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Values of isotropic quadratic forms at S-integral points

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Introduction

Let F be a non-degenerate indefinite quadratic form on $\mathbf{R}^n$ ($n \geqq 3$). A. Oppenheim conjectured and G. A. Margulis proved [M] that if F is not a multiple of a rational form, then $F(\mathbf{Z}^n)$ is not discrete around the origin. In this paper, we are concerned with a generalization of this result in a S-arithmetic setting.

In the sequel, k is a number field and $\mathfrak{o}$ the ring of integers of k. For every normalized absolute value $|.|_v$ on k, let k_v be the completion of k at v. Let S be a finite set of places of k containing the set S_∞ of archimedean ones, k_S the direct sum of the fields k_s ($s \in S$) and $\mathfrak{o}_S$ the ring of S-integers of k (i.e. of elements $x \in k$ such that $|x|_v \leqq 1$ for all $v \notin S$).

Let F be a quadratic form on k_S^n. Equivalently, F can be viewed as a collection F_s ($s \in S$), where F_s is a quadratic form on k_s^n. The form is non-degenerate if and only if each F_s is non-degenerate. We shall say that F is isotropic if each F_s is so, i.e. if there exists for each $s \in S$ an element $x_s \in k_s^n - \{0\}$ such that $F_s(x_s) = 0$. If s is a real place, this condition is equivalent to F_s being indefinite (since it is non-degenerate). The form F will be said to be *rational* (over k) if it is a multiple of a form on k^n, i.e. if there exists a form F_o on k^n and λ invertible in k_S such that $F = \lambda . F_o$, and *irrational* otherwise.

Endowed with the product topology, k_S is, with respect to the addition, a locally compact group and $\mathfrak{o}_S$ is a discrete cocompact subgroup. Similarly, $\mathfrak{o}_S^n$ is a cocompact lattice in k_S^n. If F is rational, then $F(\mathfrak{o}_S^n)$ is discrete in k_S, since we can write $F = \lambda . F_o$ ($\lambda \in k_S^*$) and may even assume that F_o has coefficients in $\mathfrak{o}$, whence $F(\mathfrak{o}_S^n) \subset \lambda \mathfrak{o}_S$. As a generalization of the Oppenheim conjecture we shall prove that if F is irrational, isotropic non-degenerate and $n \geqq 3$, then $F(\mathfrak{o}_S^n)$ is non-discrete around the origin of k_S. In fact, we shall establish a somewhat stronger statement:

THEOREM A. *Let F be as above. Assume F to be non-degenerate, isotropic and*

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$n \geqq 3$. *Then the following conditions are equivalent*:

(i) F *is irrational.*

(ii) *Given* $\varepsilon > 0$, *there exists* $x \in \mathfrak{o}_S^n$ *such that*

$$0 < \max_{s \in S} |F_s(x)|_s < \varepsilon. \tag{1}$$

(iii) *Given* $\varepsilon > 0$ *there exists* $x \in \mathfrak{o}_S^n$ *such that*

$$0 < |F_s(x)|_s < \varepsilon \quad \textit{for all } s \in S. \tag{2}$$

We just remarked that (ii) $\Rightarrow$ (i) and it is obvious that (iii) $\Rightarrow$ (ii). So the interest lies in the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If $k = \mathbf{Q}$ and $S = S_\infty$, then (ii) and (iii) are identical and (i) $\Rightarrow$ (ii) is the Oppenheim conjecture. Thus our (i) $\Rightarrow$ (ii) is a direct generalization of it, while (i) $\Rightarrow$ (iii) is a natural strengthening, which should of course be true if, as is expected, $F(\mathfrak{o}_S^n)$ is dense in k (see Section 6).

To prove Theorem A we shall first handle two main special cases:

(I) The implication (i) $\Rightarrow$ (ii) when $S = S_\infty$. The argument there is patterned after that of Margulis.[1]

(II) The implication (i) $\Rightarrow$ (iii) when at least one of the F_s is multiple of a k-rational form. The proof uses strong approximation in algebraic groups, some elementary geometry of numbers and is quite different from that of (I).

The implication (i) $\Rightarrow$ (iii) in the general case then follows easily from (I) and (II).

Margulis deduced the Oppenheim conjecture from a theorem about closures of orbits of $\mathbf{SO}(2, 1)$ in $\mathbf{SL}_3(\mathbf{R})/\mathbf{SL}_3(\mathbf{Z})$. It is easily seen that conversely the Oppenheim conjecture implies such an orbit theorem. There is a similar equivalence in the general case (Section 1). In (I), we follow Margulis by proving first an orbit theorem but, in (II), we proceed directly to the Oppenheim conjecture, so that Theorem A yields an assertion about closures of orbits in the S-arithmetic case.

The paper is organized as follows: Section 1 contains some preliminary results, in particular a reduction of the proof of Theorem A to the case $n = 3$ and a discussion of the relation, for $n = 3$, between Theorem A and assertions about closures of orbits in spaces of lattices. As a preparation to (I), we give in Section 2 more algebraic geometric proofs of some lemmas of [M] on actions of unipotent groups, so that the consequences drawn in [M] for $\mathbf{SL}_3(\mathbf{R})$ are also valid for $\mathbf{SL}_3(\mathbf{C})$. We then treat (I) in Section 3, (II) in Section 4, and the general case in Section 5. In section 6, we add some remarks and questions about a still open problem, namely whether $F(\mathfrak{o}_S^n)$ is dense when F is irrational.

The generalization from the original case to that of a number field (with

[1] This implication also follows from the validity of the Raghunathan conjecture on orbit closures, recently proved by M. Ratner [Rt], (cf. §7).

$S = S_\infty$), was proposed first in [RR], where some partial results are obtained. We have added finite places following a suggestion of G. Faltings. We thank G. A. Margulis for a simplification in the proof of I which arose in a discussion with one of us (G. P.).

The main results of this paper have been announced, with sketches of some proofs, in [BP].

1. Preliminaries

In this section, we show that it suffices to establish Theorem A for $n = 3$ and then prove that it is equivalent to some statements about closures of orbits in $\mathbf{SL}_3(k_S)/\mathbf{SL}_3(\mathfrak{o}_S)$. We first fix some notation and conventions:

1.0. In the sequel, the notation preceding the statement of Theorem A is used without further reference. We also let $S_f = S - S_\infty$.

For any subring A of k, we view A^n as diagonally embedded in k_S^n. If V is a vector subspace of k^n, then $V_S := V \otimes_k k_S$ is viewed as a k_S-submodule of k_S^n.

If G is a locally compact group, G^o denotes the connected component of the identity in G. An element $d \in G$ is said to *contract* a subgroup U if it normalizes U and $\lim_{n\to\infty} d^n.u.d^{-n} = 1$ for every $u \in U$.

1.1. PROPOSITION. *Let G be a semi-simple algebraic group defined over $\mathbf{Q}$ and $\Gamma \subset G(\mathbf{Q})$ an arithmetic subgroup. Let E be a subgroup of $G(\mathbf{R})$ generated by unipotent elements and assume that $\overline{E.\Gamma} = R\Gamma$, where R is a closed connected subgroup of $G(\mathbf{R})$ such that $R \cap \Gamma$ has finite covolume in R. Then $R = \tilde{E}(\mathbf{R})^o$, where $\tilde{E}$ is the smallest $\mathbf{Q}$-subgroup of G whose group of real points contains E.*

Proof. We note first that $\tilde{E}$ is connected in the Zariski topology and, more precisely, that $E \subset \tilde{E}(\mathbf{R})^o$. Indeed, if $u \in E$ then some power u^m of u belongs to $\tilde{E}(\mathbf{R})^o$. If u is unipotent and $\neq 1$, then the whole one-parameter unipotent subgroup $\exp(t \log u^m)$ ($t \in \mathbf{R}$), in particular u itself, belongs to $\tilde{E}(\mathbf{R})^o$.

Next we claim that the Levi subgroups of $\tilde{E}$ are semi-simple. Let C be the quotient $\tilde{E}/\mathscr{R}_u\tilde{E}$ of $\tilde{E}$ by its unipotent radical. The image of E in C is generated by unipotent elements, hence belongs to the derived group $\mathscr{D}C$ of C. The inverse image of $\mathscr{D}C$ in $\tilde{E}$ is defined over $\mathbf{Q}$ and its group of real points contains E, so it coincides with $\tilde{E}$. It follows that $\tilde{E}$ has no non-trivial rational character defined over $\mathbf{Q}$, which, as is known, implies that $\tilde{E}(\mathbf{R})^o.\Gamma$ is closed [*A*], hence also that $R \subset \tilde{E}(\mathbf{R})^o$. On the other hand, by [D: §4], the Zariski-closure $\mathscr{A}(R \cap \Gamma)$ of $R \cap \Gamma$ contains all unipotent elements of R, hence E and therefore $\tilde{E}$. But the inclusion $R \subset \tilde{E}(\mathbf{R})^o$ shows that $\tilde{E}$ contains the smallest $\mathbf{Q}$-subgroup $\tilde{R}$ containing R, hence also $\mathscr{A}(R \cap \Gamma)$. As a result

$$\tilde{E} = \mathscr{A}(\Gamma \cap R) = \tilde{R},$$

and R is contained and Zariski-dense in $\tilde{E}$. Being connected, in the ordinary

topology, it must be normal in $\tilde{E}(\mathbf{R})^o$. Its image in $C(\mathbf{R})^o$ is normal, connected, hence closed (C is semi-simple) and Zariski-dense. Therefore the image of R is the whole of $C(\mathbf{R})^o$. It also follows that the image in C of a maximal connected semi-simple subgroup M of R is equal to $C(\mathbf{R})^o$, whence $R = M.(\mathscr{R}_u\tilde{E} \cap R)$. The group M being semi-simple, linear, is of finite index in the group of real points of an algebraic $\mathbf{R}$-group. The group $\mathscr{R}_u\tilde{E} \cap R$ is connected, unipotent, hence also algebraic. As a consequence, R itself is of finite index in the group of real points of an algebraic group [B: §7]. Since R is Zariski-dense in $\tilde{E}$, the proposition is proved.

1.2. PROPOSITION. *Let $m, n \in \mathbf{N}$. For each $s \in S$, let $\mathscr{H}_s$ be a connected almost simple and isotropic k_s-subgroup of $\mathbf{SL}_n$, of dimension m. Let $H_s = \mathscr{H}_s(k_s)$ and H_S the product of the H_s ($s \in S$), viewed as a subgroup of $\mathbf{SL}_n(k_S)$. Assume that $\Gamma = H_S \cap \mathbf{SL}_n(\mathfrak{o}_S)$ is of finite covolume in H_S. Then there exists a connected k-subgroup M of $\mathbf{SL}_n/k$ such that $M(k_S) = H_S$.*

Proof. Let

$$H_\infty = \prod_{s \in S_\infty} H_s, \qquad H_f = \prod_{s \in S_f} H_s, \tag{1}$$

U be a compact open subgroup of H_f and $\Gamma_U = (U \times H_\infty) \cap \Gamma$. The group Γ_U is of finite covolume in $U \times H_\infty$ hence the projection Γ'_U of Γ_U in H_∞, which is discrete since U is compact, is of finite covolume. We view the product of the $\mathbf{SL}_n(k_s)$ ($s \in S_\infty$) as the group of real points of the $\mathbf{Q}$-group $L = R_{k/\mathbf{Q}}\mathbf{SL}_n/k$ obtained from $\mathbf{SL}_n$, viewed as a k-group, by restriction of scalars from k to $\mathbf{Q}$. Then Γ'_U is contained in an arithmetic subgroup of L_∞ and, being of finite covolume, is Zariski-dense, since H_∞ is by assumption a product of simple non-compact Lie groups. Therefore H_∞ is the group of real points of a $\mathbf{Q}$-group. By assumption, it is the product of its projections on the factors $\mathbf{SL}_n(k_s)$, therefore it is itself of the form $R_{k/\mathbf{Q}}M(\mathbf{R})$, where M is a connected k-subgroup of $\mathbf{SL}_n$, of dimension m. This already shows that $H_s = M(k_s)$ for all $s \in S_\infty$. Now Γ'_U is to be viewed as an arithmetic subgroup of $M(k)$. Let M_f be the product of the $M(k_s)$ for $s \in S_f$. It is known that Γ'_U, diagonally embedded in M_f, is dense in an open subgroup. Therefore $M_f \subset H_f$ and then $M_f = H_f$ for dimensional reasons.

