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MARIUSZ KORAS

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A characterization of $\mathbb{A}^2/\mathbb{Z}_a$

MARIUSZ KORAS
Institute of Mathematics, Warsaw University, Banacha 2, 00-913 Warszawa

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The purpose of this paper is to prove the following:

THEOREM. *Let S' be an affine, irreducible, simply connected surface with exactly one singular point q and assume the analytic type of q is that of the origin in $\mathbb{A}^2/\mathbb{Z}_a$, $a > 1$. Suppose also that $\bar{k}(S') = -\infty$, $\mathrm{Pic}(S' - q) \cong \mathbb{Z}_a$, and $\bar{k}(S' - q) < 2$. Then S' is isomorphic to $\mathbb{A}^2/\mathbb{Z}_a$.*

The main application of the theorem is to the linearization problem. Suppose that $\mathbb{C}^*$ acts on three dimensional affine space $\mathbb{A}^3$ with exactly one fixed point p. If the weights of the action at p are all positive or all negative then the action is linearizable, [2]. If the weights are $-a, b, c$ with $a, b, c > 0$, where a, b, c are pairwise relatively prime then, as is proved in [6], the action is linearizable provided the quotient space $\mathbb{A}^3/\mathbb{C}^*$ is isomorphic to $\mathbb{A}^2/\mathbb{Z}_a$. It is easy to prove that $S' = \mathbb{A}^3/\mathbb{C}^*$ satisfies all the assumptions of the theorem except perhaps $\bar{k}(S' - q) < 2$. So far we are not able to prove that this assumption is satisfied in general, we can only prove it for a wide class of actions, so-called "good" actions (see [6]).

In the proof of the theorem we consider three cases according to the value of $\bar{k}(S' - q)$.

Similar results appear to have been obtained recently by R. V. Gurjar and M. Miyanishi.

Section 1

In this section we recall some facts about $\mathbb{P}^1$-rulings. All facts below can be found in [3].

Let S be a smooth surface. Let $\bar{S}$ be a smooth compactification of S with $D = \bar{S} - S$ being a divisor with normal crossings as the only singularities. (Such a divisor we call a NC-divisor.) By $\tilde{H}_i(\bar{S}, D; Z)$ and $\hat{H}_i(\bar{S}, D; Z)$ we denote $coker(H_i(D) \to H_i(\bar{S}))$ and $ker(H_{i-1}(D) \to H_{i-1}(\bar{S}))$ respectively. By Lefschetz duality $H_j(\bar{S}, D; \mathbb{Z}) \cong H^{4-j}(S; \mathbb{Z})$. Therefore $\tilde{H}_i(\bar{S}, D; \mathbb{Z})$ and $\hat{H}_i(\bar{S}, D; \mathbb{Z})$ correspond to a subgroup and a quotient group of $H^{4-i}(S; \mathbb{Z})$ respectively.

DEFINITION. The subgroup (resp. quotient group) of $H^i(S;\mathbb{Z})$ corresponding to $\tilde{H}_{4-i}(\bar{S}, D; \mathbb{Z})$ (resp. $\hat{H}_{4-i}(\bar{S}, D; \mathbb{Z})$) will be denoted by $\hat{H}^i(S;\mathbb{Z})$ (resp. $\tilde{H}^i(S;\mathbb{Z})$). Their ranks will be denoted by $\hat{b}_i(S)$ (resp. $\tilde{b}_i(S)$).

From the exact sequence

$$0 \to \tilde{H}_j(\bar{S}, D; \mathbb{Z}) \to H_j(\bar{S}, D; \mathbb{Z}) \to \hat{H}_j(\bar{S}, D; \mathbb{Z}) \to 0$$

we obtain $b_i(S) = \tilde{b}_i(S) + \hat{b}_i(S)$.

Let $f\colon \bar{S} \to C$ be a $\mathbb{P}^1$-ruling. C is a smooth curve. An irreducible component Y of a fibre F is called a D-component if $Y \subset D$, otherwise it is called a S-component. The number of S-components of F we denote by $\sigma(F)$. Of course $\sigma(F) = 1$ for general F. The S-multipliciy of F is defined to be the greatest common divisor of the multiplicities of the S-components of F and is denoted by $\mu(F)$. When $\sigma(F) = 0$ we put $\mu(F) = \infty$. A component Y of D is called horizontal if $f(Y) = C$. A fibre F is said to be D-minimal if $F \cap D$ does not contain an exceptional curve. Let h be the number of horizontal components of D. Let $\Sigma = \Sigma_{\sigma(F)>1}(\sigma(F) - 1)$. Let v be the number of fibres with $\sigma = 0$. Then

$$h - \Sigma + v - 2 = \tilde{b}_1(S) - \hat{b}_2(S). \tag{1.1}$$

Let $F_1, \dots, F_k$ be all fibres with $\mu > 1$. Then

1.2. $\pi_1(S) = 0$ only if

(a) $k \leqslant 1$ or
(b) $k = 2$, $\mu_1 \neq \infty$, $\mu_2 \neq \infty$ and $\mathrm{GCD}(\mu_1, \mu_2) = 1$.

Now consider $\mathbb{A}^1_*$-rulings.

Let $f\colon \bar{S} \to C$ be a $\mathbb{P}^1$-ruling. f is called a $\mathbb{A}^1_*$-ruling of S iff $DF = 2$, where F is a fibre of f. If there are two distinct horizontal components of D the ruling is called a sandwich. If there is only one horizontal component the ruling is called a gyoza (we follow the terminology of Fujita). A connected component of $F \cap D$ is called a rivet if it meets horizontal components of D at more than one point or if it is a node of $H_1 \cup H_2$, the union of horizontal components of D. Let ρ be the number of rivets contained in fibres of f. Let $\varepsilon(t)$ be the function defined as follows:

$$\varepsilon(t) = \begin{cases} 0 & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{cases}.$$

1.3. Suppose that f is a gyoza. Then

$$\tilde{b}_1(S) = v - \varepsilon(v),\ \hat{b}_2(S) = \Sigma + 1 - \varepsilon(v),$$

$$\tilde{b}_2(S) = \rho + 2 + 2((g(H) - g(C))$$

where H is the horizontal component and g stands for the genus of a curve.

1.4. Suppose f is a sandwich. Then

$$\tilde{b}_1(S) - \tilde{b}_2(S) = \nu - \Sigma,\ \tilde{b}_1(S) = \nu - \varepsilon(\nu)$$

or

$$\tilde{b}_1(S) = \nu - \varepsilon(\nu) + 1,\ \hat{b}_2(S) = \Sigma - \varepsilon(\nu)$$

or

$$\hat{b}_2 = \Sigma - \varepsilon(\nu) + 1,\ \tilde{b}_2(S) = 2g(C) + \rho - \varepsilon(\rho).$$

Let $\gamma(F)$ be the number of connected components of $F \cap D$ which do not meet the horizontal components of D. Let Γ be the sum of the $\gamma(F)$ for all the fibres F of f. Then $\tilde{b}_3(S) = \Gamma$ if f is a gyoza, $\tilde{b}_3(S) = \Gamma + 1 - \varepsilon(\rho)$ if f is a sandwich.

Section 2

Let things be as in the statement of the Theorem.

2.1. Since $\mathrm{Pic}(S' - q) \cong \mathbb{Z}_a$ we may apply "the covering trick" (see [1], (§17)) and infer that there exists a X' and an unramified covering $\pi'\colon X' \to S' - q$ of degree a. It follows from the construction of X' that $\mathbb{Z}_a$ acts transitively on fibres of π'. Let B' be an open neighbourhood of q of the form $B/\mathbb{Z}_a$ where B is a ball around 0 in $\mathbb{A}^2$. We will show that $\pi'^{-1}(B' - q)$ is isomorphic to $B - 0$.

Since $B - 0$ is simply connected there exists a continous mapping φ such that the following diagram is commutative

$$\begin{tikzcd} X' \arrow[d, "\pi'"'] & B - 0 \arrow[l, hook', "\varphi"'] \arrow[d, "p"] \\ S' & B' - q \arrow[l, hook'] \end{tikzcd}$$

where p is the quotient mapping $B - 0 \to B' - q = B - 0/\mathbb{Z}_a$. It is enough to show that φ is injective. Let G be the image of $\pi_1(B' - q) \cong \mathbb{Z}_a$ in $\pi_1(S' - q)$. It is proved in lemma 2.6 that $\pi_1(S' - q)$ is normally generated by G. Since $S' - q$ admits a Galois covering of degree a it follows easily that $H_1(S' - q, \mathbb{Z}) = \mathbb{Z}_a$ and that $G \cap H = \{1\}$ where H denotes the commutator subgroup of $\pi_1(S' - q)$. It follows also that the mapping $\pi_1(B' - q) \to \pi_1(S' - q)$ is injective. Suppose

$\varphi(x) = \varphi(y) = z$ for $x, y \in B - 0$. $\pi_1(S' - q)$ acts in the standard way on X'. Let $K \subset \pi_1(S' - q)$ be the stabilizer of z. Then $\pi_1(S' - q)/K$ is isomorphic to the automorphism group of the covering π which is isomorphic to $\mathbb{Z}_a$. Hence $K = H$. Take a path l joining x and y. Then $p(l)$ is a loop in $B' - q$. Let $\alpha \in G$ be the corresponding element. α lifts to a loop $\varphi(l)$ passing through z. This implies that the automorphism of the covering π corresponding to α fixes z, hence that $\alpha \in K \cap H$. But $K \cap H = \{1\}$, hence $\alpha = 1$ in G. Then $p(l) = 1$ in $\pi_1(B' - q)$. But then then l is a loop, hence $x = y$.

Set $X = X' \cup \{0\}$ and define $\pi: X \to S'$ by $\pi(x) = \pi'(x)$ for $x \neq 0$ and $\pi(0) = q$. X is a smooth analytic surface. We'll show that X has a structure of an affine algebraic surface.

