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Foliation of phase space for the cubic non-linear Schrödinger equation

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1. Introduction and theorems

Consider the defocussing cubic non-linear Schrödinger equation (NLS)

$$i\frac{\partial\psi}{\partial t}(x, t) = -\frac{\partial^2\psi}{\partial x^2}(x, t) + 2|\psi(x, t)|^2\psi(x, t)$$

for complex valued function ψ with periodic boundary conditions $\psi(x + 1, t) = \psi(x, t)$. It is well known that (NLS) is a completely integrable infinite dimensional Hamiltonian system. The periodic eigenvalues of the corresponding self-adjoint $AKNS$-system are invariant under the flow of (NLS), where the $AKNS$-system is given by

$$(H(p, q)F)(x) = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} -q(x, t) & p(x, t) \\ p(x, t) & q(x, t) \end{pmatrix}\right] F(x)$$

with $\psi(x, t) = p(x, t) - iq(x, t)$. Define for $N \in \mathbb{N}$

$$\mathscr{H}^N = \{(p, q) \in H^N_{\mathbb{R}}([0, 1])^2 / p^{(j)}(0) = p^{(j)}(1),\ q^{(j)}(0) = q^{(j)}(1) \text{ for } j = 0, \ldots, N - 1\}.$$

For $N \geqslant 1$ the Liouville tori of (NLS) in the phase space $\mathscr{H}^N$ are the isospectral sets

$$\mathrm{Iso}_N(p, q) = \{(\tilde{p}, \tilde{q}) \in \mathscr{H}^N / H(\tilde{p}, \tilde{q}) \text{ has the same periodic spectrum as } H(p, q)\}.$$

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For every N, $\mathrm{Iso}_N(p, q)$ is compact, connected and generically an infinite product of circles.

For $(p, q) \in \mathscr{H}^N$ $(N = 0, 1)$ let $\{\lambda_k(p, q)\}_{k\in\mathbb{Z}}$ be the periodic and antiperiodic spectrum of $H(p, q)$. One knows that the gap length map γ from $\mathscr{H}^1$ into l^2_N defined as

$$(p, q) \xrightarrow{\gamma} (\gamma_k(p, q) = \lambda_{2k}(p, q) - \lambda_{2k-1}(p, q))_{k\in\mathbb{Z}}$$

is continuous (but not analytic), onto and $\gamma^{-1}(\gamma(p, q)) = \mathrm{Iso}_1(p, q)$, where $l^2_N = \{(a_k)_{k\in\mathbb{Z}} / \sum_{k\in\mathbb{Z}} k^{2N}|a_k|^2 < \infty\}$ $(N \geqslant 0)$. (see [Gre-Gui]).

In Appendix A we prove

THEOREM 1.1. (1) *The gap-length map* $\gamma: \mathscr{H}^0 \to l^2$ *is continuous and*

$$\gamma^{-1}(\gamma(p, q)) = \mathrm{Iso}_0(p, q)$$

(2) $\|(p, q)\|_{\mathscr{H}^0}$ *is a spectral invariant, i.e. constant on* $\mathrm{Iso}_0(p, q)$.

Knowing the Dirichlet-spectrum $\{\mu_k(t)\}_{k\in\mathbb{Z}}$ of the operator $H(T_t p, T_t q)$, where $(T_t f)(x) = f(x + t)$ one can reconstruct p and q by the trace formulas

$$p(t) = -\sum_{k\in\mathbb{Z}} \frac{1}{2}(\lambda_{2k} + \lambda_{2k-1}) - \tilde{\mu}_k(t)$$

$$q(t) = \sum_{k\in\mathbb{Z}} \frac{1}{2}(\lambda_{2k} + \lambda_{2k-1}) - \mu_k(t).$$

Here $\{\tilde{\mu}_k(t)\}_{k\in\mathbb{Z}}$ is the Dirichlet-spectrum of $H(T_t q, -T_t p)$. The dependence of t of $\{\mu_k(t)\}_{k\in\mathbb{Z}}$ is given (see [Gre-Gui]) by a system of singular differential equations. For finite gap potentials $\mu_k(t)$ can be explicitly calculated by geometric methods (see [Pre]). In this article we compute the image of $\mu_k(\cdot)$, or equivalently the image of the flow by translation T_t on $\mathrm{Iso}(p, q)$, for non-finite gap potentials. To do this we introduce the space

$$\mathscr{M}^N = \{(R_k)_{k\in\mathbb{Z}} / R_k \text{ is a } 2\times 2 \text{ symmetric, real, trace-free matrix with } \sum_{k\in\mathbb{Z}} k^{2N}\|R_k\|^2 < \infty\}.$$

and a map $\det_N$ from $\mathscr{M}^N$ into l^2_N defined as

$$(R_k)_{k\in\mathbb{Z}} \xrightarrow{\det_N} \{2(-\det R_k)^{1/2}\}_{k\in\mathbb{Z}}.$$

We will prove

THEOREM 1.2. *For* $N = 0, 1$ *there exists a real analytic one-to-one map* Φ *from*

$\mathscr{H}^N$ into $\mathscr{M}^N$ with $\Phi(\mathrm{Iso}_N(p, q)) = \det_N^{-1}(\det_N(\Phi(p, q)))$. For $N = 1$, Φ is onto and bianalytic.

This theorem gives a geometrical description of the "foliation" $\mathrm{Iso}_N(p, q)$ in $\mathscr{H}^N$. A similar theorem for the KdV equation has been proved by T. Kappeler in [Kp]. In section 2 we construct the map Φ using results from [Gre-Gui] and [Kp]. Theorem 1.2 follows immediately as in [Kp] using arguments from [Gar-Tru, 1, 2] and

THEOREM 1.3. *The derivative of Φ at (p, q) is an isomorphism from $\mathscr{H}^N$ to $\mathscr{M}^N$ $(N = 0, 1)$.*

Theorem 1.3 is proven in section 3.

Let $\Phi = (\Phi_k)_{k\in\mathbb{Z}}$. The above mentioned result concerning the flow by translation is now a consequence of Theorem 1.2 and proved at the end of Section 2:

THEOREM 1.4. *Suppose $(p, q)\in\mathscr{H}^0$ (resp. $\mathscr{H}^1$). Then for every k with $\lambda_{2k-1}(p, q) < \lambda_{2k}(p, q)$ there exists a continuous (resp. cont. differentiable) function $\varphi_k(\cdot)\colon \mathbb{R} \to \mathbb{R}$ such that*

$$\Phi_k(T_t p, T_t q) = \frac{\gamma_k(p, q)}{2}\begin{pmatrix}\cos 2\varphi_k(t) & \sin 2\varphi_k(t)\\ \sin 2\varphi_k(t) & -\cos 2\varphi_k(t)\end{pmatrix}$$

with $\varphi_k(t + 1) - \varphi_k(t) = k\pi$ for every $t\in\mathbb{R}$.

This shows that the image of $\mu_k(\cdot)$ by the flow of translation consists, for all $k \neq 0$, of the whole gap $[\lambda_{2k-1}(p, q), \lambda_{2k}(p, q)]$.

Similarly as in [Kp] for KdV Theorem 1.2 can be applied to the so called finite gap potentials. Define, for a finite subset $J \subseteq \mathbb{Z}$,

$$\mathrm{Gap}_J := \{(p, q)\in\mathscr{H}^0 : \lambda_{2n-1}(p, q) = \lambda_{2n}(p, q),\ n\notin J\} \text{ and}$$

$$\mathrm{Gap}_{J,r} := \{(p, q)\in\mathrm{Gap}_J : \lambda_{2n-1}(p, q) < \lambda_{2n}(p, q),\ n\in J\}.$$

Elements in $\mathrm{Gap}_{J,r}$ are called regular J-gap potentials. It is well known that the potentials in Gap_J are, in fact, real analytic. Further, observe that $\mathrm{Gap}_J = \Phi^{-1}\{R = (R_k)_{k\in\mathbb{Z}}\in\mathscr{M}^0 : R_k = 0\ \forall k\notin J\}$ and thus Gap_J is a $2N$ dimensional manifold where $N = \#J$. Clearly $\mathrm{Gap}_{J,r}$ is open in Gap_J and $\Phi(\mathrm{Gap}_{J,r}) = (\mathbb{R}^+)^N \times T^N$ (diffeomorphically) where $\mathbb{R}^+ := \{x : x > 0\}$ and T^N denotes the N-torus $(S^1)^N$. Obviously $\mathrm{Gap}_{J,r}$ is invariant by NLS. Therefore, with the symplectic structure coming from NLS, it follows from Theorem 1.2 that $(\mathbb{R}^+)^N \times T^N$ is a symplectic manifold of dimension $2N$ with a trivial fibration by Lagrangian tori T^N. We thus obtain (cf. [Dui])

COROLLARY 1.5. *When restricted to $\mathrm{Gap}_{J,r}$, NLS admits global action-angle variables.*

2. Global coordinates on $\mathscr{H}^N$

We first define the map Φ mentioned in the introduction.

If $\lambda_{2k-1}(p, q) \neq \lambda_{2k}(p, q)$ ($k \in \mathbb{Z}$) one denotes by $F_{2k-1}(\cdot\,; p, q)$ and $F_{2k}(\cdot\,; p, q)$ the two corresponding eigenfunctions of $H(p, q)$ such that, for $j = 2k-1, 2k$

(i) $\|F_j(\cdot\,; p, q)\|_{L^2([0,1])^2} = 1$
(ii) If $F_j^{(1)}(0; p, q) \neq 0$ then $F_j^{(1)}(0; p, q) > 0$
If $F_j^{(1)}(0; p, q) = 0$ then $F_j^{(2)}(0; p, q) > 0$

If $\lambda_{2k-1}(p, q) = \lambda_{2k}(p, q)$ then $F_{2k-1}(\cdot\,; p, q)$ and $F_{2k}(\cdot\,; p, q)$ are two orthonormal eigenfunctions such that

(i) $F_{2k-1}^{(1)}(0; p, q) = 0$ and $F_{2k-1}^{(2)}(0; p, q) > 0$
(ii) If $F_{2k}^{(2)}(0; p, q) \neq 0$ then $F_{2k}^{(2)}(0; p, q) > 0$
If $F_{2k}^{(2)}(0; p, q) = 0$ then $F_{2k}^{(1)}(0; p, q) > 0$

As the eigenvalues λ_j are periodic or antiperiodic one has

$$F_j(x+1; p, q) = (-1)^k F_j(x; p, q).$$

Let $E_k(p, q)$ be the two-dimensional subspace of L^2 generated by F_{2k-1} and F_{2k}.

As in [Kp], in order to introduce an orthonormal basis $(G_{2k-1}(\cdot\,; p, q), G_{2k}(\cdot\,; p, q))$ of $E_k(p, q)$ depending analytically on $(p, q) \in \mathscr{H}^0$ one needs the following lemma.

LEMMA 2.1. *For $(p, q) \in \mathscr{H}^0$ and for every $k \in \mathbb{Z}$ the map*

$$F \mapsto (F^{(1)}(0), F^{(2)}(0))$$

from $E_k(p, q)$ into $\mathbb{R}^2$ is a linear isomorphism.

Before proving Lemma 2.1, let us introduce some more notations and a few elementary results from [Gre-Gui] which will be used later.

Denote by

$$F_j(x, \lambda; p, q) = \begin{pmatrix} Y_j(x, \lambda; p, q) \\ Z_j(x, \lambda; p, q) \end{pmatrix} \qquad j = 1, 2$$

the fundamental solutions to $H(p, q)F_j = \lambda F_j$ such that

$$F_1(0, \lambda; p, q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad F_2(0, \lambda; p, q) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The $\mu_k(p, q)$'s (resp. $\nu_k(p, q)$'s) are the simple zeroes of $Z_1(1, \cdot\,; p, q)$ (resp. $Y_2(1, \cdot\,;$

$p, q)$) in $\mathbb{C}$. $(\mu_k(p, q))_{k\in\mathbb{Z}}$(resp. $(\nu_k(p, q))_{k\in\mathbb{Z}}$) is a strictly increasing sequence of real numbers.

Further

$$\lambda_{2k-1}(p, q) \leqslant \mu_k(p, q), \nu_k(p, q) \leqslant \lambda_{2k}(p, q), \quad k \in \mathbb{Z}.$$

Denote by $\Delta(\lambda)$ the discriminant

$$\Delta(\lambda) = \Delta(\lambda; p, q) = Y_1(1, \lambda; p, q) + Z_2(1, \lambda; p, q).$$

The collection of periodic and antiperiodic eigenvalues $(\lambda_k(p, q))_{k\in\mathbb{Z}}$ written in increasing order and with multiplicities have the following asymptotics

$$\lambda_{2k}(p, q) = k\pi + l^2(k)$$

and

$$\lambda_{2k-1}(p, q) = k\pi + l^2(k)$$

where the error terms are uniform on bounded sets of potentials $(p, q) \in L^2([0, 1])^2$.

It follows that for $j = 2k - 1, 2k$

$$F_1(x, \lambda_j; p, q) = \begin{pmatrix} \cos \lambda_j x \\ -\sin \lambda_j x \end{pmatrix} + l^2(k)$$

and

$$F_2(x, \lambda_j; p, q) = \begin{pmatrix} \sin \lambda_j x \\ \cos \lambda_j x \end{pmatrix} + l^2(k).$$

Finally, for $\lambda_{2k-1}(p, q) < \lambda_{2k}(p, q)$ one has ($j = 2k - 1, 2k$)

$$\begin{aligned} F_j(x; p, q) = {} & \left(\frac{-Y_2(1, \lambda_j(p, q))}{\dot{\Delta}(\lambda_j(p, q))}\right)^{1/2} F_1(x, \lambda_j(p, q); p, q) \\ & + \varepsilon_j(p, q) \left(\frac{Z_1(1, \lambda_j(p, q))}{\dot{\Delta}(\lambda_j(p, q))}\right)^{1/2} F_2(x, \lambda_j(p, q); p, q) \end{aligned}$$

where $\varepsilon_j(p, q) = \pm 1$.

Proof of Lemma 2.1.