1.3. PROPOSITION. *Assume F to be irrational and $n \geqq 3$. Then there exists a three-dimensional subspace V of k^n such that the restriction of F to V_S is non-degenerate, isotropic and irrational.*

Proof. Fix $s \in S$. Let H_s be the orthogonal group of F_s. Let M be a subspace of k_s^n on which the restriction of F_s is non-degenerate. It is well-known that the $H_s(k_s)$-orbit of M in the Grassmannian $G_{m,n}$ of m-planes ($m = \dim M$) in k_s^n is open (for the analytic topology) in $G_{m,n}(k_s)$. This follows from the fact that if l_s is an algebraically closed extension of k_s, then $H_s(l_s)(M)$ is open in $G_{m,n}(l_s)$ and from the existence of local cross-sections (in the analytic topology) for the fibration of

$H_s(k_s)$ by the isotropy group of M. In particular, if F_s is isotropic on M, then F_s is non-degenerate and isotropic on any m-dimensional subspace of k_s^n sufficiently close to M.

Assume now $n > 3$. We want to prove the existence of a subspace V of k^n of codimension one, such that the restriction of F to V_S is non-degenerate, isotropic and irrational. For each $s \in S$, fix $M_s \subset k_s^n$ of codimension one, on which F_s is non-degenerate and isotropic. By weak approximation in k_S, we may find $M' \subset k^n$, of codimension one, such that $M' \otimes_k k_s$ is arbitrarily close to M_s for each s, therefore such that the restriction of F to M'_S is non-degenerate and isotropic. There remains to show that there exists such an M' on which F is in addition irrational. Fix $e \in M'(k)$ on which no F_s is zero. Again, this exists by weak approximation. After having multiplied F by a unit in k_S we may assume that $F_s(e) = 1$ for all $s \in S$. Let $\mathscr{M}$ be the set of $(n-1)$-dimensional subspaces of k^n containing e and such that the restriction of F to M_S is non-degenerate and isotropic. Assume that for no $M \in \mathscr{M}$, $F \mid M_S$ is irrational. Then $F(x) \in k$ for all $x \in M(k)$, $M \in \mathscr{M}$. For a given s, the map $x \mapsto F_s(x)$ is a regular function on l_s^n. Since it takes rational values on the union of the $M(k)$, which is obviously Zariski-dense in l_s^n, it is defined over k, hence $F_s(x) \in k$ for all $x \in k^n$ and this implies that F_s is rational over k. Since $F_s(x)$ is independent of s for $x \in M(k)$, $M \in \mathscr{M}$, it follows that F is rational, contradiction. Therefore the restriction of F to some element in $\mathscr{M}$ is irrational, and the proposition follows by descending induction on n.

1.4. LEMMA. *Fix $s \in S$. Let F_s be a non-degenerate isotropic quadratic form on k_s^n. Given a neighborhood U of the origin in k_s^n, there exists $\varepsilon > 0$ such that $F_s(U - 0)$ contains all elements of k_s with absolute value $\leqslant \varepsilon$.*

Proof. The space k_s^n is the direct sum of a hyperbolic plane for F_s and of its orthogonal complement. This reduces us to the case $n = 2$ and $F_s = x.y$, for which our assertion is obvious.

1.5. We now come back to the situation of Theorem A for $n = 3$ and introduce some further notation. We let $G_s = \mathbf{SL}_3(k_s)$ and $G_S = \mathbf{SL}_3(k_S) = \Pi_{s \in S} G_s$. The standard S-arithmetic subgroup of $\mathbf{SL}_3$ is $\Gamma_S = \mathbf{SL}_3(\mathfrak{o}_S)$. We let $\Omega = G_S/\Gamma_S$. It is the space of free $\mathfrak{o}_S$-submodules of k_S^3 of maximal rank and determinant one. We write Λ_o for $\mathfrak{o}_S^3$. For a non-degenerate quadratic form $F = (F_s)$ on k_S^3, we let H_F be the product $\Pi_{s \in S} \mathrm{SO}(F_s)(k_s)$.

1.6. PROPOSITION. *Let $\mathscr{F}$ be the set of non-degenerate isotropic quadratic forms on k_S^3. Then the following two assertions are equivalent:*

(a) *Any $F \in \mathscr{F}$ which does not represent zero rationally either is rational or satisfies* (ii) *of Theorem A.*

(b) *If $F \in \mathscr{F}$ and $z \in \Omega$ are such that $H_F.z$ is relatively compact, then $H_F.z$ is compact.*

Proof. We first assume (b) for $z = \Lambda_o$. Note first that, in view of Mahler's criterion and the fact that H_F consists of elements of determinant one, the assumption $H_F . \Lambda_o$ relatively compact is equivalent to

(1) *There exists a neighborhood U of the origin in k_S^3 such that $H_F . \Lambda_o \cap U = \{0\}$.*

Assume that (ii) of Theorem A does not hold for some $\varepsilon > 0$. Fix U such that $\max |F_s(u_s)|_s < \varepsilon$ for $u = (u_s) \in U$. Then (1) is satisfied, hence $H_F . \Lambda_o$ is compact. By Arens theorem, $H_F . \Lambda_o$ is homeomorphic to $H_F/(H_F \cap \Gamma_S)$, hence $H_F \cap \Gamma_S$ is cocompact in H_F. By 1.2, we can find a k-subgroup M of $\mathbf{SL}_3$ such that $H_F = M(k_S)$ and this implies that F is rational.

Assume now (a). Let $F \in \mathscr{F}$ and $z \in \Omega$ be such that $H_F . z$ is relatively compact. Let $g \in G_S$ be such that $g^{-1} . z = \Lambda_o$ and let $F' = {}^t g . F . g$. Then $H_{F'} . \Lambda_o$ is relatively compact so that (1) holds for F'. In view of 1.4 and Witt's theorem, this implies that F' does not take arbitrarily small values, in absolute value, on $\Lambda_o - \{0\}$, hence also that it does not represent zero rationally; therefore, it does not fulfill (ii). By (a), F' is rational and then $H_{F'} = \mathrm{SO}(F')(k_S)$. It follows that $H_F . \Lambda_o$ is closed and therefore compact. As a consequence, $H_F . z$ is compact, too.

1.7. LEMMA. *Let $n = 3$. Then F is rational and isotropic over k if and only if it is zero on infinitely many lines in k^3.*

Proof. If F is rational over k, then $\mathrm{SO}(F)$ is defined over k. If now F is zero on some rational line, then it is zero on all its transforms under $\mathrm{SO}(F)(k)$, and those are infinite in number since $\mathrm{SO}(F)(k)$ is Zariski-dense in SO(F).

Assume now F to be zero on infinitely many rational lines in k^3 (viewed as embedded diagonally in k_S^3). Take as a universal field an algebraically closed extension K of k containing the fields k_s ($s \in S$). Then the cones $F_s = 0$ ($s \in S$) in K^3 have in common a Zariski-dense set of k-rational points. Hence they are identical, all defined over k, and then F is rational and isotropic over k.

1.8. Since $k \subset k_S$, the k_S-module k_S^n can be viewed as a vector space over k. A subset $D \subset k_S^n$ will be called a k-line if it is a one-dimensional subspace for this structure, i.e. if $D = k . x$, for some $x \in k_S^n - \{0\}$. It is rational if $D \subset k^n$. If $n = 1$, and all components x_s of $x \in k_S$ are not zero, then $\mathfrak{o}_S . x$ is discrete in k_S. However, if at least one component x_r is zero, then the set $\mathfrak{o}_S . x$ is not discrete, more precisely contains a sequence of non-zero elements accumulating to 0, since there is a sequence $u_n \in \mathfrak{o}_S - \{0\}$, $n = 1, 2 \ldots$ such that $u_{n,s} \to 0$ for all $s \neq r$.

Let now $D = k . x$ be a rational line in k_S^n. Then $D \cap \mathfrak{o}_S^n$ is a finitely generated $\mathfrak{o}_S$-module, therefore if $F_s(x) \neq 0$ for all $s \in S$, then $F(D \cap \mathfrak{o}_S^n)$ is discrete in k_S. It follows that if F satisfies (iii) of Theorem A, then it takes arbitrarily small values, no component of which is zero, on S-integral points belonging to infinitely many rational lines.

If on the other hand there exist r, $s \in S$ such that $F_r(x) = 0$ and $F_s(x) \neq 0$ for some $x \in \mathfrak{o}_S^n$, then F is obviously not rational, and moreover by the remark in the

previous paragraph, it satisfies (ii). In this case therefore, the Oppenheim condition is already fulfilled for $x \in D \cap \mathfrak{o}_S^n$.

1.9. PROPOSITION. *Let $\mathscr{F}$ be the set of all non-degenerate isotropic quadratic forms on k_S^3, and* (a), (b), (c) *the three statements*:

(a) *Any $F \in \mathscr{F}$ is either rational or satisfies* (ii) *of Theorem A.*

(b) *Any $F \in \mathscr{F}$ is either rational or satisfies* (iii) *of Theorem A.*

(c) *If $F \in \mathscr{F}$ and $z \in \Omega$ are such that we can find an open neighborhood U of the origin in k_S^3 and a finite union C of k-lines* (*see* 1.8) *satisfying the condition*

$$H_F(z - z \cap C) \cap U = \{0\}, \tag{*}$$

then $H_F . z$ is compact.

Then (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

Proof. We show first that (b) $\Rightarrow$ (c). Since $g \in G_S$ is k-linear, the statement (c) is invariant under conjugation by G_S and we may assume without loss of generality that $z = \mathfrak{o}_S^3$ in (*). Under (iii), F takes arbitrary small values on elements of $\mathfrak{o}_S^3$ belonging to infinitely many rational lines (1.8); we see therefore that (*) prevents (iii) to hold, and F is rational. If it were isotropic over k, then it would be zero on infinitely many rational lines (1.7), again contradicting (*). Therefore F is rational, anisotropic over k, hence $H_F . \mathfrak{o}_S^3$ is compact.

We now prove that (c) $\Rightarrow$ (a). Let $F \in \mathscr{F}$ and assume it is not rational. Then it can be zero on only finitely many rational lines (1.7). Let C be their union. Assume (ii) is not true for some $\varepsilon > 0$. There exists a neighborhood U of the origin in k_S^3 such that $|F_s(u_s)| \leqq \varepsilon/2$ for all $s \in S$ and $u = (u_s) \in U$. Then (*) of (c) is fulfilled for this choice of U, C and $z = \mathfrak{o}_S^3$. By (c), $H_F . \mathfrak{o}_S^3$ is compact, and then $H_F = M(k_S)$ for some k-subgroup of $\mathbf{SL}_3$ (1.2), which is equivalent to saying that F is rational.

