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Lower Bounds for Betti Numbers

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1. Statement of results

A question which has attracted attention during the last 15 years is whether the Betti numbers of a non-zero module M of finite length and finite projective dimension over a local (or graded) noetherian ring R can be bounded below by binomial coefficients in the Krull dimension d of R. More precisely, if K is the residue class field at the (irrelevant) maximal ideal of R, then $b_i^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)$ is the ith Betti number of M and it has been conjectured that always

$$b_i^R(M) \geqslant \binom{d}{i}.$$

Note that these lower bounds are "predicted" by the Betti numbers of quotients of R by systems of parameters, since R has to be Cohen-Macaulay (cf. Section 2 below). For the origin of this problem, see [B-E$_1$], [Ha]. A survey of affirmative answers for particular rings is contained in the recent preprint [C-E]. Let us just mention that for $d \leqslant 4$ the conjecture follows easily from the generalized principal ideal theorem and the vanishing of Euler-Poincaré characteristics. If R is Gorenstein, even more precise results are known, [C-E-M], showing that $b_i^R(M) = \binom{d}{i}$ for some i; $0 < i < d \leqslant 4$; implies already that M is a quotient of R by a system of parameters.

By contrast, for $d \geqslant 5$ the conjecture is still undecided, even when R is regular.

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In this paper, we concentrate on the (slightly?) simpler question whether it is always true that the *total Betti number* $\beta^R(M)$ is bounded below by 2^d:

$$\beta^R(M) := \sum_i b_i^R(M) \geqslant 2^d \ ?$$

Using the Evans-Griffith Syzygy Theorem together with older results we obtain for $\beta^R(M)$ at least a lower bound which is *quadratic* in d:

PROPOSITION 1. *Let R be an equicharacteristic local noetherian ring of Krull dimension d at least 5. For any non-zero R-module M of finite length and finite projective dimension it holds that*

$$\beta^R(M) \geqslant \tfrac{3}{2}(d-1)^2 + 8.$$

In particular, for $d = 5$ *one has* $\beta^R(M) \geqslant 32 = 2^5$.

The main part of this paper is concerned with the total Betti number of a *graded* module over a *graded* noetherian K-algebra which is generated in degree 1. There it turns out that the conjectured bound for $\beta^R(M)$ holds in at least "half" of all cases. In the extreme case of *multigraded* modules over a polynomial ring, even the binomial estimates for the individual Betti numbers are known by results of [E-G$_2$], [Sa], [Ch].

To give a precise formulation of our main results, we need some notation. As in [AC VIII. §4], we denote by $\mathbb{Z}((t)) = \mathbb{Z}[[t]][t^{-1}]$ the ring of formal Laurent series with integral coefficients. If $M = \oplus_{i\in\mathbb{Z}} M_i$ is a graded vectorspace over some field K, such that $\dim_K M_i$ is finite for all i and $M_i = 0$ for $i \ll 0$, its *Hilbert series* is the element of $\mathbb{Z}((t))$ with non negative coefficients which is given by

$$H_M(t) = \sum_i (\dim_K M_i)t^i \in \mathbb{Z}((t)).$$

If furthermore M is a non-zero, finitely generated graded module over a positively graded K-algebra $R = \oplus_{i\geqslant 0} R_i$ which is finitely generated by R_1 over $R_0 = K$, then it is well known that – see for example [AC VIII. §6. Prop. 5] – the Hilbert series of M can be written uniquely in the form

$$H_M(t) = \frac{e_M(t)}{(1-t)^{d(M)}}, \tag{1.1}$$

where $d(M)$ is the *Krull dimension* of M, $e_M(t)$ is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$, and $e_M = e_M(1)$ is a positive integer, the *multiplicity* of M.

We call $e_M(t)$ the *multiplicity polynomial* of M.

