishes the inequality of divisors $\mathcal{W} \geq 4\mathcal{K}$. In Section 3, we compute the classes of the divisors $\mathcal{W}$ and $\mathcal{K}$ and find that $\mathcal{W}$ is linearly equivalent to $4\mathcal{K}$. The desired equality of divisors then follows.

1. The cohomology of the 5-fold $X$

One of the main tools in our analysis will be the cohomology groups of a certain homogeneous variety $X$ used by Mukai [M] to study the moduli space of $K3$ surfaces of genus 10. To recall the definition, let $\mathfrak{g}$ be the complex semisimple Lie algebra attached to the exceptional root system $G_2$, let $G$ be the corresponding simply connected Lie group, and let $\rho: G \to \text{Aut}(\mathfrak{g})$ be the adjoint representation. If $v \in \mathfrak{g}$ is a lowest weight vector for $\rho$, then $X = \rho(G)v$ is the orbit of $v$. Equivalently, if $P \subseteq G$ is the maximal parabolic subgroup of $G$ associated to the longer of the two roots in a system of simple roots for $\mathfrak{g}$, then $X \cong G/P$. The homogeneous variety $X$ has dimension 5 and is naturally embedded in $\mathbb{P}(\mathfrak{g})$ as a subvariety of degree 18; its canonical bundle is isomorphic to $\mathcal{O}(-3)$ ([M],
Mukai shows that the general K3 surface of genus 10 is a codimension 3 plane section of $X$ and any abstract isomorphism between two such K3s is realized by the action of $G$ on the Grassmannian of codimension 3 planes in $\mathbb{P}(g)$ ([M], Thm. 0.2).

Recall that homogeneous vector bundles on $X$ are in one to one correspondence with finite dimensional linear representations of $P$. For example, if $\{\alpha_1, \alpha_2\}$ is a basis for the root system $G_2$ with $\alpha_i$ the shorter root, so that $P$ is the subgroup corresponding to the subalgebra whose roots are all of the negative roots together with $\alpha_1$, then the tangent bundle to $X = G/P$ corresponds to the (reducible) representation of $P$ with highest weight $w_1 = 3\alpha_1 + 2\alpha_2$. It has an irreducible rank 4 subbundle corresponding to the representation of $P$ with highest weight $\alpha_2 + 3\alpha_1$ and the quotient is isomorphic to $\mathcal{O}_X(1)$, corresponding to the irreducible representation of $P$ with highest weight $w_1$. Similarly $N_X$, the normal bundle of $X$ in $\mathbb{P}(g)$, has a composition series with quotients of rank 1, 3 and 4 corresponding to irreducible representations with highest weights 0, $4\alpha_1 + 2\alpha_2$, and $6\alpha_1 + 3\alpha_2$ respectively.

Now a theorem of Bott ([B]; see also [M], 1.6) asserts that when $E$ is an irreducible homogeneous vector bundle on a compact homogeneous variety $X = G/P$, at most one of the cohomology groups $H^i(X, E)$ is non-zero, and when non-zero, the group is an irreducible $G$-module. Moreover, he gives a recipe for calculating the index of the non-vanishing cohomology group. Application of this result to the $X$ considered above, which we leave as a pleasant exercise for the reader (compare [M], Section 1), yields the following result.

**Lemma 1.1**

1. We have $h^0(X, T_X(-1)) = 0$ and $H^0(X, T_x) \cong g$ as a $G$-module. Moreover, $h^i(X, T_X(-i)) = h^i(X, T_X(-i - 1)) = 0$ for $i = 1, 2, 3, 4$.

2. We have $H^0(X, N_X(-1)) \cong g$ as a $G$-module and $h^i(X, N_X(-i - 1)) = 0$ for $i = 1, \ldots, 4$. Also, $h^i(X, N_X(-i - 2)) = 0$ for $i = 0, \ldots, 4$.

Now suppose that $S$ is a smooth codimension 3 plane section of $X$ and that $C$ is a smooth hyperplane section of $S$; then $S$ is a K3 surface and $C$ is a canonically embedded curve of genus 10. Using Koszul resolutions of $\mathcal{O}_S$ and $\mathcal{O}_C$ as $\mathcal{O}_X$-modules, one easily checks the following assertions.

**Lemma 1.2**

1. $h^0(S, N_S(-1)) = 14$.

