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Riesz spaces and the ultrafilter theorem, I

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Abstract. We consider various known theorems on representation of Riesz spaces (= vector lattices) and extension of positive linear maps or Riesz homomorphisms. The proofs usually given depend on the Axiom of Choice. We show that the Kakutani representation theorem for order unit spaces actually is equivalent to the Ultrafilter Theorem, and that the Kantorovič and the Lipecki-Luxemburg-Schep extension theorems are equivalent to the Hahn-Banach Theorem.

Introduction

In [6] we investigated a part of Riesz space theory which holds in the axiom system of Zermelo and Fraenkel. In the present paper we take a closer look at theorems for which the existing proofs make an appeal to the Axiom of Choice (**AC**) and for which a strong set-theoretical assumption is unavoidable. One such assumption is the Ultrafilter Theorem (**UT**), or equivalently ([11], Theorem 2.2) the Boolean Prime Ideal Theorem (**BPI**). In parts of mathematics which are close to the theory of Riesz spaces, **BPI** as a tool to study the effectiveness of extension and representation theorems is well established. Indeed, Feldman and Henriksen ([8]) have recently shown that every f-ring is a subdirect product of totally ordered rings if and only if **BPI**. Luxemburg, in [25], gave a short proof of the same result and indicated that **BPI** also implies that each Riesz space contains a prime ideal.

Our curiosity about the role of **BPI** in Riesz space theory was for a part initiated by reading page 168 of [12], where it is claimed that **BPI** suffices to prove the Kakutani Representation Theorem for order unit spaces. Following the references in [12], however, one uses Alaoglu's Theorem as well as the Krein-Milman Theorem and that combination implies **AC** ([28]). Furthermore, for the existence of maximal ideals in similar settings (commutative rings with unit ([13]), distributive lattices with unit ([3] and [28])) **AC** actually is needed. Subsequently, we were motivated by the papers [5] and [6] in which we derived extension theorems for Riesz homomorphisms using **AC** ([5]) and a constructive representation theorem for small Riesz spaces ([6]).

A typical result in the present paper is the equivalence of the statement "Every Riesz space with order unit contains a maximal ideal" with **BPI**. (This justifies the claim made in [12] and referred to above.)

Our approach is in the spirit of Luxemburg's proof of the classical Hahn-Banach Theorem ([17]) and therefore contains ideas from nonstandard analysis. However, instead of constructing ultrapowers of **R**, we need to construct ultrapowers of an arbitrary Dedekind complete space. As one of the consequences we show that the Hahn-Banach Theorem for Riesz homomorphisms ([5]) implies the Kantorovic Extension Theorem (Theorem 2.8 in [1]) and the classical Hahn-Banach Theorem effectively.

Of course, we do not use **AC** in the present paper. However, in one instance (Theorem 4.1), we use the Countable Axiom of Choice.

In a subsequent paper we will discuss some more applications of **UT** in the theory of Riesz spaces, for instance in the theory of injective Banach lattices.

Notation

All of our Riesz spaces are real. Generally, our notation is derived from the standard texts [1], [23] and [33].

Let E be a vector space and F a Riesz space and let $\phi: E \to F$ be *sublinear*, i.e. subadditive and positive-homogeneous. ϕ is called a *seminorm* if $\phi(\lambda x) = |\lambda|\phi(x)$ for all $\lambda \in \mathbf{R}$ and all $x \in E$. If in addition E is a Riesz space, a seminorm ϕ is called a *Riesz seminorm* if for all $x \in E$, $\phi(x) = \phi(|x|)$ and for all $0 \leqslant x \leqslant y$, $\phi(x) \leqslant \phi(y)$. A Riesz seminorm is an *M-seminorm* if for all $x, y \in E^+$, $\phi(x \vee y) = \phi(x) \vee \phi(y)$. A seminorm ϕ is called a *norm* if $\phi(x) = 0 \Leftrightarrow x = 0$. Similarly, we define *Riesz norm* and *M-norm*.