Fix k and (p, q). It suffices to show that

$$W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))(0) \neq 0.$$

where

$$\begin{aligned} &W(F_{2k}(\cdot; p, q), F_{2k-1}(\cdot; p, q))(x) \\ &\quad = F_{2k}^{(1)}(x; p, q)F_{2k-1}^{(2)}(x; p, q) - F_{2k}^{(2)}(x; p, q)F_{2k-1}^{(1)}(x; p, q) \end{aligned}$$

is the Wronskian of F_{2k} and F_{2k-1}. Using the equation $H(p,q)F_j = \lambda_j F_j$ one derives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} W(F_{2k}, F_{2k-1})(x) \\ &\quad = (\lambda_{2k} - \lambda_{2k-1})(F_{2k}^{(1)}(x)F_{2k-1}^{(1)}(x) + F_{2k}^{(2)}(x)F_{2k-1}^{(2)}(x)) \end{aligned}$$

(cf. [Gre-Gui]).

Thus, if $\lambda_{2k} = \lambda_{2k-1}$, we conclude that $W(F_{2k}, F_{2k-1})$ is constant. As F_{2k} and F_{2k-1} are linearly independent, this constant is different from zero. In the case where $\lambda_{2k-1} < \lambda_{2k}$ we first show that $W(F_{2k}, F_{2k-1})(x)$ has at most simple zeroes. Assume that this is not the case. Then there exists $0 \leqslant x_0 \leqslant 1$ and $0 \leqslant \varphi(x_0) \leqslant 2\pi$ such that

$$\begin{aligned} &F_{2k}^{(1)}(x_0)F_{2k-1}^{(2)}(x_0) - F_{2k}^{(2)}(x_0)F_{2k-1}^{(1)}(x_0) \\ &\quad = |F_{2k}(x_0)|\,|F_{2k-1}(x_0)|\sin\varphi(x_0) = 0 \end{aligned}$$

and

$$\begin{aligned} &F_{2k}^{(1)}(x_0)F_{2k-1}^{(1)}(x_0) + F_{2k}^{(2)}(x_0)F_{2k-1}^{(2)}(x_0) \\ &\quad = |F_{2k}(x_0)|\,|F_{2k-1}(x_0)|\cos\varphi(x_0) = 0 \end{aligned}$$

where here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$.

But both $|F_{2k}(x_0)| \neq 0$ and $|F_{2k-1}(x_0)| \neq 0$ which leads to a contradiction.

Let us consider the smooth path (tp, tq) in $\mathscr{H}^0$. Denote by $t_0 = \max\{0 \leqslant t \leqslant 1;\ \lambda_{2k}(tp, tq) = \lambda_{2k-1}(tp, tq)\}$. Then $0 \leqslant t_0 < 1$. Choose L^2-normalized eigenfunctions $\tilde{F}_{2k}(\cdot, tp, tq)$ and $\tilde{F}_{2k-1}(\cdot, tp, tq)$ such that for $t = 1$, $\tilde{F}_{2k}(\cdot, p, q) = F_{2k}(\cdot, p, q)$ and $\tilde{F}_{2k-1}(\cdot, p, q) = F_{2k-1}(\cdot, p, q)$ and $\tilde{F}_{2k}$ and $\tilde{F}_{2k-1}$ are continuous in t, i.e. $\tilde{F}_{2k}$ and $\tilde{F}_{2k-1} \in C([t_0, 1], (H^1[0,1])^2)$. In particular we conclude that $\tilde{F}_{2k}(\cdot; t_0p, t_0q)$ and $\tilde{F}_{2k-1}(\cdot; t_0p, t_0q)$ are L^2-normalized orthogonal eigenfunctions for $\lambda_{2k}(t_0p, t_0q)$. We conclude that for $t = t_0$

$W(\tilde{F}_{2k}, \tilde{F}_{2k-1})$ is constant and different from zero. Clearly $W(t,x) := W(\tilde{F}_{2k}(\cdot, tp, tq), \tilde{F}_{2k-1}(\cdot, tp, tq))(x)$ is continuous in $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant 1$. To simplify notation assume that $W(t_0, x) > 0$ $(0 \leqslant x \leqslant 1)$. For fixed $t_0 \leqslant t \leqslant 1$, $W(t,x)$ can have at most simple zeroes and thus by a classical argument from homotopy theory we conclude that $W(t,x)$ can never vanish for $0 \leqslant x \leqslant 1$ and $t_0 \leqslant t \leqslant 1$ and Lemma 2.1 is proved.

We use Lemma 2.1 to define $G_{2k-1}(\cdot; p, q)$ as the unique function in $E_k(p,q)$ satisfying

(i) $\|G_{2k-1}(\cdot;\, p,\, q)\|_{L^2([0,1])^2} = 1$
(ii) $G_{2k-1}^{(1)}(0;\, p,\, q) = 0$ and $G_{2k-1}^{(2)}(0;\, p,\, q) > 0$.

$G_{2k}(\cdot;\, p,\, q)$ is then defined to be the unique function in $E_k(p,q)$ such that

(i) $\|G_{2k}(\cdot;\, p,\, q)\|_{L^2([0,1])^2} = 1$; $G_{2k}^{(1)}(0;\, p,\, q) > 0$
(ii) $(G_{2k}(\cdot;\, p,\, q),\, G_{2k-1}(\cdot;\, p,\, q))_{L^2([0,1])^2} = 0$

Clearly, G_{2k} and G_{2k-1} can be expressed in terms of F_{2k} and F_{2k-1}. There exist a unique $\theta_k(p,q) \in [0, 2\pi)$ such that

$$\begin{pmatrix} G_{2k}(\cdot;\, p,\, q) \\ G_{2k-1}(\cdot;\, p,\, q) \end{pmatrix} = \begin{pmatrix} \cos\theta_k(p,\, q) & -\sin\theta_k(p,\, q) \\ \sin\theta_k(p,\, q) & \cos\theta_k(p,\, q) \end{pmatrix} \begin{pmatrix} F_{2k}(\cdot;\, p,\, q) \\ \varepsilon_k F_{2k-1}(\cdot;\, p,\, q) \end{pmatrix}$$

where $\varepsilon_k = \operatorname{sign} W(F_{2k}(\cdot;\, p,\, q),\, F_{2k-1}(\cdot;\, p,\, q))(0)$.

Using a perturbation argument (cf. [Ka]) one proves as in [Kp] that $G_{2k}(\cdot; p, q)$ and $G_{2k-1}(\cdot; p, q)$ are both analytic functions of (p,q) as maps from $(L^2([0,1]))^2$ into $(H^1_{\mathbb{R}}([0,1]))^2$.

F_{2k} and F_{2k-1} are eigenfunctions of $H(p,q)$ but they cannot depend analytically on (p,q) due to possible multiplicity of the eigenvalue λ_{2k}. G_{2k} and G_{2k-1} are not necessarily eigenfunctions but they depend analytically on (p,q).

For $(p,q) \in \mathscr{H}^N$ $(N = 0, 1)$ and for $k \in \mathbb{Z}$ define

$$\Phi_k(p,\, q) =$$
$$\begin{pmatrix} (G_{2k}(\cdot),\, (H - \tau_k)G_{2k}(\cdot))_{L^2([0,1])^2} & (G_{2k}(\cdot),\, (H - \tau_k)G_{2k-1}(\cdot))_{L^2([0,1])^2} \\ (G_{2k-1}(\cdot),\, (H - \tau_k)G_{2k}(\cdot))_{L^2([0,1])^2} & (G_{2k-1}(\cdot),\, (H - \tau_k)G_{2k-1}(\cdot))_{L^2([0,1])^2} \end{pmatrix}$$

where $\tau_k = (\lambda_{2k} + \lambda_{2k-1})/2$. One easily shows that

$$\Phi_k(p,q) = \frac{\gamma_k(p,\, q)}{2} \begin{pmatrix} \cos 2\theta_k(p,\, q) & \sin 2\theta_k(p,\, q) \\ \sin 2\theta_k(p,\, q) & -\cos 2\theta_k(p,q) \end{pmatrix}$$

where $\gamma_k(p,\, q) = \lambda_{2k}(p,\, q) - \lambda_{2k-1}(p,\, q)$.

The matrix $\Phi_k(p,q)$ is symmetric and its trace is zero. Its eigenvalues are

$\pm[\gamma_k(p,q)/2]$. For every $k\in\mathbb{Z}$, $\Phi_k(\cdot,\cdot)$ is a compact map from $\mathscr{H}^0$ into the space of real symmetric trace free matrices. (See [Kp] for a proof.)

Furthermore it is proved in [Gre-Gui] that $(\gamma_k(p,q))_{k\in\mathbb{Z}}\in l^2(\mathbb{Z})$ (resp. $l_1^2(\mathbb{Z})$) for $(p,q)\in\mathscr{H}^0$ (resp. $\mathscr{H}^1$) and, for $N=0,\ 1$, $\sum_k \gamma_k(p,q)^2k^{2N}<\infty$ uniformly on bounded sets of potentials in $\mathscr{H}^N$.

DEFINITION 2.2. For $(p,q)\in\mathscr{H}^N$ set

$$\Phi(p,q)=(\Phi_k(p,q))_{k\in\mathbb{Z}}.$$

It follows that $\Phi(\cdot,\cdot)$ is a bounded map from $\mathscr{H}^N$ $(N=0,1)$ into $\mathscr{M}^N$.

As in [Kp] one shows that $\Phi(\cdot,\cdot)$ is real analytic. Furthermore $\Phi(\cdot,\cdot)$ preserves isospectrality in the following sense: $\Phi(p,q)$ and $\Phi(p',q')$ are isospectral, i.e., $\operatorname{spec}\Phi_k(p,q)=\operatorname{spec}\Phi_k(p',q')$ for every k, if and only if $\gamma_k(p,q)=\gamma_k(p',q')$ for every k. It is shown in [Gre-Gui] that, for (p,q) and (p',q') in $\mathscr{H}^1$, $\gamma_k(p,q)=\gamma_k(p',q')$ for every k implies $\lambda_k(p,q)=\lambda_k(p',q')$ for every k. For (p,q) and (p',q') in $\mathscr{H}^0$ the same conclusion follows from Appendix A (see Corollary A.4) by the same argument given for the case $N=1$ in [Gre-Gui].

REMARK 2.3. $\mathscr{M}^0$ (resp. $\mathscr{M}^1$) can be identified with $l^2(\mathbb{Z})$ (resp. $l_1^2(\mathbb{Z})$) by the map

$$\left(\frac{\gamma_k(p,q)}{2}\cos 2\theta_k(p,q),\frac{\gamma_k(p,q)}{2}\sin 2\theta_k(p,q)\right)$$

$$\to c_k(p,q)=\frac{\gamma_k(p,q)}{2}\,e^{2i\theta_k(p,q)}\qquad k\in\mathbb{Z}.$$

It then follows that for $(p,q)\in\mathscr{H}^N$ with $N=0,1$

$$\sum_{k\in\mathbb{Z}}k^{2N}\|\Phi_k(p,q)\|^2=\sum_{k\in\mathbb{Z}}k^{2N}|c_k|^2<\infty.$$

In particular $\Phi(\cdot,\cdot)$ coordinatizes $\mathscr{H}^N$ globally.

It follows that for $(p_0,q_0)\in\mathscr{H}^N$

$$\Phi(\mathrm{Iso}_N(p_0,q_0))=\{(c_k)_{k\in\mathbb{Z}}\in l_N^2(\mathbb{Z})\,|\,|c_k|=|c_k(p_0,q_0)|,\ k\in\mathbb{Z}\}.$$

One recovers the well-known result that $\mathrm{Iso}_N(p_0,q_0)$ is a compact set, generically an infinite product of circles, the radii of which are in $l_N^2(\mathbb{Z})$.

We now prove Theorem 1.4. Following [Kp, Thm. 4] one easily shows that there exists a continuous (resp. continuously differentiable in the case

$(p, q) \in \mathscr{H}^1$) function $\psi_k(t, s)$ such that

$$\begin{aligned} G_{2k-1}(x; sT_tp, sT_tq) &= \cos\psi_k(t, s)\tilde{F}_{2k-1}(x + t; sp, sq) \\ &\quad + \sin\psi_k(t, s)\tilde{F}_{2k}(x + t; sp, sq) \\ G_{2k}(x; sT_tp, sT_tq) &= -\sin\psi_k(t, s)\tilde{F}_{2k-1}(x + t; sp, sq) \\ &\quad + \cos\psi_k(t, s)\tilde{F}_{2k}(x + t; sp, sq) \end{aligned}$$

for $(t, s) \in [0, 1]^2$ where, for $s_0 \leqslant s \leqslant 1$, $\tilde{F}_{2k}(\cdot; sp, sq)$ and $\tilde{F}_{2k-1}(\cdot; sp, sq)$ are chosen as in the proof of Lemma 2.1 with $s_0 = \max\{0 \leqslant s < 1; \lambda_{2k}(sp, sq) = \lambda_{2k-1}(sp, sq)\}$. Taking possible crossings of the eigenvalues $\lambda_{2k}(sp, sq)$ and $\lambda_{2k-1}(sp, sq)$ into account (cf. [Ka]), $\tilde{F}_{2k}(\cdot; sp, sq)$ and $\tilde{F}_{2k-1}(\cdot; sp, sq)$ can be chosen to depend smoothly on s, $0 \leqslant s \leqslant s_0$, if one allows $\tilde{F}_{2k}(\cdot; sp, sq)$ to be either a (normalized) eigenfunction for $\lambda_{2k}(sp, sq)$ or $\lambda_{2k-1}(sp, sq)$ and similarly for $\tilde{F}_{2k-1}(\cdot; sp, sq)$.

Define $\varphi_k(t) := \psi_k(t, 1)$ and the winding numbers $h_k(s) := (\psi_k(1 + t, s) - \psi_k(t, s))/\pi$, $h_k(\cdot)$ being a continuous function of s with values in $\mathbb{Z}$. Therefore $h_k(s) = h_k(0) = k$ for every $s \in [0, 1]$ and thus $\varphi_k(1 + t) - \varphi_k(t) = k\pi$.