Note that these combinatorial invariants of M depend only on its underlying graded K-vectorspace. However, one has $d(R) \geqslant d(M)$ – and our next result will establish a further constraint for modules of finite projective dimension:

PROPOSITION 2. *With notation as just introduced, assume furthermore that M is of finite projective dimension as an R-module. Then $e_R(t)$ divides $e_M(t)$ in $\mathbb{Z}[t, t^{-1}]$. In particular, the multiplicity of R divides the multiplicity of M.*

We set $e_M^R(t) = e_M(t)/e_R(t)$ and call this Laurent polynomial over $\mathbb{Z}$ the *reduced multiplicity polynomial of M over R.*

In the non-graded case, it is not necessarily true that the multiplicity of the ring divides the multiplicity of a finitely generated module of finite projective dimension: In [D-H-M], there is constructed an artinian module A of length (=multiplicity) equal to 15 which has finite projective dimension over the homogeneous coordinate ring R of the quadric $Q = (xy - uv) \in K[x, y, u, v]$. This ring has multiplicity equal to 2. We do however not known whether over a local or graded ring R there can be a non-zero module M of finite projective dimension whose multiplicity is *smaller* than the multiplicity of R.

To formulate the main numerical result concisely, set also $\Psi_n(t) = 1 + t + \cdots + t^{n-1} \in \mathbb{Z}[t]$ for any positive integer n.

THEOREM 3. *Let K be a field, R a positively graded K-algebra finitely generated by elements of degree 1. For a non-zero, finitely generated graded R-module M of finite projective dimension and a prime $p \in \mathbb{N}$, set*

$$m = \max\{\mu \geqslant 0 \mid \Psi_{p^\mu}(t) \text{ divides } e_M^R(t) \text{ in } \mathbb{Z}[t, t^{-1}]\}.$$

The total Betti number satisfies then

$$\beta^R(M) \geqslant 2^{\frac{d(R) - d(M) + p^m - 1}{p^m(p-1)} \cdot \log_2 p}.$$

As a special case, consider those modules M as above which in addition satisfy $e_M(-1) \neq 0$. The Theorem then applies with $p = 2$ and $m = 0$ to yield the inequality $\beta^R(M) \geqslant 2^{d(R) - d(M)}$. This settles the conjecture on the total Betti number for a large class of artinian modules:

COROLLARY 4. *Let R be as above, of Krull dimension d, and let M be a graded R-module of finite length and finite projective dimension. If the alternating sum of dimensions $\Sigma_i (-1)^i \dim_K M_i$* does not vanish, then $\beta^R(M) \geqslant 2^{d(R)}$.

A statement which is weaker than the one in the theorem, but whose hypotheses might be easier to check, is obtained from the following observation. If $l = \mathrm{ord}_p(e_M/e_R)$, then $\Psi_{p^{l+1}}(t)$ does not divide $e_M^R(t)$, since otherwise $p^{l+1} = \Psi_{p^{l+1}}(1)$ divides $e_M^R(1) = e_M/e_R$. Thus the number m above is at most equal to l. As the expression $[d(R) - d(M) + p^{\mu} - 1/p^{\mu}(p - 1)]$ is decreasing in μ, we have

COROLLARY 5. *With the notation of the theorem, set* $l = \max\{\lambda \geqslant 0 \mid p^{\lambda}$ divides e_M/e_R in $\mathbb{Z}\}$. *Then*

$$\beta^R(M) \geqslant 2^{\frac{d(R)-d(M)+p^l-1}{p^l(p-1)}\cdot\log_2 p}.$$

To give a feeling for the numerical range of this estimate, we mention some special cases:

EXAMPLE 6. Let R and M be as above, but assume furthermore that M is of finite length.

(i) If $\dfrac{\dim_K M}{e(R)}$ is *odd*, then $\beta^R(M) \geqslant 2^d$.

(ii) If $\dfrac{\dim_K M}{e(R)}$ is *even* but not divisible by 6, then $\beta^R(M) \geqslant 3^{d/2} \geqslant 2^{0.79d}$.

(iii) If $\dfrac{\dim_K M}{e(R)}$ is divisible by 6, but not by 30, then $\beta^R(M) \geqslant 5^{d/4} \geqslant 2^{0.58d}$.

(iv) If $\dfrac{\dim_K M}{e(R)}$ is divisible by 30, but not by 60, then $\beta^R(M) \geqslant 2^{(d+1/2)}$.