For an algebraic variety Y let $\mathbb{C}[Y]$ denote the algebra of regular functions on Y. S' is normal, hence $S' - q \hookrightarrow S'$ induces isomorphism $\mathbb{C}[S'] \to \mathbb{C}[S' - q]$. Hence $\mathbb{C}[S' - q]$ is a finitely generated $\mathbb{C}$-algebra. Let Z be an affine normal variety such that $\mathbb{C}[Z] \approx \mathbb{C}[X']$. Consider the diagram

$$\begin{tikzcd} X' \arrow[r, "\psi"] \arrow[d, "\pi"'] & Z \arrow[d, "r"] \\ S' - q \arrow[r] & S' \end{tikzcd}$$

where ψ is induced by $\mathbb{C}[Z] \xrightarrow{\sim} \mathbb{C}[X']$ and r is induced by $\mathbb{C}[S'] \simeq \mathbb{C}[S' - q] \hookrightarrow \mathbb{C}[X'] = \mathbb{C}[Z]$. $\psi: X' \to r^{-1}(S' - q)$ is an isomorphism. ψ extends to an analytic mapping from X to Z since X is smooth at 0. Also $\psi^{-1}: r^{-1}(S' - q) \to X'$ extends to an analytic mapping from Z to X since Z is normal. Hence Z is analytically isomorphic to X.

Our goal is to show that $X \cong \mathbb{A}^2$.

Let S be a minimal desingularization of S'. It is well known that the exceptional divisor E in S is a rational linear chain, i.e. $E = E_1 + \cdots + E_r$ where $E_i \cong \mathbb{P}^1$, $i = 1, \ldots, r$, and $E_i E_{i+1} = 1$, $1 \leqslant i \leqslant r - 1$, $E_i E_j = 0$ if $|i - j| \geqslant 2$.

2.2. LEMMA. $\mathrm{Pic}(S) \cong \mathbb{Z}^r$ and is generated by $C, E_1, \ldots, E_{r-1}$ where C is an arbitrary curve such that $CE_1 = 1$, $CE_j = 0$, $1 < j \leqslant r$.

Proof. We keep the notations of 2.1. Let x, y be local parameters at 0 on X which are semi-invariant with respect to the $\mathbb{Z}_a$-action on X. Let L_x (resp. L_y) be the proper transform on S of the curve $\pi(x = 0)$ (resp. $\pi(y = 0)$). It is well known that L_x (resp. L_y) meets a terminal component of E, say E_1 (resp. E_r), transversally and does not meet any other component of E. Moreover it is easy to show that the divisors $\pi(x = 0)$ and $\pi(y = 0)$ are of order a in $\mathrm{Pic}(S' - q) = \mathrm{Pic}(S - E)$. The divisor of x^a on S is of the form $(x^a) = \Sigma_{i=1}^{r-1} a_i E_i + E_r + aL_x$. Since L_x is a generator of $\mathrm{Pic}(S - E)$ we infer that $\mathrm{Pic}(S)$ is generated by $L_x, E_1, \ldots, E_{r-1}$. It is easy to see that there are no relations between these generators. Let C be a curve such that $CE_1 = 1, CE_i = 0$ for $i > 1$. Then $C \sim bL_x + \Sigma_{i=1}^{r-1} b_iE_i$ for some integers $b, b_1, \ldots, b_{r-1}$. We obtain $b_{r-1} = CE_r = 0$, $b_{r-2} = CE_{r-1} = 0$ and so on. Hence $C = bL_x$ in Pic(S). Then $1 = CE_1 = b(L_xE_1) = b$ and $C = L_x$ in Pic(S).

By symmetry we obtain that Pic(S) is freely generated by $C', E_2, \ldots, E_r$ where $C'E_r = 1$, $C'E_i = 0$ for $1 \leqslant i \leqslant r-1$.

2.3. LEMMA. *S is simply connected.*

Proof. Apply Van Kampen's theorem to the union $(S' - q) \cup B' = S'$ where $B' = B/\mathbb{Z}_a$ and B is a small ball around 0 in $\mathbb{A}^2$. B' gives rise to a neighbourhood B'' of E in S. $\pi_1(B') \cong \pi_1(B'') = 0$. Now apply Van Kampen's theorem to $(S - E) \cup B'' = S$. Since $S - E = S' - q$ and $(S - E) \cap B'' = (S' - q) \cap B'$ we obtain $\pi_1(S) = \pi_1(S') = 0$.

Let $\bar{S}$ be a smooth compactification of S with $D = \bar{S} - S$ being a NC-divisor. $\bar{S}$ is simply connected since S is. $\bar{k}(S) = -\infty$ implies $\bar{k}(\bar{S}) = -\infty$. Therefore $\bar{S}$ is ruled. The base curve of any ruling must be simply connected, hence is isomorphic to $\mathbb{P}^1$. Thus

2.4. $\bar{S}$ (and S) is rational.

2.5. LEMMA. *Invertible regular functions on S are constant.*

Proof. Let $A(S)^*$ denote the group of units on S. There exists exact sequence $0 \to A(S)^*/\mathbb{C}^* \to H^1(S; \mathbb{Z})$ ([3], Prop. 1.18). S is simply connected, hence $A(S)^* = \mathbb{C}^*$.

2.6. LEMMA. *$\pi_1(S' - q) = \pi_1(S - E)$ is normally generated by $\mathbb{Z}_a$.*

Proof. Apply Van Kampen's theorem to the union $(S' - q) \cup B' = S'$ where $B' = B/\mathbb{Z}_a$ and B is a small ball around $0 \in \mathbb{A}^2$. $\pi_1(B' - q) = \mathbb{Z}_a$, $\pi_1(S') = 0$. The lemma follows.

2.7. COROLLARY. $b_1(S - E) = 0$.

2.8. *COROLLARY. If $\pi_1(S' - q)$ is abelian then it is isomorphic to $\mathbb{Z}_a$.*

Proof. By lemma 2.6, $\pi_1(S' - q)$ is a quotient group of $\mathbb{Z}_a$. On the other hand $S' - q$ admits the covering $X' - 0 \to S' - q$ of degree a, hence contains a subgroup of index a.

2.9. LEMMA. $b_1(S) = 0$, $\hat{b}_2(S) = r$, $b_2(S) = r$, $\tilde{b}_2(S) = 0$, $\tilde{b}_3(S) = 0$.

Proof. $b_1(S) = 0$ follows from $\pi_1(S) = 0$. Let $\bar{S}$ be a NC-compactification of S as above. $H^2(\bar{S}; \mathbb{Z}) \cong \text{Pic}(\bar{S})$ since $\bar{S}$ is rational. Hence $H^2(\bar{S}; \mathbb{Q}) \cong \text{Pic}(\bar{S}) \otimes \mathbb{Q}$. Pic($\bar{S}$) is freely generated by the irreducible components of D and free generators of Pic(S), by 2.2 and 2.5. Hence $H^2(\bar{S}; \mathbb{Q})$ is freely generated by the components of D and $E_1, \ldots, E_r$. Consider the exact sequence of homology groups with rational coefficients

$$H_3(\bar{S}, D \cup E) \to H_2(D \cup E) \to H_2(\bar{S}) \to H_2(\bar{S}, D \cup E) \to H_1(D \cup E) \to H_1(\bar{S}).$$

Since

$$H_3(\bar{S}, D \cup E) \cong H^1(S - E) \cong H^1(S' - q) = 0,$$

$$H_2(\bar{S}, D \cup E) \cong H^2(S - E),\ H_1(D \cup E) \cong H_1(D)$$

and

$$\operatorname{rank} H_2(D \cup E) = \operatorname{rank} H_2(\bar{S}),$$

we get

$$H^2(S - E) \cong 0 \quad \text{and} \quad H_1(D) = 0.$$

The latter implies that D is a rational tree, hence $H_1(D; \mathbb{Z}) = 0$. From the sequences

$$H_2(D; \mathbb{Z}) \to H_2(\bar{S}; \mathbb{Z}) \to H_2(\bar{S}, D; \mathbb{Z}) \to 0$$

$$\mathscr{L}(D) \to \operatorname{Pic}(\bar{S}) \to \operatorname{Pic}(S) \to 0$$

where $\mathfrak{L}(D)$ is the free abelian group generated by the irreducible components of D, using the natural isomorphisms

$$H_2(D; \mathbb{Z}) \cong \mathfrak{L}(D),\ H_2(\bar{S}; \mathbb{Z}) \cong H^2(\bar{S}; \mathbb{Z}) \cong \operatorname{Pic}(\bar{S}),\ H_2(\bar{S}, D; \mathbb{Z}) \simeq H^2(S; \mathbb{Z}),$$

we obtain $H^2(S; \mathbb{Z}) \cong \operatorname{Pic}(S)$. Hence $b_2(S) = r.\ \hat{b}_2(S) = r$ follows easily. $\tilde{b}_3(S) = 0$ follows from the fact that D is connected as a boundary divisor of the affine surface S'.

2.10. It follows from 2.9. and Lefschetz duality that $b_2(S'-q)=0$, $b_3(S'-q)=1$, $b_4(S'-q)=0$. Thus $\chi(S'-q)=0$ and $\chi(X-0)=0$. Therefore $\chi(X) = 1$. In particular, $b_1(X) = b_2(X)$.

Section 3

In this section we will prove the Theorem in case $\bar{k}(S' - q) = -\infty$.

Let $\bar{S}$ be a NC-compactification of S. Let $D = \bar{S} - S$. Assume that $S' - q$ is not $\mathbb{A}^1$-ruled, i.e. that $S' - q$ doesn't contain a cylinder $C \times \mathbb{A}^1$ where C is a curve. Then, by [7], there exists $p: \bar{S} \to Y$ such that:

(i) Y is a smooth surface and p is birational.
(ii) Let $B = p_*(D \cup E)$. Then $Y - B$ contains an open subset U which is $\mathbb{A}^1_*$-ruled over $\mathbb{P}^1$. More precisely, there exists a surjective map $g: U \to \mathbb{P}^1$

such that each fibre of g is irreducible and isomorphic to $\mathbb{A}^1_*$ and there are exactly three multiple fibres g_1, g_2, g_3 and the sequence of multiplicities is one of the following: $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. Such a fibration is called a Platonic fibration.

It is known [7] that the fundamental group of a Platonic fibration is finite and its universal covering is isomorphic to $\mathbb{A}^2 - 0$.