REMARK 2.4. For $(p, q) \in \mathscr{H}^1$ one shows that

$$\operatorname{sign} \frac{d\varphi_k}{dt}(t) = \operatorname{sign}(\lambda_{2k-1} + q(t)).$$

Then, for $|k|$ sufficiently large, one has

$$\frac{d\varphi_k}{dt}(t) > 0 \text{ if } k > 0 \quad \text{and} \quad \frac{d\varphi_k(t)}{dt} < 0 \text{ if } k < 0$$

i.e. $\Phi_k(T_tp, T_tq)$ winds $|k|$ times around the origin without stopping, clockwise if $k < 0$ and counterclockwise if $k > 0$.

3. The derivative of Φ

In this section we compute the derivative of Φ and show that it is a linear isomorphism from $\mathscr{H}^N$ onto $\mathscr{M}^N$ for $N = 0, 1$.

As in [Kp] it is convenient to write Φ in a slightly different form. One writes Φ as a map Ψ from $\mathscr{H}^N$ into $l^2_N(\mathbb{Z})$ (see Remark 2.3) with $\Psi(p, q) = (\Psi_k(p, q))_{k\in\mathbb{Z}}$

where

$$\Psi_{2k-1}(p, q) = (G_{2k-1}(\cdot; p, q), (H - \tau_k(p, q))G_{2k-1}(\cdot; p, q))_{L^2([0,1])^2}$$
$$\Psi_{2k}(p, q) = (G_{2k}(\cdot; p, q), (H - \tau_k(p, q))G_{2k-1}(\cdot; p, q))_{L^2([0,1])^2}.$$

Let $d_{(p,q)}\Psi_{2k}$ (resp. $d_{(p,q)}\Psi_{2k-1}$) denote the derivative of $\Psi_{2k}(\cdot, \cdot)$ (resp. $\Psi_{2k-1}(\cdot, \cdot)$).

THEOREM 3.1. *Suppose* $(u, v) \in \mathscr{H}^0$. *Then*

$$\begin{aligned}
&\mathrm{d}_{(p,q)}\Psi_{2k-1}[(u, v)] \\
&\quad = 2\Psi_{2k}(p, q) \int_0^1 \mathrm{d}_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)](x) \cdot G_{2k}(x; p, q)\,\mathrm{d}x \\
&\qquad + \frac{1}{2}\int_0^1 (G^{(2)}_{2k-1}(x; p, q)^2 - G^{(1)}_{2k-1}(x; p, q)^2 + G^{(1)}_{2k}(x; p, q)^2 \\
&\qquad - G^{(2)}_{2k}(x; p, q)^2)v(x)\,\mathrm{d}x + \int_0^1 (G^{(1)}_{2k-1}(x; p, q)G^{(2)}_{2k-1}(x; p, q) \\
&\qquad - G^{(1)}_{2k}(x; p, q)G^{(2)}_{2k}(x; p, q))u(x)\,\mathrm{d}x \\
&d_{(p,q)}\Psi_{2k}[(u, v)] \\
&\quad = -2\Psi_{2k-1}(p, q) \int_0^1 d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)](x) \\
&\qquad \cdot G_{2k}(x; p, q)\,\mathrm{d}x + \int_0^1 (-G^{(1)}_{2k}(x; p, q)G^{(1)}_{2k-1}(x; p, q) \\
&\qquad + G^{(2)}_{2k}(x; p, q)G^{(2)}_{2k-1}(x; p, q))v(x)\,\mathrm{d}x \\
&\qquad + \int_0^1 (G^{(1)}_{2k}(x; p, q)G^{(2)}_{2k-1}(x; p, q) \\
&\qquad + G^{(2)}_{2k}(x; p, q)G^{(1)}_{2k-1}(x; p, q))u(x)\,\mathrm{d}x
\end{aligned}$$

where '$\cdot$' denotes the scalar product in $\mathbb{R}^2$.

Proof of Theorem 3.1. The derivative $d_{(p,q)}\Psi_{2k-1}[(u, v)]$ is given by

$$\begin{aligned}
&d_{(p,q)}\Psi_{2k-1}[(u, v)] \\
&\quad = (d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)], (H - \tau_k)G_{2k-1}(\cdot; p, q)) \\
&\qquad + (G_{2k-1}(\cdot; p, q), (H - \tau_k)d_{(p,q)}G_{2k-1}(\cdot; p, q)[(u, v)](\cdot)) \\
&\qquad + (G_{2k-1}(\cdot; p, q), d_{(p,q)}(H - \tau_k)[(u, v)](\cdot) \cdot G_{2k-1}(\cdot; p, q)).
\end{aligned}$$

The chosen normalization of G_k imply that

$$(d_{(p,q)}G_k(\cdot;\, p,\, q),\, G_k(\cdot;\, p,\, q)) = 0, \qquad k \in \mathbb{Z}.$$

Further

$$(H - \tau_k(p,\, q))G_{2k-1}(x;\, p,\, q) = -\frac{\gamma_k(p,\, q)}{2}\cos 2\theta_k(p,\, q)G_{2k-1}(x;\, p,\, q)$$
$$+\frac{\gamma_k(p,\, q)}{2}\sin 2\theta_k(p,\, q)G_{2k}(x;\, p,\, q).$$

One then gets

$$\begin{aligned}
&d_{(p,q)}\Psi_{2k-1}[(u,\, v)] \\
&\quad = \Psi_{2k}(p,\, q)(G_{2k}(\cdot;\, p,\, q),\, d_{(p,q)}G_{2k-1}(\cdot;\, p,\, q)[(u,\, v)](\cdot)) \\
&\qquad + \Psi_{2k}(p,\, q)(d_{(p,q)}G_{2k-1}(\cdot;\, p,\, q)[(u,\, v)](\cdot),\, G_{2k}(\cdot;\, p,\, q)) \\
&\qquad + (G_{2k-1}(\cdot;\, p,\, q),\, \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix} G_{2k-1}(\cdot;\, p,\, q)) \\
&\qquad - d_{(p,q)}\tau_k[(u,\, v)].
\end{aligned}$$

Hence one finally obtains

$$\begin{aligned}
&d_{(p,q)}\Psi_{2k-1}[(u,\, v)] \\
&\quad = 2\Psi_{2k-1}(p,\, q)(G_{2k}(\cdot;\, p,\, q),\, d_{(p,q)}G_{2k-1}(\cdot;\, p,\, q)[(u,\, v)](\cdot)) \\
&\qquad + (G_{2k-1}(\cdot;\, p,\, q),\, \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix} G_{2k-1}(\cdot;\, p,\, q)) \\
&\qquad - d_{(p,q)}\tau_k[(u,\, v)].
\end{aligned}$$

Let us now compute $d_{(p,q)}\tau_k[(u,\, v)]$.

Define, for fixed $k \in \mathbb{Z}$, the open set $\mathscr{U}_k \subseteq \mathscr{H}^0$

$$\mathscr{U}_k = \{(p,\, q) \in \mathscr{H}^0;\ \lambda_{2k}(p,\, q) \text{ simple}\}.$$

$\lambda_{2k}(\cdot,\, \cdot)$ and $\lambda_{2k-1}(\cdot,\, \cdot)$ are continuously differentiable on $\mathscr{U}_k$.

Using $H(p,q)F_j = \lambda_j(p,q)F_j$ $(j = 2k-1, 2k)$ one obtains for $(p,q) \in \mathscr{U}_k$

$$d_{(p,q)}\lambda_j[(u,\, v)] = (F_j(\cdot;\, p,\, q),\, \begin{pmatrix} -v(\cdot) & u(\cdot) \\ u(\cdot) & v(\cdot) \end{pmatrix} F_j(\cdot;\, p,\, q)).$$

Thus

$$d_{(p,q)}\tau_k[(u,v)] = \frac{1}{2}\int_0^1 (F_{2k}^{(2)}(x;p,q)^2 + F_{2k-1}^{(2)}(x;p,q)^2 - F_{2k}^{(1)}(x;p,q)^2$$
$$-F_{2k-1}^{(1)}(x;p,q)^2)v(x)\,\mathrm{d}x$$
$$+\int_0^1 (F_{2k}^{(1)}(x;p,q)F_{2k}^{(2)}(x;p,q)$$
$$+F_{2k-1}^{(1)}(x;p,q)F_{2k-1}^{(2)}(x;p,q))u(x)\,\mathrm{d}x.$$

Expressed in terms of the G_k's we obtain

$$d_{(p,q)}\tau_k[(u,v)] = \frac{1}{2}\int_0^1 (G_{2k}^{(2)}(x;p,q)^2 + G_{2k-1}^{(2)}(x;p,q)^2 - G_{2k}^{(1)}(x;p,q)^2$$
$$-G_{2k-1}^{(1)}(x;p,q)^2)v(x)\,\mathrm{d}x$$
$$+\int_0^1 (G_{2k}^{(1)}(x;p,q)G_{2k}^{(2)}(x;p,q)$$
$$+G_{2k-1}^{(1)}(x;p,q)G_{2k-1}^{(2)}(x;p,q))u(x)\,\mathrm{d}x.$$

Now $\mathscr{U}_k$ is dense in $\mathscr{H}^0$ and both sides of the least equality are continuous functions of (p,q) in $\mathscr{H}^0$. Thus this equality expresses $d_{(p,q)}\tau_k$ in terms of the G_k's on $\mathscr{H}^0$. $d_{(p,q)}\Psi_{2k}$ is calculated in the same way as $d_{(p,q)}\Psi_{2k-1}$.

The derivatives $d_{(p,q)}\Psi_{2k}$ and $d_{(p,q)}\Psi_{2k-1}$ can be expressed in a slightly different way as follows.

COROLLARY 3.2. *Suppose* $(u,v)\in\mathscr{H}^0$. *Then*

$$\begin{pmatrix} d_{(p,q)}\Psi_{2k}[(u,v)] \\ d_{(p,q)}\Psi_{2k-1}[(u,v)] \end{pmatrix}$$
$$=\Bigg(\int_0^1 (F_{2k}^{(1)}(x;p,q)^2 - F_{2k-1}^{(1)}(x;p,q)^2 + F_{2k-1}^{(2)}(x;p,q)^2$$
$$-F_{2k}^{(2)}(x;p,q)^2)\,\frac{v(x)}{2}\,\mathrm{d}x\Bigg)\begin{pmatrix} -\sin 2\theta_k(p,q) \\ \cos 2\theta_k(p,q) \end{pmatrix}$$
$$+\varepsilon_k\Bigg(\int_0^1 (F_{2k}^{(2)}(x;p,q)F_{2k-1}^{(2)}(x;p,q)$$
$$-F_{2k}^{(1)}(x;p,q)F_{2k-1}^{(1)}(x;p,q))v(x)\,\mathrm{d}x\Bigg)\begin{pmatrix} \cos 2\theta_k(p,q) \\ \sin 2\theta_k(p,q) \end{pmatrix}$$

$$+\left(\int_0^1 (F^{(1)}_{2k-1}(x;\,p,\,q)F^{(2)}_{2k-1}(x;\,p,\,q)\right.$$

$$\left.-F^{(1)}_{2k}(x;\,p,\,q)F^{(2)}_{2k}(x;\,p,\,q))u(x)\,\mathrm{d}x\right)\begin{pmatrix}-\sin 2\theta_k(p,\,q)\\ \cos 2\theta_k(p,\,q)\end{pmatrix}$$

$$+\varepsilon_k\left(\int_0^1 (F^{(1)}_{2k}(x;\,p,\,q)F^{(2)}_{2k-1}(x;\,p,\,q)\right.$$

$$\left.+F^{(1)}_{2k-1}(x;\,p,\,q)F^{(2)}_{2k}(x;\,p,\,q))u(x)\,\mathrm{d}x\right)\begin{pmatrix}\cos 2\theta_k(p,\,q)\\ \sin 2\theta_k(p,\,q)\end{pmatrix}$$

$$+\gamma_k(p,\,q)\left(\int_0^1 d_{(p,q)}G_{2k-1}(\cdot\,;\,p,\,q)[(u,\,v)](x)\right.$$

$$\left.\cdot\, G_{2k}(x;\,p,\,q)\,\mathrm{d}x\right)\begin{pmatrix}\cos 2\theta_k(p,\,q)\\ \sin 2\theta_k(p,\,q)\end{pmatrix}$$

where $\varepsilon_k = \operatorname{sign} W(F_{2k}(\cdot\,;\,p,\,q),\,F_{2k-1}(\cdot\,;\,p,\,q))(0)$.

We now study the asymptotics of $d_{(p,q)}\Psi_{2k}$ and $d_{(p,q)}\Psi_{2k-1}$. First of all it will be useful to bring

$$\int_0^1 d_{(p,q)}G_{2k-1}(\cdot\,;\,p,\,q)[(u,\,v)](x)\cdot G_{2k}(x,\,p,\,q)\,\mathrm{d}x$$

into another form.

LEMMA 3.3.

$$\int_0^1 d_{(p,q)}G_{2k-1}(\cdot\,;\,p,\,q)[(u,\,v)](x)\cdot G_{2k}(x;\,p,\,q)\,\mathrm{d}x$$

$$=\sum_{j\neq 2k,2k-1} F^{(1)}_j(0)\left(F_j,\begin{pmatrix}-v & u\\ u & v\end{pmatrix}F_{2k}\right)\sin\theta_k\,\frac{1}{\lambda_{2k}-\lambda_j}$$

$$+\sum_{j\neq 2k,2k-1} F^{(1)}_j(0)\left(F_j,\begin{pmatrix}-v & u\\ u & v\end{pmatrix}F_{2k-1}\right)\varepsilon_k\cos\theta_k\,\frac{1}{\lambda_{2k-1}-\lambda_j}.$$

The proof of Lemma 3.3 follows as in [Kp; Lemma 5.3].

In order to bound $F_{2k-1}(\cdot)$ and $F_{2k}(\cdot)$ uniformly with respect to k we use the following lemma.