A final remark on the scope of the Theorem:

The function $\dfrac{d(R) + p^m - 1}{d(R)p^m(p - 1)}\cdot\log_2 p$ decreases rapidly in each of its three variables $d(R)$, p, m, so that we are still far from the expected bound $\beta^R(M) \geqslant 2^{d(R)}$ for artinian modules in all generality. Nevertheless, if we fix the class of those modules M whose length is relatively prime to a given p, we get at least an *exponential* bound in d, whereas the best known general result is the quadratic estimate in d given in Proposition 1 above.

2. Here we give the **proof of Proposition 1.**

First remark that a local noetherian ring R admits an artinian module of finite projective dimension iff it is Cohen-Macaulay: this follows from the New Intersection Theorem, [Ro]. The projective dimension of such a module is then necessarily equal to the Krull dimension d of R and we shall systematically use the fact that the R-dual of an R-free resolution of an artinian module M is an R-free resolution of $M^\vee := \mathrm{Ext}_R^d(M, R)$, which is again an artinian R-module. In particular, $b_{d-i}^R(M) = b_i^R(M^\vee)$ for all i. In the sequel we set $b_i := b_i^R(M)$.

The Generalized Principal Ideal Theorem, [E-N, Thm. 1], applied to M and $M^\vee$ yields the inequalities

$$b_1 - b_0 + 1 \geqslant d,$$
$$b_{d-1} - b_d + 1 \geqslant d.$$

If one of the inequalities becomes an equality, the module M or its dual $M^\vee$ is a *generic module* in the sense of [B-E$_2$, Thm. 5.1.]. The minimal resolution of such modules is determined in (loc. cit.) and as a consequence the Betti numbers are known:

$$b_i = \binom{b_0+i-3}{i-2}\binom{b_1}{b_0+i-1} \quad \text{for } i \geqslant 2.$$

In particular,

$$b_i \geqslant \binom{b_1}{b_0+i-1} = \binom{b_0+d-1}{b_0+i-1} \quad \text{for } i \geqslant 2.$$

So

$$b_i \geqslant \binom{d}{i}$$

for all i and we may discard this case. We hence assume from now on

$$b_1 \geqslant b_0 + d \qquad \text{and} \qquad b_{d-1} \geqslant b_d + d.$$

The key ingredient in the rest of the proof is the *Syzygy Theorem* by G. Evans, P. Griffith [E-G$_1$, Cor. 3.15], which asserts

$$\sum_{j=i}^{\delta} (-1)^{j-i} b_j^R(N) \geqslant i \quad \text{for} \quad 0 \leqslant i \leqslant \delta - 1,$$

for any finitely generated module N of finite projective dimension δ over an equicharacteristic Cohen-Macaulay local ring R of dimension d.

For $M^\vee$ and M we have $\delta = d$, and applying the inequality for $i = d - 2$ in each case, one gets

$$b_2 - b_1 + b_0 \geqslant d - 2,$$
$$b_{d-2} - b_{d-1} + b_d \geqslant d - 2.$$

Combined with the inequalities above, this yields $b_2 \geqslant 2d - 2$ and $b_{d-2} \geqslant 2d - 2$. As a consequence,

$$(b_0 + b_1 + b_2) + (b_{d-2} + b_{d-1} + b_d) \geqslant 6d - 4 + 2(b_0 + b_d).$$

Consider first the case $b_0 + b_d = 2$, which is the minimal value. Then $b_0 = b_d = 1$ and M is the quotient of R by a Gorenstein ideal I, cf. [B-E$_1$]. Then by a result of E. Kunz [Ku], as extended in [C-E-M, Lemma 2], I is generated by at least $d + 2$ many elements, whence $b_1 \geqslant d + 2$ and $b_{d-1} \geqslant d + 2$. So either $b_0 + b_d = 2$ and $b_1 + b_{d-1} \geqslant 2d + 4$ or $b_0 + b_d \geqslant 3$. In either case we get

$$(b_0 + b_1 + b_2) + (b_{d-2} + b_{d-1} + b_d) \geqslant 6d + 2.$$

For $d = 5$ we are already done. Else, the syzygy theorem, applied to M and $M^\vee$ again, implies

$$b_i \geqslant \max\{2i + 1, 2(d - i) + i\} \quad \text{for} \quad 3 \leqslant i \leqslant d - 3.$$

