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## On the mean square value of Dirichlet's $L$ -functions\*

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**Abstract.** The main purpose of this paper is to give a sharper asymptotic formula for the mean square value

$$\sum_{\chi \bmod q} L(\sigma + it, \chi)L(1 - \sigma - it, \bar{\chi})$$

where  $0 < \sigma < 1$ . This will be derived from the functional equation of Hurwitz's zeta-function and the analytic methods.

### 1. Introduction

For integer  $q > 2$ , let  $\chi$  denote a typical Dirichlet character mod  $q$ , and  $L(s, \chi)$  be the corresponding Dirichlet  $L$ -function. We define the function  $T(q, s)$  as follows:

$$T(q, s) = \sum_{\chi \bmod q} L(s, \chi)L(1 - s, \bar{\chi})$$

where the summation is over all Dirichlet characters mod  $q$ , and  $s = \sigma + it$ ,  $0 < \sigma < 1$ .

The main purpose of this paper is to study the asymptotic property of  $T(q, s)$ . We know very little at present about this problem. Although D. R. Heath-Brown [1] first introduced the function  $T(q, s)$ , he obtained an asymptotic series only for  $T(q, 1/2)$ . Enlightened by the idea in [2], this paper, using the functional equation of Hurwitz's zeta-function and the analytic method, studies the asymptotic property of  $T(q, s)$  for all  $0 < \sigma < 1$  and proves the following three theorems:

**THEOREM 1.** *Let integer  $q > 2$  and real  $t > 3$ ,  $0 < \sigma < 1$ ,  $c(\sigma) = \text{Max}(\sigma, 1 - \sigma)$ ,  $s = \sigma + it$ , then we have*

$$T(q, s) = \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} + \sum_{p|q} \frac{\ln p}{p-1} + \frac{1}{2} \frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \frac{\Gamma(s/2)}{\Gamma(s/2)} \right] + O\left[ (qt)^{c(\sigma)} \exp\left(\frac{\ln(qt)}{\ln \ln(qt)}\right) \right]$$

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where  $\gamma$  is the Euler constant,  $\sum_{p|q}$  denote the summation over all distinct prime divisors of  $q$ ,  $\Gamma(s)$  is Gamma function and  $\exp(y) = e^y$ .

**THEOREM 2.** *Let  $\text{mod } q > 2$ , then we have asymptotic formula*

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2}, \chi)|^2 = \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{8\pi}\right) + \gamma + \sum_{p|q} \frac{\ln p}{p-1} \right] + o\left[q^{1/2} \exp\left(\frac{\ln q}{\ln \ln q}\right)\right]$$

**THEOREM 3.** *The asymptotic formula*

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2} + it, \chi)|^2 = \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{qt}{2\pi}\right) + 2\gamma + \sum_{p|q} \frac{\ln p}{p-1} \right] + o(qt^{-1}) + o\left[(qt)^{1/2} \exp\left(\frac{\ln(qt)}{\ln \ln(qt)}\right)\right]$$

holds for all  $\text{mod } q$  and real  $t > 2$ .

From the theorems we may immediately deduce the following:

**COROLLARY 1.** *Let  $0 < \sigma < 1$ ,  $s = \sigma + it$ ,  $c = \text{Min}\left(\frac{\sigma}{1-\sigma}, \frac{1-\sigma}{\sigma}\right)$ , if  $1 < |t| < q^{c-\varepsilon}$ , then we have*

$$T(q, s) \sim \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} + \sum_{p|q} \frac{\ln p}{p-1} + \frac{1}{2} \frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right]$$

where  $\varepsilon$  is any fixed positive number.

**COROLLARY 2.** *If  $|t| < q^{1-\varepsilon}$ , then*

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2} + it, \chi)|^2 \sim \frac{\phi^2(q)}{q} \ln|qt| \quad \text{for all } t > 3.$$

Corollary 2 is an improvement of result of Balasubramanian [5], who gave the asymptotic formula in the range  $|t| < q^{3/4-\varepsilon}$ .

## 2. Some lemmas

In this section, we shall give some basic lemmas which are necessary in the course of proving the theorems.