LEMMA 3.4. *For* $(p,\,q)\in\mathscr{H}^0$ *and* $k\in\mathbb{Z}$ *denote* $I_k(\cdot)$ *the unique function in* $E_k(p,q)$ *such that* $\|I_k(\cdot)\|_{L^2([0,1])^2}=1$ *with* $I^{(1)}_k(0)>0$ *and* $I^{(2)}_k(0)=0$. *Then for*

$j \in \{2k-1, 2k\}$

(i) $F_1(\cdot, \lambda_j) = I_k(\cdot) + l^2(k)$ *and*
(ii) $F_2(\cdot, \lambda_j) = G_{2k-1}(\cdot) + l^2(k)$.

The error terms are uniform with respect to $0 \leqslant x \leqslant 1$ *and* (p, q) *in any bounded set of* $\mathscr{H}^0$.

REMARK. We present a proof of Lemma 3.4 which generalizes easily to a situation encountered in Lemma 3.14 below.

Proof of Lemma 3.4. (1) Assume that $j = 2k$. Observe that (see [Gre-Gui])

$$F_1(0, \lambda_{2k}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad F_1(1, \lambda_{2k}) = \begin{pmatrix} (-1)^k \\ 0 \end{pmatrix} + l^2(k).$$

Existence and uniqueness of $I_k(\cdot)$ follow from Lemma 2.1. Then there exist α_k and β_k satisfying

$$I_k(\cdot) = \alpha_k F_{2k-1}(\cdot) + \beta_k F_{2k}(\cdot)$$

with $\alpha_k^2 + \beta_k^2 = 1$.
Further

$$H(p, q) I_k(\cdot) = \lambda_{2k} I_k(\cdot) - \alpha_k \gamma_k F_{2k-1}(\cdot)$$

with $(\alpha_k \gamma_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$.
Define

$$f_k(\cdot) = I_k(\cdot) - I_k^{(1)}(0) F_1(\cdot, \lambda_{2k}).$$

Then $f_k(\cdot)$ satisfies

$$H(p, q) f_k(\cdot) = \lambda_{2k} f_k(\cdot) - \alpha_k \gamma_k F_{2k-1}(\cdot)$$

with

$$f_k(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Set

$$K(x) = \begin{pmatrix} F_1^{(1)}(x, \lambda_{2k}) & F_2^{(1)}(x, \lambda_{2k}) \\ F_1^{(2)}(x, \lambda_{2k}) & F_2^{(2)}(x, \lambda_{2k}) \end{pmatrix}.$$

We then obtain

$$f_k(x) = -\int_0^x K(x)^{-1}K(x')(\alpha_k\gamma_k F_{2k-1}(x'))\,\mathrm{d}x'.$$

It follows from the estimates of $F_1(\cdot, \lambda)$ and $F_2(\cdot, \lambda)$ in [Gre-Gui; Section 1] that there is a constant $C > 0$ independent of k such that

$$\|f_k\|_\infty \leqslant C|\alpha_k|\gamma_k \leqslant C\gamma_k.$$

Therefore we get

$$\|F_1(\cdot, \lambda_{2k})\|_{L^2([0,1])^2} I_k^{(1)}(0) = 1 + l^2(k).$$

Further we get from [Gre-Gui; Section 1]

$$\|F_1(\cdot, \lambda_{2k})\|_{L^2([0,1])^2} = 1 + l^2(k).$$

Thus

$$I_k^{(1)}(0) = 1 + l^2(k)$$

and (i) is proved with $j = 2k$. The case $j = 2k - 1$ follows exactly in the same way.

To prove (ii) remark that

$$F_2(0, \lambda_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad F_2(1, \lambda_j) = \begin{pmatrix} 0 \\ (-1)^k \end{pmatrix} + l^2(k).$$

Further

$$\|G_{2k-1}(\cdot)\|_{L^2([0,1])^2} = 1 \quad \text{and} \quad G_{2k-1}^{(2)}(0) > 0.$$

Thus (ii) follows in the same way as (i) and Lemma 3.4 is proved.

Let us deduce from Lemma 3.4 that

$$\|F_k(\cdot)\|_{L^\infty([0,1])^2} \leqslant C \tag{3.1}$$

uniformly with respect to k.

Consider F_{2k}. For $|k|$ sufficiently large it follows from Lemma 3.4 that $W(I_k, G_{2k-1})(\cdot) \neq 0$ because $W(F_1(\cdot, \lambda_{2k}), F_2(\cdot, \lambda_{2k})) = 1$.

Therefore

$$F_{2k}(\cdot) = \alpha_k I_k(\cdot) + \beta_k G_{2k-1}(\cdot), \qquad \alpha_k, \beta_k \in \mathbb{R}$$

for $|k|$ sufficiently large.

From $\|F_{2k}(\cdot)\|_{L^2([0,1])^2} = 1$ we deduce that

$$1 = \alpha_k^2 + \beta_k^2 + 2\alpha_k\beta_k(I_k(\cdot), G_{2k-1}(\cdot))_{L^2([0,1])^2}$$

with $|(I_k, G_{2k-1})| \leqslant 1$ and $(I_k(\cdot), G_{2k-1}) \in l^2(k)$ because $(F_1(\cdot, \lambda_{2k}), F_2(\cdot, \lambda_{2k})) \in l^2(k)$.

We then get

$$|\alpha_k| \leqslant C \quad \text{and} \quad |\beta_k| \leqslant C$$

uniformly with respect to k. (3.1) then follows from Lemma 3.4.

We now study the asymptotics of $d_{(p,q)}\Psi_{2k}$ and $d_{(p,q)}\Psi_{2k-1}$. One easily shows that

$$G_{2k}(x;\, p,\, q) = \begin{pmatrix} \cos k\pi x \\ -\sin k\pi x \end{pmatrix} + l^2(k)$$

$$G_{2k-1}(x;\, p,\, q) = \begin{pmatrix} \sin k\pi x \\ \cos k\pi x \end{pmatrix} + l^2(k)$$

where the error terms are uniform with respect to $0 \leqslant x \leqslant 1$. Furthermore since $G_{2k}(\cdot\,; p, q)$ and $G_{2k-1}(\cdot\,; p, q)$ are real analytic functions of (p, q) as maps from $\mathscr{H}^0$ into $H^1_{\mathbb{R}}([0, 1])^2$ it follows that $d_{(p,q)}G_{2k}(\cdot\,; p, q)$ and $d_{(p,q)}G_{2k-1}(\cdot\,; p, q)$ are bounded linear maps from $\mathscr{H}^0$ into $H^1_{\mathbb{R}}([0, 1])^2$ which are still real analytic functions of (p, q).

It follows from Lemma 3.3 and (3.1) that the norm of the linear map

$$(u, v) \mapsto \int_0^1 d_{(p,q)}G_{2k-1}(\cdot\,; p, q)[(u, v)](x) \cdot G_{2k}(x;\, p,\, q)\, \mathrm{d}x$$

is uniformly bounded with respect to (p, q) on bounded sets of $\mathscr{H}^0$ and to $k \in \mathbb{Z}$ (See [Kp; Prop. 5.4]).

It then follows from Theorem 3.1 and from the fact that $(\Psi_k(p, q))_{k\in\mathbb{Z}}$ is in $l^2(\mathbb{Z})$ that we obtain

THEOREM 3.5.

$$\begin{pmatrix} d_{(p,q)}\Psi_{2k}[(u, v)] \\ d_{(p,q)}\Psi_{2k-1}[(u, v)] \end{pmatrix} = \int_0^1 \begin{pmatrix} \cos 2k\pi x & -\sin 2k\pi x \\ \sin 2k\pi x & \cos 2k\pi x \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \mathrm{d}x + l^2(k)$$

where the error term is bounded uniformly with respect to (u, v) and (p, q) in any bounded subset of $\mathscr{H}^0$.

We need to introduce some more notation. For $(p, q) \in \mathscr{H}^0$ set

$$J = \{k \in \mathbb{Z};\ \lambda_{2k-1}(p, q) < \lambda_{2k}(p, q)\}.$$

Then, for $k \in \mathbb{Z}$, define

$$H_{2k}(x; p, q) = \begin{pmatrix} F^{(1)}_{2k-1}(x; p, q)F^{(2)}_{2k-1}(x; p, q) - F^{(1)}_{2k}(x; p, q)F^{(2)}_{2k}(x; p, q) \\ \frac{1}{2}(F^{(1)}_{2k}(x; p, q)^2 - F^{(2)}_{2k}(x; p, q)^2 + F^{(2)}_{2k-1}(x; p, q)^2 - F^{(1)}_{2k-1}(x; p, q)^2) \end{pmatrix}$$

For $k \notin J$ set

$$H_{2k-1}(x; p, q) = \varepsilon_k \begin{pmatrix} F^{(1)}_{2k}(x; p, q)F^{(2)}_{2k-1}(x; p, q) + F^{(1)}_{2k-1}(x; p, q)F^{(2)}_{2k}(x; p, q) \\ F^{(2)}_{2k}(x; p, q)F^{(2)}_{2k-1}(x; p, q) - F^{(1)}_{2k}(x; p, q)F^{(1)}_{2k-1}(x; p, q) \end{pmatrix}$$

and for $k \in J$ define

$$\begin{aligned} H_{2k-1}(x; p, q) &= \\ &= \varepsilon_k \begin{pmatrix} F^{(1)}_{2k}(x; p, q)F^{(2)}_{2k-1}(x; p, q) + F^{(1)}_{2k-1}(x; p, q)F^{(2)}_{2k}(x; p, q) \\ F^{(2)}_{2k}(x; p, q)F^{(2)}_{2k-1}(x; p, q) - F^{(1)}_{2k}(x; p, q)F^{(1)}_{2k-1}(x; p, q) \end{pmatrix} \\ &\quad + \gamma_k(p, q) \begin{pmatrix} \displaystyle\int_0^1 \left\{ G^{(1)}_{2k}(y; p, q) \frac{\partial G^{(1)}_{2k-1}}{\partial p(x)}(y; p, q) + G^{(2)}_{2k}(y; p, q) \frac{\partial G^{(2)}_{2k-1}}{\partial p(x)}(y; p, q) \right\} \mathrm{d}y \\ \displaystyle\int_0^1 \left\{ G^{(1)}_{2k}(y; p, q) \frac{\partial G^{(1)}_{2k-1}}{\partial q(x)}(y; p, q) + G^{(2)}_{2k}(y; p, q) \frac{\partial G^{(2)}_{2k-1}}{\partial q(x)}(y; p, q) \right\} \mathrm{d}y \end{pmatrix} \end{aligned}$$

Then, from Corollary 3.2, it follows that

$$\begin{aligned} &\begin{pmatrix} d_{(p,q)}\Psi_{2k}[(u, v)] \\ d_{(p,q)}\Psi_{2k-1}[(u, v)] \end{pmatrix} \\ &\quad = (H_{2k}(\cdot\,; p, q), (u(\cdot), v(\cdot))) \begin{pmatrix} -\sin 2\theta_k(p, q) \\ \cos 2\theta_k(p, q) \end{pmatrix} \\ &\qquad + (H_{2k-1}(\cdot\,; p, q), (u(\cdot), v(\cdot))) \begin{pmatrix} \cos 2\theta_k(p, q) \\ \sin 2\theta_k(p, q) \end{pmatrix}. \end{aligned}$$

THEOREM 3.6. *Suppose $(p, q) \in \mathscr{H}^0$. Then $d_{(p,q)}\Phi$ is a linear isomorphism form $\mathscr{H}^0$ onto $\mathscr{M}^0$.*

The proof of Theorem 3.6 is rather long and several steps are needed.

Theorem 3.5 shows that $d_{(p,q)}\Psi$ is a Fredholm operator of index zero. Therefore it suffices to show that $d_{(p,q)}\Psi$ is one to one in order to prove Theorem 3.6.

Assume that $d_{(p,q)}\Psi[(u, v)] = 0$ where $(u, v) \in \mathscr{H}^0$. From the above formula we conclude that $(H_k(\cdot\,; p, q), (u(\cdot), v(\cdot))) = 0$ for every $k \in \mathbb{Z}$. Therefore, in order to prove that $d_{(p,q)}\Psi$ is one to one, one must prove that $\{H_k(\cdot\,; p, q)\}_{k\in\mathbb{Z}}$ is a Riesz basis of $\mathscr{H}^0$. Using the definition of the H_k's and the asymptotic behavior of the G_k's one shows that $\{H_k(\cdot\,; p, q)\}_{k\in\mathbb{Z}}$ is quadratically close to the orthonormal basis $(T_k(\cdot\,; p, q))$ of $\mathscr{H}^0$ where

$$T_{2k}(x; p, q) = -\sin 2\theta_k(p, q)\begin{pmatrix} \cos 2k\pi x \\ -\sin 2k\pi x\end{pmatrix} + \cos 2\theta_k(p, q)\begin{pmatrix} \sin 2k\pi x \\ \cos 2k\pi x\end{pmatrix}$$

$$T_{2k-1}(x; p, q) = \cos 2\theta_k(p, q)\begin{pmatrix} \cos 2k\pi x \\ -\sin 2k\pi x\end{pmatrix} + \sin 2\theta_k(p, q)\begin{pmatrix} \sin 2k\pi x \\ \cos 2k\pi x\end{pmatrix}$$

Thus to prove that $(H_k(\cdot\,; p, q))_{k\in\mathbb{Z}}$ is a basis of $\mathscr{H}^0$ it remains to prove that the H_k's are linearly independent, i.e., if $(\alpha_k)_{k\in\mathbb{Z}}$ is a sequence of real numbers such that

(i) $\sum_{k\in\mathbb{Z}} \alpha_k^2 \,\|\, H_k(\cdot\,; p, q)\|^2_{L^2([0,1])^2} < \infty$ and
(ii) $\sum_{k\in\mathbb{Z}} \alpha_k H_k = 0$,

then $\alpha_k = 0$ for all k.