For *odd d*, this gives

$$\sum_{i=3}^{d-3} b_i \geqslant \frac{1}{2}(3d^2 - 18d + 15),$$

whereas for *even d*, one obtains

$$\sum_{i=3}^{d-3} b_i \geqslant \frac{1}{2}(3d^2 - 18d + 14).$$

Adding in the remaining Betti numbers, we find

$$\beta^R(M) \geqslant \tfrac{1}{2}(3d^2 - 6d + 19) \quad \text{for } d \text{ odd},$$
$$\beta^R(M) \geqslant \tfrac{1}{2}(3d^2 - 6d + 18) \quad \text{for } d \text{ even}.$$

Finally, note that as M is artinian, the Euler-Poincaré characteristic $\Sigma_{j=0}^{d}(-1)^j b_j^R(M)$ vanishes, hence $\beta^R(M)$ is always even whereas

$3d^2 - 6d + 18 \equiv 2 \bmod 4$ for even d. It follows that we get a common estimate

$$\beta^R(M) \geqslant \tfrac{1}{2}(3d^2 - 6d + 19) = \tfrac{3}{2}(d-1)^2 + 8$$

for all $d \geqslant 5$. □

To end this section, we note that for $d = 5$ we[(1)] cannot rule out the existence of an artinian module M with sequence of Betti numbers (1, 6, 8, 8, 7, 2) which would give $\beta^R(\mathrm{M}) = 2^5$. This is in contrast with the results in [Ch-E-M], which show that for R Gorenstein and $d \leqslant 4$, one has $\beta^R(M) \geqslant 2^d + 2^{d-1}$, unless M is the quotient of R by a system of parameters.

3. The first step towards the **proof of Proposition 2** is a combinatorial analysis of Hilbert series. It does not require any "noetherian" hypothesis. Hence we may work in the following context.

Let $R = \oplus_{j \geqslant 0} R_j$ be a commutative, positively graded ring such that $R_0 = K$ is a field and R_i is a finite dimensional K-module for each i. We will consider only those R-modules $M = \oplus_{j \in \mathbb{Z}} M_j$ which are graded and satisfy $M_j = 0$ for all sufficiently negative degrees j, M_j a finite dimensional K-vectorspace for every j. Set $\iota(M) = \inf\{j \in \mathbb{Z} \cup \{\infty\} \mid M_j \neq 0\}$. As usual, $M(k)$ denotes the same ungraded R-module as M, but with the grading shifted, $M(k)_j = M_{k+j}$ for all $j \in \mathbb{Z}$. In particular, $\iota(M(k)) = \iota(M) - k$. For the associated Hilbert series one has that $H_{M(-k)}(t) = t^k H_M(t)$ and that the *order* of $H_M(t)$ equals $\iota(M)$. The formation of Hilbert series is *additive* on exact sequences.

The next result seems to be part of the mathematical folklore – in one disguise or the other – see for example [Ko], [P-S].

LEMMA 7. *Let M, N be R-modules.*

(i) *For each i, the graded R-module* $\mathrm{Tor}_i^R(M,N) = \oplus_{j \in \mathbb{Z}} \mathrm{Tor}_i^R(M, N)_j$ *has finite dimensional homogeneous components and*

$$\iota(\mathrm{Tor}_i^R(M, N)) \geqslant i + \iota(M) + \iota(N).$$

(ii) *Set* $\chi^R(M, N)(t) = \Sigma_i (-1)^i H_{\mathrm{Tor}_i^R(M,N)}(t)$. *This is a well-defined element of* $\mathbb{Z}((t))$ *and one has an equality of formal Laurent series*

$$\chi^R(M, N)(t) = \frac{H_M(t) \cdot H_N(t)}{H_R(t)}$$

Proof. It is well known and easy to see that every R-module M as above

[(1)]See note at the end of the paper.

admits a graded free resolution

$$\mathbb{F}(M) \equiv \cdots \to F_i \to F_{i-1} \to \cdots \to F_0(\to M \to 0)$$

where $F_i \cong \oplus_{j\in\mathbb{Z}} R(-j)^{b_{ij}}$ with $b_{ij} \in \mathbb{N}$ and $b_{ij} = 0$ for $j < i + \imath(M)$. Evaluating Hilbert series of the above resolution one gets

$$H_M(t) = \sum_{i,j\in\mathbb{Z}} (-1)^i b_{ij} H_R(t) t^j. \tag{3.1}$$

The sum on the right is well defined in $\mathbb{Z}((t))$, due to the vanishing of b_{ij} for $j < i + \imath(M)$.