LEMMA 1. Let integer  $q > 2$ , then for any  $0 < \sigma < 1$  and  $s = \sigma + it$  we have

$$T(q, s) = \frac{\phi(q)}{q} \sum_{d|q} \mu(d) \sum_{n=1}^{q/d} \zeta\left(s, \frac{n}{q/d}\right) \zeta\left(1-s, \frac{n}{q/d}\right)$$

where  $\zeta(s, \alpha)$  is Hurwitz zeta-function,  $\phi(q)$  is Euler function and  $\mu(n)$  is Möbius function.

*Proof.* From the orthogonality of Dirichlet characters and

$$L(s, \chi) = \frac{1}{q^s} \sum_{1 \leq a \leq q} \chi(a) \zeta(s, a/q)$$

we may get

$$\begin{aligned} T(q, s) &= \frac{1}{q} \sum_{\chi_q} \left( \sum_{a=1}^q \chi(a) \zeta\left(s, \frac{a}{q}\right) \right) \left( \sum_{a=1}^q \bar{\chi}(a) \zeta\left(1-s, \frac{a}{q}\right) \right) \\ &= \frac{\phi(q)}{q} \sum_{1 \leq a \leq q, (a,q)=1} \zeta\left(s, \frac{a}{q}\right) \zeta\left(1-s, \frac{a}{q}\right) \\ &= \frac{\phi(q)}{q} \sum_{d|q} \mu(d) \sum_{n=1}^{q/d} \zeta\left(s, \frac{n}{q/d}\right) \zeta\left(1-s, \frac{n}{q/d}\right) \quad \square \end{aligned}$$

Let

$$\begin{aligned} F(w, s, k) &= \frac{\Gamma\left(\frac{1-s+w}{2}\right) \Gamma\left(\frac{s+w}{2}\right)}{(\pi k)^w} \cdot \frac{\sin(\pi s) + \sin(\pi w)}{2} \\ &\quad \times \sum_{h=1}^k \zeta\left(s+w, \frac{h}{k}\right) \zeta\left(1-s+w, \frac{h}{k}\right). \end{aligned}$$

We then have the following.

LEMMA 2. If integer  $k > 2$ ,  $\text{Re}(w) \geq 1$ , then

$$\begin{aligned} F(-w, s, k) &= F(w, s, k) - \frac{\Gamma\left(\frac{1-s+w}{2}\right) \Gamma\left(\frac{s+w}{2}\right)}{(\pi k)^w} \times \frac{\sin(\pi w)}{2} \\ &\quad \times \left[ \sum_{h=1}^k \zeta\left(s+w, \frac{h}{k}\right) \zeta\left(1-s+w, \frac{h}{k}\right) + \zeta(s+w) \zeta(1-s+w) \right. \\ &\quad \left. + \sum_{\gamma=1}^{k-1} \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{k-\gamma}{k}\right) \right] \end{aligned}$$

*Proof.* From the functional equation of Hurwitz zeta-function (See [3], theorem 12.8) and the property of Gamma function we know that

$$\begin{aligned} & \zeta\left(1-s, \frac{h}{k}\right) \pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right) \sin \frac{\pi}{2}(1-s) \\ &= \frac{\Gamma(s/2) \pi^{-(s/2)}}{2k^s} \sum_{\gamma=1}^k \left( \exp\left(\frac{2\pi i \gamma h}{k} - \frac{\pi i s}{2}\right) + \exp\left(\frac{\pi i s}{2} - \frac{2\pi i \gamma h}{k}\right) \right) \zeta\left(s, \frac{\gamma}{k}\right) \end{aligned}$$

holds for all integers  $1 \leq h \leq k$ .