2. $h^0(C, T_X(-1)|_C) = 0$ and $h^0(C, T_X|_C) = 14$.

3. $h^0(C, N_C(-2)) = 0$ and $h^0(C, N_C(-1)) = 14$.

(Here $N_C$ and $N_S$ are the normal bundles to $C$ and $S$ in the projective spaces they span in $\mathbb{P}(g)$; the last part also uses the standard isomorphism $N_X|_C \cong N_C$.)
2. The corank of the Wahl map

We retain the notations of the introduction.

**Proposition 2.1.** Suppose $S$ is a general K3 surface of genus 10. Then $h^1(S, T_S(-1)) = 3$ and $h^2(S, T_S(-1)) = 1$.

*Proof.* Consider the exact sequence

$$0 \rightarrow T_S(-1) \rightarrow T_P(-1)|_S \rightarrow N_S(-1) \rightarrow 0$$

where $S \subseteq P = P^{10}$ is the given embedding. The long exact sequence of cohomology yields

$$0 \rightarrow H^0(S, T_P(-1)|_S) \rightarrow H^0(S, N_S(-1)) \rightarrow H^1(S, T_S(-1))$$

$$\rightarrow H^1(S, T_P(-1)|_S).$$

But the Euler sequence for $T_P|_S$ implies that $h^0(T_P(-1)|_S) = 11$ and $h^1(T_P(-1)|_S) = 0$. Indeed, we have

$$0 \rightarrow H^0(S, \mathcal{O}_S)^{11} \rightarrow H^0(S, T_P(-1)|_S) \rightarrow H^1(S, \mathcal{O}_S(-1))$$

$$\rightarrow H^1(S, \mathcal{O}_S)^{11} \rightarrow H^1(S, T_P(-1)|_S) \rightarrow H^2(S, \mathcal{O}_S(-1)) \rightarrow H^2(S, \mathcal{O}_S)^{11}$$

with $H^1(S, \mathcal{O}_S) = 0$ (S is a K3) and $H^1(S, \mathcal{O}_S(-1)) = 0$ ([K], Thm. 2.5); moreover, the map $H^2(S, \mathcal{O}_S(-1)) \rightarrow H^2(S, \mathcal{O}_S)^{11}$ is injective by duality and the projective normality of $S$ ([Ma], Prop. 2). By Lemma 1.2, $h^0(S, N_S(-1)) = 14$, so $h^1(S, T_S(-1)) = 3$. As $h^0(S, T_S(-1)) = 0$, Riemann-Roch implies $h^2(S, T_S(-1)) = 1$.

**Proposition 2.2.** The locus $\mathcal{H} \subseteq M_{10}$ is a divisor.

*Proof.* First we need some deformation theory. Generally, given a smooth complete curve $C$ in a smooth complete surface $S$, we have the tangent sheaf $T_S$ of $S$, the tangent sheaf $T_C$ of $C$ and the restriction $T_{S|C} = T_S \otimes \mathcal{O}_C$. Extending the latter two sheaves by 0 on $S$, we can define a coherent sheaf $F$ on $S$ as the fiber product

$$F \rightarrow T_C$$

$$\downarrow \quad \downarrow$$

$$T_S \rightarrow T_{S|C}.$$

The sheaf $F$ is locally free of rank 2 and sits in exact sequences

$$0 \rightarrow T_S(-C) \rightarrow F \rightarrow T_C \rightarrow 0 \quad (2.3)$$

and

$$0 \rightarrow F \rightarrow T_S \rightarrow N_{C|S} \rightarrow 0. \quad (2.4)$$
It is easy to check that the space of first order deformations of the pair $C \subseteq S$ is isomorphic to $H^1(S, F)$.

Returning to the case where $S$ is a general $K$-3 of genus 10 and $C$ is a smooth plane section of $C$, the long exact cohomology sequence of (2.3) gives

$$0 \to H^1(S, T_S(-C)) \to H^1(S, F) \to H^1(C, T_C) \to H^2(S, T_S(-C))$$

$$\to H^2(S, F) \to 0$$

and by Proposition 2.1, $h^2(S, T_S(-C)) = 1$. But $H^1(S, F) \to H^1(C, T_C)$ cannot be surjective as the locus of curves on $K$-3s has codimension at least one in $\mathcal{M}_{10}$. Thus $h^2(S, F) = 0$, $h^1(S, F) = 29$ and the codimension of the image of $H^1(S, F) \to H^1(C, T_C)$ is exactly 1. But this last map is the differential of the map $\mu$ of the Introduction, so the image of $\mu$ actually fills out a divisor.