First, let us recall that the set $\mathrm{Iso}_0(p, q)$ of isospectral potentials is a countable intersection of manifolds and that one can define the normal space $N(p, q)$ and the tangent space $T(p, q)$ of $\mathrm{Iso}_0(p, q)$ at (p, q). Using results of [Gre-Gui], an easy computation shows that $\{H_{2k}(\cdot\,; p, q)\}_{k\in\mathbb{Z}}$ and $\{H_{2k-1}(\cdot\,; p, q)\}_{k\notin J}$ belong to the normal space $N(p, q)$ of the isospectral set $\mathrm{Iso}_0(p, q)$ at (p, q).

Set for $k' \in \mathbb{Z}$

$$(p_{k'}, q_{k'}) = (\nabla_{(p,q)}\Delta(\lambda; p, q)|_{\lambda = v_{k'}(p,q)})^{\perp} \tag{3.2}$$

where $(a, b)^{\perp} = (-b, a)$, $(v_{k'}(p, q))_{k'\in\mathbb{Z}}$ is one of the two Dirichlet auxiliary spectra defined in section 2.

Clearly $(p_{k'}, q_{k'})$ is in the tangent space $T(p, q)$ of $\mathrm{Iso}_0(p, q)$ at (p, q). Hence it follows that for every k'

$$\begin{aligned} 0 &= \sum_{k\in\mathbb{Z}} \alpha_k(H_k(\cdot\,; p, q), (p_{k'}(\cdot), q_{k'}(\cdot))), \\ &= \sum_{k\in J} \alpha_{2k-1}(H_{2k-1}(\cdot\,; p, q), (p_{k'}(\cdot), q_{k'}(\cdot))). \end{aligned} \tag{3.3}$$

The proof of Theorem 3.6 consists of three steps. In the first one we show that

$\alpha_{2k-1} = 0$ for $k \in J$. In the second one we prove that $\alpha_{2k} = \alpha_{2k-1} = 0$ for $k \notin J$ and in the third one we finally show that $\alpha_{2k} = 0$ for every k in J.

3.1. The first step

Let us begin with a computational lemma.

LEMMA 3.7. *If* $(u, v) \in T(p, q)$ *and* k *in* J *such that* $\lambda_{2k-1}(p, q) < \nu_k(p, q) < \lambda_{2k}(p, q)$, *then*

$$\begin{aligned}(H_{2k-1}(\cdot\,; p, q), (u(\cdot)), v(\cdot))) &= -\frac{\gamma_k(p, q)}{2}(G_{2k}^{(1)}(0; p, q))^{-1}\varepsilon_k \cos\theta_k(p, q) F_{2k-1}^{(1)}(0; p, q)\\ &\cdot \sum_{j\in\mathbb{Z}} \left(\frac{1}{\nu_j(p, q) - \lambda_{2k-1}(p, q)} - \frac{1}{\nu_j(p, q) - \lambda_{2k}(p, q)}\right)\\ &\cdot (\nabla_{(p,q)}\nu_j(p, q), (u, v)).\end{aligned}$$

Proof of Lemma 3.7. We first prove that for $(u, v) \in T(p, q)$

$$\gamma_k(p, q) d_{(p,q)}\theta_k[(u, v)] = (H_{2k-1}(\cdot\,; p, q), (u(\cdot), v(\cdot))) \tag{3.4}$$

as follows:

$$\begin{aligned}&\int_0^1 d_{(p,q)}G_{2k-1}(\cdot\,; p, q)[(u, v)](x) \cdot G_{2k}(x; p, q)\,dx\\ &= d_{(p,q)}\theta_k[(u, v)] + \varepsilon_k \cos\theta_k(p, q) \int_0^1 d_{(p,q)}F_{2k-1}(\cdot\,; p, q)[(u, v)](x)\\ &\cdot G_{2k}(x; p, q)\,dx + \sin\theta_k(p, q) \int_0^1 d_{(p,q)}F_{2k}(\cdot\,; p, q)[(u, v)](x)\\ &\cdot G_{2k}(x; p, q)\,dx\\ &= d_{(p,q)}\theta_k[(u, v)] + \varepsilon_k \int_0^1 d_{(p,q)}F_{2k-1}(\cdot\,; p, q)[(u, v](x)\\ &\cdot F_{2k}(x; p, q)\,dx.\end{aligned}$$

Using $H(p, q)F_j = \lambda_j F_j$ one gets

$$\begin{aligned}&(d_{(p,q)}F_{2k-1}(\cdot\,; p, q)[(u, v)](\cdot), F_{2k}(\cdot\,; p, q))\\ &= -\frac{1}{\gamma_k(p, q)}\left(F_{2k-1}(\cdot\,; p, q), \begin{pmatrix} -v(\cdot) & u(\cdot)\\ u(\cdot) & v(\cdot)\end{pmatrix} F_{2k}(\cdot\,; p, q)\right).\end{aligned}$$

Thus (3.4) follows from the definition of H_{2k-1}. To compute $d_{(p,q)}\theta_k[(u,v)]$ take the derivative of $0 = G^{(1)}_{2k-1}(0) = \sin\theta_k F^{(1)}_{2k}(0) + \varepsilon_k \cos\theta_k F^{(1)}_{2k-1}(0)$ and use a similar argument as in [Kp, Lemma 6.8] to obtain

$$
\begin{aligned}
&-G^{(1)}_{2k}(0;\, p,\, q) d_{(p,q)}\theta_k[(u,\, v)]\\
&= \frac{1}{2}\varepsilon_k \cos\theta_k(p,\, q) F^{(1)}_{2k-1}(0;\, p,\, q)\\
&\quad\times \sum_{j\in\mathbb{Z}} \left(\frac{1}{\nu_j(p,q) - \lambda_{2k-1}(p,q)} - \frac{1}{\nu_j(p,q) - \lambda_{2k}(p,q)}\right)\\
&\quad\cdot (\nabla_{(p,q)}\nu_j,\, (u,\, v)).
\end{aligned}
$$

In the case where $\nu_k(p,\, q) \in \{\lambda_{2k}(p,\, q),\, \lambda_{2k-1}(p,q)\}$ the following result holds.

LEMMA 3.8. *If* $k\in J$ *with* $\nu_k(p,q)\in\{\lambda_{2k}(p,\, q),\, \lambda_{2k-1}(p,\, q)\}$, *then, for* $k'\in\mathbb{Z}$,

$$
(H_{2k'-1}(\cdot;\, p,\, q),\, (p_k(\cdot),\, q_k(\cdot))) = \delta_{k'k} c_k \quad \textit{with } c_k \neq 0.
$$

The proof of Lemma 3.8 follows as in [Kp, Lemma 6.10], once the following result is proved:

"Every $(p,q)\in\mathscr{H}^0$ with $\nu_k(p,q)\in\{\lambda_{2k}(p,q),\, \lambda_{2k-1}(p,q)\}$, for some $k\in J$, is the limit of a sequence $(p_j,\, q_j)_{j\in\mathbb{N}}$ in $\mathrm{Iso}_0(p,q)$ with $\lambda_{2k-1}(p,q) < \nu_k(p_j,\, q_j) < \lambda_{2k}(p,q)$."

This result easily follows from Appendix A.

Thus using (3.3) and Lemma 3.8 one gets $\alpha_{2k-1} = 0$ for every $k\in J - J_1$ where $J_1 = \{k\in\mathbb{Z};\, \lambda_{2k-1}(p,q) < \nu_k(p,q) < \lambda_{2k}(p,q)\}$. We now prove that $\alpha_{2k-1} = 0$ for $k\in J_1$. For that purpose define

$$
A_{k',k} = (H_{2k-1}(\cdot;\, p,\, q),\, (p_{k'},\, q_{k'})), \quad k,\, k'\in J_1
$$

where $(p_{k'},\, q_{k'})$ is given by (3.2). Define

$$
\begin{aligned}
B_{k',k} &= A_{k',k} - A_{k',k}\delta_{k'k}\\
C_{k',k} &= A_{k',k}\delta_{k',k}
\end{aligned}
$$

where $\delta_{k',k}$ denotes the Kronecker delta function.

Let A (resp. B, C) be the linear operator associated with the matrix $(A_{k',k})_{(k',k)\in J_1\times J_1}$ (resp. $(B_{k'k})$, $(C_{k'k})$). Then A (resp. B, C) $\in\mathscr{B}(l^2(J_1))$ has the following properties.

LEMMA 3.9.
(i) *B is of trace class.*
(ii) *C is invertible with a bounded inverse.*
(iii) *A is one-to-one.*

It then follows that $\alpha_{2k-1} = 0$ for $k \in J_1$ since

$$\sum_{k \in J_1} \alpha_{2k-1}(H_{2k-1}(\cdot;\, p, q), (p_{k'}, q_{k'})) = \sum_{k \in J_1} \alpha_{2k-1} A_{kk'}, \quad k' \in J_1.$$

Proof of Lemma 3.9. Use [Gre, part II Chap 3 Th. 5] to conclude that

$$(\nabla_{p,q} v_k, (p_{k'}, q_{k'})) = \delta_{kk'}(Z_2(1, v_{k'}) - Y_1(1, v_{k'})).$$

From Lemma 3.7, it follows that

$$A_{k',k} = \frac{1}{2}(G^{(1)}_{2k}(0))^{-1} \varepsilon_k \cos\theta_k(p, q) F^{(1)}_{2k-1}(0;\, p, q)(Z_2(1, v_{k'}) - Y_1(1, v_{k'})) \cdot \frac{\lambda_{2k}(p, q) - \lambda_{2k-1}(p, q)}{(v_{k'}(p, q) - \lambda_{2k-1}(p, q))(\lambda_{2k}(p, q) - v_{k'}(p, q))}. \tag{3.5}$$

Moreover as we have already observed

$$(G^{(1)}_{2k}(0;\, p, q))^{-1} = 1 + l^2(k), \qquad G^{(1)}_{2k-1}(0;\, p, q) = l^2(k)$$

as well as $\cos^2\theta_k = F^{(1)}_{2k}(0)^2/(F^{(1)}_{2k}(0)^2 + F^{(1)}_{2k-1}(0)^2)$, we conclude that

$$\begin{aligned}
&|\cos\theta_k(p, q) F^{(1)}_{2k-1}(0;\, p, q)| \\
&= \frac{|F^{(1)}_{2k}(0;\, p, q) F^{(1)}_{2k-1}(0;\, p, q)|}{(F^{(1)}_{2k}(0;\, p, q)^2 + F^{(1)}_{2k-1}(0;\, p, q)^2)^{1/2}} \\
&= \frac{|F^{(1)}_{2k}(0;\, p, q) F^{(1)}_{2k-1}(0;\, p, q)|}{(G^{(1)}_{2k}(0;\, p, q)^2 + G^{(1)}_{2k-1}(0;\, p, q)^2)^{1/2}} \\
&= |F^{(1)}_{2k}(0;\, p, q) F^{(1)}_{2k-1}(0;\, p, q)|(1 + l^2(k)) \\
&= \left(-\frac{Y_2(1, \lambda_{2k}(p, q))}{\dot{\Delta}(\lambda_{2k}(p, q))}\right)^{1/2} \left(-\frac{Y_2(1, \lambda_{2k-1}(p, q))}{\dot{\Delta}(\lambda_{2k-1}(p, q))}\right)^{1/2} (1 + l^2(k))
\end{aligned}$$

(see the beginning of section 2). Using Lemma B.3 (Appendix B) we then obtain the estimate

$$\begin{aligned}
&|\cos\theta_k(p, q) F^{(1)}_{2k-1}(0;\, p, q)| \\
&= \frac{((\lambda_{2k})(p, q) - v_k(p, q))^{1/2}(v_k(p, q) - \lambda_{2k-1}(p, q))^{1/2}}{\lambda_{2k}(p, q) - \lambda_{2k-1}(p, q)} (1 + l^2(k)).
\end{aligned}$$

Further (cf. [Gre, Part II, Ch. 3, Th. 5])

$$
\begin{aligned}
&|Z_2(1, \nu_{k'}(p, q)) - Y_1(1, \nu_{k'}(p, q))| \\
&\quad = (\Delta^2(\nu_{k'}(p, q)) - 4)^{1/2} \\
&\quad = 2(\lambda_{2k'}(p, q) - \nu_{k'}(p, q))^{1/2}(\nu_{k'}(p, q) - \lambda_{2k'-1}(p, q))^{1/2}(1 + l^2(k'))
\end{aligned}
$$

where we used for the last equality the representation of $\Delta^2 - 4$ by an infinite product (cf. Appendix B). Thus, from (3.5), one obtains that $|A_{k'k}|$ is given by

$$
\frac{(\lambda_{2k'} - \nu_{k'})^{1/2}(\nu_{k'} - \lambda_{2k'-1})^{1/2}(\lambda_{2k} - \nu_k)^{1/2}(\nu_k - \lambda_{2k-1})^{1/2}}{(\nu_{k'} - \lambda_{2k-1})(\lambda_{2k} - \nu_{k'})}(1 + l^2(k))(1 + l^2(k')). \tag{3.6}
$$

From the asymptotic behavior of the λ_k's and ν_k's it follows that

$$
B_{k',k} = \frac{a_{k'}b_k}{(k - k')^2}
$$

where $(a_{k'})_{k' \in J_1}$ and $(b_k)_{k \in J_1}$ are in $l^2(J_1)$. To prove (i) one must show that

$$
\sum_{\substack{k,k' \in J_1 \\ k \neq k'}} |B_{k',k}| < +\infty .
$$

By well known properties of the convolution this follows from the estimate

$$
\sum_{\substack{k,k' \in J_1 \\ k \neq k'}} |B_{k',k}| \leqslant \sum_{k' \in J_1} |a_{k'}| \sum_{\substack{k \in J_1 \\ k \neq k'}} \frac{|b_k|}{(k - k')^2} .
$$

From (3.6) we learn that

$$
|A_{kk}| = 1 + l^2(k).
$$

Furthermore A_{kk} is different from zero for any $k \in J_1$. Thus (ii) follows.