From above and notice that

$$\sin \frac{\pi}{2}(1-s+w) \sin \frac{\pi}{2}(s+w) = \frac{\sin(\pi s) + \sin(\pi w)}{2},$$

$$\sum_{\gamma=1}^k \exp(2\pi i n \gamma / k) = \begin{cases} k & \text{if } k/n \\ 0 & \text{if } k \nmid n. \end{cases}$$

we may immediately get

$$\begin{aligned} F(-w, s, k) &= \sum_{h=1}^k \frac{\Gamma\left(\frac{1-s-w}{2}\right) \Gamma\left(\frac{s-w}{2}\right) \pi^{1/2}}{k^{-w} \pi^{\frac{1-s-w}{2}} \pi^{\frac{s-w}{2}}} \times \frac{\sin(\pi s) - \sin(\pi w)}{2} \\ & \quad \times \zeta\left(1-s-w, \frac{h}{k}\right) \zeta\left(s-w, \frac{h}{k}\right) \\ &= \frac{\Gamma\left(\frac{1-s+w}{2}\right) \Gamma\left(\frac{s+w}{2}\right)}{4k(\pi k)^w} \sum_{\gamma=1}^k \sum_{\gamma_1=1}^k \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{\gamma_1}{k}\right) \\ & \quad \times \sum_{h=1}^k \left( \exp\left(-\frac{\pi i}{2}(s+w) + \frac{2\pi i \gamma h}{k}\right) + \exp\left(\frac{\pi i}{2}(s+w) - \frac{2\pi i \gamma h}{k}\right) \right) \\ & \quad \times \left( \exp\left(-\frac{\pi i}{2}(1-s+w) + \frac{2\pi i \gamma_1 h}{k}\right) + \exp\left(\frac{\pi i}{2}(1-s+w) - \frac{2\pi i \gamma_1 h}{k}\right) \right) \\ &= \frac{\Gamma\left(\frac{1-s+w}{2}\right) \Gamma\left(\frac{s+w}{2}\right)}{(\pi k)^w} \\ & \quad \times \left[ \frac{\sin(\pi s)}{2} \sum_{\gamma=1}^k \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{\gamma}{k}\right) \right. \\ & \quad \left. - \frac{\sin(\pi w)}{2} \left( \zeta(s+w) \zeta(1-s+w) + \sum_{\gamma=1}^{k-1} \zeta\left(s+w, \frac{\gamma}{k}\right) \right) \right] \end{aligned}$$

$$\begin{aligned} & \times \zeta \left( 1 - s + w, \frac{k - \gamma}{k} \right) \Big] \\ & = F(w, s, k) - \frac{\Gamma \left( \frac{1 - s + w}{2} \right) \Gamma \left( \frac{s + w}{2} \right)}{(k\pi)^w} \cdot \frac{\sin(\pi w)}{2} \\ & \quad \times \left[ \sum_{\gamma=1}^k \zeta \left( s + w, \frac{\gamma}{k} \right) \zeta \left( 1 - s + w, \frac{\gamma}{k} \right) + \zeta(s + w) \zeta(1 - s + w) \right. \\ & \quad \left. + \sum_{\gamma=1}^{k-1} \zeta \left( s + w, \frac{\gamma}{k} \right) \zeta \left( 1 - s + w, \frac{k - \gamma}{k} \right) \right] \end{aligned}$$

This completes the proof of the lemma 2. □

LEMMA 3. Let integer  $k > 2$ ,  $s = \sigma + it$ ,  $0 < \sigma < 1$ , then

$$\begin{aligned} & \sum_{\gamma=1}^k \zeta \left( s, \frac{\gamma}{k} \right) \zeta \left( 1 - s, \frac{\gamma}{k} \right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w)}{g(s, 0)} \left( 2 + \frac{\sin(\pi w)}{\sin(\pi s)} \right) \\ & \quad \times \sum_{\gamma=1}^k \zeta \left( s + w, \frac{\gamma}{k} \right) \zeta \left( 1 - s + w, \frac{\gamma}{k} \right) \frac{e^{w^2}}{w} dw \\ & \quad - \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w)}{g(s, 0)} \frac{\sin(\pi w)}{\sin(\pi s)} \left( \sum_{\gamma=1}^{k-1} \zeta \left( s + w, \frac{\gamma}{k} \right) \zeta \left( 1 - s + w, \frac{k - \gamma}{k} \right) \right. \\ & \quad \left. + \zeta(s + w) \zeta(1 - s + w) \right) \frac{e^{w^2}}{w} dw + O \left( \frac{k^\sigma + k^{1-\sigma}}{|s|} \right) \end{aligned}$$

where

$$g(s, w) = \Gamma \left( \frac{1 - s + w}{2} \right) \Gamma \left( \frac{s + w}{2} \right) / (\pi k)^w$$