A Riesz subspace D of a Riesz space E is called *majorizing* if for each $x \in E^+$ there exists a $y \in D$ with $y \geqslant x$.

1. Ultrapowers and the infinitesimal kernel

1.1. Let E be a Riesz space, not necessarily Archimedean. Let $\mathscr{J}$ be the set of all Riesz ideals J of E for which the quotient Riesz space E/J is Archimedean. By the *infinitesimal kernel* of E we mean the kernel of the natural Riesz homomorphism

$$E \to \prod_{J \in \mathscr{J}} E/J.$$

As the latter product space is Archimedean, the infinitesimal kernel of E is itself an element of $\mathscr{J}$, and is, in fact, the intersection of $\mathscr{J}$. (See [23], 60.2 and 62.1.)

1.2. LEMMA. *Let E be a Riesz space with infinitesimal kernel K_E.*

(i) *Let F be a Riesz space with infinitesimal kernel K_F, and $\phi: E \to F$ either a Riesz homomorphism or a Riesz seminorm. Then $\phi(K_E) \subset K_F$.*

(ii) *Let D be a majorizing Riesz subspace of E with infinitesimal kernel K_D. Then $D \cap K_E = K_D$. In particular, $D \cap K_E = \{0\}$ if D is Archimedean.*

Proof. (i) Let P be the quotient map $F \to F/K_F$. Then $\psi := P \circ \phi$ is a Riesz homomorphism or a Riesz seminorm on E with values in an Archimedean Riesz space, and we are done if $K_E \subset \operatorname{Ker}\psi$. This will be true if $E/\operatorname{Ker}\psi$ is Archimedean. Thus, let $a, b \in E^+$ be such that $(na - b)^+ \in \operatorname{Ker}\psi$ for all $n \in \mathbf{N}$: we prove $a \in \operatorname{Ker}\psi$. Now for every n we have $na \leqslant b + (na - b)^+$; the given property of ψ implies that then $n\psi(a) \leqslant \psi(b)$. Hence, $\psi(a) = 0$.

(ii) $D/D \cap K_E$ is (isomorphic to) a Riesz subspace of E/K_E and therefore is Archimedean. Thus, $D \cap K_E \supset K_D$. For the reverse inclusion, let F be the Dedekind completion of the Archimedean Riesz space D/K_D and Q the natural map $D \to F$. As D is majorizing in E we can define a Riesz seminorm $\phi: E \to F$ by

$$\phi(x) := \inf\{Qd : d \in D^+,\ d \geqslant |x|\} \quad (x \in E).$$

Observe that $\phi(x) = Q(|x|)$ for $x \in D$. Applying (i) we obtain $K_D = \operatorname{Ker} Q = D \cap \operatorname{Ker}\phi = D \cap \phi^{-1}(K_F) \supset D \cap K_E$ and we are done. □

In the balance of this section I is a non-empty set, $\mathscr{U}$ a filter of subsets of I.

1.3. Let F be an Archimedean Riesz space. We define as follows the *ultrapower* $F_{\mathscr{U}}$ of F relative to $\mathscr{U}$.

First, let $F_1 := \{f \in F^I : f(I)$ is an order bounded subset of $F\}$ and $F_0 := \{f \in F_1 : f = 0\ \mathscr{U}$-almost everywhere$\}$. In a natural way F_1 is an Archimedean Riesz space, F_0 a Riesz ideal in F_1, and F_1/F_0 a Riesz space, possibly not Archimedean. Let K be the infinitesimal kernel of F_1/F_0. We define $F_{\mathscr{U}} := (F_1/F_0)/K$.

