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## On Tunnell's formula for characters of $GL(2)$

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### 1. Introduction

For a local field  $F$  of characteristic 0, let  $\pi$  an irreducible admissible representation of  $GL_2(F)$  of infinite dimension, and  $\omega_\pi$  its central character,  $\chi_\pi$  its character respectively. For each quadratic extension  $L$  of  $F$ , we fix an embedding of  $L^\times$  into  $GL_2(F)$ , and consider  $L^\times$  as the set of  $F$ -rational points of a Cartan subgroup of  $GL_2(F)$ . In [T], Tunnell gave an expression for the restriction of  $\chi_\pi$  to  $L^\times$  as a sum of quasicharacters of  $L^\times$ , in which the coefficients are written in terms of  $\varepsilon$ -factors of the base change lifting of  $\pi$  to  $L$ . His proof is based on the case by case computation of  $\varepsilon$ -factors and characters and is not transparent. Furthermore the case of residual characteristic 2 was only partially treated. The purpose of this paper is to give a natural proof of the formula of Tunnell including the case of residual characteristic 2.

We state the theorem more exactly. Let  $\psi$  be a nontrivial additive character of  $F$ , and set  $\psi_L = \psi \circ tr_{L/F}$ , where  $tr_{L/F}$  is the trace from  $L$  to  $F$ . Let  $\Pi$  be the base change lifting of  $\pi$  to  $L$  (cf. [L]). Then the central character  $\omega_\Pi$  of  $\Pi$  is given by  $\omega_\pi \circ n_{L/F}$ , where  $n_{L/F}$  is the norm from  $L$  to  $F$ . For each quasicharacter  $\lambda$  of  $L^\times$ , let  $\varepsilon(\Pi \otimes \lambda, \psi_L)$  be the  $\varepsilon$ -factor of the representation  $\Pi \otimes \lambda$  of  $GL_2(L)$  with respect to  $\psi_L$ . For  $\lambda$  whose restriction to  $F^\times$  is  $\omega_\pi$ , it is shown in [T] that  $\varepsilon(\Pi \otimes \lambda, \psi_L)$  is independent of  $\psi$  and is equal to 1 or  $-1$ . In this notation, we can state the result as follows.

**THEOREM.** *Let  $\pi$  be an infinite dimensional irreducible admissible representation of  $GL_2(F)$ , and  $\chi_\pi$  its character. For a quadratic extension  $L$  of  $F$ , let  $\Pi$  be the base change lifting of  $\pi$  to  $L$ . Then one has*

$$\chi_\pi|_{L^\times} = \sum_{\lambda} \frac{1 + \varepsilon(\Pi \otimes \lambda, \psi_L)\omega_\pi(-1)}{2} \lambda,$$

where  $\lambda$  runs through all quasicharacters of  $L^\times$  whose restriction to  $F^\times$  coincide with  $\omega_\pi$ .

Here the functions on both sides are considered as continuous functions on

$L_{\text{reg}}^\times = L^\times - F^\times$ , and the sum on the right-hand side is computed by partial summation with respect to the conductors of quasicharacters.

The idea of the proof is simple. Let  $\sigma$  be the generator of the Galois group  $\text{Gal}(L/F)$ . For  $g \in GL_2(L)$ , we set

$$N(g) = g^\sigma g.$$

Here  $g^\sigma$  denotes the componentwise action of  $\sigma$  on  $g$ . For  $a \in L^\times$ , to distinguish components of  $GL_2(L)$  and elements of  $GL_2(F)$  under the embedding fixed above, we write  $\tilde{a}$  when considered as an element of  $GL_2(F)$ . Then we have

$$N\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w\right) = \begin{pmatrix} a & 0 \\ 0 & \sigma a \end{pmatrix} \sim \tilde{a} \in GL_2(F), \quad (1.1)$$

where  $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , and  $\sim$  denotes the conjugacy in  $GL_2(L)$ . The idea is to reduce the calculation of  $\chi_\pi|_{L^\times}$  to that in  $GL_2(L)$  by means of (1.1) and the theory of base change lifting.