*Proof.* Let

$$I = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} F(w, s, k) \frac{e^{w^2}}{w} dw,$$

moving the line of integration in  $I$  to  $\text{Re}(w) = -1$ , in this time the integrand has one order pole at the point  $w = 0, s$  and  $1 - s$  with the residues  $F(0, s, k)$  and

$$\begin{aligned} & \frac{\Gamma(\frac{1}{2})\Gamma(s)}{(\pi k)^s} \sin(\pi s) \frac{e^{s^2}}{s} \sum_{\gamma=1}^k \zeta \left( 2s, \frac{\gamma}{k} \right) \ll \frac{k^\sigma}{|s|}, \\ & \frac{\Gamma(\frac{1}{2})\Gamma(1-s)}{(\pi k)^{1-s}} \sin(\pi(1-s)) \frac{e^{(1-s)^2}}{1-s} \sum_{\gamma=1}^k \zeta \left( 2-2s, \frac{\gamma}{k} \right) \ll \frac{k^{1-\sigma}}{|s|}. \end{aligned}$$

Thus from lemma 2 and the above we may get

$$\begin{aligned}
 I &= F(0, s, k) - \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} F(-w, s, k) \frac{e^{w^2}}{w} dw + O\left(\frac{k^\sigma + k^{1-\sigma}}{|s|}\right) \\
 &= \frac{1}{2} \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{s}{2}\right) \sin(\pi s) \sum_{\gamma=1}^k \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1-s, \frac{\gamma}{k}\right) \\
 &\quad - \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} F(w, s, k) \frac{e^{w^2}}{w} dw + \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w)}{2} \sin(\pi w) \\
 &\quad \times \sum_{\gamma=1}^k \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{\gamma}{k}\right) \frac{e^{w^2}}{w} dw \\
 &\quad + \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w) \sin(\pi w)}{2} \left[ \sum_{\gamma=1}^{k-1} \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{k-\gamma}{k}\right) \right. \\
 &\quad \left. + \zeta(s+w) \zeta(1-s+w) \right] \frac{e^{w^2}}{w} dw + O\left(\frac{k^\sigma + k^{1-\sigma}}{|s|}\right)
 \end{aligned}$$

by the definition of  $I$  and  $F(w, s, k)$ , and the above we immediately deduce lemma 3.  $\square$

LEMMA 4. For real number  $t > 3$  and  $x > 0$ , we have

$$\begin{aligned}
 w(x) &= \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} x^{-w} \frac{g(s, w)}{g(s, 0)} \left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right) \frac{e^{w^2}}{w} dw \\
 &\ll \begin{cases} t/kx, & \text{if } x \geq (t/2\pi k), \\ 1, & \text{if } x < (t/2\pi k). \end{cases}
 \end{aligned}$$

*Proof.* From the Stirling Formula we know that

$$|\Gamma(\beta + it)| = |t|^{\beta-1/2} e^{-(\pi/2)|t|} \sqrt{2\pi} \left\{1 + O\left(\frac{1}{t}\right)\right\}, \quad t \rightarrow \infty. \quad (1)$$

For  $w = \gamma + iy$ , by (1) we may estimate

$$\left| \frac{g(s, w)}{g(s, 0)} \right| \ll (|t| + |y|)^\gamma e^{\pi|y|} k^{-\gamma} \quad (2)$$

If  $x \geq t/(2\pi k)$ , then by (2) we may get trivial estimate

$$\begin{aligned}
 w(x) &\ll \int_{-\infty}^{+\infty} (xk)^{-1} (|t| + |y|) e^{\pi|y|} \left(2 + \frac{e^{\pi|y|}}{e^{\pi|t|}}\right) \frac{e^{-y^2}}{|y| + 1} dy \\
 &\ll t/kx \int_0^{+\infty} e^{2\pi y - y^2} dy \ll t/kx.
 \end{aligned}$$