REMARK 2.5. Let $\mu : \mathcal{P} \to \mathcal{M}_{10}$ be the rational moduli map as in the Introduction. If $\mathcal{K}$ is the closure of the image of $\mu$ and $N$ is the normal bundle of $\mathcal{K}$ in $\mathcal{M}_{10}$ then it follows from the long exact cohomology sequence of (2.4) and the analysis above that the fiber at $(C, S) \in \mathcal{P}$ (for $C$ a curve in the $K$-3 surface $S$) of the bundle $\mu^*(N)$ is the one dimensional vector space $H^2(S, T_S(-C))$.

PROPOSITION 2.6. If $C$ is a smooth codimension 4 plane section of $X$, then $\text{Corank } \Phi(C) = 4$. For every $C$ in $\mathcal{K}$, $\text{Corank } \Phi(C) \geq 4$.

Proof. By [B-E-L] (2.11), $\text{Corank } \Phi(C) = h^0(C, N_C(-1)) - g$ where $N_C$ is the normal bundle to $C$ in its canonical embedding. But by Lemma 1.2, $h^0(C, N_C(-1)) = 14$ for a smooth codimension 4 plane section of $X$. The second assertion follows by semi-continuity.

REMARKS 2.7. (a) If $C$ is any smooth codimension 4 plane section of $X$ then the Clifford index of $C$ is at least 3: if $\text{Cliff}(C) \leq 2$, $C$ is either hyperelliptic, trigonal, or a degeneration of a smooth plane sextic and in all these cases, the corank of $\Phi(C)$ is strictly greater than 4.

(b) It is possible to give (at least) two other proofs of the inequality $\text{Corank } \Phi(C) \geq 4$: if $C$ has $\text{Cliff}(C) \geq 3$, it follows from results in [B-E-L] that $h^0(N_C(-2)) = 0$ where $N_C$ is the normal bundle to $C$ in its canonical embedding. On the other hand, a smooth codimension 4 plane section $C$ of $X$ is clearly 4-extendable, so applying a theorem of Zak (described in [B-E-L]) and [B-E-L], 2.11, we find $\text{Corank } \Phi(C) \geq 4$.

(c) For a third proof, let $C$ be a smooth codimension 4 plane section of $X$ and consider the commutative diagram

$$\begin{array}{ccc}
\bigwedge^2 H^0(X, \mathcal{O}_X(1)) & \xrightarrow{\delta} & H^0(X, \Omega_X^1(2)) \\
\downarrow h & & \downarrow e \\
\bigwedge^2 H^0(C, \mathcal{O}_C(1)) & \xrightarrow{\delta} & H^0(C, \mathcal{O}_C(3))
\end{array}$$

$$\begin{array}{c}
H^0(C, \Omega_X^1(2)) \xrightarrow{f} H^0(C, \Omega_C^1(2))
\end{array}$$
Here the horizontal maps are the Wahl maps for $\mathcal{O}(1)$ and the other maps are the natural restrictions. Now $b$ is clearly surjective, so the image of $d = \Phi(C)$ is contained in the image of $f$. We claim that $f$ has corank $4$: the exact sequence of cohomology of $0 \to N_{C1x}^*) \to \Omega^1_X(2) \to \Omega^1_C(2) \to 0$ gives

\[ H^0(C, \Omega^1_X(2)|_C) \to H^0(C, \Omega^1_C(2)) \to H^1(C, N_{C1x}^*) \to H^1(C, \Omega^1_X(2)|_C) \]

and the claim follows by observing that $h^1(N_{C1x}^*) = h^1(\mathcal{O}_C(-1)^\oplus 4(2)) = 4$ and that $H^1(\Omega^1_X(2)|_C) = H^0(T_X(-1)|_C)^* = 0$ (Lemma 1.2).