Let things be as above. Then $p\colon p^{-1}(U) \to U$ is an isomorphism since it is a birational map and $S' - q$ does not contain a compact curve. Thus we may find an open $V \subset S' - q$ which has a structure of Platonic fibration. There exists a proper unramified map $\alpha\colon \mathbb{A}^2 - 0 \to V$. Assume that $\dim(S' - q) - V \geqslant 1$. Then, since $\mathrm{Pic}(S' - q)$ is a torsion group, there exists a nontrivial invertible function on V. Such a function would induce a nontrivial invertible function on $\mathbb{A}^2 - 0$. Therefore $\dim(S' - q) - V = 0$ or $S' - q = V$. The covering map $\alpha\colon \mathbb{A}^2 - 0 \to V \subset S' - q$ extends to a finite map $\alpha\colon \mathbb{A}^2 - \alpha(\mathbb{A}^2) \subset S'$. The image of α is affine since it is isomorphic to $\mathbb{A}^2/\pi_1(V)$. It follows easily that $\alpha(\mathbb{A}^2) = S'$ and $\alpha(0) = q$. Consider the following diagram

$$\begin{array}{ccc} & & X - 0 \\ & & \downarrow \\ \alpha\colon \mathbb{A}^2 - 0 & \to & S' - q \end{array}$$

$(\mathbb{A}^2 - 0) \times_{S'-q} (X - 0)$ must split into components, each of them isomorphic to $\mathbb{A}^2 - 0$. We get a finite map $\mathbb{A}^2 - 0 \to X - 0$ which extends to a map $\beta\colon \mathbb{A}^2 \to X$. β is unramified over $X - 0$ and totally ramified over 0. Since X is smooth it follows that $\deg(\beta) = 1$. Therefore $X \cong \mathbb{A}^2$ and $S' \cong \mathbb{A}^2/\mathbb{Z}_a$, which implies that $S' - q$ is $\mathbb{A}^1$-ruled.

We have proved that if $\bar{k}(S' - q) = -\infty$ then $S' - q$ is $\mathbb{A}^1$-ruled.

Let $f\colon \bar{S} \to \mathbb{P}^1$ be a $\mathbb{A}^1$-ruling of S which extends a $\mathbb{A}^1$-ruling of $S' - q = S - E$. Then E is contained in a fibre F_E. By (1.1) we have $\Sigma = r - 1 + v$. The fibre F_E must contain at least one S-component different from the E_i. Hence $\sigma(F_E) \geqslant r + 1$ and $\Sigma \geqslant r$. By (1.2), $v \leqslant 1$. Thus $\sigma(F_E) = r + 1$, $\Sigma = r$ and $v = 1$. Hence f has one fibre F_0 with $\sigma(F_0) = 0$ and the fibre F_E contains E and one more component C. By (1.2) we infer that F_0 is the unique multiple fibre. Hence $S - F_E$ contains $\mathbb{A}^1_* \times \mathbb{A}^1$. Hence $\pi_1(S' - q)$ is abelian and $\pi_1(S' - q) \cong \mathbb{Z}_a$ and $\pi_1(X) = 0$. Since $b_1(X) = b_2(X) = 0$, $\mathrm{Pic}(X) = 0$ and invertible functions on X are constant ([3], Prop. 2.5). X contains a covering of $\mathbb{A}^1_* \times \mathbb{A}^1$, which is isomorphic to $\mathbb{A}^1_* \times \mathbb{A}^1$. One then sees that the complement of $\mathbb{A}^1_* \times \mathbb{A}^1$ in X consists of one curve L. If $h = 0$ is an equation of L, then $h\colon\to \mathbb{A}^1$ gives a structure of $\mathbb{A}^1$-fibration. Hence $X \cong \mathbb{A}^2$.

One can also argue as follows: $\bar{k}(X) = \bar{k}(X - 0) = \bar{k}(S - E) = -\infty$,

$\operatorname{Pic}(X) = 0$ and there are no nonconstant invertible functions on X. By [8], $X \cong \mathbb{A}^2$.

Section 4

In this section we recall some facts concerning NC-divisors on smooth compact surfaces. We follow the terminology of Fujita [3], see also Tsunoda [10].

Let D be a reduced NC-divisor on a smooth surface M. Assume that each component of D is isomorphic to $\mathbb{P}^1$. Let Y be an irreducible component of D. We put $\beta_D(Y) = Y(D - Y)$ and we will call this number the branching number of Y. Y is called a tip of D if $\beta_D(Y) = 1$. A sequence $C_1, \dots, C_r$ of irreducible components of D is called a twig of D if $\beta_D(C_1) = 1$, $\beta_D(C_j) = 2$ for $2 \leqslant j \leqslant r$. C_1 is called the tip of this twig, T say. Since $\beta_D(C_r) = 2$ there exists a component C of D, not in T, such that $C_r C = 1$. If C is a tip of D then $T' = T + C$ is a connected component of D and will be called a club of D. A component Y such that $\beta_D(Y) = 0$ also will be called a club of D. A component D_1 such that $\beta_D(D_1) \geqslant 3$ will be called a branching component of D. Let $T = C_1 + \cdots + C_r$ be a maximal twig of D. T is called a contractible twig if the intersection matrix $[C_i C_j]$ is negative definite. In this case let $Bk(T) = \Sigma_{i=1}^r a_i C_i$ be the $\mathbb{Q}$-divisor such that $Bk(T)C_1 = -1$, $Bk(T)C_j = 0$ for $j \geqslant 2$. $Bk(T)$ is called the bark of T. For a contractible club T of D, $T = C_1 + \cdots + C_r + C$, its bark is defined by $Bk(T)C_1 = Bk(T)C = -1$, $Bk(T)C_j = 0$ for $2 \leqslant j \leqslant r$. For an isolated Y its bark is defined to be $(-2/Y^2)Y$. In all cases we have $Bk(T)C = (K + D)C$ for any component C of T.

Let $T = C_1 + \cdots + C_r$ be a twig such that $C_i^2 \leqslant -2$ for $1 \leqslant i \leqslant r$. Such a twig will be called an admissible twig. We define $d(T) = \det[-C_i C_j]_{1 \leqslant i,j \leqslant r}$. Let $\bar{T} = C_2 + \cdots + C_r$. We define $e(T) = d(\bar{T})/d(T)$. Then $d(\bar{T})$ and $d(T)$ are relatively prime integers and $d(\bar{T}) < d(T)$.

4.1. PROPOSITION ([3], Cor. 3.8). *$T \to e(T)$ defines a* 1-1 *correspondence between all the admissible twigs and all rational numbers in the interval* (0, 1).

Let $T = C_1 + \cdots + C_r$ be an admissible twig of D. Let $Bk(T) = \Sigma\, n_i C_i$. Then $n_1 = e(T)$, $n_r = d(T)^{-1}$, $(Bk(T))^2 = -n_1 = -e(T)$, $0 < n_i < 1$ for $1 \leqslant i \leqslant r$. If T is an admissible club of D then $0 < n_i < 1$ except in the case in which $C_i^2 = -2$ for every i. (Then $Bk(T) = T$). Also $n_1 = e(T) + d(T)^{-1}$. By definition of $Bk(T)$ we have $(Bk(T))^2 = -n_1 - n_r$.

Section 5

Let things be as in the Theorem.

5.1. PROPOSITION. *If $\bar{k}(S' - q) \geqslant 0$, then $S' - q$ does not contain an open U which is $\mathbb{A}^1_*$-ruled.*

Proof. Assume that there exists U open in $S' - q$ and a map $f: U \to \mathbb{P}^1$ for which a general fibre is isomorphic to $\mathbb{A}^1_*$. f induces a rational map $\hat{f}: X \to \mathbb{P}^1$. Suppose that $\hat{f}$ is not defined at some $x \in X$. Let $\beta: \tilde{X} \to X$ be a modification of X such that $\hat{f} \circ \beta$ is defined everywhere on $\tilde{X}$. A general fibre of $\hat{f} \circ \beta$ contains $\mathbb{A}^1$. Hence $\bar{k}(\tilde{X}) = \bar{k}(X) = -\infty$, which implies $\bar{k}(S' - q) = -\infty$.

So $\hat{f}$ is defined on X, hence f is defined on S and E is contained in a fibre of f. f extends to a $\mathbb{P}^1$-ruling of $\bar{S}$, some suitable compactification of S.

We consider two cases:

Case I. $f: \bar{S} \to \mathbb{P}^1$ is a gyoza.

In this case, by (1.3), $v = \varepsilon(v)$, $r = \Sigma + 1 - \varepsilon(v)$, $0 = \rho$. Let F_E be a fibre containing E. Then $\sigma(F_E) \geqslant r + 1$. Hence $\Sigma \geqslant r$ and $\varepsilon(v) \geqslant 1$. Therefore $\varepsilon(v) = v = 1$ and $\Sigma = r$. There is one fibre F_0 with $\sigma(F_0) = 0$ and $\sigma(F_E) = r + 1$. Let H be the horizontal component of f.

5.2. LEMMA. *Let C be the S-component of F_E not contained in E. Then C meets a terminal component of E.*

Proof. Assume that $CE_i = 1$, $1 < i < r$. Suppose that L is an exceptional curve in $F_E \cap D$. If $\beta_D(L) \leqslant 2$ we may contract L. Therefore we may assume that $\beta_D(L) \geqslant 3$. Since $\beta_{F_E}(L) \leqslant 2$, H meets L. Thus $HL = 1$, otherwise H meets L in two distinct points (D is a NC-divisor) and there would be a loop in D. The multiplicity of L is equal to 2, otherwise $\beta_{F_E}(L) = 1$ and $\beta_D(L) = 2$. L meets two D-components D_1, D_2 of F_E. D_1, D_2 have multiplicity 1. Also C does not meet H. L is the unique exceptional curve in $F_E \cap D$.

Let $p: \bar{S} \to p(\bar{S})$ be the blowing down of L. Then $p(H)p(D_1) = p(H)p(D_2) = 1$. Both $p(D_1)$ and $p(D_2)$ have multiplicity 1 with respect to the induced ruling. Assume that $p(D_1)$ is exceptional. Let $q: \bar{S} \to q(\bar{S})$ be the composition of the contractions of first L and then $p(D_1)$. $q(H)$ has contact of order 2 with $q(D_2)$. It is known that by successive blowings down we may contract the fibre to a $\mathbb{P}^1$. $q(D_2)$ cannot be contracted during this process. Also the curve E_i cannot be contracted since at each stage it meets three other components of the fibre. Therefore $p(D_i)$, $i = 1, 2$, is not exceptional. Hence $p(C)$ is the unique exceptional curve in $p(F_E)$. $p(C)$ meets exactly one component D_0 of $p(D) \cap p(F_E)$. Let $D_0 + \cdots + D_s$ be the maximal linear chain in $p(D) \cap p(F_E)$ such that $\beta_{p(F_E)}(D_i) = 2$ for $i = 0, \ldots, s - 1$. We shrink successively $p(C)$, $D_0, \ldots, D_{s-1}$. If $\beta_{p(F_E)}(D_s) \geqslant 3$ then there is no possibility of shrinking D_s and E_i. Hence $p(D) \cap p(F_E)$, is a linear chain. After shrinking $p(C)$ and then $p(D) \cap p(F_E)$, the image of H has contact of order 2 with the image of E_i, which therefore has multiplicity 1. But this is impossible since it is exceptional and meets two other components of the fibre.