1.10. REMARK. By definition here, F is isotropic if each F_s is so. This is stronger than the usual requirement: $F(x) = 0$ for some non-zero vector in k_S^n. However, it is necessary here because if we allow one F_s to be anisotropic, Theorem A cannot hold. To see this, observe first that if F_s and F_s' are anisotropic over k_s^n, then $|F_s(x)|_s / |F_s'(x)|_s$ is bounded from above and away from zero on any compact subset of k_s^n not containing the origin, whence the existence of a constant $c > 1$ such that

$$c^{-1} |F_s(x)|_s \leqq |F_s'(x)|_s \leqq c \,.\, |F_s(x)|_s \quad \text{for all } x \in k_s^n. \tag{1}$$

Let now F be a rational form; assume F_r to be anisotropic over k_r^n for some $r \in S$, and let $F' = (F_s')$ be such that $F_s' = F_s$ for $s \neq r$ but F_r' is an anisotropic form on k_r^n which is not a multiple of F_r. Then F' is irrational, but in view of (1), $F'(\mathfrak{o}_S^n)$ is still discrete, since $F(\mathfrak{o}_S^n)$ is so.

2. Geometric lemmas

In this section, unless otherwise stated, the field K is either **R** *or* **C**.

2.1. Let U be a connected unipotent K-group, d its dimension. As a variety, U is K-isomorphic to the d-dimensional affine space under "log". Let u_i be coordinates in the latter and use multiexponential notation for monomials, i.e. if $\alpha = (\alpha, \ldots, \alpha_d) \in \mathbf{N}^d$, then u^α stands for $u_1^{\alpha_1} \cdots u_d^{\alpha_d}$ and $|\alpha| = \Sigma \alpha_i$. If $u = (u_i)$, then λu is the element with coordinates λu_i ($\lambda \in K$).

LEMMA. *Let U be as before, E a finite dimensional vector space defined over K and $\sigma: U \to \mathrm{GL}(E)$ a rational representation of U in E, defined over K. Let F be the fixed point set of $\sigma(U)$ in E. Choose $c_o \in F(K)$ and let $c_i \in E(K) - F(K)$ ($i = 1, 2, \ldots$) be a sequence of elements tending to c_o as $i \to \infty$. Then there exists a non-constant K-morphism of varieties $\varphi: U \to F$, mapping the identity onto c_o, such that $\varphi(U(K))$ belongs to the closure of the union of the orbits $\sigma(U(K)).c_i$ ($i = 1, 2, \ldots$). If U is commutative, then there exist a sequence $\lambda_i \in K^*$ and a subsequence $\{c_{r_i}\}$ of $\{c_i\}$, such that $|\lambda_i| \to \infty$ and $\varphi(u) = \lim_i \sigma(\lambda_i . u).c_{r_i}$ for all $u \in U$.*

Proof. Fix a basis (e_i) of $E(K)$. Then $\sigma(u) = (\sigma(u)_{p,q})$ is represented by a matrix whose coefficients are polynomials on U, with coefficients in K. Let $(c_{i,q})$ be the coordinates of c_i. The pth coordinate of $\sigma(u).c_i$ is then

$$(\sigma(u).c_i)_p = \sum_q \sigma(u)_{p,q}.c_{i,q} = \sum_\alpha a_{p,i,\alpha} u^\alpha \quad (a_{p,i,\alpha} \in K). \tag{1}$$

For a given i, the $a_{p,i,\alpha}$ ($|\alpha| \neq 0$) are not all zero since, by assumption, c_i is not fixed under $\sigma(U)$. We can find $\lambda_i \in K^*$ such that

$$|a_{p,i,\alpha} \lambda_i^{|\alpha|}| \leqq 1 \quad \text{for all } p, q, \alpha \quad \text{with } |\alpha| \neq 0 \tag{2}$$

$$|a_{p_o,i,\alpha_o} \lambda_i^{|\alpha_o|}| = 1 \text{ for some choice } p_o, \alpha_o \text{ of } p \quad \text{and} \quad \alpha \ (|\alpha_o| \neq 0). \tag{3}$$

Passing to a subsequence, if necessary, we may arrange that p_o, α_o are independent of i. Again going over to a subsequence, we may assume that the $a_{p,i,\alpha} \lambda_i^{|\alpha|}$ converge for $|\alpha| \neq 0$. But this is automatic if $|\alpha| = 0$, since the sequence c_i converges. Let then

$$a_{p,\alpha} = \lim_{i \to \infty} a_{p,i,\alpha} \lambda_i^{|\alpha|}. \tag{4}$$

We define $\psi: U \to E$ by

$$\psi(u)_p = \sum_\alpha a_{p,\alpha} u^\alpha. \tag{5}$$

i.e. by

$$\psi(u) = \lim_{i \to \infty} \sigma(\lambda_i . u).c_i. \tag{6}$$

If $u = 1$, it is represented by zero in our coordinates, hence $\lambda_i . u = u$ and

$$\psi(1) = \lim_i c_i = c_o. \tag{7}$$

The map ψ is not constant since by our construction $|a_{p,\alpha}| = 1$ for some choice of p, $\alpha(|\alpha| \neq 0)$. This implies that $|\lambda_i| \to \infty$, because otherwise a subsequence of $\{\lambda_i . u\}$ would converge to some element $v(u) \in U$ and $\lim_i \sigma(\lambda_i . u) . c_{r_i}$ would be equal to $v(u) . c_o$, hence to c_o since $c_o \in F$, and ψ would be a constant map.

Assume now U to be commutative. We claim that in this case $\psi(U) \subset F$, hence that $\varphi = \psi$ fulfills our conditions. We have to show

$$v . \psi(u) = \psi(u) \quad \text{for all } u, v \in U. \tag{8}$$

We have $v . \psi(u) = \lim_i v . \sigma(\lambda_i . u) . c_{r_i}$ hence also, since U is assumed to be commutative

$$v . \psi(u) = \lim_i \sigma(\lambda_i . u + v) . c_{r_i}. \tag{9}$$

But $\lambda_i . u + v = \lambda_i . (u + v/\lambda_i)$ and, by (1),

$$(\sigma(\lambda_i . (u + v/\lambda_i) c_{r_i}))_p = \sum_\alpha a_{p,i,\alpha} \lambda_i^{|\alpha|} (u + v/\lambda_i)^\alpha. \tag{10}$$

Since $|\lambda_i| \to \infty$ we see that (8) follows from (4) and (5).

We now drop the assumption that U is commutative and write c_i for c_{r_i}. To prove the first assertion, we proceed by induction on the codimension of F in E. There is nothing to prove if $F = E$, so we assume the first assertion established if the c_i belong to a proper U-stable subspace defined over K of E and containing F.

Let now L be a K-subspace of E of codimension one, stable under U and containing F. Since U is unipotent, this always exists and moreover U acts trivially on E/L. Let $\pi: E \to E/L$ be the canonical projection. The relation $c_i \to c_o$ implies $\pi(c_i) \to 0$, hence also $\pi(\sigma(\lambda_i . u) . c_i) \to 0$, which shows that $\psi(U) \subset L$. If now some neighborhood of the identity in $U(K)$ has its image in $F(K)$, then $\psi(U) \subset F$. Otherwise, there is a sequence $c'_j \in L(K) - F(K)$, belonging to $\psi(U(K))$, which converges to c_o. By the induction assumption, there is a non-constant K-morphism of varieties $\varphi: U \to F$, sending 1 onto c_o, such that $\varphi(U(K))$ is in the closure of the union of the orbits $\sigma(U(K)) . c'_j$. By construction (see (6)), c'_j belongs to the closure of the union of the orbits $\sigma(U(K)) . c_i$. Then so does $\sigma(U(K)) . c'_j$, hence also $\varphi(U(K))$.

2.2. In the sequel, there is some interplay between the Zariski topology and the analytic topology. We first settle a minor technical point.

Let G be a K-group, U_i $(i \in I)$ a finite set of smooth irreducible K-varieties, $u_i \in U_i(K)$ and $f_i: U_i \to G$ a K-morphism sending u_i onto the identity $(i \in I)$. It is standard that the $f_i(U_i)$ and their inverses generate a K-subgroup L [B:2.2].

Assume now that $U_i(K)$ is Zariski-dense in $U_i (i \in I)$. We claim that if V_i is any neighborhood of u_i in $U_i(K)$ (in the K-topology), then the topological identity component $L(K)^o$ of $L(K)$, viewed as a Lie group over K, is generated by the $f_i(V_i)$ and their inverses.

By [B: 2.2], there exist $n \in \mathbf{N}$, a sequence $i(j)$ of elements in I $(j = 1, \ldots, n)$ and $\varepsilon_j = \pm 1$ such that the image of $Y = \Pi_j U_{i(j)}$ under the map

$$\mu: (x_1, \ldots, x_n) \mapsto f_{i(1)}(x_1)^{\varepsilon_1} \cdots f_{i(n)}(x_n)^{\varepsilon_n}$$

is the K-group L. We may assume the V_i's to be open. Then $Z = \Pi_j V_{i(j)}$ contains a point at which d_μ is surjective, since the set of such points is Zariski-open in Y, so the image of Z contains a non-empty open set in $L(K)$ and the group it generates contains $L(K)^o$.

2.3. PROPOSITION. *Let G be a connected K-group and U a unipotent K-subgroup. Let M be a subset of $G(K) - \mathcal{N}_G(U)(K)$ whose closure contains the identity. Then there is a K-morphism of varieties $\psi: U \to \mathcal{N}_G(U)$, mapping the identity onto the identity, and a Zariski K-open neighborhood V of 1 in U such that $\psi(V(K)) \subset \overline{U(K).M.U(K)}$ and $\psi(V(K))$ meets any left-coset of $U(K)$ in at most one point. There is in $\mathcal{N}_G(U)(K)$, viewed as a Lie group over K, a connected Lie subgroup L, containing $U(K)$ strictly and belonging to the subgroup generated by $\mathcal{N}_G(U)(K) \cap \overline{U(K).M.U(K)}$.*

Proof. There exists a finite dimensional vector space E defined over K with a line C and a rational representation $\sigma: G \to \mathrm{GL}(E)$ defined over K such that U is the subgroup of G leaving fixed any point of C. Fix $c_o \in C$. We have then $G.c_o \cong G/U$ and since U is unipotent, also $G(K).c_o \cong G(K)/U(K)$. Let F be the fixed point set of U in E. It contains C and it is elementary that

$$F \cap G(K).c_o = \mathcal{N}_G(U)(K).c_o.$$

Let now (m_i) $(i = 1, 2, \ldots)$ be a sequence of elements in M tending to the identity and let $c_i = m_i.c_o$. Then $c_i \in E(K) - F(K)$ and $c_i \to c_o$. We may apply 2.1 to get a non-constant K-morphism of varieties $\varphi: U \to F$ such that $\varphi(1) = c_o$ and $\varphi(U(K))$ is contained in the intersection of $F(K)$ with the closure of $\bigcup_i \sigma(U(K)).c_i$, hence, a fortiori, in

$$C = F(K) \cap \overline{\sigma(U(K).M).c_o}. \tag{1}$$

The orbit $G.c_o$ is Zariski-open in its closure, which contains C. Therefore U contains a Zariski K-open subset V such that $\varphi(V(K))$ belongs to $C \cap G.c_o$.