Tensoring the resolution above with N over R yields the complex of R-modules

$$\mathbb{F}(M) \bigotimes_R N \equiv \cdots \to \bigoplus_{j\in\mathbb{Z}} N(-j)^{b_{ij}} \to \bigoplus_{j\in\mathbb{Z}} N(-j)^{b_{i-1,j}} \to \cdots.$$

Since $\operatorname{Tor}_i^R(M, N)_k = H_i(\mathbb{F}(M) \otimes_R N)_k$, we get

$$\begin{aligned}\dim_K \operatorname{Tor}_i^R(M, N)_k &\leqslant \dim_K\left(\left(\bigoplus_{j\in\mathbb{Z}} N(-j)^{b_{ij}}\right)_k\right)\\ &= \sum_{j\in\mathbb{Z}} b_{ij} \dim_K N_{k-j} < \infty,\end{aligned}$$

the last sum being finite as $b_{ij} = 0$ for $j < i + \imath(M)$ and $\dim_K N_{k-j} = 0$ for $k - j < \imath(N)$.

In particular, one sees that $\imath(\operatorname{Tor}_i^R(N, N)) \geqslant i + \imath(M) + \imath(N)$. This proves (i) and shows also that $\chi^R(M, N)(t)$ is indeed well defined.

Taking the alternating sum of Hilbert series of $\mathbb{F}(M) \otimes_R N$, we obtain from (3.1)

$$\sum_{i,j\in\mathbb{Z}} (-1)^i b_{ij} H_N(t) t^j = \frac{H_M(t) \cdot H_N(t)}{H_R(t)}.$$

By the invariance of Euler-Poincaré characteristics, this alternating sum equals the corresponding sum of the Hilbert series of the homology of the complex, that is, it equals $\chi^R(M, N)(t)$. □

Specializing to the case where $N = K$, the augmentation module of R, we set

$$\chi_M^R(t) = \chi^R(M, K)(t) = \sum_{i,j\in\mathbb{Z}} (-1)^i b_{ij} t^j$$

and obtain

$$H_M(t) = \chi_M^R(t) \cdot H_R(t). \tag{3.2}$$

This result can be found in many places, e.g. [Sm].

Proposition 2 is now a special case of the second part of the next result.

PROPOSITION 8. *Let R be generated by its elements of degree* 1.

(i) *For finitely generated R-modules M, N one has the following relation between multiplicity polynomials:*

$$e_M(t) \cdot e_N(t) = e_R(t) \cdot \chi^R(M, N)(t) \cdot (1 - t)^{d(M)+d(N)-d(R)}.$$

(ii) *If M is non-zero, finitely generated and of finite projective dimension over R, then $\chi_M^R(t)$ is a Laurent polynomial, $e_R(t)$ divides $e_M(t)$ in $\mathbb{Z}[t, t^{-1}]$ and with $e_M^R(t) = e_M(t)/e_R(t)$, one has an equality of Laurent polynomials,*

$$\chi_M^R(t) = e_M^R(t) \cdot (1 - t)^{d(R)-d(M)}.$$

Proof. (i) follows by comparing Lemma 7(ii) and (1.1). Taking $N = K$ in (i), we get

$$(1 - t)^{d(R)-d(M)} e_M(t) = e_R(t) \cdot \chi_M^R(t).$$

As each $\mathrm{Tor}_i^R(M, k)$ is a finite dimensional graded K-vector space and $\mathrm{Tor}_i^R(M, k) = 0$ for $i > \mathrm{projdim}_R M$, it follows that $\chi_M^R(t)$ is a Laurent polynomial as soon as M is of finite projective dimension. Since $d(R) \geqslant d(M)$ and $e_R(1) \neq 0$, assertion (ii) now follows from unique factorization in $\mathbb{Z}[t, t^{-1}]$. □

4. The **proof of Theorem 3** relies upon a more thorough investigation of multiplicity polynomials, using basic arithmetic.