If  $x < t/(2\pi k)$ , then we move the line of integration to

$$\operatorname{Re}(w) = -\min\left(\frac{\sigma}{2}, \frac{1-\sigma}{2}\right),$$

in this time the integrand has one order pole at the point  $w = 0$  with the residues 2, from this and (2) we can deduce that

$$\begin{aligned} w(x) &\ll 1 + \left(\frac{t}{kx}\right)^{-\min\{\sigma/2, (1-\sigma)/2\}} \int_{-\infty}^{+\infty} e^{2\pi|y|-y^2} dy \\ &\ll 1 + (t/kx)^{-\min\{\sigma/2, (1-\sigma)/2\}} \ll 1. \end{aligned}$$

Combining above two cases we immediately deduce the lemma 4. □

LEMMA 5. For integer  $k$  and real  $t > 2$ , we have

$$\begin{aligned} \bar{M}_1 &\equiv \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} k^{1+2w} \frac{g(s, w)}{g(s, 0)} \left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right) \zeta(1+2w) \frac{e^{w^2}}{w} dw \\ &= k \left[ \ln\left(\frac{k}{\pi}\right) + \frac{1}{2} \left( \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right) + \frac{\pi}{2 \sin(\pi s)} + 2\gamma \right] \\ &\quad + O(k^\sigma) + O(k^{1-\sigma}) \end{aligned}$$

*Proof.* Moving the line of integration in  $\bar{M}_1$  to  $\operatorname{Re}(w) = -1$ , this time the integrand has two order poles at point  $w = 0$  and the one order pole at the points  $w = -s$  and  $w = -(1-s)$  with the residues:

$$\begin{aligned} \operatorname{Res}_{w=0} &\left[ k^{1+2w} \frac{g(s, w)}{g(s, 0)} \left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right) w \zeta(1+2w) e^{w^2} \right], \\ &= k \left[ \ln\left(\frac{k}{\pi}\right) + \frac{1}{2} \left( \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} \right] \end{aligned}$$

and the residues:

$$\begin{aligned} 2k^{1-2s}(k\pi)^s \Gamma\left(\frac{1}{2}-s\right) \zeta(1-2s) e^{s^2}/(-s \cdot g(s, 0)) &\ll k^{1-\sigma}, \\ 2 \cdot k^{1-2(1-s)}(k\pi)^{1-s} \Gamma\left(s-\frac{1}{2}\right) e^{(1-s)^2} \zeta(2s-1)/(-(1-s) \cdot g(s, 0)) \\ &\ll k^\sigma. \end{aligned}$$



For  $w = -1 + iy$ ,  $|y| < t/2$ , from the estimate (2) we may get

$$\begin{aligned} & \left| k^{1+2w} \frac{g(s, w)}{g(s, 0)} \left( 2 + \frac{\sin(\pi w)}{\sin(\pi s)} \right) \zeta(1+2w) \frac{e^{w^2}}{w} \right| \\ & \ll k^{-1} \cdot \left( \frac{k}{t} \right) \cdot |\zeta(-1+2iy)| \cdot (1+|y|)^{-1} e^{\pi|y|-y^2} \\ & \ll e^{2\pi|y|-y^2}. \end{aligned} \quad (3)$$

It is clear that the estimate (3) also holds for  $|y| > t/2$ . From the residues and estimates (3), and notice that

$$\int_{-\infty}^{+\infty} e^{2\pi|y|-y^2} dy \ll 1$$

we may immediately deduce lemma 5. □

LEMMA 6. For any fixed  $0 < \sigma < 1$  and real number  $t > 2$ , we have the asymptotic formula

$$\frac{\Gamma'(\sigma + it)}{\Gamma(\sigma + it)} = \ln t + \frac{\pi}{2}i + O\left(\frac{1}{t}\right)$$

*Proof.* (See [4], Lemma 3). □

LEMMA 7. For integer  $k$  and real number  $t > 2$ , let  $0 < \sigma < 1$ ,  $s = \sigma + it$ , then we have the asymptotic formula

$$\begin{aligned} & \sum_{\gamma=1}^k \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1-s, \frac{\gamma}{k}\right) = k \left[ \ln\left(\frac{k}{\pi}\right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} \right. \\ & \left. + \frac{1}{2} \left( \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right) \right] + O((kt)^{c(\sigma)} \ln t) \end{aligned}$$

where  $c(\sigma) = \max(\sigma, 1 - \sigma)$ .