Thus we may assume that there is no exceptional curve in $D \cap F_E$, i.e. C is the unique such curve in the fibre. As above we show that $D \cap F_E$ is a linear chain and that the multiplicity of E_i is $\geqslant 2$. The chain is obtained by successive blowings up performed over E_i. Hence the multiplicity of every component of $D \cap F_E$ is at least 2. Therefore H meets one of these components transversally. This must be the terminal component. Otherwise, after contraction of $D \cap F_E$, the image of H would have contact of order 2 with E_i, which implies $mult(E_i) = 1$.

It is clear that H meets F_0 at one point, otherwise there would be a loop in D.

Consider $f: H \to \mathbb{P}^1$. From the Hurwitz formula we infer that f has exactly two ramification points. It follows that H meets every fibre different from F_E, F_0 in two distinct points.

Suppose that F_1 is a singular fibre different from F_E, F_0. F_1 contains exactly one S-component G and the multiplicity of G is 1. Indeed, the surface $\tilde{S} = S \cup (H - D)$ is simply connected, and thus by (1.2) every fibre has $\tilde{S}$-multiplicity equal to 1. G cannot be the only exceptional curve in F_1. Let L be an exceptional curve in $F_1 \cap D$. We may assume that H meets L. H meets L transversally at one point since D is a NC-divisor and there are no loops in D. The multiplicity of L is 2 otherwise we could shrink L. Therefore $HL = 1$ and H meets F_1 at one point; contradiction.

Therefore the only singular fibres are F_0, F_E. Thus H is contained in some rational maximal twig of D. Hence by ([3], (6.13)) H is contained in the negative part $(K + D + E)^-$ of the Zarisski decomposition of the divisor $K + D + E$. Hence $F(K + D + E)^- > 0$, where F is the general fibre. But $F(K + D + E)^+ \geqslant 0$ and $F(K + D + E) = 0$; contradiction. The lemma is proved.

We go back to the proof of 5.1. in case I. By lemma 5.2. C meets a terminal component of E, say E_1. Then, by (2.2), $C, E_1, \ldots, E_{r-1}$ form a basis of Pic(S). Let L be a section of the ruling f. Then, in Pic($\bar{S}$), $L = xH +$ (combination of prime divisors contained in fibres of f). Taking intersection index of both sides with a fibre F we get $1 = 2x$; contradiction.

Now we consider

Case II. $f: \bar{S} \to \mathbb{P}^1$ is a sandwich.

By (1.4.) there are two possibilities.

IIa. $\nu = 1, \Sigma = r + 1$,

IIb. $\nu = 0, \Sigma = r, \delta = 1$.

Let H_1, H_2 be the horizontal components of D.

We contract all exceptional D-components in singular fibres and assume that f is D-minimal. Assume IIa. There exists a fibre F_0 such that $\sigma(F_0) = 0$; E is contained in a fibre F_E. Suppose that F_E contains only one S-component C not contained in E. Then C is the unique exceptional curve in F_E. This implies that $F_E \cap D$ is connected. Neither H_1 nor H_2 meets C, otherwise mult $C = 1$ and there would be another exceptional curve in F_E. Therefore both H_1, H_2 meet $F_E \cap D$ and we have a loop in D (since H_1, H_2 meet F_0). Thus $\sigma(F_E) = r + 2$.

Suppose that there exists a singular fibre F_1 different from F_0 and F_E. Then the unique S-component of F_1 is the only exceptional curve in F_1. But the multiplicity of this component is 1 by (1.2). This implies that F_1 contains another exceptional curve. So any fibre different from F_0 and F_E is isomorphic to $\mathbb{P}^1$ and the horizontal components meet it in two different points, since otherwise there would be a loop in D. Hence $S - F_E \subset S - E$ contains $\mathbb{A}^1_* \times \mathbb{A}^1_*$. Again, $\pi_1(S - E)$ is abelian and, by (2.8), $\pi_1(S - E) \simeq \mathbb{Z}_a$ and X is simply connected. $H^2(X; \mathbb{Z}) = 0 = \mathrm{Pic}(X)$, invertible functions on X are constant and X contains $\mathbb{A}^1_* \times \mathbb{A}^1_*$.

Suppose that the complement of $\mathbb{A}^1_* \times \mathbb{A}^1_*$ in X contains 3 curves. Let h_1, h_2, h_3 be the equations of these curves. Then h_i, $i = 1, 2, 3$, is invertible on $\mathbb{A}^1_* \times \mathbb{A}^1_*$, hence there are integers k_1, k_2, k_3, not all of them are equal to 0, such that $h_1^{k_1} = h_2^{k_2} \cdot h_3^{k_3}$. Then $k_1(h_1) = k_2 \cdot (h_2) + k_3(h_3)$. But the divisors (h_i) are distinct.

Suppose that there is one curve in the complement. Let x, y be coordinates on $\mathbb{A}^1_* \times \mathbb{A}^1_* \subset X$. Then there exist m, n such that $m(x) = n(y)$. But then $x^m = cy^n$ for some $c \neq 0$.

Hence the complement of $\mathbb{A}^1_* \times \mathbb{A}^1_*$ in X contains exactly two curves. Let h_1, h_2 be their equations. Let as before x, y be coordinates on $\mathbb{A}^1_* \times \mathbb{A}^1_*$. Then there exist m, n such that $(x) = m(h_1) + n(h_2)$. Hence $x = ch_1^m h_2^n$ for some constant c. Similarly $y = c_1 h_1^{m_1} h_2^{n_1}$. Hence h_1, h_2 can be taken as coordinates on $\mathbb{A}^1_* \times \mathbb{A}^1_*$.

In particular a curve $h_1 = a \neq 0$ is isomorphic to $\mathbb{A}^1_*$. Let $C = C_1 \cup C_2$ be the complement of $\mathbb{A}^1_* \times \mathbb{A}^1_*$ in X. One can easily compute that $b_0(C) = 1, b_1(C) = 0$. In particular $C_1 \cap C_2 \neq \emptyset$, which implies that C_2 is not contained in a fibre of the mapping $h_1: X \to \mathbb{A}^1$. Hence the general fibre meets C_2 and therefore is isomorphic to $\mathbb{A}^1$. So $\bar{k}(X) = -\infty$, which implies $\bar{k}(S - E) = -\infty$.

Now we consider the possibility IIb.

If we add the horizontal components to our S we obtain a simply connected surface S^0. By (1.2.) there are at most two multiple fibres of the induced ruling $f_0: S^0 \to \mathbb{P}^1$. Of course $\mathrm{mult}_f = \mathrm{mult}_{f_0}$. Hence there are atmost two multiple fibres of the ruling $f: S \to \mathbb{P}^1$. Assume that $\mathrm{mult}_S(F_E) \geqslant 2$ or that there exists only one S-multiple fibre F_1. Consider a singular fibre F different from F_E and F_1. F contains exactly one S-component G. It follows that G is the unique exceptional curve in F. But this is not possible since $\mathrm{mult}(G) = 1$. Therefore any fibre different from F_E and F_1 is isomorphic to $\mathbb{P}^1$. As in case IIa $F_E \cap D$ is connected and both H_1, H_2 meet $F_E \cap D$. Therefore H_1, H_2 meet any fibre F different from F_E and F_1 in two distinct points. It follows that $S - E$ contains $\mathbb{A}^1_* \times \mathbb{A}^1_*$. As in IIa, $\bar{k}(X) = -\infty$ and $\bar{k}(S - E) = -\infty$.

Thus we may assume that $\mathrm{mult}_s(F_E) = 1$ and that there exist two multiple fibres F_0, F_1; of course $\sigma(F_E) = r + 1$, $\sigma(F) = 1$ for any other fibre F. Let us examine F_E. Let C be the unique S-component of F_E not contained in E. Then C is the unique exceptional curve in F_E. It follows that C must meet E in a terminal component, say E_r. It is easy to see that mult(E_1) divides the multiplicities of $E_2, \ldots, E_r, C$. Thus mult(E_1) = 1. Let $T = E_1 + \cdots + E_r + C + D_1 + \cdots + D_s$ be the maximal linear chain in F_E. Suppose $D_s \cdot D_{s+1} = 1$, D_{s+1} is a branching component of F_E. The chain T is obtained by successive blowings up over D_{s+1}. Hence mult $D_{s+1} = 1$, otherwise the multiplicities of all components of T are greater than 1. After shrinking T, D_{s+1} becomes an exceptional curve of multiplicity 1. This is impossible since, in the new fibre, D_{s+1} still meets at least two other components. Hence F_E is a rational linear chain with top component D_s. E_1 and D_s are the only components of F_E of multiplicity 1. It implies that both H_1, H_2 meet D_s and that, in the process of shrinking F_E to P_1, E_1 and D_s become exceptional on the last but one stage. It follows that G is the unique exceptional curve in F. But this is not possible since mult(G) = 1. Therefore any fibre different from F_E and F_1 is isomorphic to $\mathbb{P}^1$. As in IIa, $F_E \cap D$ is connected and both H_1, H_2 meet $F_E \cap D$. Therefore H_1, H_2 meet any fibre F different from F_E and F_1 in two distinct points. It follows that $S - E$ contains $\mathbb{A}^1_* \times \mathbb{A}^1_*$. As in IIa, $\bar{k}(X) = -\infty$ and $\bar{k}(S - E) = -\infty$.