## 2. Proof of the theorem

Besides the notations introduced above, we use the following ones. For a local field  $L$ , let  $\mathfrak{o}_L$  be the maximal compact ring of  $L$ , and  $\mathfrak{p}_L$  the maximal ideal of  $\mathfrak{o}_L$ . Let  $v_L$  be the additive valuation of  $L$  such that  $v_L(\varpi_L) = 1$  for a prime element  $\varpi_L$  of  $L$ . Let  $|\mathfrak{o}_L/\mathfrak{p}_L| = q_L$ , and let  $|\cdot|$  be the absolute value of  $L$  such that  $|\varpi_L| = q_L^{-1}$ . For a quasicharacter  $\lambda$  of  $L^\times$ , we denote by  $f(\lambda)$  the exponent of the conductor of  $\lambda$ . For quasicharacters  $\lambda_1, \lambda_2$  of  $L^\times$ , we define  $\lambda_1 \sim \lambda_2$  if  $\lambda_1 \lambda_2^{-1}$  is unramified. Let  $L$  and  $F$  be as in Section 1, and let  $n(\psi_L)$  the largest integer which satisfies  $\psi_L(\mathfrak{p}_L^{-n(\psi_L)}) = \{1\}$ . For a positive integer  $n$ , set

$$\Gamma_n = \begin{pmatrix} 1 + \mathfrak{p}_L^n & \mathfrak{p}_L^n \\ \mathfrak{p}_L^n & 1 + \mathfrak{p}_L^n \end{pmatrix} \cap GL_2(\mathfrak{o}_L).$$

For an irreducible admissible representation  $\Pi$  of  $GL_2(L)$ , let  $p_L^{f(\Pi)}$  the conductor of  $\Pi$ . For  $\Pi$ , let  $L(s, \Pi)$  and  $\varepsilon(s, \Pi, \psi_L)$  be as in [J-L], and set  $\varepsilon(\Pi, \psi_L) = \varepsilon(1/2, \Pi, \psi_L)$ .

The proof of the theorem in the cases of principal series and special representations is easy and treated completely in [T], and in the following we assume  $\pi$  is supercuspidal. Let  $\mathscr{W}(\Pi)$  be the Whittaker model of  $\Pi$  with respect to the additive character  $\psi_L$ . Then  $\mathscr{W}(\Pi)$  consists of functions  $W$  on  $GL_2(L)$

which satisfy

$$W\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \psi_L(x)W(g), \quad x \in L,$$

and the right translation  $\rho$  gives  $\Pi$ . Let  ${}^\sigma\Pi$  be the representation defined by  ${}^\sigma\Pi(g) = \Pi({}^\sigma g)$  for  $g \in GL_2(L)$ . We set

$${}^\sigma\mathcal{W}(\Pi) = \{W({}^\sigma g) \mid W \in \mathcal{W}(\Pi)\},$$

and  $(I_\sigma W)(g) = W({}^{\sigma^{-1}}g) = W({}^\sigma g)$ . Then  $I_\sigma$  gives an isomorphism from  ${}^\sigma\mathcal{W}(\Pi)$  to  $\mathcal{W}(\Pi)$ , and for  $W' \in {}^\sigma\mathcal{W}(\Pi)$ , we see  $W'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}g\right) = \psi_L(x)W'(g)$  and

$$\rho(g)W' = I_\sigma^{-1}\Pi({}^\sigma g)I_\sigma W'.$$

By the uniqueness of Whittaker models, we have  $\mathcal{W}({}^\sigma\Pi) = {}^\sigma\mathcal{W}(\Pi)$ . Since  $\Pi$  is the base change lifting of  $\pi$  to  $L$ ,  $\Pi$  is equivalent to  ${}^\sigma\Pi$  and  $\mathcal{W}(\Pi) = \mathcal{W}({}^\sigma\Pi) = {}^\sigma\mathcal{W}(\Pi)$ . We see also that  $I_\sigma$  gives an intertwining operator of  $\Pi$  to  ${}^\sigma\Pi$  which satisfies  $I_\sigma^2 = 1$ , and fixes the linear form  $L(W) = W(1)$  on  $\mathcal{W}(\Pi)$ . Setting  $\Pi((g, \sigma)) = \Pi(g)I_\sigma$ , we can extend  $\Pi$  to the semidirect product of  $GL_2(L)$  and  $\text{Gal}(L/F)$  by the action of  $\text{Gal}(L/F)$  on  $GL_2(L)$ . Let  $\chi_{\Pi, \sigma}$  be the twisted character of this representation, namely, the distribution on  $C_c^\infty(GL_2(L))$  defined by