Since U is unipotent, there exists a section $s: G/U \to G$, defined over K, of the fibration of G by U. Then $\psi = s_o\varphi$ is a non-trivial K-morphism $U \to G$ mapping 1 onto 1, whose image meets any left coset of U in at most one point. The set $\psi(V(K))$ is contained in the inverse image of C, which is obviously equal to $\mathcal{N}_G(U)(K) \cap \overline{U(K).M.U(K)}$. This proves our first assertion.

The second assertion now follows from 2.2, applied to the case where $\{V_i\}$ consists of U and V.

2.4. NOTATION. We recall and adapt to our framework some notation of [M]. The field K being understood, we let $G = \mathbf{SL}_3(K)$,

$$v_1(s) = \begin{pmatrix} 1 & s & s^2/2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \qquad v_2(s) = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad d(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \tag{1}$$

($s \in K$, $t \in K^*$), V_1 (resp. V_2, resp. D) the group generated by the $v_1(s)$ (resp. $v_2(s)$, resp. $d(t)$), $V = V_1 . V_2$ and if $K = \mathbf{R}$,

$$V_2^+ = \{v_2(s) \mid s > 0\}, \quad V_2^- = \{v_2(s) \mid s < 0\}, \quad D^o = \{d(t) | t > 0\}. \tag{2}$$

Thus our notation deviates from [M] only in that our D^o is D there. If $K = \mathbf{C}$, then $D^o = D$.

As pointed out in [M], it is easily checked that

$$\mathcal{N}_G(V_1) = D . V. \tag{3}$$

$H \subset \mathbf{SL}_3(K)$ denotes the special orthogonal group of the form $2x_1x_3 - x_2^2$. The group $D . V_1$ is then maximal solvable in H.

2.5. PROPOSITION. *Let $M \subset G - \mathcal{N}_G(V_1)$ be a subset whose closure contains the identity. Then the subgroup C generated by $\mathcal{N}_G(V_1) \cap \overline{V_1 . M . V_1}$ contains either V or a subgroup of the form $v . D^o . V_1 . v^{-1}$ for some $v \in V$.*

Proof. By 2.3, the subgroup C has a connected Lie subgroup L containing V_1 strictly. Since $\mathcal{N}_G(V_1) = D . V$, the quotient $\mathcal{N}_G(V_1)/V_1$ is isomorphic to $D . V_2$ (i.e. to the affine group of the line). It is elementary that its only connected Lie subgroups over K are V_2 and the V_2-conjugates of D^o, whence the proposition.

2.6. PROPOSITION. *Let M be a subset of $G - H$ whose closure contains the identity. Then $\overline{H . M . D^o . V_1}$ contains V_2 if $K = \mathbf{C}$ and either V_2^+ or V_2^- if $K = \mathbf{R}$.*

Proof. For $K = \mathbf{R}$, this is Lemma 7 in [M]. Our proof is basically the same and we only point out the modification allowing us to include the case $K = \mathbf{C}$.

There is no change in the argument of p. 394 in [M] until the last five lines. There, instead of Lemma 13(i), we invoke 2.1 above and conclude that there is a non-constant K-morphism of varieties of V_1 into $\mathfrak{v}_2$ whose image contains the origin. If $K = \mathbf{C}$, it is then surjective. If $K = \mathbf{R}$, its image contains at least one of the half-lines $\mathfrak{v}_2^+$, $\mathfrak{v}_2^-$. Then the relation on the last line of p. 394 in [M] shows that $\overline{H . M . D^o . V_1}$ contains V_2 if $K = \mathbf{C}$ and either V_2^+ or V_2^- if $K = \mathbf{R}$, under the assumption that $M \cap V_2$ is empty. If it is not, we see from the argument on p. 395 that $H . M . D^o . V_1$ contains all elements $v_2(s) \in V_2$ with $s = t^2 . s_o$, for some $s_o \neq 0$

and all $t \in K^*$. If $K = \mathbf{C}$, this contains $V_2 - 0$, hence the closure contains V_2. In the real case, it contains either V_2^+ or V_2^-.

2.7. LEMMA. *Assume $K = \mathbf{R}$. Let X be a second countable locally compact space on which V_2 operates continuously and $x \in X$. If $V_2^+ . x$ (resp. $V_2^- . x$) is relatively compact, then its closure C contains a non-empty subset invariant under V_2.*

Proof. Let $\varphi(x)$ be the set of limit points of sequences $v_2(t_n).x$, where $t_n \to \infty$ (resp. $t_n \to -\infty$). It is contained in C, and is not empty since C is compact. We claim that it is invariant under V_2. Let $y \in \varphi(x)$ and write it as $\lim_{n\to\infty} v_2(t_n).x$, where $\{t_n\}$ is as before. For $t \in \mathbf{R}$, we have

$$v_2(t).y = \lim_{n\to\infty} v_2(t).v_2(t_n).x = \lim_{n\to\infty} v_2(t+t_n).x.$$

But $t + t_n$ tends to ∞ or to $-\infty$ with t_n, hence $v_2(t).y \in \varphi(x)$.

2.8. REMARKS. (1) 2.1 and 2.3 are valid over a non-archimedean local field K and the proof are identical, with one minor modification to take into account the fact that the set of absolute values is discrete, more precisely the set of powers of some real number $c \in (0, 1)$. Now in 2.1 (2), (3) we let the absolute value in K be the normalized absolute value, keep (2) and replace (3) by

$$|a_{p_o,i,\alpha_o} \lambda_i^{|\alpha_o|}| \geqq c^{|\alpha_o|}. \tag{3}$$

(2) In [M], Lemmas 1 and 4 are for general locally compact second countable groups and we shall be able to use them without any modification. Our 2.3 is a replacement for Lemma 5 of [M] and our 2.1 one for Lemma 13(i) of [M]. The formulation of the latter was suggested to one of us by P. Deligne. We have already pointed out that 2.6 generalizes Lemma 7 of [M]. Over $\mathbf{R}$, Proposition 2.5 is a weaker version of Lemma 8(ii) of [M], weaker in the sense that we assume $M \subset G - \mathcal{N}_G(V_1)$ rather than in $G - V_1$. But thanks to 1.1 and 3.1, we shall not need this stronger statement in the proof of 3.4. Finally, 2.7 is just a more general formulation of the argument in Lemma 12 of [M].

3. The implications (i) ⇒ (ii) in the archimedean case

From 3.4 on, it is assumed in this section that $S = S_\infty$.

3.1. LEMMA. *Let T be a finite set, Q_t ($t \in T$) a locally compact second countable group, Q the product of the Q_t, U a closed subgroup of Q and pr_t the projection of Q onto Q_t. Fix $s \in T$ and $q \in Q$. Let Γ be a discrete subgroup of Q on which pr_s is injective. Assume D_s is a subgroup of Q_s containing an element d which contracts $pr_s U$ and such that the image of $D_s . q$ in Q/Γ is relatively compact. Then $U \cap q.\Gamma.q^{-1} = \{1\}$.*

Proof. The projection pr_s is also injective on $q.\Gamma.q^{-1}$, so we may (and do)

replace Γ by $q.\Gamma.q^{-1}$. As D_s is now relatively compact modulo Γ, there exists a sequence of elements $\gamma_n \in \Gamma$ such that the set $\{d^n.\gamma_n\}$ $(n \in \mathbf{N})$ is bounded in Q. Let $\gamma \in U \cap \Gamma$ and $\gamma_t = pr_t(\gamma)$. By assumption,

$$\lim_{n\to\infty} d^n.\gamma_s.d^{-n} = 1,$$

therefore $d^n.\gamma.d^{-n}$ converges to some element $\delta \in Q$ such that $pr_s\delta = 1$ and $pr_t\delta = \gamma_t$ for $t \neq s$. But we can write

$$d^n.\gamma.d^{-n} = (d^n.\gamma_n).(\gamma_n^{-1}.\gamma.\gamma_n).(d^n.\gamma_n)^{-1},$$

and, as $d^n\gamma_n$ has a convergent subsequence, we see that a subsequence of $\{\gamma_n^{-1}.\gamma.\gamma_n\}$ converges to an element of Q whose image under pr_s is the identity. Since Γ is discrete and $\gamma_n^{-1}.\gamma.\gamma_n \in \Gamma$, the subsequence is eventually constant, therefore $pr_s(\gamma_n^{-1}.\gamma.\gamma_n) = 1$ for some n, whence $pr_s\gamma = 1$ and $\gamma = 1$ in view of our assumption on Γ.

3.2. We now revert to the setup of Theorem A, assume $n = 3$ and introduce some further notation, extending that of 2.4 to the S-arithmetic case.

We let $G_s = \mathbf{SL}_3(k_s)$ and G or G_S be the product of the G_s $(s \in S)$. Let pr_s be the projection of G onto G_s $(s \in S)$. For $g \in G$, the element $pr_s(g)$ is also called the s-component of g. For $s \in S$, we let $V_{1,s}$, $V_{2,s}$, V_s and D_s be the groups defined as V_1, V_2, V, D in 2.4, except that now the entries are in k_s for the first three groups, in k_s^* for the last one.

We let $\Gamma = \mathbf{SL}_3(\mathfrak{o}_S)$ and view it as a subgroup of G via the diagonal embedding. It is discrete, of finite covolume, but not cocompact. Let $\Omega = G/\Gamma$.

Given a quadratic form $F = \{F_s\}$ on k_S^3 we let $H_s = \mathrm{SO}(F_s)(k_s)$ and H_F be the product of the H_s's. Recall that F is always assumed to be non-degenerate and isotropic.

If $S = S_\infty$, then, as in 1.2, we view G as the group of real points of the $\mathbf{Q}$-group $L = R_{k/\mathbf{Q}}\mathbf{SL}_3$ obtained from $\mathbf{SL}_3$ by restriction of scalars from k to Q.

3.3. PROPOSITION. *Let $r \in S$ and $y \in \Omega$. Then $D_r^o.V_r.y$ and, for r real, $D_r^o.V_{1,r}.V_{2,r}^+.y$ and $D_r^o.V_{1,r}.V_{2,r}^-.y$, are not relatively compact.*

We first consider $D_r^o.V_r.y$ (over $\mathbf{R}$ or $\mathbf{C}$) and choose $g \in G$ such that $g.\mathfrak{o}^3 = y$.

Given a plane E in k_r^3, there exists a sequence $b_n \in \mathfrak{o}^3$ such that the s-component $b_{n,s} \to 0$ for $s \neq r$ while the r-component $b_{n,r}$ is not contained in E. Therefore we can find a sequence $a_n \in g.\mathfrak{o}^3$ such that $a_{n,s} \to 0$ for $s \neq r$ and that the third coordinate $a_{n,r}^3$ of $a_{n,r}$ is not zero. It is then elementary to choose first $v_n \in V_r$ such that the first two coordinates of $v_n.a_{n,r}$ are zero and then $d_n \in D_r^o$ such that $d_n.v_n.a_{n,r} \to 0$, whence also $d_n.v_n.a_n \to 0$. The lemma for $D_r^o.V_r$ now follows from Mahler's criterion.

Let r be real and assume that the closure C of $D_r^o.V_{1,r}.V_{2,r}^+.y$ (resp. $D_r^o.V_{1,r}.V_{2,r}^-.y$) is compact. By 2.7, C contains a non-empty subset Q invariant

under V_{2r}. Then $D_r^o V_r . Q$ is contained in C. Since C is assumed to be compact, this contradicts what has already been proved.

REMARK. 3.3 extends to our case Lemmas 11 and 12 of [M]. We have used a simplification of the proof of Lemma 11 proposed by B. J. Birch.

We recall that, up to the end of this section, we assume that $S = S_\infty$.