Let n be any positive integer, $\zeta_n \in \mathbb{C}$ a primitive nth root of unity, and $\Phi_n(t) = \prod_{\gcd(i,n)=1} (t - \zeta_n^i)$ the nth cyclotomic polynomial. Recall that $\Phi_n(t)$ is an irreducible polynomial in $\mathbb{Z}[t]$ of degree $\varphi(n) = n(1 - (1/p_1)) \cdots (1 - (1/p_k))$, where p_i; $i = 1, \ldots, k$; are the different primes dividing n. Note that the prime factorization of $\Psi_n(t) = (t^n - 1)/(t - 1)$ in $\mathbb{Z}[t]$ is given by

$$\Psi_n(t) = \prod_{\substack{1 \neq d \\ d \mid n}} \Phi_d(t). \tag{4.1}$$

If $f(t) \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial, we define its nth *cyclotomic norm* as

$$N_n(f(t)) = \prod_{\gcd(i,n)=1} f(\zeta_n^i).$$

As $N_n(f(t)) = N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}(f(\zeta_n))$, and the norm of the algebraic integer $f(\zeta_n)$ in $\mathbb{Q}(\zeta_n)$ is a rational integer, N_n is a multiplicative function from $\mathbb{Z}[t, t^{-1}]$ to $\mathbb{Z}$. Note that $N_n(f(t)) = 0$ iff $f(t)$ is a multiple of $\Phi_n(t)$, due to the irreducibility of the cyclotomic polynomials in $\mathbb{Z}[t, t^{-1}]$.

The next Lemma contains those arithmetical results which will be needed to establish the Theorem.

LEMMA 9. *Let m be a nonnegative integer, $n, m_1, \dots, m_k$ positive integers, $p, p_1, \dots, p_k$ primes, and let $f(t) \neq 0$ be a Laurent polynomial over $\mathbb{Z}$.*

(i) $\Phi_n(1) = \begin{cases} p; & \textit{if } n = p^m \textit{ for some } m \geqslant 1, \\ 1; & \textit{otherwise.} \end{cases}$

(ii) $N_{p^{m+1}}(\Psi_{p^m}(t)) = p^{p^m - 1}$.

(iii) *If* $m = \max\{\mu \in \mathbb{Z} \mid \Psi_{p^\mu}(t) \text{ divides } f(t)\}$, *then* $p^{p^m - 1}$ *divides* $N_{p^{m+1}}(f(t))$ *and* $N_{p^{m+1}}(f(t)) \neq 0$.

Proof. (i) If $n = p$, then $\Phi_p(t) = 1 + t + \cdots + t^{p-1}$ and hence $\Phi_p(1) = p$. If $n = p_1^{m_1} \cdots p_k^{m_k}$ is the prime factorization of n, with different primes p_i, we argue by induction on $m = \Sigma_{i=1}^k m_i$. Write the product formula for $\Psi_n(t)$ as

$$\Psi_n(t) = \Phi_n(t) \cdot \prod_{\substack{1 \neq d \neq n \\ d \mid n}} \Phi_d(t),$$

set $t = 1$ and use the induction hypothesis.

(One may also use Moebius inversion directly on $n = \prod_{\substack{1 \neq d \\ d \mid n}} \Phi_d(1)$.)