$$\chi_{\Pi, \sigma}(\varphi) = \text{trace}(\Pi(\varphi)I_\sigma),$$

for  $\varphi \in C_c^\infty(GL_2(L))$ . Then the twisted character is given by a function which is locally integrable and locally constant on the set of  $\sigma$ -regular elements of  $GL_2(L)$  and satisfies

$$\chi_{\Pi, \sigma}(g) = \chi_\pi(Ng),$$

for  $\sigma$ -regular elements (cf. [L], [A-C]). By (1.1), to obtain  $\chi_\pi|_{L^\times}$ , it is enough to compute  $\chi_{\Pi, \sigma}\left(\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix}w\right)\right)$ . We carry out this calculation in the Kirillov model

$\mathcal{K}(\Pi)$  of  $\Pi$ , that is, on  $\mathcal{K}(\Pi) = \left\{W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \mid W \in \mathcal{W}(\Pi)\right\}$ . An element  $f$  of  $\mathcal{K}(\Pi)$  is a locally constant function on  $L^\times$  which satisfies

$$f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}x\right) = \psi_L(bx)f(ax),$$

and the action of  $I_\sigma$  can be written as  $(I_\sigma f)(x) = f(\sigma^{-1}x) = f(\sigma x)$ .

First we treat the case where  $\Pi$  is supercuspidal. Then  $\mathcal{K}(\Pi)$  coincides with the space  $\mathcal{S}(L^\times)$  of Schwartz-Bruhat functions on  $L^\times$  and a basis of this space is given by the set of the following functions:

$$\xi_\lambda^{(n)} = \begin{cases} \lambda(x) & \text{if } v_L(x) = -n, \\ 0 & \text{otherwise.} \end{cases}$$

Here  $n$  is extended over all integers and  $\lambda$  is extended over a complete system of representatives of all quasicharacters of  $L^\times$  modulo  $\sim$  defined above. Later in the proof we choose representatives suitably. On this basis, the action of  $w$  is described by means of  $\varepsilon$ -factor as follows.

**LEMMA 2.1.** *Let  $\Pi$  be a supercuspidal representation of  $GL_2(L)$  and  $\xi_\lambda^{(n)}$  as above. Then one has*

$$\Pi(w)\xi_\lambda^{(n)} = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\xi^{(m)}\omega_\Pi\lambda^{-1},$$

where  $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n$ .

This is the formula (9) of [Y] and can be deduced from local functional equations of  $GL_2(L)$ .

We determine the subspace  $\mathcal{K}(\Pi)^n$  of  $\mathcal{K}(\Pi)$  consisting of elements invariant under  $\Gamma_n$ . Since an element of the form  $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ ,  $a \in \mathcal{O}_L^\times$ , normalizes  $\Gamma_n$ ,  $\mathcal{K}(\Pi)^n$  has a basis consisting of elements of the form

$$v = \sum_m a_m \xi_\lambda^{(m)},$$

with  $\lambda$  such that  $f(\lambda) \leq n$ . For  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(L)$  with  $d \neq 0$ , we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (ad-bc)/d & b \\ 0 & d \end{pmatrix} w^{-1} \begin{pmatrix} 1 & -c/d \\ 0 & 1 \end{pmatrix} w.$$