Thus we may assume that mult(F_E) = 1 and that there exist two multiple fibres F_0, F_1; of course $\sigma(F_E) = r + 1$, $\sigma(F) = 1$ for any other fibre F. Let us examine F_E. Put $S_0 = (S \cup \bigcup_{i=1}^{s-1} D_i) - D_s$. Let $p\colon \bar{S} \to p(\bar{S})$ be the contraction of $E_2 + \cdots + D_{s-1}$. Put $S_1 = p(S_0) - p(D_s)$. We will show that $\bar{k}(S_1) = -\infty$. S_1 is obtained from S_0 by successive contractions of curves in F_E. Of course $\bar{k}(S_0) = -\infty$ since $S \subset S_0$. So long as we contract curves entirely contained in S_0 the Kodaira dimension does not change. Suppose that we come to a situation when we have to contract a curve C_1 meeting D_s. Let $\tilde{S}$ be the surface obtained from S_0 at this stage, i.e. just before contracting C_1. Let $\tilde{D}$ be the boundary divisor of $\tilde{S}$. Let $\tilde{S}_1 = \tilde{S} - C_1$. Suppose that $nK + n\tilde{D} + mC_1 \geqslant 0$, where $m > 0$ and K is the canonical divisor on the compactification of $\tilde{S}$. Then $C_1(nK + n\tilde{D} + mC_1) = -m < 0$. Hence C_1 is a fixed component of $nK + n\tilde{D} + mC_1$. It follows that $nK + n\tilde{D} \geqslant 0$; contradiction since $\bar{k}(\tilde{S}) = -\infty$. Thus $\bar{k}(\tilde{S}_1) = -\infty$. Therefore we may shrink C_1 and get a surface with Kodaira dimension $-\infty$. The curve $p(E_1)$ is an exceptional curve meeting the boundary divisor of S_1 transversally in one point. Repeating the argument above we infer that $\bar{k}(S_1 - p(E_1)) = -\infty$. But $S_1 - p(E_1) = S - E - C \subset S' - q$; contradiction.

Section 6

It follows from (5.1) that $\bar{k}(S' - q) \neq 1$. Otherwise, by virtue of ([3], (6.11)) there would exist a ruling of $S' - q$ with general fibre isomorphic to an elliptic curve or $\mathbb{A}^1_*$. From now on we assume that $\bar{k}(S' - q) = 0$. We will show that this also leads to contradiction.

We will use the following known fact ([3], proof of Thm. (8.5)).

6.1. LEMMA. *Let* Y *be a smooth surface with* $\bar{k}(Y) = 0$. *Suppose that there exists a nonconstant invertible function on* Y *and that* Y *contains only finitely many compact curves. Then* Y *is* $\mathbb{A}^1_*$*-ruled.*

Proof. We get a dominant morphism $f: Y \to \mathbb{A}^1_*$. Let $f': Y \to C \to \mathbb{A}^1_*$ be its Stein factorization. For generic $c \in C$, by Kawamata's Addition Theorem [4], we obtain $\bar{k}(Y) \geqslant \bar{k}(f'^{-1}(c)) + \bar{k}(C)$ and $\bar{k}(C) \geqslant \bar{k}(\mathbb{A}^1_*) = 0$. Thus $\bar{k}(C) = 0$ and $\bar{k}(f'^{-1}(c)) = 0$ which implies $f'^{-1}(c) \cong \mathbb{A}^1_*$.

In the process of constructing the relatively minimal model of $S - E$, ([3] or [10]), we may have to contract some exeptional curves in $\bar{S}$ not contained in $D \cup E$ for which $\bar{k}(S - E - C) = \bar{k}(S - E) = 0$. Suppose C is such a curve. Then, since $\mathrm{Pic}(S - E)$ is torsion, there exists an invertible function on $S - E - C$. But then, by lemma 6.1., $S - E - C$ is $\mathbb{A}^1_*$-ruled which is impossible by virtue of 5.1. Hence we may assume that $(\bar{S}, D \cup E)$ is a relatively minimal model of $S - E$, i.e. that the negative part $(K + D + E)^-$ in the Zariski decomposition of $K + D + E$ does not contain an exceptional curve.

Fujita ([3], Thm. 8.8.) classifies the connected components of the boundary divisor of a relatively minimal surface with Kodaira dimension 0. In our case there can be only three possibilities for D.

6A. D is a rational tree with precisely two branching components and four tips T_1, T_2, T_3, T_4 with $T_i^2 = -2$.

6B. D is a rational tree with three twigs T_1, T_2, T_3 and their common branching component. In this case $\Sigma\, d(T_j)^{-1} = 1$.

6C. D is a rational tree with four tips T_1, T_2, T_3, T_4 and their common branching component. Moreover, $T_i^2 = -2$.

In all these cases $(K + D + E)^- = Bk(D) + Bk(E)$.

6.2. REMARK. Since $\bar{k}(S - E) = 0$, $(K + D + E)^+ \approx 0$, ([3], 6.11). Hence $K + D + E \approx (K + D + E)^- = Bk(D) + Bk(E)$. Here $\approx$ stands for numerical equivalence.

We consider the three cases separately.

Case 6A. We need some elementary facts about determinants. Let $D = C_1 + \cdots + C_n$ be a connected rational tree on a projective smooth surface. We denote $d(D) = \det[-C_i \cdot C_j]_{1 \leqslant i,j \leqslant n}$. Let $C_1 \cap C_2 = \{p\}$. Let $\bar{D}_1$ be the sum of components C_j contained in the connected component of $D - \{p\}$ containing $C_1 - \{p\}$. Similarly we define $\bar{D}_2$. Let $D_1 = \bar{D}_1 + C_1$, $D_2 = \bar{D}_2 + C_2$. D_1, D_2 are subtrees of D and $D = D_1 + D_2$. Then one can show that

$$d(D) = d(D_1) \cdot d(D_2) - d(\bar{D}_1) \cdot d(\bar{D}_2). \tag{*}$$

(We put $d(\bar{D}_i) = 1$ if $\bar{D}_i = \varnothing$).

In our case let $D = B_1 + B_2 + \cdots + B_s + T_1 + T_2 + T_3 + T_4$ where B_1, B_s are the branching components, T_1, T_2 meet B_1 and T_3, T_4 meet B_s. We may assume that $B_i^2 \leqslant -2$ for $1 < i < s$.

Suppose that $B_1^2 \leqslant -2$. Then one can show, using (*) above and Sylvester's criterion, that the intersection matrix of $B_1 + \cdots + B_{s-1} + T_1 + T_2 + T_3 + T_4$ is negative definite. Thus $d(D) < 0$. Otherwise the intersection matrix of D is negative definite which is not the case since D is a boundary divisor of the affine surface S'. Applying (*) for $C_1 = B_1, C_2 = B_2$ we obtain $d(D^{(2)}) < d(D^{(3)})$, where $D^{(i)} = B_i + \cdots + B_s + T_3 + T_4$.

Next apply (*) for $C_1 = B_2$, $C_2 = B_3$, $D = D^{(2)}$. We get $d(D^{(3)}) < d(D^{(4)})$. By induction we obtain $d(D^{(s-1)}) < d(D^{(s)})$.

Now $d(D^{(s-1)}) = 4(B_{s-1}^2 \cdot B_s^2 + B_{s-1}^2 - 1)$, $d(D^{(s)}) = 4(-B_s^2 - 1)$. Hence $B_{s-1}^2 \cdot B_s^2 + B_{s-1}^2 - 1 < -B_s^2 - 1$. This implies $(-B_{s-1}^2 - 1)(-B_s^2 - 1) < 1$. But $-B_{s-1}^2 \geqslant 2$. Therefore $B_s^2 \geqslant -1$. So we proved that $B_1^2 \geqslant -1$ or $B_s^2 \geqslant -1$. Assume $B_1^2 \geqslant -1$. After blowing up successively over $B_1 \cap B_2$ we may assume that $B_1^2 = -1$. Then $F = T_1 + 2B_1 + T_2$ gives an $\mathbb{A}_*^1$-ruling of $S - E$.

6.3. LEMMA. *a divides* $\det[D_iD_j]$, *where* $D = D_1 + \cdots + D_n$.

Proof. Take a small tubular neighborhood U of D. Then (Mumford [9]) the order of $H_1(U - D\colon \mathbb{Z})$ is equal to $|\det[D_iD_j]|$. We know that $U - D$ admits a cyclic covering of degree a (note that X is affine, hence connected at infinity). Therefore there exists a surjective homomorphism $\pi_1(U - D) \to \mathbb{Z}_a$. It factors through $\pi_1(U - D) \to H_1(U - D)$. Hence $\mathbb{Z}_a$ is a quotient group of $H_1(U - D; \mathbb{Z})$.

Actually the stronger fact is true:

6.4. LEMMA. $a = |\det[D_iD_j]$.

Proof. Let L be an irreducible curve such that $LE_1 = 1, LE_j = 0$ for $j \geqslant 2$. Then $L, D_1, \ldots, D_n, E_1, \ldots, E_{r-1}$ are free generators of $\mathrm{Pic}(\bar{S})$. In particular, the determinant of the intersection matrix of this configuration equals ∓ 1. We know that the divisor aL is supported on $D \cup E$. Thus there exist integers k_i, e_j such that $aL \sim \Sigma_{i=1}^n k_iD_i + \Sigma_{j=1}^r e_jE_j$. Hence $aLD_j = \Sigma k_iD_iD_j$, $j = 1, \ldots, n$.

Let A be the intersection matrix of $D_1, \ldots, D_n$; let B be the intersection matrix of $L, D_1, \ldots, D_n, E, \ldots, E_{r-1}$. Let A_i denote the matrix obtained from A by replacing the i-th column by $(LD_1, \ldots, LD_n)^T$. By Cramer's rule, $k_i \det A = a \det A_i$, $i = 1, \ldots, n$. Let $\det A = ka$. Then

6.5. k divides $\det A_i$, $i = 1, \ldots, n$.

Expanding $\det B$ along the first row we have

$$\pm 1 = \det B = \left(L^2 \det A + \sum_{i=1}^{n} (-1)^{i+1} LD_i \det A_i\right) \det[E_iE_j]_{1 \leqslant i, j \leqslant r}$$
$$+ (-1)^{n+1} \det A \det[E_iE_j]_{2 \leqslant i, j \leqslant r}.$$

Every term in this expression is divisible by k.

Case 6*B*. We have $D = B + T_1 + T_2 + T_3$, where B is a common branching component of the twigs T_1, T_2, T_3. Since $\Sigma_{i=1}^3 d(T_j)^{-1} = 1$, the triplet $(d(T_1), d(T_2), d(T_3))$ is, up to permutation, one of the following: $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$.