*Proof.* From the definition of  $W(x)$  and  $\zeta(s, \alpha)$ , and apply lemma 3 we may get

$$\begin{aligned} & \sum_{\gamma=1}^k \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1-s, \frac{\gamma}{k}\right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w)}{g(s, 0)} \left( 2 + \frac{\sin(\pi w)}{\sin(\pi s)} \right) \\ & \quad \times \sum_{\gamma=1}^k \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{\gamma}{k}\right) \frac{e^{w^2}}{w} dw \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w)}{g(s, 0)} \frac{\sin(\pi w)}{\sin(\pi s)} \left[ \zeta(s+w)\zeta(1-s+w) + \right. \\
 & \left. + \sum_{\gamma=1}^{k-1} \zeta\left(s+w, \frac{\gamma}{k}\right) \zeta\left(1-s+w, \frac{k-\gamma}{k}\right) \right] \frac{e^{w^2}}{w} dw \\
 & + O(k^\sigma) + O(k^{1-\sigma}) \\
 \equiv & A(k, s) - B(k, s) + O(k^{c(\sigma)}) \tag{4}
 \end{aligned}$$

Now we estimate  $A(k, s)$  and  $B(k, s)$  respectively, we have

$$\begin{aligned}
 A(k, s) &= \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} k^{1+2w} \frac{g(s, w)}{g(s, 0)} \left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right) \zeta(1+2w) \frac{e^{w^2}}{w} dw \\
 &+ \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \sum_{\gamma=1}^k \frac{1}{\left(m + \frac{\gamma}{k}\right)^{1-s} \left(n + \frac{\gamma}{k}\right)^s} w \left( \left(n + \frac{\gamma}{k}\right) \left(m + \frac{\gamma}{k}\right) \right) \\
 \equiv & M_1 + M_2 \tag{5}
 \end{aligned}$$

Let  $c(\sigma) = \max(\sigma, 1 - \sigma)$ , by lemma 4 we may get

$$\begin{aligned}
 M_2 &\ll \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \sum_{\gamma=1}^k \frac{1}{\left(n + \frac{\gamma}{k}\right)^\sigma \left(m + \frac{\gamma}{k}\right)^{1-\sigma}} \min\left(1, \frac{t}{k \left(n + \frac{\gamma}{k}\right) \left(m + \frac{\gamma}{k}\right)}\right) \\
 &\ll \sum_{n=1}^{\infty} \sum_{\gamma=1}^k \left[ \frac{k^{1-\sigma}}{n^\sigma \gamma^{1-\sigma}} + \frac{k^\sigma}{n^{1-\sigma} \gamma^\sigma} \right] \times \min\left(1, \frac{t}{\gamma n}\right) \\
 &+ k \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^\sigma m^{1-\sigma}} \min\left(1, \frac{t}{kmn}\right) \\
 &\ll \sum_{\gamma n \leq t} \left( \frac{k^{1-\sigma}}{n^\sigma \gamma^{1-\sigma}} + \frac{k^\sigma}{n^{1-\sigma} \gamma^\sigma} \right) + \sum_{\substack{mn \leq t/k \\ m \neq n}} \frac{k}{m^{1-\sigma} n^\sigma} \\
 &+ \sum_{\substack{n=1 \\ \gamma n > t}}^{\infty} \sum_{\gamma=1}^k \left( \frac{k^{1-\sigma}}{n^\sigma \gamma^{1-\sigma}} + \frac{k^\sigma}{n^{1-\sigma} \gamma^\sigma} \right) \frac{t}{\gamma n} \\
 &+ k \sum_{\substack{mn > t/k \\ m \neq n}} \frac{t}{m^{2-\sigma} n^{1+\sigma} k} \ll (kt)^{c(\sigma)} \ln t \tag{6}
 \end{aligned}$$