Hence for  $n$  such that  $f(\omega_\Pi) \leq n$ ,  $v$  is invariant under  $\Gamma_n$  if and only if  $v$  and  $\Pi(w)v$  are invariant under  $\begin{pmatrix} 1 & p_L^n \\ 0 & 1 \end{pmatrix}$ . This condition is equivalent to that the supports of  $v$  and  $\Pi(w)v$  are contained in  $p_L^{-n(\psi_L)-n}$ , and that  $a_m = 0$  unless  $f(\Pi \otimes \lambda^{-1}) + n(\psi_L) - n \leq m \leq n(\psi_L) + n$  by the above lemma. Let  $B_n$  be the set of  $\xi_\lambda^{(m)}$  such that  $f(\lambda) \leq n$  and  $f(\Pi \otimes \lambda^{-1}) + n(\psi_L) - n \leq m \leq n(\psi_L) + n$ . Then  $B_n$

gives a basis of  $\mathcal{K}(\Pi)^n$  for  $n$  sufficiently large and the union  $\bigcup_n B_n$  gives a basis of  $\mathcal{K}(\Pi)$ . Let  $P_n$  be the projection of  $\mathcal{K}(\Pi)$  onto  $\mathcal{K}(\Pi)^n$  defined by

$$\int_{\Gamma_n} \Pi(g)dg \Big/ \int_{\Gamma_n} dg,$$

where  $dg$  is a Haar measure on  $GL_2(L)$ . Then we can calculate the value of  $\chi_{\Pi,\sigma}(\tilde{a})$  as trace  $\left( \Pi \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w \right) I_\sigma P_n \right)$  with respect to this basis for a sufficiently large  $n$  (cf. [T] Lemma 2.2).

By the above lemma and the relation  $\varepsilon(\Pi \otimes \sigma \lambda^{-1}, \psi_L) = \varepsilon(\sigma \Pi \otimes \lambda^{-1}, \psi_L) = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)$ , which follows from  $\sigma \Pi \sim \Pi$ , we have

LEMMA 2.2. *The notation being as above, for  $a \in L_{\text{reg}}^\times$  one has*

$$\Pi \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w \right) I_\sigma \xi_\lambda^{(n)} = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \omega_\Pi^\sigma \lambda^{-1}(-a) \zeta_{\omega_\Pi^\sigma \lambda^{-1}}^{(m+v_L(a))}.$$

Here  $m = f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n$ .

Hence if  $\xi_\lambda^{(n)}$  contributes to the  $\chi_{\Pi,\sigma}(\tilde{a})$ , then it holds

- (a)  $n = n(\psi_L) + \frac{1}{2}(f(\Pi \otimes \lambda^{-1}) + v_L(a))$ ,
- (b)  $\lambda|_{o_L^\times} = \omega_\Pi^\sigma \lambda^{-1}|_{o_L^\times}$ .

First we assume  $L/F$  is unramified. Then (b) implies that  $\lambda|_{o_F^\times} = \omega_\pi|_{o_F^\times}$ . As a representative of the class of  $\lambda$ , we take  $\lambda$  such that  $\lambda(\varpi_F) = \omega_\pi(\varpi_F)$ . Then we have  $\omega_\Pi^\sigma \lambda^{-1} = \lambda$ . Let  $\lambda'$  be the quasicharacter of  $L^\times$  defined by the condition  $\lambda|_{o_L^\times} = \lambda'|_{o_L^\times}$  and  $\lambda(\varpi_F) = -\lambda'(\varpi_F)$ . Then we see the contribution to  $\chi_{\Pi,\sigma}$  of  $\xi_\lambda^{(n)}$  for the above  $\lambda$  is equal to

$$\begin{cases} \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \lambda(-a) & \text{if } v_L(a) \equiv f(\Pi \otimes \lambda^{-1}) \pmod{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and noticing  $\varepsilon(\Pi \otimes \lambda'^{-1}, \psi_L) = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) (-1)^{f(\Pi \otimes \lambda^{-1})}$ , we see this is equal to

$$\frac{1}{2} (\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \lambda(-a) + \varepsilon(\Pi \otimes \lambda'^{-1}, \psi_L) \lambda'(-a)).$$

Let  $\chi_{L/F}$  be the unramified character of  $F^\times$  corresponding to the quadratic

extension  $L/F$ . Then we have

$$\begin{aligned}
 \chi_\pi(\tilde{a}) &= \chi_{\Pi, \sigma} \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w \right) = \sum_{\lambda|_{F^\times} = \omega_\pi} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a) \\
 &\quad + \sum_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a) \\
 &= \sum_{\lambda|_{F^\times} = \omega_\pi} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \omega_\pi(-1)}{2} \lambda(a) \\
 &\quad + \sum_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \omega_\pi(-1)}{2} \lambda(a),
 \end{aligned}$$

since  $\lambda(-1) = \omega_\pi(-1)$  and  $\sum_{\lambda|_{F^\times} = \omega_\pi} \lambda = 0$ ,  $\sum_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \lambda = 0$ . We note  $\chi_\pi(r\tilde{a}) = \omega_\pi(r) \chi_\pi(\tilde{a})$  for  $r \in F^\times$ . Therefore the second factor of the above sum vanishes and our assertion has been proved in this case.