Towards (iii) we first observe that $C^{-1}A = \mathrm{Id} + C^{-1}B$ is a Fredholm operator of index zero. Thus in order to prove the first step we must show that $C^{-1}A$ is one to one, or equivalently, that the Fredholm determinant of $C^{-1}A$ is different from zero. Let $\det C^{-1}A$ be this Fredholm determinant which is a limit of determinants of finite matrices, i.e., $\det C^{-1}A = \lim_{J_2 \to J_1} \det(C^{-1}A)_{J_2}$ where $(C^{-1}A)_{J_2}$ denotes the $J_2 \times J_2$ matrix $(C^{-1}A)_{k,k' \in J_2}$ with J_2 a finite subset of J_1. As

C^{-1} is diagonal, one has

$$\det(C^{-1}A)_{J_2} = \frac{\det A_{J_2}}{\det C_{J_2}} = \det\left(\frac{1}{v_{k'} - \lambda_{2k-1}} - \frac{1}{v_{k'} - \lambda_{2k}}\right)_{k',k\in J_2} \cdot \left[\prod_{k\in J_2}\left(\frac{1}{v_k - \lambda_{2k-1}} - \frac{1}{v_k - \lambda_{2k}}\right)\right]^{-1}.$$

As in [Kp] one considers the sequence $x = (x_k)_{k\in J_2}$ with $x_k \in \{-\lambda_{2k-1}, -\lambda_{2k}\}$ and $\varepsilon = (\varepsilon_k)_{k\in J_2}$ with $\varepsilon_k = 0$ if $x_k = -\lambda_{2k-1}$ and $\varepsilon_k = 1$ if $x_k = -\lambda_{2k}$. From [P-S p. 98] (cf. also [Mck-Tru, p. 207]) it follows that

$$\det\left(\frac{1}{v_{k'} - \lambda_{2k-1}} - \frac{1}{v_{k'} - \lambda_{2k}}\right)_{k',k\in J_2} = \sum_x (-1)^{|\varepsilon|} \det\left(\frac{1}{v_{k'} + x_k}\right)_{k',k\in J_2}$$
$$= \sum_x (-1)^{|\varepsilon|} \frac{\prod_{k'>k}(v_{k'} - v_k)\prod_{k'>k}(x_{k'} - x_k)}{\prod_{k,k'}(x_k + v_{k'})}$$

where $|\varepsilon| = \sum_{k\in J_2} \varepsilon_k$.

Then

$$\det\left(\frac{1}{v_{k'} - \lambda_{2k-1}} - \frac{1}{v_{k'} - \lambda_{2k}}\right)_{k',k\in J_2}$$
$$= \sum_x \left(\prod_{k'\in J_2} \frac{1}{|v_{k'} + x_{k'}|}\right) \prod_{k'\in J_2} \prod_{\substack{k>k'\\ k\in J_2}} \left(1 - \frac{x_k + v_k}{x_k + v_{k'}}\right)\left(1 - \frac{x_k + v_k}{x_{k'} + v_k}\right)$$
$$= \sum_x \left(\prod_{k'\in J_2} \frac{1}{|v_{k'} + x_{k'}|}\right) \prod_{\substack{k,k'\in J_2\\ k>k'}} \left(1 - \frac{(x_k + v_k)(x_{k'} + v_{k'})}{(v_{k'} + x_k)(x_{k'} + v_k)}\right). \tag{3.7}$$

Note that

$$1 - D_{k,k'} = 1 - \frac{(x_k + v_k)(x_{k'} + v_{k'})}{(x_k + v_{k'})(x_{k'} + v_k)} > 0 \quad \text{for } k \neq k'.$$

Furthermore $D_{kk'}$ is of the form

$$D_{k,k'} = \frac{a_k b_{k'}}{(k - k')^2}$$

with $(a_k)_{k\in\mathbb{Z}}$ and $(b_{k'})_{k'\in\mathbb{Z}}$ in $l^2(\mathbb{Z})$. Thus

$$\sum_{\substack{k,k'\in\mathbb{Z}\\ k\neq k'}} D_{k,k'} < \infty$$

and there exists an integer $N > 0$ independent of J_2 such that

$$\Sigma_N = \sum_{\substack{|k|,|k'|\geqslant N\\ k\neq k'\in J_2}} D_{k,k'} < \frac{1}{2}.$$

One deduces that

$$\prod_{\substack{k,k'\in J_2\\ k\neq k'\\ |k|,|k'|\geqslant N}} (1 - D_{k,k'}) \geqslant 1 - \sum_{j\geqslant 1} (\Sigma_N)^j = K' > 0.$$

On the other hand one has

$$\prod_{\substack{k,k'\in J_2\\ k>k'\\ |k|,|k'|<N}} (1 - D_{k,k'}) \geqslant K'' > 0.$$

These two estimates lead to

$$\prod_{\substack{k,k'\in J_2\\ k>k'}} (1 - D_{k,k'}) \geqslant K = K'K'' > 0 \tag{3.8}$$

where K does not depend on the finite subset J_2 of J_1. Moreover

$$\det C_{J_2} = \sum_{x} \prod_{k\in J_2} \frac{1}{|\nu_k + x_k|}.$$

This implies together with (3.7) and (3.8) that $\det(C^{-1}A)_{J_2} \geqslant K$ uniformly with respect to $J_2 \subset J_1$. Thus $\det C^{-1}A \geqslant K > 0$ and A is one-to-one.

3.2. *The second step*

We must show that $\alpha_{2k} = \alpha_{2k-1} = 0$ for every $k \notin J$.

The main ingredient of the proof is the following

LEMMA 3.10. (i) $(H_{2k}(\cdot; p, q), H_{2k'}(\cdot; p, q)^{\perp}) = 0$, $k, k' \in \mathbb{Z}$.
(ii) *For* $k \notin J$ *and* $k' \in \mathbb{Z}$

$$(H_{2k-1}(\cdot; p, q), H_{2k'}(\cdot\, p, q)^{\perp}) = -\tfrac{1}{2}\delta_{kk'}W(F_{2k}, F_{2k-1})(0).$$

Proof of Lemma 3.10. The proof is the same as in [Gre-Gui, Th. 1.7, assertions (i) and (ii)].

To prove Step 2 we argue as follows. For $k' \notin J$ one deduces from the first step and Lemma 3.10 that

$$\begin{aligned} 0 &= \sum_{k\in\mathbb{Z}} \alpha_{2k}(H_{2k}(\cdot\,; p, q), H_{2k'}(\cdot\,; p, q)^{\perp}) \\ &\quad + \sum_{k\notin J} \alpha_{2k-1}(H_{2k-1}(\cdot\,; p, q), H_{2k'}(\cdot\,; p, q)^{\perp}) \\ &= -\tfrac{1}{2}\alpha_{2k'-1} W(F_{2k'}, F_{2k'-1})(0). \end{aligned}$$

As $W(F_{2k'}, F_{2k'-1})(0) \neq 0$ (Lemma 2.1) we conclude that $\alpha_{2k'-1} = 0$ for every $k' \in J$.

Next, again for $k' \notin J$

$$\begin{aligned} 0 &= \sum_{k\in\mathbb{Z}} \alpha_{2k}(H_{2k}(\cdot\,; p, q), H_{2k'-1}(\cdot\,; p, q)^{\perp}) \\ &= -\sum_{k\in\mathbb{Z}} \alpha_{2k}(H_{2k'-1}(\cdot\,, p, q), H_{2k}(\cdot\,; p, q)^{\perp}) \\ &= \tfrac{1}{2}\alpha_{2k'} W(F_{2k'}, F_{2k'-1})(0) \end{aligned}$$

and therefore $\alpha_{2k'} = 0$ for $k' \notin J$. Thus step 2 is proved.

3.3. *The third step*

Here we show that $\alpha_{2k} = 0$ for every $k \in J$. One already knows that

$$\sum_{k\in J} \alpha_{2k} H_{2k}(\cdot\,; p, q) = 0. \tag{3.9}$$

Thus it suffices to show that $\{H_{2k}(\cdot\,; p, q)\}_{k\in J}$ is linearly independent. Note that $H_{2k}(x; T_t p, T_t q) = H_{2k}(x + t; p, q)$. Therefore it suffices to prove that $(H_{2k}(\cdot, T_t p, T_t q))_{k\in J}$ is linearly independent for some t. The following result is easy to prove.

LEMMA 3.11. *There exists t_0 such that for all $k \in J$*

$$\lambda_{2k-1}(p, q) < \nu_k(T_{t_0} p, T_{t_0} q) < \lambda_{2k}(p, q).$$

To make notation easier, we assume that $t_0 = 0$.

It remains to prove that $\alpha_{2k} = 0$ for $k \in J_1 = \{k \in \mathbb{Z};\ \lambda_{2k-1}(p,q) < \nu_k(p,q) < \lambda_{2k}(p,q)\}$.

Define

$$A_{k',k} = \frac{1}{2} \frac{\frac{\partial Y_2}{\partial \lambda}(1, \nu_k)(\lambda_{2k} - \lambda_{2k-1})}{(\lambda_{2k} - \nu_k)^{1/2}(\nu_k - \lambda_{2k-1})^{1/2}} (H_{2k'}(\cdot\,; p, q)^{\perp}, \nabla_{(p,q)}\nu_k), \quad k, k' \in J_1.$$

A straightforward computation using [Gre-Gui] and [Gre] leads to

$$A_{k',k} = \frac{(\Delta(\nu_k)^2 - 4)^{1/2}(\lambda_{2k} - \lambda_{2k-1})}{2(\lambda_{2k} - \nu_k)^{1/2}(\nu_k - \lambda_{2k-1})^{1/2}} \cdot \left(\frac{F^{(1)}_{2k'-1}(0)^2 F^{(2)}_{2k'-1}(0)^2}{\nu_k - \lambda_{2k'-1}} - \frac{F^{(1)}_{2k'}(0)^2 F^{(2)}_{2k'}(0)^2}{\nu_k - \lambda_{2k'}} \right). \tag{3.10}$$

Define

$$B_{k',k} = A_{k',k} - A_{k',k}\delta_{k'k}$$
$$C_{k',k} = A_{k',k}\delta_{k'k}.$$

Let A (resp. B, C) denote the linear operator associated with the matrix $(A_{k',k})_{(k',k) \in J_1 \times J_1}$ (resp. $(B_{k',k})$, $(C_{k',k})$). Then A (resp. B, C) $\in \mathscr{B}(l^2(J_1))$. The proof of the third step follows from

LEMMA 3.12.
(i) *B is a Hilbert-Schmidt operator.*
(ii) *C is invertible with a bounded inverse.*
(iii) *A is one-to-one.*

Proof of Lemma 3.12. Clearly

$$F^{(1)}_{2k'-1}(0)F^{(2)}_{2k'-1}(0) + F^{(1)}_{2k'}(0)F^{(2)}_{2k'}(0)$$
$$= G^{(1)}_{2k'-1}(0)G^{(2)}_{2k'-1}(0) + G^{(1)}_{2k'}(0)G^{(2)}_{2k'}(0) = l^2(k').$$

Thus

$$(F^{(1)}_{2k'-1}(0)F^{(2)}_{2k'-1}(0))^2 = (F^{(1)}_{2k'}(0)F^{(2)}_{2k'}(0))^2 + l^2(k')$$

and $A_{k',k}$ is given by

$$\frac{1}{2}\frac{(\lambda_{2k}-\lambda_{2k-1})(\Delta(\nu_k)^2-4)^{1/2}}{(\lambda_{2k}-\nu_k)^{1/2}(\nu_k-\lambda_{2k-1})^{1/2}}\Bigg[(F^{(1)}_{2k'}(0)F^{(2)}_{2k'}(0))^2 \times\left(\frac{1+l^2(k')}{\nu_k-\lambda_{2k'-1}}-\frac{1}{\nu_k-\lambda_{2k'}}\right)+\frac{l^2(k')}{\nu_k-\lambda_{2k'-1}}\Bigg]. \tag{3.11}$$

Using formulas expressing the F_k's in terms of F_1 and F_2 (see the beginning of Section 2) and Appendix B one shows that

$$\begin{aligned}(F^{(1)}_{2k'}(0)F^{(2)}_{2k'}(0))^2 &= -\frac{Y_2(1,\lambda_{2k'})Z_1(1,\lambda_{2k'})}{(\dot\Delta(\lambda_{2k'}))^2}\\ &=\frac{(\lambda_{2k'}-\nu_{k'})(\lambda_{2k'}-\mu_{k'})}{(\lambda_{2k'}-\lambda_{2k'-1})^2}(1+l^2(k')).\end{aligned}$$

Further

$$(\Delta(\nu_k)^2-4)^{1/2}=2(\lambda_{2k}-\nu_k)^{1/2}(\nu_k-\lambda_{2k-1})^{1/2}(1+l^2(k))$$

and hence

$$\begin{aligned}A_{k',k}=&\frac{\lambda_{2k}-\lambda_{2k-1}}{(\lambda_{2k'}-\lambda_{2k'-1})^2}(\lambda_{2k'}-\nu_{k'})(\lambda_{2k'}-\mu_{k'}).\\ &\times\left\{\frac{\lambda_{2k'}-\lambda_{2k'-1}}{(\lambda_{2k}-\nu_k)(\nu_k-\lambda_{2k'-1})}+\frac{l^2(k')}{\nu_k-\lambda_{2k'-1}}\right\}(1+l^2(k))(1+l^2(k'))\\ &+\frac{\lambda_{2k}-\lambda_{2k-1}}{\nu_k-\lambda_{2k'-1}}l^2(k').\end{aligned}$$

It follows from the asymptotic behavior of λ_k, μ_k and ν_k for large $|k|$ that for $k'\neq k$

$$\begin{aligned}|A_{k',k}|\leqslant&\left(\frac{(\lambda_{2k}-\lambda_{2k-1})(\lambda_{2k'}-\lambda_{2k'-1})}{(k-k')^2\pi^2}+\frac{(\lambda_{2k}-\lambda_{2k-1})}{|k'-k|\pi}l^2(k')\right)\\ &\times(1+l^2(k))(1+l^2(k')).\end{aligned}$$

Thus, for $k'\neq k$, we obtain

$$|A_{k',k}|\leqslant\frac{l^2(k)l^2(k')}{(k-k')^2}+\frac{l^2(k)l^2(k')}{|k-k'|}(1+l^2(k))$$

and therefore

$$\sum_{k',k\in J_1} |B_{k',k}|^2 = \sum_{\substack{k',k\in J_1\\ k'\neq k}} |A_{k',k}|^2 < \infty.$$

Thus (i) is proved.