6.6. Suppose that $(d(T_1), d(T_2), d(T_3)) = (3,3,3)$. Then

$$e(T_i) = \frac{d(\bar{T}_i)}{d(T_i)} = \frac{1}{3}$$

or

$$e(T_i) = \frac{2}{3},\ i = 1, 2, 3.$$

$$\text{Let } K + D + E \approx BkD + BkE = \sum_{i=1}^{3} Bk(T_i) + \sum_{j=1}^{r} n_j E_j.$$

Every coefficient in $Bk(T_i)$, $i = 1,2,3$, equals $\frac{1}{3}$ or $\frac{2}{3}$. Let C be an irreducible curve which meets E_1 transversally once and does not meet $E_2 \cup \cdots \cup E_r$. Then

$$(K + D + E)C = \left(\sum_{i=1}^{3} Bk(T_i)\right) C + n_1.$$

Hence $n_1 = k/3$ for some integer k. But $0 < n_1 < 1$ except in the case where $E_j^2 = -2, j = 1,\ldots,r$. Then $n_j = 1, j = 1,\ldots,r$. In this case one checks easily that sE is a fixed component of $s(K + D + E)$ and hence $\bar{k}(S - E) = \bar{k}(S) = -\infty$. We infer that $n_1 = \frac{1}{3}$ or $n_1 = \frac{2}{3}$. Similarly, $n_r = \frac{1}{3}$ or $n_r = \frac{2}{3}$. Also $(K + D + E)^2 = \Sigma\,(BkT_i)^2 + (BkE)^2 = -e(T_1) - e(T_2) - e(T_3) - n_1 - n_r$. The sum on the right-hand side is an integer, each summand is equal to 1/3 or 2/3. Assume that $e(T_1) \leqslant e(T_2) \leqslant e(T_3)$ and $n_1 \leqslant n_r$. We have only the following possibilities:

	$e(T_1)$	$e(T_2)$	$e(T_3)$	n_1	n_r
(a)	1/3	1/3	1/3	1/3	2/3
(b)	1/3	1/3	2/3	1/3	1/3
(c)	1/3	2/3	2/3	2/3	2/3
(d)	2/3	2/3	2/3	1/3	2/3
(e)	1/3	1/3	2/3	$r = 1$,	$E^2 = -3$
(f)	1/3	2/3	2/3	$r = 1$,	$E^2 = -6$

Now we list all possible configurations of E. Consider for instance the case

$$n_1 = \frac{1}{3}, \; n_r = \frac{2}{3}.$$

By definition of the bark

$$E_1 \cdot BkE = E_r \cdot BkE = -1, \; E_i \cdot BkE = 0, \; 1 < i < r.$$

Hence

$$-1 = n_1 E_1^2 + n_2, \; -1 = n_{r-1} + n_r E_r^2, \; 0 = n_{i-1} + n_i E_i^2 + n_{i+1}, \; 1 < i < r. \qquad (*)$$

Thus

$$n_2 = -1 - n_1 E_1^2 = -1 - \frac{1}{3} E_1^2 = \frac{k_2}{3}$$

where k_2 is an integer. By induction

$$n_i = \frac{k_i}{3}, \; k_i \in \mathbb{Z}, \; i = 1, \ldots, r.$$

Hence

$$n_i = \frac{1}{3} \quad \text{or} \quad n_i = \frac{2}{3}$$

for every i. Also

$$n_{i-1} + n_{i+1} = n_i(-E_i^2) \geqslant 2n_i \Rightarrow n_{i+1} - n_i \geqslant n_i - n_{i-1} \quad \text{for } 1 < i < r.$$

Let

$$\frac{1}{3} = n_1 = n_2 \cdots = n_{s-1} n_s = \frac{2}{3}.$$

Then

$$n_{s+1} \geqslant n_s + \frac{1}{3} \geqslant 1.$$

Therefore $s = r$. From (*) we can compute $E_i^2 i = 1, \ldots, r$. Similar reasoning applies in every case. We indicate below the dual graphs of E. The numbers above a graph are the corresponding n_i, the numbers below are the corresponding self-intersection indices. We indicate the relation between a and r.

$$\begin{array}{ccc} 1/3 & & 2/3 \\ \bullet & \text{---} & \bullet \\ -5 & & -2 \end{array}$$

$$\begin{array}{ccccc} 1/3 & & 1/3 & & 2/3 \\ \bullet & \text{---} & \bullet & \text{---} & \bullet \\ -4 & & -3 & & -2 \end{array}$$

$$\begin{array}{ccccccccccc} 1/3 & & 1/3 & & & & 1/3 & & 1/3 & & 2/3 \\ \bullet & \text{---} & \bullet & & \cdots & & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ -4 & & -2 & & & & -2 & & -3 & & -2 \end{array}$$

In the three cases above $a = 9(r - 1)$.

$$\begin{array}{ccccccc} 1/3 & & 1/3 & & & 1/3 & & 1/3 \\ \bullet & \text{---} & \bullet & \cdots & \bullet & \text{---} & \bullet \\ -4 & & -2 & & -2 & & -4 \end{array}$$

Here $a = 3(3r - 1)$.

$$\begin{array}{ccccc} 2/3 & & 1/3 & & 2/3 \\ \bullet & \text{---} & \bullet & \text{---} & \bullet \\ -2 & & -4 & & -2 \end{array}$$

$$\begin{array}{ccccccccccc} 2/3 & & 1/3 & & 1/3 & & 1/3 & & 1/3 & & 2/3 \\ \bullet & \text{---} & \bullet & \text{---} & \bullet & \cdots & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ -2 & & -3 & & -2 & & -2 & & -3 & & -2 \end{array}$$

In the two cases above $a = 9r - 15$.

Now we consider cases according to the above table.

6.6(a). In this case

$$(K + D + E)^2 = -\frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{2}{3} = -2.$$

Therefore

$$K(K + D + E) = -2 - D(K + D) - E(K + E) = -2 + 2 + 2 = 2.$$

The T_i are tips of D, $T_i^2 = -3$, $i = 1,2,3$.

$$(K + D + E) \approx \frac{1}{3}(T_1 + T_2 + T_3) + \frac{1}{3}E' + \frac{2}{3}E_r$$

where

$E' = E_1 + \cdots + E_{r-1}$. From this

$$3K + 3B + 2(T_1 + T_2 + T_3) + 2E' + E_r = 0. \tag{*}$$

From Riemann-Roch

$$h^0(2K + D + E) + h^0(-K - D - E) \geqslant 1.$$

If $-K - D - E \geqslant 0$, then

$$K + D + E = 0 \quad \text{since } \exists_n n(K + D + E) \geqslant 0.$$

But $(K + D + E)^2 = -2$. Hence $2K + D + E \geqslant 0$. E_r is a fixed component of $2K + D + E$ hence $2K + D + E' \geqslant 0$. Now, from (*) we get that

$$6K + 6B + 4(T_1 + T_2 + T_3) + 4E' + 2E_r = 0.$$

But

$$3(2K + D + E') = 6K + 3B + 3(T_1 + T_2 + T_3) + 3E' \geqslant 0.$$

We get $3B + T_1 + T_2 + T_3 + E' + 2E_r \leqslant 0$, contradiction.

6.6(b). $T_1^2 = -3 = T_2^2$, $T_3 = T_{31} + T_{32}$, $T_{31}^2 = T_{32}^2 = -2$.

$d(D) = 9(-3B^2 - 4)$, $a = 3(3r - 1)$. $d(D) \leqslant 0$, otherwise the intersection matrix of D is negative definite. By virtue of (6.4), $-9(-3B^2 - 4) = 3(3r - 1)$. Contradiction.

6.6(c). T_1 is a tip with $T_1^2 = -3$, $T_i = T_{i1} + T_{i2}$, $T_{i1}^2 = T_{i2}^2 = -2$, $i = 2,3$. By an easy computation, $d(D) = -9(3B^2 + 5)$. By virtue of lemma (6.4), $a = 9r - 15 = 9(3B^2 + 5)$, which is absurd.

6.6(d). In this case

$$T_i = T_{i1} + T_{i2},\ T_{i1}^2 = T_{i2}^2 = -2 \quad (i = 1,2,3.)$$

T_{i1} is the tip of the twig T_i. $(K + D + E)^2 = -3$, hence $K(K + D + E) = 1$. Furthermore $KD = -2-B^2$, $KE = 3$. Thus $K^2 = B^2$. In view of Noether's formula, $K^2 + b_2(\bar{S}) = B^2 + r + 7 = 10$, and hence $B^2 = 3 - r$. By (6.4),

$$a = 9r - 9 = -d(D) = 27(B^2 + 2) = 27(5 - r),$$

which implies $r = 4$ and

$$B^2 = -1,\ K^2 = -1.\ K + D + E \approx Bk(D) + Bk(E)$$
$$= 2/3(T_{11} + T_{21} + T_{31} + E_4) + 1/3(T_{12} + T_{22} + T_{32} + E'),$$

where $E_4^2 = -2$, $E' = E_1 + E_2 + E_3$. Suppose that C is an exceptional curve in $\bar{S}$. Then

$$C(K + D + E) = 2/3(T_{11} + T_{21} + T_{31} + E_4)C + 1/3(T_{12} + T_{22} + T_{32} + E')C,$$

which gives

$$CB - 1 + 1/3(T_{11} + T_{21} + T_{31} + E_4)C + 2/3(T_{12} + T_{22} + T_{32} + E')C = 0.$$

If $CB = 1$ then $C(D - B) = 0$ and $CE = 0$. But then

$$\bar{k}(S - E - C) = \bar{k}(S - E) = 0$$

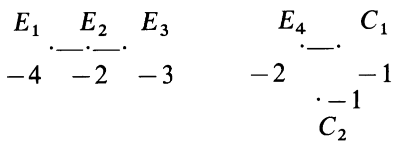
and there exists an invertible nonconstant function on $S - E - C$, which is impossible by virtue of (6.1). Thus $CB = 0$. We have two cases:

(i) $C(T_{11}+T_{21}+T_{31}+E_4)=1$, $C(T_{12}+T_{22}+T_{32}+E')=1$
(ii) $C(T_{11}+T_{21}+T_{31}+E_4)=3$, $C(T_{12}+T_{22}+T_{32}+E')=0$.

Consider $F = 2B + T_{12} + T_{22}$. We have $F^2 = 0$, $p_a(F) = 0$. Therefore F defines a $\mathbb{P}^1$-ruling of $\bar{S}$. We have $FT_{11} = FT_{21} = 1$, $FT_{32} = 2$, $FT_{31} = 0$, $FE = 0$. E is contained in a fibre F_E. There are three horizontal D-components: T_{11}, T_{21}, T_{32}. One fibre is F and $\sigma(F) = 0$. Any other fibre contains at least one S-component. By virtue of (1.1) we obtain $3-\Sigma+1-2=-4$, i.e. $\Sigma=6$.