Now assume  $L/F$  is ramified. Let  $\varpi_F$  be a prime element of  $F$  which is contained in the norm of  $L$ . Let  $\chi_{L/F}$  the quadratic character of  $F^\times$  corresponding to the extension  $L/F$  as above. In this case the condition (b) implies only that  $\lambda|_{n_{L/F} o_L^\times} = \omega_\pi|_{n_{L/F} o_L^\times}$ , hence that  $\lambda|_{o_F^\times} = \omega_\pi|_{o_F^\times}$  or  $\omega_\pi \chi_{L/F}|_{o_F^\times}$ . In the class of  $\lambda$  satisfying (b), there exists exactly two characters satisfying  $\lambda_i(\varpi_F) = \omega_\pi(\varpi_F)$ ,  $i = 1, 2$ , and they satisfy  $\omega_\pi^\sigma \lambda_i^{-1} = \lambda_i$  and  $\lambda_1(\varpi_L) = -\lambda_2(\varpi_L)$ . In the same way as in the unramified case, we obtain

$$\begin{aligned}
 \chi_\pi(a) &= \sum_{\lambda|_{F^\times} = \omega_\pi} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \lambda(-1)}{2} \lambda(a) \\
 &\quad + \sum_{\lambda|_{F^\times} = \omega_\pi \chi_{L/F}} \frac{1 + \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L) \lambda(-1)}{2} \lambda(a).
 \end{aligned}$$

Also as in the unramified case, the second sum vanishes and for  $\lambda$  in the first sum, one has  $\lambda(-1) = \omega_\pi(-1)$ . This completes the proof of the theorem in the case where  $\Pi$  is supercuspidal.

Next assume  $\Pi$  is a principal series representation  $\pi(\mu_1, \mu_2)$ . Since  $\Pi$  is a base change lifting and  $\pi$  is neither principal series nor special, we have  $\mu_2 = \sigma \mu_1$ . In this case, we have to take care of the difference between  $\mathcal{H}(\Pi)$  and  $\mathcal{S}(L^\times)$ . As a basis of  $\mathcal{H}(\Pi)$ , we employ the following:  $\{\zeta_\lambda^{(n)}\} \cup \{\eta_1^{(n)}, \eta_2^{(n)}\}$ . Here  $n \in \mathbf{Z}$  and  $\lambda$  is extended over all classes of quasicharacters with respect to  $\sim$  which do not contain  $\mu_1$  nor  $\mu_2$ . The element  $\eta_i^{(n)}$ , for  $i = 1, 2$ , is defined by

$$\eta_i^{(n)}(x) = \begin{cases} \mu_i(x) |x|^{1/2} & \text{if } v_L(x) \geq -n \\ 0 & \text{otherwise.} \end{cases}$$

We note  $\omega_\Pi = \omega_\pi \circ n_{L/F} = \mu_1 \mu_2 = \mu_1 \circ n_{L/F} = \mu_2 \circ n_{L/F}$ . Hence we have  $\mu_i = \omega_\pi$  or  $\omega_\pi \chi_{L/F}$  on  $F^\times$ . The action of  $w$  on this basis can be described by

LEMMA 2.3. *The notation being as above, let  $\mu = \mu_1 \mu_2^{-1}$ . For  $\lambda \nmid \mu_1, \mu_2$ , one has*

$$\Pi(w)\xi_\lambda^{(n)} = \varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\xi_{\omega_\Pi \lambda^{-1}}^{(f(\Pi \otimes \lambda^{-1}) + 2n(\psi_L) - n)}.$$