**COROLLARY 2.8.** We have an inequality of divisors $\mathcal{W} \geq 4\mathcal{K}$.

**Proof.** Let $\mathcal{M} = \mathcal{M}_{10}$ denote the moduli space of smooth automorphism-free genus 10 curves over the complex numbers, $\pi: \mathcal{C} \to \mathcal{M}$ the universal curve, $\omega = \Omega^1_{\mathcal{C}/\mathcal{M}}$ the sheaf of relative differentials and $\lambda = \det(\pi_*^\#(\omega)) \in \text{Pic}(\mathcal{M})$. We have the relative Wahl map

\[ \Phi: \pi_*^2(\omega) = \pi_*^\#(\omega)^\otimes 3 \]

which is a map of bundles of rank 45; let $\mathcal{W}$ denote its degeneracy locus. By [C-H-M] the support of $\mathcal{W}$ is a proper subvariety of $\mathcal{M}$ and hence $\mathcal{W}$ is a divisor.

By Proposition 2.6, the universal Wahl map $\Phi$ has corank at least 4 at each point of $\mathcal{K}$. It follows that $\det(\Phi)$ vanishes to order at least 4 along $\mathcal{K}$. Indeed, take a small arc $\{C_t\}$ crossing $\mathcal{K}$ transversally at a general point $C_0 \in \mathcal{K}$ and apply the following observation: if $\{M_t\}$ is a one parameter family of square matrices then $\text{ord}_{t=0} \det(M_t) \geq \text{dim ker}(M_0)$; this is easily seen by diagonalizing the matrix $\{M_t\}$ over the discrete valuation ring of convergent power series in $t$.

### 3. The classes of $\mathcal{W}$ and $\mathcal{K}$

We continue to use the notations of the Introduction and Section 2. For divisors $D$ and $E$, linear equivalence will be denoted $D \sim E$. If $L$ is a line bundle, we write $D \sim L$ to mean that the line bundles $\mathcal{O}(D)$ and $L$ are isomorphic. We will show that $\mathcal{W} \sim 28\lambda$ and that $\mathcal{K} \sim 7\lambda$. The divisor $\mathcal{W} - 4\mathcal{K}$ is then linearly equivalent to zero and by Corollary 2.8 it is effective. But in the variety $\mathcal{M} = \mathcal{M}_{10}$ the only effective divisor $D$ linearly equivalent to zero is $D = 0$: since $\mathcal{M}$ has a projective compactification with boundary of codimension 2, if $D$ were not zero, there would exist a complete curve $T \subset \mathcal{M}$ not contained in $D$ and intersecting $D$; since $D \sim 0$ we have $D \cdot T = \text{deg}(\mathcal{O}(D)|_T) = 0$, a contradiction. It follows that $\mathcal{W} = 4\mathcal{K}$.

**PROPOSITION 3.1.** $\mathcal{W} \sim 28\lambda$.

**Proof.** Since $\mathcal{W}$ is the divisor of zeros of the section $\det(\Phi)$, $\mathcal{W}$ belongs to the
class $c_1(\omega^{\otimes 3}) - c_1(\int_2 \omega)$. From [Mu], 5.10, $c_1(\pi_*(\omega^{\otimes 3})) \sim 37\lambda$. By the splitting principle if $E$ is a bundle of rank $r$ then $c_1(\int_2 E) = (r - 1)c_1(E)$, so $c_1(\int_2 \omega) \sim 9$ and the result follows.

Computing the class of $\mathcal{X}$ will require some more preparation. We start with some enumerative formulas. If $f : X \to B$ is a flat family of curves, where $X$ and $B$ are smooth and complete and $\dim(B) = 1$, it follows from the Leray spectral sequence that $\chi(X, \mathcal{O}_X) = \chi(B, \mathcal{O}_B) - \chi(B, R^1f_*\mathcal{O}_X)$. Applying Riemann-Roch and duality to $E = R^1f_*\mathcal{O}_X$, we obtain $\chi(E) = \deg(E) + \text{rk}(E)\chi(B)$ and $R^1f_*\mathcal{O}_X = (f_*\omega_{X/B})^*$ so

$$\deg(\chi_{X/B}) = \chi(X, \mathcal{O}_X) - \chi(B, \mathcal{O}_B)\chi(C, \mathcal{O}_C)$$

where we write $\chi_{X/B}$ for $\det(f_*\omega_{X/B})$ and where $C$ is a general fiber of $f$.