REMARK. It follows from the expression for $K + D + E$ that there is no curve C in $\bar{S}$ such that $C^2 \leqslant -2$ except for those contained in $D \cup E$.

Suppose that $T_{31} \not\subset F_E$. F_E contains E and at least one S-component C not contained in E. If there is only one such C then C is exceptional by the Remark above. But this is not possible since C is neither of the type (i) nor (ii) as one can easily see. Hence F_E contains at least 6 S-components. Then $\sigma(F_E) \geqslant 6$. The fibre F_1 which contains T_{31} must contain at least two S-components. Therefore $\sigma(F_1) \geqslant 2$. Since $\Sigma = 6$ we have $\sigma(F_E) = 6$ and $\sigma(F_1) = 2$. Then $F_E = E \cup C_1 \cup C_2$. Both C_1 and C_2 are exceptional by the Remark above and both meet E. T_{32} meets $C_1 \cup C_2$. Let $T_{32}C_1 \geqslant 1$. Then C_1 is of the type (i). In particular $C_1T_{32} = 1$, $C_1(T_{12} + T_{22} + E') = 0$, hence $C_1E_4 = 1$ and $C_1(T_{11} + T_{21} + T_{31}) = C_1(T_{11} + T_{21}) = 0$. Both sections T_{11} and T_{21} meet C_2. Thus C_2 is of the type (ii). In particular $C_2T_{32} = 0 = C_2E'$ and $C_2E_4 = 1$. Hence the dual graph of F_E looks like



This is impossible. Such a configuration cannot occur as the fibre of a $\mathbb{P}^1$-ruling. Therefore $T_{31} \subset F_E$. If $F_E = E \cup T_{31} \cup C$ then $T_{11}C = T_{21}C = 1$. If follows that $\mathrm{mult}(C) = 1$. But C is exceptional and meets two components of the fibre which implies $\mathrm{mult}(C) \geqslant 2$.

Suppose $F_E = E \cup C_1 \cup C_2 \cup T_{31}$. Both C_1 and C_2 are exceptional. Let $C_1T_{31} = 1$, $C_1E = 1$. Then $\mathrm{mult}(C_1) \geqslant 2$. Hence $T_{11}C_1 = T_{21}C_1 = 0$, $T_{11}C_2 = T_{21}C_2 = 1$. Therefore C_2 is of type (ii). In particular $C_2T_{32} = 0$. Hence the horizontal component T_{32} meets C_1. It follows that C_1 is of type (i). Thus $C_1E' = 0$ and, from the first statement in (i), $C_1E_4 = 0$ since C_1 meets T_{31}. It follows that C_1 does not meet E; contradiction.

Hence $\sigma(F_E) = 7$. $F_E = E \cup C_1 \cup C_2 \cup C_3 \cup T_{31}$; the C_i are exceptional. Let $C_1T_{31} = CE = 1$. Then $T_{11}C_1 = T_{21}C_1 = 0$ since $\mathrm{mult}(C_1) \geqslant 2$. C_1 is of type (i), otherwise $C_1(T_{31} + E_4) = 3$. In particular $C_1E_4 = 0$ which implies $C_1E' = 1$. Therefore $C_1T_{32} = 0$. T_{32} meets the union $C_2 \cup C_3$. Let $T_{32}C_2 = 1$. $T_{31}C_2 = 0$, otherwise the subtree $C_1 + T_{31} + C_2$ contracts down to a curve with selfintersection index equal to 0. Similarly $T_{31}C_3 = 0$. C_2 is of type (i) since it meets T_{32}. Therefore $C_2E' = 0$. But C_2 meets E, otherwise C_2 is an isolated component of the fibre. Hence $C_2E_4 = 1$ and $C_2(T_{11} + T_{21}) = 0$. Therefore both horizontal components T_{11}, T_{21} meet C_3, which implies that C_3 is of type (ii). It follows that $C_3E_4 = 1$. This is impossible since then we could shrink $C_2 + E_4 + C_3$ to a 0-curve.

6.6(e). $d(D) = 9(-3B^2-4)<0$. We get $9(-3B^2-4)=-3$; absurd.

6.6(f). In this case

$$D=B+T_1+T_{21}+T_{22}+T_{31}+T_{32},\ T_1^2 = -3,\ T_{ij}^2=-2.\ d(D)=9(5+3B^2).$$

By virtue of (6.4), $6 = 9(5 + 3B^2)$; absurd.

6.7. Suppose that $(d(T_1), d(T_2), d(T_3)) = (2, 4, 4)$.

Then $e(T_1) = 1/2$, $e(T_i) = 1/4$ or $3/4$, $i = 1, 2$. Again, if $Bk(E) = \Sigma n_i E_i$, then $n_i = 1/4$ or $2/4$ or $3/4$, $i = 1, \ldots, r$. We have the following possibilities:

	$e(T_1)$	$e(T_2) \leqslant$	$e(T_3)$	$n_1 \leqslant$	n_r
(a)	1/2	1/4	1/4	1/2	1/2
(a_1)	1/2	1/4	1/4	1/4	3/4
(b)	1/2	1/4	3/4	1/4	1/4
(b_1)	1/2	1/4	3/4	3/4	3/4
(b_2)	1/2	1/4	3/4	$r=1$	$E^2=-4$
(c)	1/2	3/4	3/4	1/4	3/4
(c_1)	1/2	3/4	3/4	1/2	1/2

Now we list all possible configurations of E.

$$\begin{array}{ccc} 3/4 & 1/2 & 3/4 \\ \cdot & \cdot & \cdot \\ -2 & -3 & -2 \end{array} \qquad a = 8$$

$$\begin{array}{ccccccccc} 3/4 & 1/2 & 1/4 & 1/4 & & 1/4 & 1/4 & 1/2 & 3/4 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ -2 & -2 & -3 & -2 & & -2 & -3 & -2 & -2 \end{array}$$

$$\begin{array}{ccccc} 3/4 & 1/2 & 1/4 & 1/2 & 3/4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -2 & -2 & -4 & -2 & -2 \end{array}$$

In the two cases above $r \geqslant 5$ and $a = 16r - 56$.

$$\begin{array}{ccc} 1/4 & 1/2 & 3/4 \\ \cdot & \cdot & \cdot \\ -6 & -2 & -2 \end{array}$$

$$\begin{array}{cccc} 1/4 & 1/4 & 1/2 & 3/4 \\ \cdot & \cdot & \cdot & \cdot \\ -5 & -3 & -2 & -2 \end{array}$$

$$\begin{array}{ccccccc} 1/4 & 1/4 & & 1/4 & 1/4 & 1/2 & 3/4 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot \\ -5 & -2 & & -2 & -3 & -2 & -2 \end{array}$$

In the three cases above $r \geqslant 3$ and $a = 16r - 32$.

$$\begin{matrix} 1/2 & 1/2 & & 1/2 & 1/2 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -3 & -2 & & -2 & -3 \end{matrix} \quad a = 4r.$$

$$\begin{matrix} 1/4 & 1/4 & 1/4 & & 1/4 & 1/4 & 1/4 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ -5 & -2 & -2 & & -2 & -2 & -5 \end{matrix} \quad a = 16r - 8$$

There is no E such that $n_1 = 1/2$, $n_r = 3/4$.

6.7(a).

$$D = B + T_1 + T_2 + T_3,\ T_1^2 = -2,\ T_2^2 = -4 = T_3^2.$$

$(K + D + E)^2 = -2$ hence $K(K + D + E) = 2$. By Riemann-Roch,

$$h^0(2K + D + E) \geqslant 1/2(2K + D + E)(K + D + E) + 1 = 1.$$

Therefore

$$2K + D + E \geqslant 0.\ K + D + E \approx BkD + BkE = \frac{1}{2}T_1 + \frac{1}{4}T_2 + \frac{1}{4}T_3 + \frac{1}{2}E.$$

It follows that $4K + 4B + 2T_1 + 3T_2 + 3T_3 + 2E = 0$.
Then

$$\begin{aligned} 0 &= 4K + 4B + 2T_1 + 3T_2 + 3T_3 + 2E \\ &= 2(2K + D + E) + 2B + T_2 + T_3. \end{aligned}$$

Contradiction since $2(2K + D + E) \geqslant 0$.

6.7(a_1). In a similar way:

$$2K + D + E \geqslant 0;\ K + D + E \approx \frac{1}{2}T_1 + \frac{1}{4}T_2 + \frac{1}{4}T_3 + \frac{1}{4}E' + \frac{1}{2}E_{r-1} + \frac{3}{4}E_r$$

where $E' = E_1 + \cdots + E_{r-2}$. Hence

$$4K + 4B + 2T_1 + 3T_2 + 3T_3 + 3E' + 2E_{r-1} + E_r = 0.$$

Notice that, since

$$E_r^2 = -2, (2K + D + E)E_r = -1.$$

Hence $2K + D + E' + E_{r-1} \geqslant 0$. Now

$$0 = 2(2K + D + E' + E_{r-1}) + 2B + T_2 + T_3 + E' + E_r$$

contradiction.

6.7(b) $d(D) = 16(-2B^2 - 3) < 0$, $a = 16r - 8$. We get $2(2B^2 + 3) = 2r - 1$; impossible.

6.7(b_1) $d(D)$ as in 6.7(b); $a = 8$ or $a = 16r - 56$. Again contradiction with (6.2).

6.7(b_2) $|d(D)| = 16(2B^2 + 3) > 4$.

6.7(c) Then T_1 consists of single component and

$$T_1^2 = -2;\ T_i = T_{i1} + T_{i2} + T_{i3},\ i = 1, 2,\ T_{ij}^2 = -2.$$

$d(D) = -32(B^2 + 2)$. For any configuration of E, $KE = 4$. Applying Noether's formula, we obtain $B^2 + r = 3$. Hence $16r - 32 = 32(5 - r)$. It follows that $r = 4$, $B^2 = -1$, $K^2 = -2$.

$$Bk(D) = 1/2T_1 + 1/4(3T_{21} + 2T_{22} + T_{23} + 3T_{31} + 2T_{32} + T_{33}).$$

$Bk(E) = 1/4E_1 + 1/4E_2 + 1/2E_3 + 3/4E_4$. As in 6.6(d) we may assume that every exceptional curve in $\bar{S}$ does not meet B.