(ii) Multiplicativity of the norm shows

$$N_{p^{m+1}}(\Psi_{p^m}(t)) = N_{p^{m+1}}(t^{p^m} - 1) \cdot [N_{p^{m+1}}(t - 1)]^{-1}. \tag{4.2}$$

As $\zeta_{p^{m+1}}^{p^m} - 1 = \zeta_p - 1$ is already in $\mathbb{Q}(\zeta_p)$, considering the tower of field extensions $\mathbb{Q}(\zeta_{p^m}) \supseteq \mathbb{Q}(\zeta_p) \supseteq \mathbb{Q}$, we get

$$\begin{aligned}
N_{p^{m+1}}(t^{p^m} - 1) &= N_{\mathbb{Q}(\zeta_{p^{m+1}})/\mathbb{Q}}(\zeta_{p^{m+1}}^{p^m} - 1) && \text{by definition,} \\
&= N_{\mathbb{Q}(\zeta_{p^{m+1}})/\mathbb{Q}}(\zeta_p - 1) \\
&= N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(N_{\mathbb{Q}(\zeta_{p^{m+1}})/\mathbb{Q}(\zeta_p)}(\zeta_p - 1)) && \text{by transitivity,} \\
&= N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}((\zeta_p - 1)^{p^m}) && \text{as } [\mathbb{Q}(\zeta_{p^{m+1}}) : \mathbb{Q}(\zeta_p)] = p^m, \\
&= (N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\zeta_p - 1))^{p^m} && \text{by multiplicativity,} \\
&= p^{p^m} && \text{as } \Phi_p(1) = p, \text{ by (i).}
\end{aligned}$$

For the second factor in (4.2), note that for any $v \geqslant 1$, $N_{\mathbb{Q}(\zeta_{p^v})/\mathbb{Q}}(\zeta_{p^v} - 1) = \Phi_{p^v}(1) = p$, again by (i). Now (ii) follows.

(iii) Writing $f(t) = \Psi_{p^m}(t) \cdot g(t)$ with $g(t) \in Z[t, t^{-1}]$, we conclude from (ii) and from the multiplicativity of the norm that $p^{p^m - 1}$ divides $N_{p^{m+1}}(f(t))$. As $\Psi_{p^{m+1}}(t) = \Psi_{p^m}(t) \cdot \Phi_{p^{m+1}}(t)$ by the product formula (4.1), $\Phi_{p^{m+1}}(t)$ does not divide $g(t)$. Hence $N_{p^{m+1}}(g(t)) \neq 0$ and the result follows. □

Now we can finish the proof of the theorem. Taking norms in Proposition 8(ii) for some positive integer n, we get

$$N_n(1 - t)^{d(R) - d(M)} \cdot N_n(e_M^R(t)) = N_n(\chi_M^R(t)).$$

But $N_n(1 - t) = \Phi_n(1)$ and

$$\begin{aligned} |N_n(\chi_M^R(t))| &= \prod_{\gcd(v,n)=1} \left| \sum_{i,j \in \mathbb{Z}} (-1)^i \dim_K(\operatorname{Tor}_i^R(M, K))_j \zeta_n^{jv} \right| \\ &\leqslant \prod_{\gcd(v,n)=1} \left| \sum_{i,j \in \mathbb{Z}} \dim_K(\operatorname{Tor}_i^R(M, K))_j \right| \\ &= \dim_K(\operatorname{Tor}_*^R(M, K))^{\varphi(n)} = \beta^R(M)^{\varphi(n)}. \end{aligned}$$

Thus we obtain the inequality

$$\beta^R(M) \geqslant \Phi_n(1)^{\frac{d(R) - d(M)}{\varphi(n)}} \cdot |N_n(e_M^R(t))|^{\frac{1}{\varphi(n)}}.$$

For $n = p^{m+1}$, where p and m are chosen as in the statement of the theorem, one has $\varphi(n) = p^m(p - 1)$, $\Phi_n(1) = p$ by Lemma 9(i), and $N_n(e_M^R(t)) = p^{p^m - 1} \cdot a$ with a non-zero integer a by Lemma 9(iii). This establishes the desired lower bound.

□

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(1) As M. Miller informed us after reading the preprint, he is able to rule out the sequence of Betti numbers (1, 6, 8, 8, 7, 2) by more or less the same argument as in [C-E-M], using double linkage and the multiplicative structure on $\operatorname{Tor}_*^R(R/I, k)$, where $M = R/I$ is the suspected module. The first case he believes one cannot rule out yet is the sequence (1, 6, 9, 10, 8, 2) with total Betti number equal to 36. This sequence still falls short of both $b_2^R(M) \geqslant \binom{5}{2} = 10$ and $\beta^R(M) \geqslant 2^5 + 2^4 = 48$.

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