For example, if $C \subset S$ is a smooth curve on a smooth surface which moves in a pencil, consider $f : \bar{S} \to \mathbb{P}^1$ where $\bar{S}$ is the blow-up of $S$ at the base locus of the pencil. Then $\deg(\chi_{\bar{S}}) = \chi(\bar{S}, \mathcal{O}_\bar{S}) - 1 + g_C = \chi(S, \mathcal{O}_S) - 1 + g_C$ since $\chi$ is a birational invariant. In particular, if $S$ is a K3 surface,

$$\deg(\chi) = 1 + g_C.$$ (3.2)

If $C$ is a very ample smooth curve on a smooth complete surface $S$, let $\mathcal{O} = |C|$ denote the discriminant hypersurface, consisting of singular members of the complete linear system $|C|$. If we consider a general (Lefschetz) pencil in $|C|$ and apply the Leray spectral sequence to the constant sheaf $\mathcal{C}$ this time, we may count the number of singular fibers and obtain (see [G-H], pp. 508–510 for details) $\deg(\mathcal{O}) = 4(g_C - 1) + C^2 + \chi_{\text{top}}(S)$. In particular, if $S$ is a K3 surface,

$$\deg(\mathcal{O}) = 6(g_C + 3).$$ (3.3)

**Lemma 3.4.** If $S$ is a general K3 surface of genus 10, then

(a) only finitely many smooth curves $C$ in the linear series $|\mathcal{O}_S(1)|$ have automorphisms.

(b) The linear series $|\mathcal{O}_S(1)|$ contains at most a 2 dimensional family of curves with a single node and with automorphisms.

(c) $S$ carries a Lefschetz pencil consisting entirely of curves without automorphisms.

**Proof.** (a) Consider a 19 dimensional family $\mathcal{F}$ of K3 surfaces of genus 10 in $\mathbb{P}^{10}$ which dominates $\mathcal{F}_{10}$ (see, e.g., [M] for a construction) and let $\mathcal{P}$ be the canonical $\mathbb{P}^{10}$ bundle over $\mathcal{F}$ (whose fiber at $S$ is $|\mathcal{O}_S(1)|$). Let $k$ be the dimension, for a general $S$ in $\mathcal{F}$, of the subset of $|\mathcal{O}_S(1)|$ representing smooth curves with nontrivial automorphisms. We want to show that $k \leq 0$. By the definition of $k$
there exists a subvariety $\mathcal{A} \subset \mathcal{P}$ of dimension $19 + k$ consisting of smooth curves with automorphisms, such that $\mathcal{A}$ dominates $\mathcal{F}$. Let $\mu : \mathcal{A} \to \mathcal{M}_{10}$ be the moduli map.

As $S$ is general, its Picard group is isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}_S(C)$. It then follows immediately from the main theorem of [G-L] that $S$ contains no $n$-gonal curves for $n \leq 5$. But the largest component of curves with automorphisms in $\mathcal{M}_{10}$ which are not of this type has dimension 16 and consists of curves with an involution such that the quotient has genus 3. Thus the fibers of $\mu$ are at least $k + 3$-dimensional.

On the other hand the dimension of the fibers of $\mu$ is constant in a linear series $|\mathcal{O}_S(1)|$ and generically this dimension is 3 (as follows from the proof of Proposition 2.2). Thus $k \leq 0$ as was to be shown.

(b) The argument in this case is similar, except that we work in $\Delta_0 \subset \mathcal{M}_{10}$, the boundary component of $\mathcal{M}_{10}$ representing curves of arithmetic genus 10 with one node. Here the locus of curves with non-trivial automorphisms has dimension 17, consisting of hyperelliptic curves of (geometric) genus 9 with two points conjugate under the involution identified. We find $k \leq 2$. (Perhaps a more refined analysis would improve this estimate.)

(c) This is an immediate consequence of (a) and (b).

**PROPOSITION 3.5.** $\mathcal{K} \sim 7\lambda$.

**Proof.** Fix a general $S \in \mathcal{F}_{10}$, and let $C \subset S$ be a smooth genus 10 curve. Consider a general Lefschetz pencil $l \subset |C|$. By Lemma 3.4 $\mu(l) \subset \bar{\mathcal{M}}$, where $\bar{\mathcal{M}}$ is the moduli space of stable genus 10 curves without automorphisms. The Picard group of the smooth variety $\bar{\mathcal{M}}$ is freely generated by $\lambda$ and the classes of the divisors $\Delta_0$, $\Delta_2$, $\Delta_3$, $\Delta_4$, $\Delta_5$ where for $i > 0$, $\Delta_i$ consists of stable curves with a node that separates the curve into components of genus $i$ and $10-i$, and $\Delta_0$ is the divisor of stable curves with a singular irreducible component (as follows from [A-C] Section 4 and [C] Section 1.3).