For  $\operatorname{Re}(w) = 1$ , we have trivial estimate

$$\begin{aligned} & \sum_{\gamma=1}^{k-1} \zeta\left(s+w, \frac{k-\gamma}{k}\right) \zeta\left(1-s+w, \frac{\gamma}{k}\right) \\ & \ll \sum_{\gamma=1}^{k-1} \left(\frac{k}{k-\gamma}\right)^{1+\sigma} \left(\frac{\gamma}{k}\right)^{1-\sigma+1} \\ & \ll k^{1+c(\sigma)} \end{aligned} \quad (7)$$

from (2), (7) and the definition of  $B(k, s)$  we get

$$B(k, s) \ll \int_{-\infty}^{\infty} \frac{t e^{\pi|y|}}{k e^{\pi|t|}} k^{1+c(\sigma)} \frac{e^{-y^2}}{1+|y|} dy \ll k^{c(\sigma)} \quad (8)$$

Combining (4), (5), (6), (8) and lemma 5 we may obtain

$$\begin{aligned} & \sum_{\gamma=1}^k \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1-s, \frac{\gamma}{k}\right) \\ & = k \left[ \ln\left(\frac{k}{\pi}\right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} + \frac{1}{2} \left( \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right) \right] \\ & \quad + O((qt)^{c(\sigma)} \ln t) \end{aligned}$$

This completes the proof of the lemma 7. □

### 3. Proof of the theorems

In this section, we shall give the proof of the theorems. First we prove theorem 1; by lemma 1 and lemma 7 we may get

$$\begin{aligned} T(q, s) &= \frac{\phi(q)}{q} \sum_{d|q} \mu(d) \sum_{n=1}^{q/d} \zeta\left(s, \frac{n}{q/d}\right) \zeta\left(1-s, \frac{n}{q/d}\right) \\ &= \frac{\phi(q)}{q} \sum_{d|q} \mu(d) \left\{ \frac{q}{d} \left[ \ln\left(\frac{q}{\pi d}\right) + \frac{\pi}{2 \sin(\pi s)} + 2\gamma \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \left( \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right) \right] + 0 \left( \left( \frac{qt}{d} \right)^{c(\delta)} \ln t \right) \right\} \\ &= \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} + \frac{1}{2} \left( \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} \right) \right] \\ & \quad - \frac{\phi(q)}{q} q \sum_{p|q} \frac{\mu(d) \ln d}{d} + 0 \left[ (qt)^{c(\sigma)} \ln t \sum_{d|q} |\mu(d)| \right] \end{aligned}$$

Notice that

$$\sum_{d|q} \frac{\mu(d) \ln d}{d} = -\frac{\phi(q)}{q} \sum_{p|q} \frac{\ln p}{p-1}, \quad \sum_{d|q} |\mu(d)| \ll \exp\left(\frac{\ln q}{\ln \ln q}\right)$$

From (9) we may immediately obtain

$$T(q, s) = \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2 \sin(\pi s)} + \sum_{p|q} \frac{\ln p}{p-1} \right. \\ \left. + \frac{1}{2} \left( \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)} \right) \right] + O\left[ (qt)^{c(\sigma)} \exp\left(\frac{\ln(qt)}{\ln \ln(qt)}\right) \right]$$

This completes the proof of the theorem 1.

Notice that  $L(\frac{1}{2} + it, \chi)L(\frac{1}{2} - it, \bar{\chi}) = |L(\frac{1}{2} + it, \chi)|^2$ , from theorem 1 and lemma 6 we can easily deduce theorem 3.

From the properties of Gamma function we may get

$$\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \frac{\Gamma'(\frac{3}{4})}{\Gamma(\frac{3}{4})} = -\pi \quad \text{and} \quad \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} + \frac{\Gamma'(\frac{3}{4})}{\Gamma(\frac{3}{4})} = -2\gamma - 2 \ln 8$$

Applying the method of proving theorem 1 and above we can deduce

$$\sum_{\chi \pmod q} |L(\frac{1}{2}, \chi)|^2 = \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2} + \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} + \sum_{p|q} \frac{\ln p}{p-1} \right] \\ + O\left[ q^{1/2} \exp\left(\frac{\ln q}{\ln \ln q}\right) \right] \\ = \frac{\phi^2(q)}{q} \left[ \ln\left(\frac{q}{8\pi}\right) + \gamma + \sum_{p|q} \frac{\ln p}{p-1} \right] + O\left[ q^{1/2} \exp\left(\frac{\ln q}{\ln \ln q}\right) \right]$$

This completes the proof.

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