3.4. THEOREM. *Let $z \in \Omega$ and assume that $H_F . z$ is relatively compact. Then $H_F . z$ is compact.*

Proof. This statement is invariant under conjugation in G. Since a non-degenerate isotropic quadratic form on k_s^3 is equivalent to a multiple of the standard form $2x_1x_3 - x_2^2$, there is no loss in generality in assuming that F_s is equal to the latter form, hence that $H_s = \mathrm{SO}(2x_1x_3 - x_2^2)$ for all s. We shall do so. We write H for H_F.

Let $Z = \overline{Hz}$. Let $X \subset Z$ be a minimal H-invariant closed subset and $Y \subset X$ a minimal V_1-invariant closed subset of X (these exist since Z is compact). We shall first show that Y is D^o-invariant.

Fix $r \in S$. Let Y_o be a minimal closed $V_{1,r}$-invariant subset of Y and let R be the identity component of the group of elements $g \in \mathcal{N}_G(V_{1,r})$ such that $g . Y_o = Y_o$. Choose $y \in Y_o$. We claim that $R \cap G_y$ is not cocompact in R. Assume to the contrary that it is. Then $R . y$ is closed, homeomorphic to $R/(R \cap G_y)$, contained in Y_o and containing $V_{1,r} . y$. The latter being dense in Y_o, we see that $Y_o = R . y$ and we are in the situation of 1.1, up to conjugacy, but with the present group L (see 3.2) playing the role of G there. Let $g \in G$ be such that $g . y = \mathfrak{o}^3$. Then, by 1.1, $g . R . g^{-1} = \tilde{E}(\mathbf{R})^o$, where $\tilde{E}$ is the smallest $\mathbf{Q}$-subgroup of L containing $g . V_{1,r} . g^{-1}$. The group $pr_r(R)$ is contained in the G_r-normalizer of $V_{1,r}$, which is equal to $D_r . V_r$, hence solvable, so $\tilde{E}$ is solvable. Since it is the smallest $\mathbb{Q}$-subgroup containing the unipotent group $g . V_1$, $r \cdot g^{-1}$, it is in fact unipotent. Since any unipotent subgroup of $D_r . V_r$ is contained in V_r, we see that $pr_r(R) \subset V_r$. Consequently, D_r^o contains an element d which contracts $pr_r(R)$. The discrete group G_y is conjugate to Γ, hence pr_r is injective on G_y. Moreover, $D_r^o . y$, being contained in Z, is relatively compact. Therefore 3.1 holds and shows that $R \cap G_y = \{1\}$, a contradiction which implies that $R \cap G_y$ is not cocompact in F. Lemma 4 of [M] now shows that

$$M_o = \{g \in G - R \mid gY_o \cap Y_o \neq \varnothing\}$$

contains 1 in its closure. On the other hand, since $M_o \cap R$ is empty, Lemma 2 in [M] implies that $M_o \subset G - \mathcal{N}_G(V_{1,r})$. By Lemma 3 of [M], Y_o is stable under the subgroup Q generated by $\mathcal{N}_G(V_{1,r}) \cap \overline{V_{1,r} . M_o . V_{1,r}}$. As

$$M_o \subset G - \mathcal{N}_G(V_{1,r}) = G - \mathcal{N}_{G_r}(V_{1,r}) \prod_{s \neq r} G_s,$$

it follows from 2.5 that Q contains either V_r or $v . D_r^o . V_{1,r} . v^{-1}$ for some $v \in V_r$. An

elementary computation shows that if $v \in V_r - V_{1,r}$ then

$$D_r^o . V_{1,r} v . D_r^o v^{-1}$$

contains $D_r V_r$ if r is complex and either $D_r^o . V_{1,r} . V_{2,r}^+$ or $D_r^o . V_{1,r} . V_{2,r}^-$ if r is real. The group Q leaves Y_o stable. Therefore, if either $Q \supset V_r$ or $v \in V_r - V_{1,r}$ then X contains an orbit of $D_r . V_r$ if r is complex and the set of transforms of a point under $D_r^o . V_{1,r}$, $V_{2,r}^+$ or $D_r^o . V_{1,r} . V_{2,r}^-$ if r is real. Since X is compact, this contradicts 3.3. Consequently $v \in V_{1,r}$ and Q contains $D_r^o V_{1,r}$, i.e. Y_o is stable under $D_r^o . V_{1,r}$. As Y is minimal closed invariant under V_1 and D_r^o normalizes V_1, Lemma 2 of [M] shows that Y is stable under D_r^o. This being true for every $r \in S$, we see that Y is D^o-invariant, as claimed.

Choose $y \in Y$ and let

$$M = \{g \in G - H \mid g . y \in Z\}. \tag{1}$$

Assume first that $1 \in \bar{M}$. For some $s \in S$ we can find a sequence $\{m_i\} \subset M$ tending to 1, such that $pr_s m_i \in G_s - H_s$ for all i's. Since H, D, V_1 are products of their projections on the G_s's, it follows from 2.6 that $\overline{HMD^o V_1}$ contains $V_{2,s}$ if s is complex, and either $V_{2,s}^+$ or $V_{2,s}^-$ if s is real. But Y is a minimal closed $D^o . V_1$ invariant set. By Lemma 1 of [M], $g . Y \subset Z$ for any $g \in \mathcal{N}_G(V_1) \cap \overline{HMD^o V_1}$. Therefore we have $V_{2,s} Y \subset Z$ if s is complex and either $V_{2,s}^+ . Y \subset Z$ or $V_{2,s}^- . Y \subset Z$ if s is real. As a consequence, for $y \in Y$,

$$D_s^o . V_s . y = D_s^o . V_{1,s} V_{2,s} . y \subset Z \quad \text{if } s \text{ is complex}$$

and

$$\textit{either } D_s^o . V_{1,s} . V_{2,s}^+ . y \subset Z \quad \textit{or} \quad D_s^o . V_{1,s} . V_{2,s}^- . y \subset Z \quad \text{if } s \text{ is real.}$$

Since Z is compact this contradicts 3.3 and shows that $1 \notin \bar{M}$. As $y \in X$ and X is a minimal compact H-invariant set, Lemma 4 of [M] implies that $H/(H \cap G_y)$ is compact. We claim that $z \in Hy$. By assumption $y \in Z$, therefore y is a limit of elements $h_n . z$ ($h_n \in H$). We can write $h_n . z = c_n . y$ with $c_n \in G$, $c_n \to 1$. But then $c_n . y \in Z$ i.e. $c_n \in M$. Since $1 \notin \bar{M}$, we have $c_n \in H$ for n big enough, whence our contention. But then $H . z = H . y$ is compact.

3.5. We shall say that a sequence $t_n = (t_{n,s})_{s \in S}$ in k_S ($t_{n,s} \in k_s$), tends to infinity if $|t_{n,s}|_s \to \infty$ for s complex and $t_{n,s} \to \infty$ for s real.

Given $y \in \Omega$, we let $\phi(y)$ be the set of limit points of sequences $v_1(t_n) . y$, where $t_n \to \infty$ in k_S.

LEMMA. *Let $y, z \in \Omega$. Assume that $H_F . z$ is compact, $y \notin H_F . z$ and $V_1 . y$ is relatively compact. Then $\phi(y) \not\subset H_F . z$.*

For $k = \mathbf{Q}$, this is Lemma A of [M]. The proof in the present case is so similar that we shall not repeat it in detail. Assume this assertion to be false, we find as

in *loc. cit.* $p \in G/H$, different from the origin $q = H$ in G/H such that $q \in \overline{V_1 . p}$. Now G/H is the product of the G_s/H_s ($s \in S$) and $V_1 . p$ the product of the $V_{1,s} . p_s$, so that for each s, the point $q_s = H_s$ should be in the closure of $V_{1,s} . p_s$. Since $p_s \neq q_s$ for at least one s this is impossible because G_s/H_s is an affine variety (or at any rate open and closed in one if s is real), and any orbit of a unipotent group in an affine variety is closed, hence $V_{1,s} . p_s$ is also closed in G_s/H_s in the ordinary topology.

3.6. THEOREM. *Let $z \in \Omega$. Assume there exist a finite union C of k-lines* (cf. 1.8) *and an open neighborhood U of the origin in k_S^3 such that*

$$H_F(z - (z \cap C)) \cap U = \{0\}. \tag{1}$$

Then $H_F . z$ is compact.

Proof. Let $g \in G$ be such that $g . \mathfrak{o}^3 = z$. After having replaced F by ${}^t g . F . g$ and U, C by $g^{-1} . U$, $g^{-1} . C$, we are reduced to the case where $z = \mathfrak{o}^3$. Of course, $\mathfrak{o}^3 \cap k . x$ ($x \in k_S^3$) is $\neq \{0\}$ if and only if $x \in k^n$. We may assume that C consists of finitely many rational lines.

In H_{F_s} we choose a unipotent one-dimensional subgroup Q_s which does not fix any line in C and let Q be the product of the Q_s.

We claim there exists a neighborhood U' of the origin in k_S^3 such that

$$Q . x \cap U' = \varnothing \quad \text{for every non-zero} \quad x \in C \cap \mathfrak{o}^3. \tag{2}$$

Let $x \in C \cap \mathfrak{o}^3 - \{0\}$. For every s, $x_s \neq 0$ and, by our choice of Q, the element x_s is not fixed under Q_s. Therefore $q \mapsto q . x_s$ is injective. Since Q_s is unipotent, $Q_s . x_s$ is closed in k_s^3 and does not contain the origin, hence $Q . x$ does not meet some polydisc $U_d = \{u = (u_s) \mid |u_s|_s < d\}$.

But then the same is true for all elements $\lambda . x$ ($\lambda \in \mathfrak{o} - \{0\}$). Indeed, we have

$$|q_s . \lambda . x|_s = |\lambda|_s |q_s . x_s|_s.$$

and at least one of $|\lambda_s|$ is $\geqq 1$, since $\lambda \in \mathfrak{o}$, $\lambda \neq 0$.

As we remarked in 1.8, if L is a rational line, then $L \cap \mathfrak{o}^3$ is a finitely generated module. There exists therefore a finite subset E of $C \cap \mathfrak{o}^3 - \{0\}$ such that $C \cap \mathfrak{o}^3 = \mathfrak{o} . E$. The claim (2) now follows by choosing a constant d suitable for every $e \in E$. In the sequel, we assume, as we may, that U in (1) is chosen small enough to be contained in U'. In view of the assumption (1), we have now

$$Q . \mathfrak{o}^3 \cap U = \{0\}, \tag{4}$$

therefore $Q . \mathfrak{o}_3$ is relatively compact in Ω by Mahler's criterion. From then on, the argument proceeds as in Section 4 of [M]. Since our framework and formulation are somewhat different, we describe it for the sake of completeness.