Denote $\bar{\mathcal{K}}$ the closure of $\mathcal{K}$ in $\bar{\mathcal{M}}$. Then we have a relation

$$\bar{\mathcal{K}} \sim a \cdot \lambda - b_0 \cdot \Delta_0 - b_2 \cdot \Delta_2 - b_3 \cdot \Delta_3 - b_4 \cdot \Delta_4 - b_5 \cdot \Delta_5$$

(3.6)

with $a, b_i \in \mathbb{Z}$. Now we pull-back (3.6) to $l$ in order to determine $a$. Since the surface $S$ is general, its Picard group is generated by the class of $C$ and then there are no reducible curves in $|C|$. This implies that $\Delta_i \cdot l = 0$ for $i > 0$ (notice that since $l$ is general its singular members have only nodes as singularities). From (3.3), $\Delta_0 \cdot l = 78$ (notice that $\tilde{S}$, the blow-up of $S$ along the base locus of the pencil $l$, is smooth and hence $\mu(l)$ is transverse to $\Delta_0$) and from (3.2) we obtain $\lambda \cdot l = 11$.

To find $\bar{\mathcal{K}} \cdot l = \deg \mu^*(N_{\mathcal{K}/\bar{\mathcal{M}}})|_l$, we need to compute the degree of the line bundle over $l$ with fiber $H^2(S, T_{\tilde{S}}(-C))$ for $C \in l$ (Remark 2.5). More precisely,
suppose \( l \) is spanned by \( C_0 = \{ s_0 = 0 \} \) and \( C_1 = \{ s_1 = 0 \} \) for \( s_0, s_1 \in H^0(S, L) \) (we write \( L = \mathcal{O}_S(C) \)). We have a diagram

\[
\begin{array}{ccc}
\tilde{S} & \xrightarrow{g} & S \\
\downarrow f & & \\
\mathbb{P}^1 & & \\
\end{array}
\]

and \( \tilde{S} = \{(x, t_0, t_1)|t_0 \cdot s_0(x) + t_1 \cdot s_1(x) = 0\} \subset S \times \mathbb{P}^1 \) is the zero set of a section of \( f^* \mathcal{O}_\mathbb{P}^1(1) \otimes g^* L \). Then

\[
\tilde{X}.l = \deg R^2 f_*(T_S \otimes \mathcal{O}(\tilde{S})) \\
= \deg R^2 f_*(g^* T_S \otimes g^*(L^*) \otimes f^* \mathcal{O}_\mathbb{P}(-1)) \\
= \deg R^2 f_*(g^* T_S \otimes \mathcal{O}(\tilde{S})) \otimes \mathcal{O}_\mathbb{P}(-1)
\]

which equals (by base change and cohomology) \( \deg H^2(S, T_S \otimes L^*) \otimes \mathcal{O}_\mathbb{P}(-1) = -1 \).

Combining these results we obtain the relation

\[
-1 = 11a - 78b._\tag{3.7}
\]

The integral solutions to this equation area \( a = 7 + 78k, b = 1 + 11k \) for \( k \in \mathbb{Z} \).

We know (2.8) that \( \mathcal{W} \geq 4\mathcal{X} \) and (3.1) that \( \mathcal{W} \sim 28\mathcal{L} \). Hence \( 0 \leq a \leq 7 \) and so \( k = 0, a = 7, \) as desired.

As explained at the beginning of this section, the linear equivalence \( \mathcal{W} \sim 4\mathcal{X} \) together with the inequality \( \mathcal{W} \geq 4\mathcal{X} \) implies \( \mathcal{W} = 4\mathcal{X} \); this completes the proof of the main theorem.

REMARK 3.8. Note that our computation of the class of \( \mathcal{X} \) in \( \text{Pic}(\mathcal{M}) \) uses the inequality \( a \leq 7 \) (coming from Corollary 2.8 and Proposition 3.1) and the equality 3.7, together with the fact that the coefficients \( a \) and \( b \) in 3.7 are integral. This integrality is why we work in the smooth variety \( \mathcal{M}_{10} \). A more traditional approach, which we were unable to carry out, would proceed by writing down several pencils of genus 10 curves, computing their intersections with \( \tilde{X}, \lambda, \) and the \( \Delta_i \), and then solving the resulting system of linear equations over \( \mathbb{Q} \).

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