To show (ii) observe that

$$\frac{(F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2}{\nu_k - \lambda_{2k-1}} - \frac{(F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2}{\nu_k - \lambda_{2k}} = \frac{1}{\lambda_{2k} - \lambda_{2k-1}}(1 + l^2(k)).$$

Hence

$$A_{k,k} = 1 + l^2(k).$$

As A_{kk} is different from zero for every $k \in J_1$, (ii) follows.

In order to prove (iii) we must show that $C^{-1}A$ is one-to-one. Lemma 3.10 shows that $C^{-1}A = \mathrm{Id} + C^{-1}B$ where $C^{-1}B$ is a Hilbert-Schmidt operator. In order to show that $C^{-1}A$ is one-to-one it suffices to prove that the regularized determinant $\det_2 C^{-1}A$ is different from zero (see [Sim] for the definition and properties of $\det_2$). As in the first step one estimates $\det_2 C^{-1}A$ by the regularized determinants of finite matrices $(C^{-1}A)_{J'}$ associated with a finite subset J' of J_1.

First, recall that

$$\det{}_2(C^{-1}A)_{J'} = \det(C^{-1}A)_{J'} e^{-\mathrm{Tr}(C^{-1}B)_{J'}} = \det(C^{-1}A)_{J'}$$

because $\mathrm{Tr}(C^{-1}B)_{J'} = 0$ by the definition of B. Further

$$\det(C^{-1}A)_{J'} = \det\left(\frac{(F^{(1)}_{2k'-1}(0)F^{(2)}_{2k'-1}(0))^2}{\nu_k - \lambda_{2k'-1}} + \frac{(F^{(1)}_{2k'}(0)F^{(2)}_{2k'}(0))^2}{\lambda_{2k'} - \nu_k}\right)_{(k',k)\in J'\times J'}$$
$$\cdot \prod_{k\in J'}\left(\frac{(F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2}{\nu_k - \lambda_{2k-1}} + \frac{(F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2}{\lambda_{2k} - \nu_k}\right)^{-1} \tag{3.12}$$

and, similar as above,

$$\det\left(\frac{(F^{(1)}_{2k'-1}(0)F^{(2)}_{2k'-1}(0))^2}{\nu_k - \lambda_{2k'-1}} + \frac{(F^{(1)}_{2k'}(0)F^{(2)}_{2k'}(0))^2}{\lambda_{2k'} - \nu_k}\right)_{k',k\in J'\times J'}$$
$$= \sum_{x} (-1)^{|\varepsilon|} \prod_{x_k = -\lambda_{2k}} (F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2 \prod_{x_k = -\lambda_{2k-1}} (F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2.$$
$$\cdot \det\left(\frac{1}{\nu_k + x_{k'}}\right)_{(k',k)\in J'\times J'} \tag{3.13}$$

where $x = (x_k)_{k\in J'}$, $\varepsilon = (\varepsilon_k)_{k\in J'}$ and $|\varepsilon|$ are defined as in the first step.

For $\det C_{J'}$ we obtain the following expression

$$\prod_{k\in J'}\left(\frac{(F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2}{\nu_k-\lambda_{2k-1}}+\frac{(F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2}{\lambda_{2k}-\nu_k}\right)$$
$$=\sum_x(-1)^{|\varepsilon|}\prod_{x_k=-\lambda_{2k}}(F^{(1)}_{2k}(0)F^{(2)}_{2k}(0))^2\prod_{x_k=-\lambda_{2k-1}}(F^{(1)}_{2k-1}(0)F^{(2)}_{2k-1}(0))^2\prod_{k\in J'}\frac{1}{\nu_k+x_k}. \tag{3.14}$$

As in the first step using (3.12)–(3.14) we conclude

$$\det(C^{-1}A)_{J'} = \det{}_2(C^{-1}A)_{J'} \geqslant K > 0$$

for every finite subset $J' \subset J_1$, where K is independent of J'. Therefore

$$\det{}_2 C^{-1}A \geqslant K > 0.$$

Theorem 3.6 can be improved in the case where $(p, q)\in\mathscr{H}^1$.

THEOREM 3.13. *For $(p, q)\in\mathscr{H}^1$ $d_{(p,q)}\Phi$ is a linear isomorphism form $\mathscr{H}^1$ onto $\mathscr{M}^1$.*

For this purpose we need the following

LEMMA 3.14. *If $(p, q)\in\mathscr{H}^1$ then*

$$\begin{aligned}G_{2k-1}(x) = &\begin{pmatrix}\sin k\pi x\\ \cos k\pi x\end{pmatrix}+\frac{1}{2\pi k}\begin{pmatrix}-q(x)\sin k\pi x+\cos k\pi x(p(x)-p(0))\\ \sin k\pi x(p(0)+p(x))+q(x)\cos k\pi x\end{pmatrix}\\ &+\frac{1}{2k\pi}\left(\int_0^x(p(t)^2+q(t)^2)\,\mathrm{d}t - x\int_0^1(p(t)^2+q(t)^2)\,\mathrm{d}t\right)\\ &\times\begin{pmatrix}-\cos k\pi x\\ \sin k\pi x\end{pmatrix}+l_1^2(k)\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}G_{2k}(x) = &\begin{pmatrix}\cos k\pi x\\ -\sin k\pi x\end{pmatrix}+\frac{1}{2\pi k}\begin{pmatrix}(p(0)-p(x))\sin k\pi x-q(x)\cos k\pi x\\ -q(x)\sin k\pi x+(p(x)+p(0))\cos k\pi x\end{pmatrix}\\ &+\frac{1}{2k\pi}\left(\int_0^x(p(t)^2+q(t)^2)\,\mathrm{d}t - x\int_0^1(p(t)^2+q(t)^2)\,\mathrm{d}t\right)\\ &\times\begin{pmatrix}\sin k\pi x\\ \cos k\pi x\end{pmatrix}+l_1^2(k)\end{aligned} \tag{3.16}$$

where the error terms are uniformly bounded in $0 \leqslant x \leqslant 1$ *and with respect to* (p, q) *in any bounded set of* $\mathscr{H}^1$.

Proof of Lemma 3.14. From [Gre-Gui; Section 1] we get for $j \in \{2k-1, 2k\}$

$$F_1(x, \lambda_j) = \begin{pmatrix} \cos k\pi x \\ -\sin k\pi x \end{pmatrix} + \frac{1}{2k\pi} \begin{pmatrix} -(p(x) + p(0)) \sin k\pi x + (q(0) - q(x)) \cos k\pi x \\ -(q(x) + q(0)) \sin k\pi x + (p(x) - p(0)) \cos k\pi x \end{pmatrix}$$
$$+ \frac{1}{2k\pi} \left(\int_0^x (p(t)^2 + q(t)^2) \, \mathrm{d}t - x(\|p\|^2 + \|q\|^2) \right) \begin{pmatrix} \sin k\pi x \\ \cos k\pi x \end{pmatrix} + l_1^2(k) \tag{3.17}$$

and

$$F_2(x, \lambda_j) = \begin{pmatrix} \sin k\pi x \\ \cos k\pi x \end{pmatrix} + \frac{1}{2k\pi} \begin{pmatrix} (p(x) - p(0)) \cos k\pi x - (q(x) + q(0)) \sin k\pi x \\ (q(x) - q(0)) \cos k\pi x + (p(x) + p(0)) \sin k\pi x \end{pmatrix}$$
$$+ \frac{1}{2k\pi} \left(\int_0^x (p(t)^2 + q(t)^2) \, \mathrm{d}t - x(\|p\|^2 + \|q\|^2) \right)$$
$$\times \begin{pmatrix} -\cos 2k\pi x \\ \sin 2k\pi x \end{pmatrix} + l_1^2(k) \tag{3.18}$$

Then for $j \in \{2k-1, 2k\}$ and for $k \neq 0$

$$F_1(0, \lambda_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad F_1(1, \lambda_j) = \begin{pmatrix} (-1)^k \\ 0 \end{pmatrix} + l_1^2(k),$$
$$\|F_1(\cdot, \lambda_j)\|_{L^2([0,1])^2} = 1 + \frac{q(0)}{k\pi} + l_1^2(k) \tag{3.19}$$

and

$$F_2(0, \lambda_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad F_2(1, \lambda_j) = \begin{pmatrix} 0 \\ (-1)^k \end{pmatrix} + l_1^2(k),$$
$$\|F_2(\cdot, \lambda_j)\|_{L^2([0,1])^2} = 1 - \frac{q(0)}{k\pi} + l_1^2(k). \tag{3.20}$$

Further

$$(F_1(\cdot, \lambda_j), F_2(\cdot, \lambda_j))_{L^2([0,1])^2} = -\frac{p(0)}{k\pi} + l_1^2(k). \tag{3.21}$$

Following the proof of Lemma 3.4 we now obtain for $j \in \{2k-1, 2k\}$

$$I_k(\cdot) = \frac{F_1(\cdot, \lambda_j)}{\|F_1(\cdot, \lambda_j)\|_{L^2([0,1])^2}} + l_1^2(k) \tag{3.22}$$

$$G_{2k-1}(\cdot) = \frac{F_2(\cdot, \lambda_j)}{\|F_2(\cdot, \lambda_j)\|_{L^2([0,1])^2}} + l_1^2(k). \tag{3.23}$$

The error terms are in $l_1^2(\mathbb{Z})$ because, for $(p, q) \in \mathscr{H}^1$, $(\gamma_k(p, q))_{k \in \mathbb{Z}} \in l_1^2(\mathbb{Z})$.

Define for $|k|$ sufficiently large

$$L_k(\cdot) = \frac{\|F_1(\cdot, \lambda_{2k-1})\| I_k(\cdot) + (p(0)/k\pi) G_{2k-1}(\cdot)}{\| \|F_1(\cdot, \lambda_{2k-1})\| I_k(\cdot) + (p(0)/k\pi) G_{2k-1}(\cdot)\|}. \tag{3.24}$$

Thus $L_k(\cdot) \in E_k(p, q)$ and $\|L_k(\cdot)\|_{L^2([0,1])^2} = 1$. It follows from (3.19), (3.21), (3.22) and (3.24) that

$$(G_{2k-1}(\cdot), L_k(\cdot))_{L^2([0,1])^2} = l_1^2(k) \tag{3.25}$$

for $|k|$ sufficiently large.

Thus for $|k|$ sufficiently large, there exist α_k and β_k such that

$$G_{2k}(\cdot) = \alpha_k L_k(\cdot) + \beta_k G_{2k-1}(\cdot).$$

From $\|G_{2k}(\cdot)\| = 1$ and $(G_{2k}(\cdot), G_{2k-1}(\cdot)) = 0$ we deduce that

$$1 = \alpha_k^2 + \beta_k^2 + 2\alpha_k \beta_k (L_k(\cdot), G_{2k-1}(\cdot))$$

and

$$0 = \alpha_k (L_k(\cdot), G_{2k}(\cdot)) + \beta_k.$$

It then follows from (3.25) that

$$\beta_k = l_1^2(k) \quad \text{and} \quad \alpha_k = 1 + l_1^1(k).$$

We then obtain

$$G_{2k}(\cdot) = L_k(\cdot) + l_1^2(k). \tag{3.26}$$

Finally (3.15) and (3.16) are deduced from (3.17)–(3.23) and (3.26) and Lemma 3.14 is proved.

We then obtain

LEMMA 3.15. *If* $(p, q) \in \mathscr{H}^1$ *and* $(u, v) \in \mathscr{H}^0$ *then*

$$d_{(p,q)}\Psi_{2k}[(u, v)] = -\int_0^1 \sin 2k\pi x \; v(x)\,\mathrm{d}x + \int_0^1 \cos 2k\pi x \; u(x)\,\mathrm{d}x + l_1^2(k)$$

$$d_{(p,q)}\Psi_{2k-1}[(u, v)] = \int_0^1 \cos 2k\pi x \; v(x)\,\mathrm{d}x + \int_0^1 \sin 2k\pi x \; u(x)\,\mathrm{d}x + l_1^2(k)$$

where the error terms are uniform with respect to (u, v) *on any bounded set of* $\mathscr{H}^0$.

Proof of Lemma 3.15. As $(p, q) \in \mathscr{H}^1$, the gap sequence $(\gamma_k)_{k\in\mathbb{Z}}$ is in $l_1^2(\mathbb{Z})$. Lemma 3.15 then follows from Theorem 3.1 and the asymptotic estimates (3.15) and (3.16).

Proof of Theorem 3.13. It follows from Theorem 3.6 that $d_{(p,q)}\Phi$ is one-to-one. To prove that $d_{(p,q)}\Phi$ is onto it is equivalent to show that the linear map $d_{(p,q)}\Psi$ from $\mathscr{H}^1$ into $l_1^2(\mathbb{Z}) \times l_1^2(\mathbb{Z})$ given by

$$d_{(p,q)}\Psi[(u, v)] = (d_{(p,q)}\Psi_{2k}[(u, v)],\, d_{(p,q)}\Psi_{2k-1}[(u, v)])_{k\in\mathbb{Z}}.$$

is onto.

Let $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ be in $l_1^2(\mathbb{Z})$. From Theorem 3.6 it follows that there exist $u(\cdot)$ and $v(\cdot)$ in $L^2([0, 1])$ such that

$$d_{(p,q)}\Psi[(u, v)] = (a_k, b_k)_{k\in\mathbb{Z}}.$$

It is to prove that (u, v) is in $\mathscr{H}^1$. Lemma 3.15 shows that each of the sequences

$$\left(\int_0^1 \cos 2n\pi x \;\; v(x)\,\mathrm{d}x\right)_{n\in\mathbb{N}}, \qquad \left(\int_0^1 \cos 2n\pi x \;\; u(x)\,\mathrm{d}x\right)_{n\in\mathbb{N}}$$

$$\left(\int_0^1 \sin 2n\pi x \;\; v(x)\,\mathrm{d}x\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\int_0^1 \sin 2n\pi x \;\; u(x)\,\mathrm{d}x\right)_{n\in\mathbb{N}}$$

are in $l_1^2(\mathbb{N})$. Then, as in the proof of Theorem I.18 of [Gre-Gui], this implies that $u(\cdot)$ and $v(\cdot)$ are in $H^1([0, 1])$ with $u(1) - u(0) = v(1) - v(0) = 0$.