If  $\mu$  is ramified, one has

$$\Pi(w)\eta_1^{(n)} = \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)|\varpi_L|^{f(\Pi \otimes \mu_1^{-1})/2 + n(\psi_L) - n} \eta_2^{(f(\Pi \otimes \mu_1^{-1}) + 2n(\psi_L) - n)},$$

$$\Pi(w)\eta_2^{(n)} = \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)|\varpi_L|^{f(\Pi \otimes \mu_2^{-1})/2 + n(\psi_L) - n} \eta_1^{(f(\Pi \otimes \mu_2^{-1}) + 2n(\psi_L) - n)},$$

and if  $\mu$  is unramified, one has

$$\begin{aligned} \Pi(w)\eta_1^{(n)} = & \frac{1}{2} |\varpi_L|^{n(\psi_L) - n} \{ \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)(\eta_2^{(2n(\psi_L) - n)} + |\varpi_L|\eta_2^{(2n(\psi_L) - n + 1)}) \\ & + (-1)^n (\eta_1^{(2n(\psi_L) - n)} - |\varpi_L|\eta_1^{(2n(\psi_L) - n + 1)}) \}, \end{aligned}$$

$$\begin{aligned} \Pi(w)\eta_2^{(n)} = & \frac{1}{2} |\varpi_L|^{n(\psi_L) - n} \{ \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)(\eta_1^{(2n(\psi_L) - n)} + |\varpi_L|\eta_1^{(2n(\psi_L) - n + 1)}) \\ & + (-1)^n (\eta_2^{(2n(\psi_L) - n)} - |\varpi_L|\eta_2^{(2n(\psi_L) - n + 1)}) \}. \end{aligned}$$

*Proof.* The case of  $\xi_\lambda^{(n)}$  can be proved in the same way as in the case of cuspidal representations, since the support of  $\Pi(w)\xi_\lambda^{(n)}$  is also compact. To treat the case of  $\eta_i^{(n)}$ , we recall the construction of Kirillov models in the case of principal series representations ([G]). Let  $\mathcal{F}_\mu$  be the space of locally constant functions  $\phi$  on  $L$  such that  $\phi(x)\mu(x)|x|$  is constant for large  $|x|$ . For  $\phi \in \mathcal{F}_\mu$ , we set

$$\hat{\phi}(x) = \sum_{m \in \mathbf{Z}} \int_{v_L(y)=m} \phi(y) \bar{\psi}_L(xy) dy,$$

where  $dy$  is the Haar measure of  $L$  such that  $\int_{o_L} dy = |\varpi_L|^{n(\psi_L)/2}$ . Then the map  $\phi \mapsto \mu_2(x)|x|^{1/2} \hat{\phi}$  gives an isomorphism from  $\mathcal{F}_\mu$  to  $\mathcal{K}(\Pi)$ . We denote this isomorphism by  $\mathcal{F}$ . The action of  $w$  in  $\mathcal{F}_\mu$  is given by  $\Pi(w)\phi(x) = \mu(-x^{-1})|x|^{-1} \phi(-x^{-1})$ . Let

$$\phi_1^{(n)}(x) = \begin{cases} 1 & v_L(x) \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_\mu^{(n)}(x) = \begin{cases} \mu^{-1}(x)|x|^{-1} & v_L(x) \leq -n, \\ 0 & \text{otherwise.} \end{cases}$$

Then these functions belong to  $\mathcal{F}_\mu$  and satisfy  $\Pi(w)\phi_1^{(n)} = \mu(-1)\phi_\mu^{(n)}$ .



First assume  $\mu$  is ramified. Then by some calculations, we obtain

$$\begin{aligned} \mathcal{F}(\phi_1^{(n)}) &= |\varpi_L|^{n(\psi_L)/2 + n} \eta_2^{(n + n(\psi_L))}, \\ \mathcal{F}(\phi_\mu^{(n)}) &= a \eta_1^{(f(\mu) + n(\psi_L) - n)}, \quad a = \int_{v_L(y) = -n(\psi_L) - f(\mu)} \mu^{-1}(y) \bar{\psi}_L(y) d^\times y, \end{aligned}$$

where  $d^\times y = |y|^{-1} dy$ . Hence we have

$$\Pi(w) \eta_2^{(n)} = a \mu(-1) |\varpi_L|^{n(\psi_L)/2 - n} \eta_1^{(f(\mu) + 2n(\psi_L) - n)}.$$