This *$F_{\mathscr{U}}$ is an Archimedean Riesz space.* For $x \in F$ let x_I be the constant map $I \to F$ with value x. Then $\{x_I \bmod F_0 : x \in F\}$ is a majorizing Riesz subspace of F_1, Riesz isomorphic to F. By Th. 1.2(ii) it has trivial intersection with K. Hence, if for $x \in F$ we define $x^\Delta \in F_{\mathscr{U}}$ by

$$x^\Delta := (x_I \bmod F_0) \bmod K,$$

then *$x \mapsto x^\Delta$ is a Riesz isomorphism of F onto a majorizing Riesz subspace F^Δ of $F_{\mathscr{U}}$.*

1.4. Now suppose we have two Archimedean Riesz spaces, F and G. Make F_1, F_0, $F_{\mathscr{U}}$, x^Δ (for $x \in F$), F^Δ as above, and analogously G_1, G_0, $G_{\mathscr{U}}$, x^Δ (for $x \in G$) and G^Δ.

Let $T: F \to G$ be a Riesz seminorm [or a positive linear map]. In a natural way, T induces a Riesz seminorm [a positive linear map, respectively] of F_1/F_0 into G_1/G_0, and thereupon by 1.2(i) a map $T_{\mathscr{U}}: F_{\mathscr{U}} \to G_{\mathscr{U}}$ for which the diagram

$$\begin{tikzcd} F \arrow[r, leftrightarrow] \arrow[d, "T"] & F^{\Delta} \arrow[r, phantom, "\subset"] & F_{\mathscr{U}} \arrow[d, "T_{\mathscr{U}}"] \\ G \arrow[r, leftrightarrow] & G^{\Delta} \arrow[r, phantom, "\subset"] & G_{\mathscr{U}} \end{tikzcd}$$

is commutative, i.e.,

$$T_{\mathscr{U}}(x^{\Delta}) = (Tx)^{\Delta} \quad (x \in F).$$

If T is positive linear (or a Riesz homomorphism, a Riesz seminorm, an M-seminorm), then so is $T_{\mathscr{U}}$.

1.5. In particular, let $\mathscr{U}$ be an ultrafilter. Then $\mathbf{R}_{\mathscr{U}} = \mathbf{R}^{\Delta}$. We identify $\mathbf{R}^{\Delta}$ with $\mathbf{R}$. Let again F be an Archimedean Riesz space and take $G = \mathbf{R}$. Then for every $f \in E^{\sim}$ the above produces an extension $f_{\mathscr{U}} \in (F_{\mathscr{U}})^{\sim}$:

$$\begin{tikzcd} F \arrow[r, leftrightarrow] \arrow[dr, "f"'] & F^{\Delta} \arrow[r, phantom, "\subset"] & F_{\mathscr{U}} \arrow[dl, "f_{\mathscr{U}}"] \\ & \mathbf{R} & \end{tikzcd}$$

Without the Axiom of Choice (but given an ultrafilter) we thus obtain a "simultaneous extension map" $f \mapsto f_{\mathscr{U}}$ of $F^{\sim}$ into $(F_{\mathscr{U}})^{\sim}$.

1.6. THEOREM. *In the situation sketched above ($\mathscr{U}$ being an ultrafilter) the map $f \mapsto f_{\mathscr{U}}$ is an injective Riesz homomorphism $F^{\sim} \to (F_{\mathscr{U}})^{\sim}$.*

Proof. The injectivity is plain from the diagram; from the construction one sees that the map is linear and increasing. To prove that it is a Riesz homomorphism, take $f, g \in F^{\sim +}$, $f \perp g$. Then the restriction of $f_{\mathscr{U}} \wedge g_{\mathscr{U}}$ to F^{Δ} is an element of $F^{\Delta \sim +}$ that (in $F^{\Delta \sim}$) is less than or equal to the restrictions of both $f_{\mathscr{U}}$ and $g_{\mathscr{U}}$. Consequently, $f_{\mathscr{U}} \wedge g_{\mathscr{U}}$ vanishes on F^{Δ}. But F^{Δ} majorizes $F_{\mathscr{U}}$. Therefore, $f_{\mathscr{U}} \wedge g_{\mathscr{U}} = 0$. □

2. Extension

2.1. Let E be a vector space, F an Archimedean Riesz space, ϕ a map $E \to F$. Let D be a linear subspace of E and T a linear map $D \to F$ that is *ϕ-dominated*, i.e. $Tx \leqslant \phi(x)$ for all $x \in D$. Note that we automatically have $Tx \geqslant -\phi(-x)$ for all $x \in D$.