We identify k_s with Q_s, hence k_S with Q and let $q(t)$ be the element of Q corresponding to $t = \{t_s\} \in k_S$. Let Y be the set of accumulation points of sequences $q(t_n) . \mathfrak{o}^3$, where $t_n \to \infty$ in the sense of 3.5. Since $Q . \mathfrak{o}^3$ is relatively

compact, Y is compact and not empty. The argument of 2.7 shows that it is invariant under Q. We claim that it is connected. Assume it is not. Then it is the union of two disjoint non-empty compact subsets Y_0, Y_1. There exists on Ω a continuous compactly supported real valued function, with values in $[0, 1]$, equal to 0 on Y_0 and to 1 on Y_1. Let $a \in Y_0$ and $b \in Y_1$. Then $a = \lim_n q(a_n) . \mathfrak{o}^3$ and $b = \lim_n q(b_n) . \mathfrak{o}^3$, where $\{a_n\}$ and $\{b_n\}$ are sequences in k_S tending to infinity (3.5). We may assume.

$$f(q(a_n)(\mathfrak{o}^3)) < 1/2, \qquad f(q(b_n) . (\mathfrak{o}^3)) > 1/2.$$

We can find a curve $C_n = \{c_n(t) \mid t \in [0, 1]\}$ in k_S such that $c_n(0) = a_n$, $c_n(1) = b_n$ and

$$c_{n,s}(t) \geqq \min(a_{n,s}, b_{n,s}) \quad (t \in [0, 1]) \text{ if } s \text{ is real,}$$

$$|c_{n,s}(t)| \geqq \min(|a_{n,s}|, |b_{n,s}|) \quad (t \in [0, 1]) \text{ if } s \text{ is complex.}$$

There exists then $c_n \in C_n$ such that $f(q(c_n) . \mathfrak{o}^3) = 1/2$. The elements $q(c_n) . \mathfrak{o}^3$ have an accumulation point $x \in \Omega$ by compactness. But, clearly, $c_n \to \infty$ in the sense of 3.5, hence $x \in Y$. Since f takes the value $1/2$ on x this is a contradiction, proving that Y is connected.

Let now $y \in Y$. We want to show that $H . y$ is relatively compact. By construction, there exists a sequence $\{t_n\} \subset k_S$ tending to infinity in the sense of 3.5 such that $q(t_n) . \mathfrak{o}^3 \to y$. Let x be an element of the lattice y. There is then a sequence $x_n \in \mathfrak{o}^3$ such that $q(t_n) . x_n \to x$ in k_S^3. We claim that for n big enough, $x_n \notin C$. Assume the contrary. Passing to a subsequence, we may further assume, in the above notation, that $x_n = \lambda_n . e$ for some fixed element $e \in E$ and $\lambda_n \in \mathfrak{o}$. For each $s \in S$, the orbit $Q_s . e_s$ is closed and homeomorphic to Q_s. Therefore $q_s(t_{n,s}) . e_s$ diverges. Again passing to a subsequence, we may assume that $|\lambda_n|_s \geqq 1$ for some fixed s and all n's, and then it is clear that

$$q_s(t_{n,s}) . x_{n,s} = \lambda_n . q_s(t_{n,s}) . e_s$$

diverges, too. Hence $q(t_n) . x_n$ diverges. Therefore $x_n \notin C$ for n big enough. Then, for $h \in H_F$, we have

$$h . x = \lim_n h . q(t_n) . x_n \quad (x_n \notin C).$$

By (1), $h . q(t_n) . x_n \notin U$, hence $h . x \notin U$. This shows that $H_F . y \cap U = \{0\}$ hence, by Mahler's criterion, that $H_F . y$ is relatively compact for every $y \in Y$. By 3.4, $H_F . y$ is then compact. Let us show that H_F has at most countably many compact orbits in Ω. Assume that $H_F . a . \mathfrak{o}^3$ is compact for some $a \in G$. Then $a^{-1} . H_F . a . \mathfrak{o}^3$ is compact and, by 1.2, $a^{-1} . H_F . a$ is the group of k_S-points of some k-subgroup of $\mathbf{SL}_3$. But $\mathbf{SL}_3$ has only countably many k-subgroups. Therefore the groups $a^{-1} . H_F . a$, where $a \in G$ is such that $H_F . a . \mathfrak{o}^3$ is compact, form a countable set. But H_F is its own normalizer in G, therefore if $a^{-1} . H_F . a = b^{-1} . H_F . b$ $(a, b \in G)$, then $a \in H_F . b$ and $H_F . a . \mathfrak{o}^3 = H_F . b . \mathfrak{o}^3$,

whence our statement. It now follows that Y is the union of at most countably many disjoint closed sets of the form $Y \cap H_F.y$. Since it is compact and connected, as we saw earlier, it must be contained in one of them, say $Y \subset H.z$, for some $z \in Y$. Then the lemma in 3.5 shows that $\mathfrak{o}^3 \in Y$; therefore $H_F.\mathfrak{o}^3$ is compact, and the theorem is proved.

3.7. COROLLARY. *The conditions* (i) *and* (ii) *of Theorem A for* $S = S_\infty$ *are equivalent.*

By 1.3, it suffices to prove this for $n = 3$. Our assertion now follows from 3.6 and the implication (c) $\Rightarrow$ (a) of 1.9.

4. Proof of Theorem A when F is rational for some $s \in S$

4.1. In this section we assume $|S| \geqq 2$, F irrational, the existence of a form F_o on k^n and of $q \in S$ such that $F_q = c_q.F_o$ for some $c_q \in k_q^*$. Let S' be the set of $s \in S$ such that $F_s = c_s.F_o$ with $c_s \in k_s^*$. The set S' contains q and, since F is irrational, $S' \neq S$. Let $T = S - S'$.

Let Q_o be the k-variety defined by $F_o = 0$ and $Q_{o,s} = Q_o \times k_s$. Similarly let Q_s be the k_s-variety defined by $F_s = 0$. It characterizes F_s up to a multiple. We have $Q_{o,s} = Q_s$ for $s \in S'$ but $Q_{o,t} \neq Q_t$ for $t \in T$. In particular, for $t \in T$, $\dim(Q_{o,t} \cap Q_t) < n-1$ and $Q_{o,t}(k_t) \neq Q_t(k_t)$. [Recall that F_t is isotropic, hence $Q_t(k_t)$ is Zariski dense in Q_t.]

For $t \in T$, let L_t be a fixed open subgroup of $\mathrm{SO}(F_o)(k_t)$ and M_t be the set of vectors in $k_t^n - Q_o(k_t)$, which can be mapped into $Q_t(k_t)$ by an element of L_t. Then M_t is a non-empty open subset of k_t^n and clearly $k_t^*.M_t = M_t$. We fix an element $e_{t,1}$ of M_t. Then the line $k_t.e_{t,1}$ is in M_t, except for the origin. Complete $e_{t,1}$ to a basis $(e_{t,i})$ of k_t^n. Let us denote by $D_{t,r}$ the disc $|x|_t \leqslant r$ in k_t ($r > 0$, real).

For $r > 0$, let $B_{t,r}$ and $C_{t,r}$ be the products of the discs $D_{t,r}.e_{t,j}$ ($j = 2, \ldots, n$), and $D_{t,r}.e_{t,j}$ ($j = 1, 2, \ldots, n$) respectively. We choose $a \geqslant b > 0$ so small that for all $t \in T$,

(i) $e_{t,1} + C_{t,a}$ is contained in M_t. (We note here that then $(D_{t,m} - D_{t,1})e_{t,1} + C_{t,a}$ is contained in M_t for all $m > 1$).

(ii) The sum of any $|T|$ elements of $C_{t,b}$ is contained in $C_{t,a}$.

We shall now use elementary geometry of numbers to prove:

(*) *Given a polydisc* $U = \Pi_{s \in S'} U_s$ *centered on the origin in* $k_{S'}^n$, *there exists* $x \in \mathfrak{o}_S^n$ *such that* $x_s \in U_s$ *for* $s \in S'$ *and* $x_t \in M_t$ *for* $t \in T$.

Proof. Let $V = \Pi_{s \in S'} V_s$ be a bounded polydisc centered on the origin such that the sum of any $2\,|T|$ elements of V is contained in U. Let $t \in T$. We shall show

$(*)_t$ *There exists* $y(t) \in \mathfrak{o}_S^n$ *and* $m \geqslant 2$ *such that* $y(t)_s \in V_s + V_s$, *if* $s \in S'$, $y(t)_t \in (D_{t,m} - D_{t,1})e_{t,1} + B_{t,b}$ *and* $y(t)_{t'} \in C_{t',b}$ *if* $t' \in T - \{t\}$.

Assume this to be established. Then in view of the conditions (i) and (ii), the sum of the $y(t)$, $t \in T$, satisfies $(*)$.

We now fix $t \in T$ and prove $(*)_t$. For a positive real number r, let

$$\Omega_r = V \times (D_{t,r} \cdot e_{t,1} + B_{t,b/2}) \prod_{t' \in T - \{t\}} C_{t',b/2}.$$

On k_S^n take the product of the usual Haar measures on the k_s^n and let c be the volume of $k_S^n/\mathfrak{o}_S^n$. The set $\mathfrak{o}_S^n$ is discrete in k_S^n, *hence* $(\Omega_1 + \Omega_1) \cap \mathfrak{o}_S^n$ is a finite set, say with q elements. There exists $m \geqslant 2$ such that the volume of $\Omega_{m/2}$ is $> (q+1)c$. Therefore at least one fibre of the projection onto $k_S^n/\mathfrak{o}_S^n$, restricted to $\Omega_{m/2}$, has more than $(q+1)$ elements, so we can find $y_o, \ldots, y_{q+1} \in \Omega_{m/2}$ such that the differences $x_i = y_o - y_i$ $(i = 1, \ldots, q+1)$ are distinct elements of $\mathfrak{o}_S^n$ and at least one of these, say x_1 will then be outside $\Omega_1 + \Omega_1$. Let $y(t) = x_1$, then $y(t)$ satisfies $(*)_t$.

4.2. We now prove Theorem A assuming $|S| \geqq 2$ and at least one of the F_s to be rational. From the Introduction and Section 1, we know that it suffices to show that (i) $\Rightarrow$ (iii) when $n = 3$. We use the notation and assumptions of 4.1 and write H_o for $\mathrm{SO}(F_o)$; let $\tilde{H}_o$ be its universal covering and $\sigma\colon \tilde{H}_o \to H_o$ the canonical isogeny. Let $\Gamma \subset H_o(k)$ be the stabilizer of $\mathfrak{o}_S^3$. It is an S-arithmetic subgroup, which we view, as usual, as a discrete subgroup of $H_o(k_S)$. Let Γ_T be its projection on $H_o(k_T)$. We claim first there are open subgroups of finite index $L_t \subset H_o(k_t)$ $(t \in T)$ such that the product L_T of the L_t's is contained in the closure of Γ_T. Let $\tilde{\Gamma}$ be an S-arithmetic subgroup of $\tilde{H}_o(k)$. By assumption, $\tilde{H}_o$ and H_o are isotropic over k_s for $s \in S'$. Therefore, by strong approximation [P], $\tilde{\Gamma}_T$ is dense in an open subgroup of $\tilde{H}_o(k_T)$, hence $\sigma(\tilde{\Gamma}_T)$ is dense in an open subgroup of $H_o(k_T)$. As $\sigma(\tilde{\Gamma}_T)$ is commensurable with Γ_T, our assertion is clear.

Let $\varepsilon > 0$. We choose first the U_s $(s \in S')$ small enough so that $|F_s(y)|_s \leqq \varepsilon$ for $y \in U_s$. Then, using the group L_t just defined, we construct M_t as in 4.1 $(t \in T)$. With respect to those choices, we have seen in 4.1 we may find $x \in \mathfrak{o}_S^n$ satisfying the conditions of $(*)$ there. There exists then $g_t \in L_t$ such that $y_t = g_t \cdot x_t \in Q_t(k_t)$ for every $t \in T$; we can find $z_t \in L_t \cdot x_t$ close to y_t such that $0 < |F_t(z_t)|_t \leqslant \varepsilon/2$. The closure of Γ_T contains $L_T = \Pi_{t \in T} L_t$, we can therefore find $\gamma \in \Gamma$ such that

$$0 < |F_t((\gamma \cdot x)_t)|_t \leqslant \varepsilon \quad (t \in T).$$

By assumption, $F_{0,t}(x_t) \neq 0$. But $F_{0,s}(x_s)$ is conjugate to $F_{0,t}(x_t)$, $(s \in S')$ hence is also $\neq 0$. By construction, for $s \in S'$, $|F_{0,s}(x_s)|_s = |F_s(x_s)|_s \leqslant \varepsilon$ and we have just seen that it is not zero. Since $\gamma \in H_0$, we have $F_s((\gamma \cdot x)_s) = F_s(x_s)$ hence $|F_s(\gamma \cdot x)_s|_s \leqslant \varepsilon$ for $s \in S'$. Thus the element $\gamma \cdot x$ satisfies our conditions.