To determine the relation between  $a$  and  $\varepsilon$ -factors, we use local functional equations. By [J-L], we have

$$\begin{aligned} & \int_{L^\times} (\Pi(w)\xi)(x) \omega_{\Pi}^{-1}(x) \chi^{-1}(x) |x|^{1/2 - s} d^\times x \\ &= \varepsilon(s, \Pi \otimes \chi, \psi_L) \frac{L(1-s, \widetilde{\Pi \otimes \chi})}{L(s, \Pi \otimes \chi)} \int_{L^\times} \xi(x) \chi(x) |x|^{s-1/2} d^\times x. \end{aligned}$$

for  $\xi \in \mathcal{K}(\Pi)$  and a quasicharacter  $\chi$  of  $L^\times$ . Here  $dx$  is the Haar measure of  $L^\times$  such that  $\int_{\mathcal{o}_L^\times} d^\times x = 1$ . We take  $\xi = \eta_2^{(0)}$ ,  $\chi = \mu_2^{-1}$ . Then we see the second integral is equal to  $(1 - |\varpi_L|^s)^{-1} = L(s, \Pi \otimes \mu_2^{-1})$  and the first one is equal to

$$\begin{aligned} & a \mu(-1) |\varpi_L|^{n(\psi_L)/2 - (f(\mu) + 2n(\psi_L))(1-s)} (1 - |\varpi_L|^{1-s})^{-1} \\ &= a \mu(-1) |\varpi_L|^{n(\psi_L)/2 - (f(\mu) + 2n(\psi_L))(1-s)} L(1-s, \widetilde{\Pi \otimes \mu_2^{-1}}). \end{aligned}$$

Since  $f(\mu) = f(\Pi \otimes \mu_i^{-1})$  for  $i = 1, 2$ , we obtain

$$\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) = a \mu(-1) |\varpi_L|^{-(f(\Pi \otimes \mu_2^{-1}) + n(\psi_L))/2}.$$

This shows our result on  $\eta_2^{(n)}$ . Interchanging  $\mu_1$  and  $\mu_2$ , we obtain the equation for  $\eta_1^{(n)}$ .

Next assume  $\mu$  is unramified and  $\mu = |\cdot|^{s_0}$ . If  $L/F$  is unramified, then  $\mu(\varpi_F) = 1$ , and  $\mu_1 = \mu_2 = \sigma \mu_1$ . Hence  $\Pi$  is a lifting of a principal series representation of  $GL_2(F)$ , which is contrary to our assumption. Therefore  $L/F$  is ramified, and by the relation  $\mu(\varpi_L) = |\varpi_L|^{s_0}$ , we see  $|\varpi_L|^{2s_0} = 1$ . If  $|\varpi_L|^{s_0} = 1$ , then again  $\Pi$  is a lifting of a principal series representation. Hence we may assume  $\mu(\varpi_L) = -1$ . For  $\phi_1^{(n)}$ , in the same way as above, we have

$$\mathcal{F}(\phi_1^{(n)}) = |\varpi_L|^{n(\psi_L)/2 + n} \eta_2^{(n + n(\psi_L))}.$$

By (131) of [G], we have

$$\mathcal{F}(\phi_\mu^{(n)}) = |\varpi_L|^{n(\psi_L)/2} |x|^{1/2} \mu_2(x) (F_{s_0}^{(n)}(x) - |\varpi_L| F_{s_0}^{(n)}(\varpi_L x)),$$

where

$$\begin{aligned} F_{s_0}^{(n)}(x) &= \sum_{-n(\psi_L) - v_L(x) \leq m \leq -n} |\varpi_L|^{-ms_0} \\ &= \frac{1}{2}((-1)^{n(\psi_L)} \mu(x) + (-1)^n \phi_1^{(n-n(\psi_L))}(x)). \end{aligned}$$

Since  $\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) = \varepsilon(1, \psi_L) \varepsilon(\mu, \psi_L) = \mu(\varpi_L^{n(\psi_L)}) = (-1)^{n(\psi_L)}$ , we have

$$|x|^{1/2} \mu_2(x) F_{s_0}^{(n)}(x) = \frac{1}{2}(\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) \eta_1^{(n(\psi_L)-n)} + (-1)^n \eta_2^{(n(\psi_L)-n)}).$$

From these formulas, we obtain our formula for  $\Pi(w)\eta_2^{(n)}$ . The case of  $\eta_1^{(n)}$  can be proved in the same way. This completes the proof of Lemma 2.3.