Appendix A

In this appendix we generalize Theorem 3.7 of [Gre-Gui].

Let $\pi(\cdot, \cdot)$ be the map from $\mathscr{H}^0$ into $\mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$ defined by

$$\pi(p, q) = ((\mu_k(p, q))_{k\in\mathbb{Z}}, (\chi_k(p, q))_{k\in\mathbb{Z}})$$

where the $\mu_k(p,q)$'s are the zeroes of the map $\lambda \to Z_1(1,\lambda;p,q)$ and $\chi_k(p,q) = \log\{(-1)^k Y_1(1,\mu_k(p,q))\}$. Let for $(p,q)\in\mathscr{H}^0$

$$\mathscr{T}_{(p,q)} = \left\{((\xi_k)_{k\in\mathbb{Z}}, (\eta_k)_{k\in\mathbb{Z}}) \in \left(\prod_{k\in\mathbb{Z}} [\lambda_{2k-1}(p,q), \lambda_{2k}(p,q)]\right) \times \mathbb{R}^{\mathbb{Z}}; \right.$$
$$\left. \Delta(\xi_k; p, q) = 2(-1)^k \cosh\eta_k,\ k\in\mathbb{Z}\right\}.$$

THEOREM A.1. *Suppose $(p_0, q_0)\in\mathscr{H}^0$. Then $\pi(\cdot,\cdot)$ is a homeomorphism from* $\mathrm{Iso}_0(p_0,q_0)$ *onto* $\mathscr{T}_{(p_0,q_0)}$.

In [Gre-Gui] Theorem A.1 is proved for $(p_0,q_0)\in\mathscr{H}^1$ using the isospectral flows $(k\in\mathbb{Z})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p(\cdot,t) \\ q(\cdot,t)\end{pmatrix} = V_k(p(\cdot,t), q(\cdot,t))$$
$$p(x,0) = p_0(x) \quad \text{and} \quad q(x,0) = q_0(x) \tag{A.1}$$

where

$$V_k(p(\cdot), q(\cdot)) = \begin{pmatrix} \dfrac{\partial\Delta}{\partial q(\cdot)}(\lambda; p(\cdot), q(\cdot))|_{\lambda=\mu_k(p(\cdot),q(\cdot))} \\ -\dfrac{\partial\Delta}{\partial p(\cdot)}(\lambda; p(\cdot), q(\cdot))|_{\lambda=\mu_k(p(\cdot),q(\cdot))}\end{pmatrix}.$$

According to [Gre-Gui], the ordinary differential equation (A.1) has a unique solution in $H^1([-t_0,t_0], \mathscr{H}^0)$ for initial values in $\mathscr{H}^0$ with $t_0>0$ chosen sufficiently small, and for this solution to exist globally in t, it suffices to prove the following

LEMMA A.2. *Let $(p(\cdot,t), q(\cdot,t))$ be a solution of (A.1) defined on a compact interval $I\subseteq\mathbb{R}$, $0\in I$, in $H^1(I;\mathscr{H}^0)$. Then*

$$\|p(\cdot,t), q(\cdot,t)\|_{\mathscr{H}^0} = \|p_0(\cdot), q_0(\cdot)\|_{\mathscr{H}^0}, \quad t\in I.$$

REMARK A.3. If the potentials $(p_0(\cdot), q_0(\cdot))\in\mathscr{H}^1$, it is easy to show that $\|(p(\cdot,t), q(\cdot,t))\|_{\mathscr{H}^0}$ is independent of t as this quantity is a spectral invariant appearing in the asymptotic expansion of the λ_k's (cf. [Gre-Gui]).

Proof of Lemma A.2. Define $u(x,t) = (p(x,t), q(x,t))$ and $u_0(x) = (p_0(x), q_0(x))$. Choose a sequence $(u_0^{(n)})_{n\geq 0}$ in $\mathscr{H}^1$ which converges to u_0 in $\mathscr{H}^0$. According to [Gre-Gui] there exists a unique solution $u^{(n)}(x,t)$ of (A.1) in $H^1(\mathbb{R};\mathscr{H}^1)$. Moreover these solutions satisfy for a.e t:

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t} u^{(n)}(\cdot,t)\right\|_{\mathscr{H}^0} \leq \beta(\|u^{(n)}(\cdot,0)\|_{\mathscr{H}^0})$$

where $\beta(\cdot)$ is a positive function on $\mathbb{R}$ which is independent of n and t. (See [Gre; Thm. 2, p. 132]).

Thus $(u^{(n)})_{n\geqslant 0}$ is a bounded sequence in $H^1(I;\mathscr{H}^0)$. Hence there exists a subsequence, again denoted by $(u^{(n)})_{n\geqslant 0}$, which converges weakly in $H^1(I,\mathscr{H}^0)$ to a function $v\in H^1(I;\mathscr{H}^0)$, i.e.,

$$\lim_{n\to\infty}\frac{\mathrm{d}^j}{\mathrm{d}t^j}u^{(n)}=\frac{\mathrm{d}^jv}{\mathrm{d}t^j}\text{ weakly in }L^2(I,\mathscr{H}^0)\text{ for }j=0,1.$$

Furthermore it follows from [Gre, Part II, Chap. 3, Th. 2] and [Pö-Tru] that the vector fields V_k are compact on $\mathscr{H}^0$. Thus $(V_k(u^{(n)}))_{n\geqslant 1}$ converges strongly to $V_k(v)$ in $L^2(I,\mathscr{H}^0)$. Hence

$$\frac{\mathrm{d}v}{\mathrm{d}t}=V_k(v)\text{ in }L^2(I,\mathscr{H}^0). \tag{A.2}$$

The trace theorem guarantees the weak-convergence of $(u^{(n)}(\cdot,0))_{n\geqslant 0}$ weakly in $\mathscr{H}^0$ to $v(\cdot,0)$ as n tends to infinity and $(u^{(n)}(\cdot,0))_{n\geqslant 0}=(u_0^{(n)}(\cdot))_{n\geqslant 0}$ converges to $u_0(\cdot)$ strongly in $\mathscr{H}^0$. Thus $v(x,0)=u_0(x)$ for a.e. x in $[0,1]$.

By the uniqueness of the solution to (A.1) we get $u(x,t)=v(x,t)$ for a.e. $x\in[0,1]$ and for every $t\in I$. Since $(u^{(n)}(\cdot,t))_{n\geqslant 0}$ converges to $u(\cdot,t)$ weakly in $\mathscr{H}^0$ and $\left(\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(\cdot,t)\right)_{n\geqslant 0}$ converges to $\frac{\mathrm{d}u}{\mathrm{d}t}(\cdot,t)$ strongly in $\mathscr{H}^0$ for every $t\in I$,

$$\left\{\left(u^{(n)}(\cdot,t),\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(\cdot,t)\right)\right\}_{n\geqslant 0}\text{ converges to }\left(u(\cdot,t),\frac{\mathrm{d}u}{\mathrm{d}t}(\cdot,t)\right)$$

for a.e. t in I.

Furthermore

$$\left(u^{(n)}(\cdot,t),\frac{\mathrm{d}}{\mathrm{d}t}u^{(n)}(\cdot,t)\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^{(n)}(\cdot,t)\|^2_{\mathscr{H}^0}$$

and it follows from Remark A.3 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u^{(n)}(\cdot,t)\|^2_{\mathscr{H}^0}=0\text{ for every }n\in\mathbb{N}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2_{\mathscr{H}^0}=0\quad\text{for every }t\text{ in }I$$

and Lemma A.2 is proved.

As a corollary we obtain the following generalization of Theorem 3.7 in [Gre-Gui].

COROLLARY A.4. *Suppose that* $(p, q) \in \mathscr{H}^0$. *Then*

(i) $\mathrm{Iso}_0(p, q) = \{(p', q') \in \mathscr{H}^0; \gamma_k(p', q') = \gamma_k(p, q), k \in \mathbb{Z}\}$

(ii) $\|(p, q)\|_{\mathscr{H}^0}$ *is a spectral invariant, i.e. is constant on* $\mathrm{Iso}_0(p, q)$.

In particular, this proves Theorem 1.1 as stated in the introduction.

Appendix B

In this appendix we prove the asymptotic expansions used in the proof of Theorem 3.4. The first result concerns certain asymptotic properties of the discriminant $\Delta(\lambda)$.

LEMMA B.1. *Suppose* (p, q) *in* $\mathscr{H}^0$. *Then, for every* $k \in \mathbb{Z}$,

(i) $\dot{\Delta}(\lambda_{2k}(p, q)) = (-1)^{k+1}\gamma_k(p, q)(1 + l^2(k))$

(ii) $\dot{\Delta}(\lambda_{2k-1}(p, q)) = (-1)^k\gamma_k(p, q)(1 + l^2(k))$.

Proof of Lemma B.1. We only prove (i). Assertion (ii) follows by a similar argument. In [Gre-Gui] it is shown that

$$\Delta(\lambda)^2 - 4 = -4(\lambda_0 - \lambda)(\lambda_{-1} - \lambda) \prod_{k \in \mathbb{Z}^*} \frac{(\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda)}{k^2\pi^2}$$

where $\prod_{k \in \mathbb{Z}^*} a_k$ means $\prod_{k \in \mathbb{N}^*} a_k \cdot a_{-k}$.
Thus, for $k \in \mathbb{Z}^*$,

$$2\Delta(\lambda_{2k})\dot{\Delta}(\lambda_{2k}) = -4(\lambda_0 - \lambda_{2k})(\lambda_{-1} - \lambda_{2k}) \frac{\gamma_k}{k^2\pi^2} \cdot \prod_{\substack{l \in \mathbb{Z}^* \\ l \neq k}} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k})}{l^2\pi^2}.$$

Since $\Delta(\lambda_{2k}) = 2(-1)^k$ this leads to

$$\dot{\Delta}(\lambda_{2k}) = (-1)^{k+1}\gamma_k(1 + l^2(k)) \prod_{\substack{l \in \mathbb{Z}^* \\ l \neq k}} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k})}{l^2\pi^2}.$$

Further, using that the Hilbert transform is a bounded operator on $l^2(\mathbb{Z})$,

$$\prod_{\substack{l \in \mathbb{Z}^* \\ l \neq k}} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k-1})}{l^2\pi^2} = \prod_{\substack{l \in \mathbb{Z}^* \\ l \neq k}} \frac{(l\pi - \lambda_{2k})^2}{l^2\pi^2}(1 + r(k, l))$$

where the error term satisfies $|r(k,l)| \leqslant l^2(k)$ for every $l \in \mathbb{Z}^*$, $l \neq k$. Using the well known product formula

$$\frac{\sin\lambda}{\lambda} = \prod_{l\geqslant 1} \frac{l^2\pi^2 - \lambda}{l^2\pi^2}$$

we finally obtain

$$\prod_{l\in\mathbb{Z}^*, l\neq k} \frac{(\lambda_{2l} - \lambda_{2k})(\lambda_{2l-1} - \lambda_{2k})}{l^2\pi^2}$$
$$= \left(\frac{\sin\lambda_{2k}}{\lambda_{2k}} \frac{k\pi}{k\pi - \lambda_{2k}}\right)^2 (1 + l^2(k)) = 1 + l^2(k).$$

LEMMA B.2. *Let (p, q) be in $\mathscr{H}^0$. For every $k \in \mathbb{Z}$*
(i) $Y_2(1, \lambda_{2k}(p, q)) = (-1)^k(\lambda_{2k}(p, q) - \nu_k(p, q))(1 + l^2(k))$
(ii) $Y_2(1, \lambda_{2k-1}(p, q)) = (-1)^k(\lambda_{2k-1}(p, q) - \nu_k(p, q))(1 + l^2(k))$.

Proof of Lemma B.2. In [Gre-Gui] it is proved that

$$Y_2(1, \lambda; p, q) = (\lambda - \nu_0(p, q)) \prod_{m\in\mathbb{Z}^*} \frac{\nu_m(p, q) - \lambda}{m\pi}.$$

Thus for $k \in \mathbb{Z}^*$ and $j \in \{2k-1, 2k\}$ we obtain

$$Y_2(1, \lambda_j(p, q); p, q)$$
$$= -\frac{(\lambda_j(p,q) - \nu_0(p,q))}{2\pi} (\lambda_j(p,q) - \nu_k(p,q)) \prod_{\substack{m\in\mathbb{Z}^* \\ m\neq k}} \frac{(\nu_m(p,q) - \lambda_j(p,q))}{m\pi}$$
$$= (-1)^k(\lambda_j(p,q) - \nu_k(p,q)) \left| \frac{(\lambda_j(p,q) - \nu_0(p,q))}{k\pi} \prod_{\substack{m\in\mathbb{Z}^* \\ m\neq k}} \frac{(\nu_m(p,q) - \lambda_j(p,q))}{m\pi} \right|$$

from which one deduces Lemma B.2, using similar arguments as in the proof of Lemma B.1.

Combining the two lemmas we obtain

LEMMA B.3. *Let (p, q) be in $\mathscr{H}^0$. Then for every k with $\lambda_{2k-1} < \lambda_{2k}$,*

(i) $$-\frac{Y_2(1, \lambda_{2k}(p, q))}{\dot{\Delta}(\lambda_{2k}(p, q))} = \frac{\lambda_{2k}(p, q) - \nu_k(p, q)}{\gamma_k(p, q)} (1 + l^2(k))$$

(ii) $$-\frac{Y_2(1, \lambda_{2k-1}(p, q))}{\dot{\Delta}(\lambda_{2k-1}(p, q))} = \frac{\nu_k(p, q) - \lambda_{2k-1}(p, q)}{\gamma_k(p, q)} (1 + l^2(k))$$

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