5. Proof of Theorem A

We now revert to the assumptions of Theorem A. As pointed out in the Introduction and Section 1, to prove Theorem A, it suffices to show that (i) $\Rightarrow$ (iii) when $n = 3$.

By 4.2, this is true if at least one F_s is rational. There remains to consider the case where no F_s is so. Then, by 1.7, the set of $x \in k^n$ for which $F_s(x) = 0$ for some $s \in S$ is contained in finitely many lines.

Our present assumption also implies that $F \mid k^3 \otimes_{\mathbf{Q}} \mathbf{R}$ is irrational hence it satisfies (ii) for $x \in \mathfrak{o}^n$ by 3.7. More precisely 3.6 and 3.7 show that (ii) is satisfied by elements $x \in \mathfrak{o}^3$ which belong to infinitely many lines. In view of the above remark, it follows that (iii) is fulfilled, also by elements belonging to infinitely many lines. In particular (ii) $\Rightarrow$ (iii) is proved if $S = S_\infty$.

Let now $S \neq S_\infty$. We have just seen that (iii) is true if $s \in S_\infty$ for $x \in \mathfrak{o}^n$ on infinitely many lines.

Let $s \in S_f$. The set $\mathfrak{o}^n$ is bounded in k_s^n. Therefore given any neighborhood U of the origin in k_s^n, we can find $a_s \in \mathfrak{o}_s$ (integers of k_s) such that $a_s . \mathfrak{o}^n \subset U$. As a consequence, given $\varepsilon > 0$, there exists $a_s \in \mathfrak{o}_s$ such that $|F_s(a_s \cdot x)|_s \leqslant \varepsilon$ for all $x \in \mathfrak{o}^n$.

By strong approximation, there exists $a \in \mathfrak{o}$ such that $|a|_s = |a_s|_s$ for $s \in S_f$. Let $b = \max_{s \in S_\infty} |a|_s$. We can find $x \in \mathfrak{o}^n$ such that

$$0 < |F_s(x)|_s \leqslant \varepsilon / b^2 \quad (s \in S_\infty).$$

We have then

$$0 < |F_s(a \cdot x)|_s \leqslant |a|_s^2 \varepsilon / b^2 \leqslant \varepsilon \quad \text{if } s \in S_\infty$$

and

$$|F_s(a \cdot x)|_s = |a|_s^2 |F_s(x)|_s = |a_s|_s^2 |F_s(x)|_s = |F_s(a_s \cdot x)|_s \leqslant \varepsilon \quad \text{if } s \in S_f.$$

This is true for x belonging to infinitely many lines. On the other hand, given $s \in S$, the form $F_s(x)$ can be zero on only finitely many rational lines, under our present assumptions. Therefore (iii) is already fulfilled by elements of $\mathfrak{o}^n$ in this case. This completes the proof of (i) $\Rightarrow$ (iii).

REMARK. Combined with 1.6 and 1.9, Theorem A yields some assertions about orbits in $\mathbf{SL}_3(k_S)/\mathbf{SL}_3(\mathfrak{o}_S)$. For instance, we see from 1.6 that if F is a non-degenerate isotropic quadratic form on k_S^3 which does not represent zero rationally, then any relatively compact orbit of H_F in $\mathbf{SL}_3(k_S)/\mathbf{SL}_3(\mathfrak{o}_S)$ is compact.

6. Some remarks on the density problem

6.1. The quotient $q_s = k_s^*/k_s^{*2}$ is of order one if $k_s = \mathbf{C}$, of order two if $k_s = \mathbf{R}$ and

of order 4 or 8 if k_s is non-archimedean. Let us consider the following strengthening of (ii):

(iv) Let $\varepsilon > 0$ and $\bar{c}_s \in q_s$. Then there exists $x \in \mathfrak{o}_S^n$, which satisfies (iii) and such that $F_s(x) \in \bar{c}_s$ for all $s \in S$.

We note first that (iv) is equivalent to

(v) The set $F(\mathfrak{o}_S^n)$ is dense in k_S.

Proof. It is obvious that (v) $\Rightarrow$ (iv). Assume now (iv) to hold. Let $r = (r_s) \in k_S$; let $\bar{c}_s$ be the class of r_s in q_s if $r_s \neq 0$, and be any element of q_s otherwise. We let c_s stand for a representative of $\bar{c}_s$. Note that if $x \in \bar{c}_s$, then $c_s^{-1} . x$ is a square. We denote by $(c_s^{-1} . x)^{1/2}$ any square root. Let $x \in \mathfrak{o}_S^n$ be such that $F_s(x) \in \bar{c}_s$ for all $s \in S$. Since $\mathfrak{o}_s$ is a lattice in k_S, there exists a universal constant d and $u(x) \in \mathfrak{o}_S$ such that

$$|(c_s^{-1} r_s / c_s^{-1} F_s(x))^{1/2} - u(x)|_s \leqslant d \quad (s \in S), \tag{1}$$

which can be written

$$|(c_s^{-1} r_s)^{1/2} - u(x)_s (c_s^{-1} F_s(x))^{1/2}|_s \leqslant d\, |(c_s^{-1} F_s(x))^{1/2}|_s. \tag{2}$$

By (iv) we can find a sequence $x_j \in \mathfrak{o}_S^n$ such that

$$F(x_j)_s \in \bar{c}_s \quad (s \in S), \qquad F(x_j) \to 0.$$

Therefore

$$u(x_j)_s (c_s^{-1} F_s(x_j))^{1/2} \to (c_s^{-1} r_s)^{1/2}.$$

Taking the squares and dividing by c_s^{-1}, we get

$$\lim_{j \to \infty} u(x_j)_s^2 F_s(x_j) = r_s,$$

hence also

$$\lim_{j \to \infty} F(u(x_j) . x_j) = r.$$

6.2. It is of course conjectured that (i) $\Rightarrow$ (v), hence that the five conditions (i)–(v) are equivalent. In view of Theorem A and the above, this would follow from (iii) $\Rightarrow$ (iv).

It was already known in the 1950s that in the original case ($k = \mathbf{Q}$, $S = S_\infty$), the truth of the Oppenheim conjecture implies the density of $F(\mathbf{Z}^n)$. A stronger result, namely that the values on the primitive vectors are dense, was proved in [DM]. The truth of the Raghunathan conjecture [Rt] also implies the density of the values on primitive vectors, in particular that (i) $\Rightarrow$ (v) for any k if $S = S_\infty$ (§7). The argument of Section 4 above proves it if the set S' there consists of complex places. Whether (i) $\Rightarrow$ (v) is true in general seems to be open at present. Of course,

it would again be true in a stronger form if the Raghunathan conjecture would hold when non-archimedian places are allowed.

7. Density in the Archimedean case

(Added January 1991)

In this section, we want to show how a recent result of M. Ratner [Rt] yields the implication (i) $\Rightarrow$ (v), in fact a much stronger statement (see 7.9), in the archimedean case. We shall use a special case of the following theorem:

7.1. THEOREM (M. Ratner [Rt]). *Let $\mathscr{G}$ be a connected semisimple* **Q***-group, $\Gamma \subset \mathscr{G}(\mathbf{Q})$ an arithmetic subgroup and $\Omega = \mathscr{G}(\mathbf{R})/\Gamma$. Let H be a closed connected subgroup of $\mathscr{G}(\mathbf{R})$ generated by unipotent elements and $z \in \Omega$. Then there exists a closed connected subgroup L of $\mathscr{G}(\mathbf{R})$ such that $\overline{H.z} = L.z$ and $L/(L \cap \mathscr{G}_z)$, where $\mathscr{G}_z$ is the isotropy group of z in $\mathscr{G}(\mathbf{R})$, has finite volume.*

This is in turn a special case of Corollary B in [Rt], but it will suffice for our needs. From this result and 1.1 we derive first:

7.2. COROLLARY. *Let $g \in \mathscr{G}(\mathbf{R})$ be such that $z = g.o$, where o is the coset Γ in Ω. Then $g^{-1}.L.g = \mathscr{H}_g(\mathbf{R})^o$, where $\mathscr{H}_g$ is the smallest* **Q***-subgroup in $\mathscr{G}$ whose group of real points $\mathscr{H}_g(\mathbf{R})$ contains $g^{-1}.H.g$.*

Proof. After having replaced H and L by $g^{-1}.H.g$ and $g^{-1}.L.g$, respectively, we may assume that $z = o$, $g = 1$. Since Γ is the isotropy group of o, the assumption is equivalent to $L.\Gamma = \overline{H.\Gamma}$. Moreover, $\Gamma \cap L$ is of finite covolume in L by 7.1. Therefore we may apply 1.1 and 7.2 follows.

7.3. In the sequel, $S = S_\infty$, therefore $\mathfrak{o}_S = \mathfrak{o}$. Also $k_S = k \otimes_{\mathbf{Q}} \mathbf{R}$ and, to make it clear that we deal with the case $S = S_\infty$ only, we shall from now on write $k_{\mathbf{R}}$ for k_S. As in 1.2, we view the group

$$G = \mathbf{SL}_n(k_{\mathbf{R}}) = \prod_{s \in S_\infty} \mathbf{SL}_n(k_s)$$

as the group of real points of the **Q**-group $\mathscr{G} = R_{k/\mathbf{Q}}\mathscr{G}'$, where $\mathscr{G}' = \mathbf{SL}_n$, viewed as a k-group.

PROPOSITION. *Let H_s be a closed subgroup of $G_s = \mathbf{SL}_n(k_s)$ $(s \in S_\infty)$ and H the product of the H_s. Then the smallest* **Q***-subgroup $\mathscr{L}$ of $\mathscr{G}$ whose group of real points contains H is of the form $\mathscr{L} = R_{k/\mathbf{Q}}\mathscr{L}'$, where $\mathscr{L}'$ is a connected k-subgroup of $\mathscr{G}'$.*

This is a simple property of the restriction of scalars, for which we have unfortunately no ready reference. A proof will be given at the end of this section (7.12).

Let now $n \geqq 3$ and $F = (F_s)$ be as in Theorem A. Let H_s be the special orthogonal group of F_s and $H_F = \prod_s H_s$. Then we have:

7.4. COROLLARY. *If $H = H_F$ then either F is rational or the smallest* **Q**-*subgroup containing H is $\mathscr{G}$.*

Proof. We note first that H_s is maximal among proper closed connected subgroups of G_s. This amounts to say that the representation of the Lie algebra $L(H_s)$ of H_s in $L(\mathbf{SL}_n(k_s))/L(H_s)$ is irreducible. This representation is the differential of the natural representation of an orthogonal group in the space of symmetric matrices of trace zero. Its irreducibility (for $n \geqq 3$) is classical. It is another way to state that $\mathbf{SL}_n(k_s)/H_s$ is an irreducible symmetric space, also a standard fact.