By this lemma, we can proceed in the same way as in the case where  $\Pi$  is supercuspidal. We give a proof for the case where  $\mu$  is unramified. The contribution from  $\xi_\lambda^{(n)}$  for  $\lambda \not\sim \mu_1, \mu_2$  is equal to

$$\sum_{\substack{\lambda|_F \times = \omega_\pi \text{ or } \omega_\pi \lambda_{L,F} \\ \lambda \not\sim \mu_1, \mu_2}} \frac{\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)}{2} \lambda(-a). \tag{2.1}$$

For  $\eta_1^{(n)}$ , we have

$$\begin{aligned} &\Pi\left(\begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} w\right) I_\sigma \eta_1^{(n)} \\ &= \frac{|\varpi_L|^{n(\psi_L)-n}}{2} \{ \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L) \mu_1(-a) |a|^{1/2} \\ &\quad \times (\eta_1^{(2n(\psi_L)-n+v_L(a))} + |\varpi_L| \eta_1^{(2n(\psi_L)-n+1+v_L(a))}) \\ &\quad + (-1)^n \mu_2(-a) |a|^{1/2} (\eta_2^{(2n(\psi_L)-n+v_L(a))} - |\varpi_L| \eta_2^{(2n(\psi_L)-n+1+v_L(a))}) \}, \end{aligned}$$

Hence if  $\eta_i^{(n)}$  contributes to the trace,  $n$  is equal to  $n(\psi_L) + v_L(a)/2$  or  $n(\psi_L) + (v_L(a) + 1)/2$ , and the contribution is equal to

$$\begin{cases} \frac{1}{2} \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L) \mu_1(-1) \mu_1(a) & \text{if } v_L(a) \text{ is even,} \\ \frac{1}{2} \varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L) \mu_1(-1) |\varpi_L|^{1/2} \mu_1(a) & \text{if } v_L(a) \text{ is odd.} \end{cases}$$

In the same way the contribution from the elements of the form  $\eta_2^{(n)}$  is equal to

$$\begin{cases} \frac{1}{2}\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\mu_2(a) & \text{if } v_L(a) \text{ is even,} \\ \frac{1}{2}\varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)|\varpi_L|^{1/2}\mu_2(a) & \text{if } v_L(a) \text{ is odd.} \end{cases}$$

The sum of these contributions is equal to

$$\frac{1}{2}(\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L)\mu_1(-1)\mu_1(a) + \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)\mu_2(-1)\mu_2(a)). \quad (2.2)$$

Here we used the fact that when  $v_L(a)$  is odd,  $\mu_1(a) + \mu_2(a) = 0$  and that  $\varepsilon(\Pi \otimes \mu_1^{-1}, \psi_L) = \varepsilon(\Pi \otimes \mu_2^{-1}, \psi_L)$ . The sum of (2.1) and (2.2) can be transformed into the form in the theorem as in the supercuspidal case. The case  $\Pi$  special does not occur, since  $\pi$  is supercuspidal. This completes the whole proof of the theorem.

As a corollary of the proof, we see

**COROLLARY 2.4.** *Let  $\pi$  be a supercuspidal representation of  $GL_2(F)$  with the central character  $\omega_\pi$ , and let  $\Pi$  be the base change lifting of  $\pi$  to  $GL_2(L)$ . Then for quasicharacters  $\lambda$  of  $L^\times$  which satisfy  $\lambda|_{F^\times} = \omega_\pi \chi_{L/F}$ ,  $\varepsilon(\Pi \otimes \lambda^{-1}, \psi_L)\lambda(-1)$  is independent of  $\lambda$ .*

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