Consider $F = T_1 + 2B + T_{23}$. As in 6.6(d), F defines a $\mathbb{P}^1$-ruling of $\bar{S}$. T_{22} and T_{33} are the only horizontal components, T_{21} and $T_{31} \cup T_{32}$ are contained in fibres. There exists only one fibre F with $\sigma(F) = 0$, $\Sigma = 5$. E is contained in a fibre F_E.

Assume that $\sigma(F_E) = 5$. Let C be the S-component not contained in E. C must be exceptional and $F_E = E \cup C \cup T_{21}$ or $F_E = E \cup C \cup T_{31} \cup T_{32}$. One can easily check that in both cases F_E cannot be contracted to a 0-curve. Hence $\sigma(F_E) = 6$. Assume that T_{21} is not contained in F_E. Then the fibre F_1 containing T_{21} must contain precisely one S-component C' and, therefore, it must contain also $T_{31} \cup T_{32}$. But then C' meets two -2-curves in F_1, which is impossible. Therefore $T_{21} \subset F_E$. In a similar way $T_{31} \cup T_{32} \subset F_E$. Hence

$$F_E = E \cup T_{31} \cup T_{32} \cup T_{21} \cup C_1 \cup C_2.$$

Both C_1 and C_2 are exceptional (see Remark 6.6(d)). One can check that such a F_E cannot be the support of a fibre of a $\mathbb{P}^1$-ruling.

6.7(c_1). In this case $B^2 + r = 1$, $d(D) = -32(B^2 + 2) < 0$. $a = 4r$. We get $24 = g_r$, contradiction.

6.8. $(d(T_1), d(T_2), d(T_3)) = (2, 3, 6)$.

Then $e(T_1) = 1/2$, $e(T_2) = 1/3$ or $2/3$, $e(T_3) = 1/6$ or $5/6$. As before we obtain that $n_i = k_i/6$, $1 \leqslant k_i \leqslant 5$, where $\Sigma n_i E_i = Bk(E)$. One can check that there is no E with $n_1 = \frac{1}{2}$, $n_r = \frac{5}{6}$ or $n_1 = \frac{1}{6}$, $n_r = \frac{1}{2}$. (This follows from the fact that if $n_1 = \frac{1}{2}$ then all n_i equal $\frac{1}{2}$). We have the following possibilities to consider:

	$e(T_1)$	$e(T_2)$	$e(T_3)$	$n_1 \leqslant n_r$	
(a)	1/2	1/3	1/6	1/6	5/6
(a_1)	1/2	1/3	1/6	2/6	4/6
(a_2)	1/2	1/3	1/6	3/6	3/6
(b)	1/2	1/3	5/6	1/6	1/6
(b_1)	1/2	1/3	5/6	$r = 1$	$E_1^2 = -6$
(b_2)	1/2	1/3	5/6	4/6	4/6
(c)	1/2	2/3	1/6	2/6	2/6
(c_1)	1/2	2/3	1/6	$r = 1$	$E_1^2 = -3$
(c_2)	1/2	2/3	1/6	5/6	5/6
(d)	1/2	2/3	5/6	1/6	5/6
(d_1)	1/2	2/3	5/6	2/6	4/6
(d_2)	1/2	2/3	5/6	3/6	3/6

Now we list possible configurations of E (we omit the configurations described in (6.6) and 6.7)):

$$\begin{array}{ccccc} 1/6 & 1/6 & & 1/6 & 1/6 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -7 & -2 & & -2 & -7 \end{array}$$

Here $a = 36r - 24$.

$$\begin{array}{cccccccc} 1/6 & 1/6 & & 1/6 & 2/6 & 3/6 & 4/6 & 5/6 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot \\ -7 & -2 & & -3 & -2 & -2 & -2 & -2 \end{array}$$

$$\begin{array}{ccccc} 1/6 & 2/6 & 3/6 & 4/6 & 5/6 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -8 & -2 & -2 & -2 & -2 \end{array}$$

In the two cases above $a = 36r - 144$.

$$\begin{array}{ccccccc}
5/6 & 4/6 & 3/6 & 2/6 & 3/6 & 4/6 & 5/6 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-2 & -2 & -2 & -3 & -2 & -2 & -2
\end{array} \qquad \text{Here } a = 24.$$

$$\begin{array}{ccccccccccccc}
5/6 & 4/6 & 3/6 & 2/6 & 1/6 & 1/6 & & 1/6 & 1/6 & 2/6 & 3/6 & 4/6 & 5/6 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-2 & -2 & -2 & -2 & -3 & -2 & & -2 & -3 & -2 & -2 & -2 & -2
\end{array}$$

In this case $a = 36r - 264$.

6.8(a) $T_1^2 = -2$, $T_2^2 = -3$, $T_3^2 = -6$; $B^2 = 4$, $r = 9$. $(K + D + E)^2 = -2$, thus $K(K + D + E) = 2$ and, from Riemann-Roch, $2K + D + E \geqslant 0$.

$$K + D + E \approx BkD + BkE = \frac{1}{2}T_1 + \frac{1}{3}T_2 + \frac{1}{6}T_3 + \frac{1}{6}E' + \frac{2}{6}E_6 + \frac{3}{6}E_7 + \frac{4}{6}E_8 + \frac{5}{6}E_9$$

where $E' = E_1 + \cdots + E_5$. Hence

$$\begin{aligned}
0 &= 6K + 6B + 3T_1 + 4T_2 + 5T_3 + 5E' + 4E_6 + 3E_7 + 2E_8 + E_9 \\
&= 3(2K + D + E' + E_6 + E_7) + 3B + T_2 + 2T_3 + 2E' + E_6 + 2E_8 + E_9.
\end{aligned}$$

We get a contradiction since $2K + D + E' + E_6 + E_7 \geqslant 0$ (because $E_8 + E_9$ is a fixed component of $2K + D + E$).

6.8(a_1) $6K + 6B + 3T_1 + 4T_2 + 5T_3 + 4E' + 2E_r = 0$,

where $E' = E_1 + \cdots + E_{r-1}$. E_r is a fixed component of $2K + D + E$. Hence $2K + D + E' \geqslant 0$. We get contradiction as in 6.8(a).

6.8(a_2) $6K + 6B + 3T_1 + 4T_2 + 5T_3 + 3E = 0$. But $2K + D + E \geqslant 0$; contradiction.

6.8(b) $T_1^2 = -2$, $T_2^2 = -3$, $T_3 = T_{31} + \cdots + T_{35}$, $T_{3i}^2 = -2$, $i = 1, \ldots, 5$. $d(D) = 12(-3B^2 - 5) < 0$, $a = 12(3r - 2)$. We have $3B^2 + 5 = 3r - 2$; contradiction.

6.8(b_1) $|d(D)| = 12(3B^2 + 5) > 6$.

6.8(b_2) We get $12(3B^2 + 5) = 9r - 15$. This cannot happen.

6.8(c) $T_1^2 = -2$, $T_2 = T_{21} + T_{22}$, $T_{21}^2 = T_{22}^2 = -2$, $T_3^2 = -6$.

$d(D) = 12(-3B^2 - 4) < 0$, $a = 3(3r - 1)$. We get $12(3B^2 + 4) = 3(3r - 1)$. This is impossible.

6.8(c_1) $|d(D)| > 3$.

6.8(c_2) We get $12(3B^2+4)=36r-264$ or $12(3B^2+4)=24$. This is impossible.

6.8(d)

$$T_1^2=-2,\ T_2=T_{21}+T_{22},\ T_3=T_{31}+\cdots+T_{35},$$

$$T_{ij}^2=-2,\ i=2,\ 3j=1,\ldots,5.$$

$d(D)=36(-B^2-2)<0$; $a=36r-144$. By standard arguments $B^2=-1$; $r=5$. Let $F=T_1+2B+T_{22}$. F defines a $\mathbb{P}^1$-ruling of $\bar{S}$. E is contained in a fibre F_E, F is the only fibre with $\sigma_S(F)=0$. There are two horizontal components T_{21}, T_{35}. Therefore $\Sigma=6$. Suppose that $T_{31}\cup\cdots\cup T_{34}$ is not contained in F_E. Then F_E must contain at least two S-components not contained in E (if F_E contains only one such curve then there is no possibility to shrink F_E to a 0-curve). Hence $\sigma(F_E)\geqslant 7$, which implies $\sigma(F_E)=7$. Then the fibre F_1 containing $T_{31}\cup\cdots\cup T_{34}$ contains exactly one S-component which must be an exceptional curve. Of course this is impossible. Thus $T_{31}\cup\cdots\cup T_{34}\subset F_E$. Suppose that $\sigma(F_E)=6$. Let C be the unique exceptional curve in F_E. C must meet E and it must meet $T_{31}\cup\cdots\cup T_{34}$. Hence mult$(C)\geqslant 2$. But this is impossible since the section T_{22} must meet C. Therefore $F_E=E\cup T_{31}\cup\cdots\cup T_{34}\cup C_1\cup C_2$. Both C_1 and C_2 are exceptional (see Remark (6.6.d)). It is not difficult to check that such a F_E cannot be the support of a fibre of a $\mathbb{P}^1$-ruling.

6.8(d_1) $KE=3$, $K^2=B^2$, $B^2+r=1$. Also, $9r-9=36(B^2+2)$. We get $13=5r$; contradiction.

6.8(d_2) $KE=2$, $a=4r$, $B^2+r=0$. $4r=36(B^2+2)$ and $4r=36(-r+2)$. This is impossible.

Case 6C. $D=B+T_1+T_2+T_3+T_4$, $T_i^2=-2$, $i=1,\ldots,4$.

$BkD=\frac{1}{2}(T_1+T_2+T_3+T_4)$. Hence $n_1=n_r=\frac{1}{2}$ and $BkE=\frac{1}{2}E$, $a=4r$. $(K+D+E)^2=-3$, thus $K(K+D+E)=1$. $KE=2$, hence $K(K+D)=-1$. In view of Noether's formula, $K^2+5+r=10$. Also $d(D)=16(-B^2-2)<0$. We get $16(B^2+2)=4r$, $K^2+KB=K^2-2-B^2=-1$, $K^2+r=5$. Then $B^2+r=4$ and $4(B^2+2)=r$. We get $24=5r$; contradiction.

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