Let now $\mathscr{L}$ be the smallest **Q**-subgroup of G such that $\mathscr{L}(\mathbf{R}) \supset H$. By 7.2, it is of the form $R_{k/\mathbf{Q}}\mathscr{L}'$ where $\mathscr{L}'$ is a k-subgroup of $\mathscr{G}'$. In particular $\mathscr{L}(\mathbf{R}) = \prod \mathscr{L}'(k_s)$ is the product of its intersections with the G_s's. Of course $\dim \mathscr{L}'(k_s)$ is independent of s and $\mathscr{L}'(k_s) \supset H_s$. Therefore we have either $\mathscr{L}'(k_s)^o = H_s$ or $\mathscr{L}'(k_s) = G_s$ for every s. In the former case, $\mathscr{L}' = \mathrm{SO}(F_o)$ for a suitable quadratic form F_o on k^n and then $F = c.F_o$ for some $c \in k_{\mathbf{R}}^*$, since a quadratic form is determined up to a multiple by the group leaving it invariant, or its identity component.

From this elementary remark and 7.2 we now deduce immediately:

7.5. THEOREM. *Let F be irrational. Then the orbit $H_F.o$ of the origin in Ω is dense in Ω.*

In fact, if F is irrational, then in 7.3 we have $\mathscr{L}(\mathbf{R}) = G$ hence $\overline{H_F.o} = \Omega$.

7.6. An element $x \in \mathfrak{o}^n$ is primitive if a relation $a.y = x$ $(a \in \mathfrak{o}, y \in \mathfrak{o}^n)$ implies that a is a unit in $\mathfrak{o}$. A m-tuple $(x_1, \ldots, x_m)(m \leqq n)$ of elements in $\mathfrak{o}^n$ will be said to be primitive if it is part of a basis of $\mathfrak{o}^n$ over $\mathfrak{o}$ or, equivalently, if it spans a direct summand of $\mathfrak{o}^n$. This is the case if and only if there exists $g \in \mathbf{GL}_n(\mathfrak{o})$ such that $g.e_i = x_i (1 \leqq i \leqq m)$, where (e_i) is the canonical basis of $\mathfrak{o}^n$. If $m < n$, we can then also assume $g \in \mathbf{SL}_n(\mathfrak{o})$. Any primitive m-tuple consists of primitive vectors.

It is clear that a subset $(y_1, \ldots, y_m)$ $(m \leqq n)$ of $k_{\mathbf{R}}^n$ is free (over $k_{\mathbf{R}}$) if and only for each s, the s-components $y_{1,s}, \ldots, y_{m,s}$ are linearly independent over k_s. In particular, any free subset is part of a basis. The n-tuple $(y_1, \ldots, y_n)$ is a basis if and only if there exists $g \in \mathbf{GL}_n(k_{\mathbf{R}})$ such that $g.e_i = y_i (i = 1, \ldots, n)$. We shall say that the basis (y_i) is *unimodular* if such a g can be chosen in $\mathbf{SL}_n(k_{\mathbf{R}})$. If $m < n$, any free m-tuple is part of a unimodular basis.

In the next corollaries, we let B_F be the bilinear symmetric form associated to F. We have then

$$2.B_F(x, y) = F(x + y) - F(x) - F(y), \tag{1}$$

$$F(x) = B_F(x, x) \qquad (x, y \in k_{\mathbf{R}}^n).$$

7.7. COROLLARY. *Assume F to be irrational. Let $(y_1, \dots, y_n)$ be a unimodular basis of $k_{\mathbf{R}}^n$ over $k_{\mathbf{R}}$. Then there exists a sequence $(x_{j,1}, \dots, x_{j,n})$ $(j = 1, 2, \dots)$ of bases of $\mathfrak{o}^n$ over $\mathfrak{o}$ such that*

$$B_F(y_a, y_b) = \lim_{j\to\infty} B_F(x_{j,a}, x_{j,b}) \quad (1 \leqq a, b \leqq n). \tag{1}$$

Proof. By assumption, there exists $g \in G$ such that $g.e_i = y_i$ $(1 \leqq i \leqq n)$. Since F is irrational, $H_F.o$ is dense in Ω by 7.5. This is equivalent to saying that $H_F.\mathbf{SL}_n(\mathfrak{o})$ is dense in G, (in the product of the Lie group topologies). There exist therefore sequences $h_j \in H_F$ and $\gamma_j \in \mathbf{SL}_n(\mathfrak{o})$ such that $h_j.\gamma_j \to g$. Since h_j preserves F, we have

$$\begin{aligned} B_F(y_a, y_b) = B_F(g.e_a, g.e_b) &= \lim_j B_F(h_j.\gamma_j.e_a, h_j.\gamma_j.e_b) \\ &= \lim_j B_F(\gamma_j.e_a, \gamma_j.e_b). \end{aligned}$$

Since $(\gamma_j e_1, \dots, \gamma_j e_n)$ is a basis of $\mathfrak{o}^n$ for all j's, this proves our assertion, with

$$x_{j,i} = \gamma_j.e_i \quad (i = 1, \dots, n; j = 1, 2, \dots).$$

7.8. COROLLARY. *Assume F to be irrational. Let $m < n$ and $y_1, \dots, y_m$ be a free subset of $k_{\mathbf{R}}^n$. Then there exists a sequence $(x_{j,1}, \dots, x_{j,m})$ $(j = 1, \dots)$ of primitive m-tuples of $\mathfrak{o}^n$ such that*

$$B_F(y_a, y_b) = \lim_{j\to\infty} B_F(x_{j,a}, x_{j,b}) \quad (1 \leqq a, b \leqq m) \tag{1}$$

This follows from 7.7, once it is noted that $(y_1, \dots, y_m)$ is part of a unimodular basis.

REMARK. 7.5 for $n = 3$ is already proved in [DM], where the consequence 7.8 for $m = 2$ is also drawn.

7.9. COROLLARY. *Assume F to be irrational. Let $\lambda_1, \dots, \lambda_{n-1} \in k_{\mathbf{R}}$. Then there exists a sequence of primitive $(n-1)$-tuples $(x_{j,1}, \dots, x_{j,n-1})$ $(j = 1, 2, \dots)$ in $\mathfrak{o}^n$ such that*

$$\lambda_i = \lim_{j\to\infty} F(x_{j,i}) \quad (i = 1, \dots, n-1). \tag{1}$$

In particular the set of values of F on the primitive elements of $\mathfrak{o}^n$ is dense in $k_{\mathbf{R}}$.

Proof. We note first that, since F_s is non-degenerate, its special orthogonal group is an irreducible linear subgroup of $\mathbf{SL}_n(k_s)$. In particular, it does not leave any hyperplane invariant. The form F_s, being moreover isotropic, assumes all values in k_s. Let $\lambda_{i,s}$ be the s-component of λ_i. The previous remarks show that

the level surface $F_s(x) = \lambda_{i,s}$ $(x \in k_s^n)$ is not contained in any hyperplane. It is then elementary that we can find a free subset $(c_{s,1}, \ldots, c_{s,n-1})$ of k_s^n such that

$$F_s(c_{s,i}) = \lambda_{i,s} \quad (i = 1, \ldots, n-1,\ s \in S_\infty). \tag{2}$$

The vectors $c_i = (c_{s,i})$ $(i = 1, \ldots, n-1)$ then form a free subset of $k_{\mathbf{R}}^n$ and we have

$$F(c_i) = \lambda_i \quad (i = 1, \ldots, n-1). \tag{3}$$

7.9 now follows from 7.8 for $m = n-1$, $a = b$, and 7.6(1).

7.10. There remains to justify 7.3. We assume familiarity with the functor of restriction of scalars [W: §1]. Let E be a perfect field and F a finite separable extension of E. Let $\sum$ be the set of embeddings of F in $\bar{E}$. Let $\mathscr{G}'$ be a connected F-group and $\mathscr{G} = R_{F/E}\mathscr{G}'$ be the E-group obtained by restriction of scalars from $\mathscr{G}'$. The Galois group Γ_E of $\bar{E}$ over E operates on $\sum$ by $\sigma \mapsto \gamma \circ \sigma$. We have

$$\mathscr{G}(\bar{E}) = \prod_{\sigma \in \Sigma} {}^{\sigma}\mathscr{G}'(\bar{E}),$$

where ${}^{\sigma}\mathscr{G}'$ is the group obtained from $\mathscr{G}'$ by the base change $F \to {}^{\sigma}F$.

7.11. PROPOSITION. *Let $\mathscr{L}$ be a connected E-subgroup of $\mathscr{G}$. Then there exists a connected F-subgroup $\mathscr{M}'$ of $\mathscr{G}'$ such that $\mathscr{M} = R_{F/E}\mathscr{M}' \subset \mathscr{L}$ and ${}^{\sigma}\mathscr{M}' = \mathscr{L} \cap {}^{\sigma}\mathscr{G}'$ for every $\sigma \in \Sigma$.*

Proof. Let $\mathscr{M}_\sigma = \mathscr{L} \cap {}^{\sigma}\mathscr{G}'$ and $\mathscr{M}$ be the product of the $\mathscr{M}_\sigma$. We claim that it is defined over E. The group $\mathscr{M}_\sigma$ is defined over ${}^{\sigma}F$, clearly. If $\gamma \in \Gamma$ and $\sigma \in \Sigma$, then $\sigma' = \gamma \circ \sigma \in \Sigma$. Since ${}^{\gamma}\mathscr{L} = \mathscr{L}$, we have therefore ${}^{\gamma}(\mathscr{L} \cap {}^{\sigma}\mathscr{G}') = \mathscr{L} \cap {}^{\sigma'}\mathscr{G}'$, hence $\mathscr{M}$ is a $\bar{E}$-subgroup whose group of $\bar{E}$-points is stable under Γ. Therefore it is defined over E [B: AG 14.4]. By construction, it is the product of its intersections with the ${}^{\sigma}\mathscr{G}'$, therefore $\mathscr{M} = R_{F/E}\mathscr{M}'$, with $\mathscr{M}' \subset \mathscr{G}'$ defined over F [BT: 6.18].

7.12. *Proof of* 7.3. Let $\mathscr{L}$ be the smallest $\mathbf{Q}$-subgroup of $\mathscr{G}$ such that $\mathscr{L}(\mathbf{R}) \supset H$. By 7.11, there exists a k-subgroup $\mathscr{M}'$ of $\mathscr{G}' = \mathbf{SL}_n$ such that $\mathscr{M} = R_{k/\mathbf{Q}}\mathscr{M}' \subset \mathscr{L}$ and ${}^{\sigma}\mathscr{M}' = \mathscr{L} \cap {}^{\sigma}\mathscr{G}'$ for all $\sigma \in \Sigma$. Let Σ be the set of embeddings of k into $\bar{\mathbf{Q}}$, fix an embedding of $\bar{\mathbf{Q}}$ into $\mathbf{C}$ and view the elements of Σ as embeddings of k into $\mathbf{C}$. We may identify S_∞ to a subset of Σ consisting of the real embeddings and one representative of each pair of complex conjugate embeddings. Then ${}^{s}\mathscr{G}'(k_s)$ is our group G_s, and $\mathscr{M}(\mathbf{R})$ is the product of the groups $M_s = {}^{s}\mathscr{M}'(k_s)$, $(s \in S_\infty)$.

By assumption $H_s \subset \mathscr{L}(\mathbf{R}) \cap G_s = M_s$, $(s \in S_\infty)$, therefore $H \subset \mathscr{M}(\mathbf{R})$ and then $\mathscr{L} = \mathscr{M}$ since by assumption, $\mathscr{L}$ is the smallest $\mathbf{Q}$-subgroup whose group of real points contains H. This proves 7.3.

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