Our purpose in this section is, roughly speaking, under suitable conditions to find an Archimedean Riesz space $\bar{F}$, containing F as a majorizing Riesz subspace, and a linear map $\bar{T}: E \to \bar{F}$ that extends T and is again ϕ-dominated.

To this end we apply the ultrapower construction described above. Let $\mathscr{X}$ be the set of all finite subsets of E. *Suppose* that for every $X \in \mathscr{X}$ we are given a linear subspace D_X of E containing X and a nonempty set $\mathscr{T}_X$ of linear maps $D_X \to F$ such that

$$T'x = Tx \text{ if } x \in X \cap D,\ T' \in \mathscr{T}_X,$$

$$T'x \leqslant \phi(x) \text{ for all } x \in X,\ T' \in \mathscr{T}_X.$$

(Observe that the foregoing takes as a starting point maps $X \mapsto D_X$ and $X \mapsto \mathscr{T}_X$. We take these steps so carefully to avoid the Axiom of Choice later on.)

Let I be the set of all pairs (X, T') where $X \in \mathscr{X}$, $T' \in \mathscr{T}_X$.

For $x \in E$ put $I_x := \{(X, T') \in I : x \in X\}$. Let $\mathscr{U}$ be a filter of subsets of I such that $I_x \in \mathscr{U}$ for every $x \in E$. (Such a filter exists.) Now form the ultrapower $F_{\mathscr{U}}$ as in the preceding section. We follow notations introduced there.

Define $\tau: E \to F^I$ by

$$\tau(x)(X, T') := \begin{cases} T'x & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases} \quad (x \in E, (X, T') \in I).$$

For all x we have $|\tau(x)(X, T')| \leqslant \phi(x) \vee \phi(-x)$ $((X, T') \in I)$, so that τ maps E into F_1. Define $\bar{T}: E \to F_{\mathscr{U}}$ by

$$\bar{T}x := (\tau(x) \bmod F_0) \bmod K \quad (x \in E).$$

If $x \in E$, then $\tau(x)(X, T') = T'x$ for all $(X, T') \in I_x$, hence for $\mathscr{U}$-almost all (X, T') in I. Therefore, if x, $y \in E$, then $\tau(x + y) = \tau(x) + \tau(y)$ $\mathscr{U}$-almost everywhere on I. Consequently, $\bar{T}(x + y) = \bar{T}x + \bar{T}y$.

In a similar way we infer:

(i) $\bar{T}$ is a linear map $E \to F_{\mathscr{U}}$.
(ii) $\bar{T}x \leqslant \phi(x)^{\Delta}$ for all $x \in E$.
(iii) $\bar{T}x = (Tx)^{\Delta}$ for all $x \in D$.

In a diagram, we have

$$\begin{tikzcd} & D \arrow[dl, "T"'] \arrow[dr, "\bar{T}"] & \subset E \\ F \arrow[r, leftrightarrow] & F^{\Delta} \arrow[r, phantom, "\subset"] & F_{\mathscr{U}} \end{tikzcd}$$

In our applications E is a Riesz space and the elements of the sets $\mathscr{T}_x$ have further properties. It is clear how such properties will be inherited by $\bar{T}$. E.g., if all spaces D_X are Riesz spaces and every $\mathscr{T}_X$ consists of Riesz homomorphisms, then $\bar{T}$ is a Riesz homomorphism.

We use this construction to prove:

2.2. THEOREM. *Let F be a Dedekind complete Riesz space.*

(i) *Let D be a linear subspace of a vector space E, ϕ a sublinear map $E \to F$ and T a ϕ-dominated linear map $D \to F$. Then there exist a set I, a filter $\mathscr{U}$ in I and a linear map $\bar{T}: E \to F_{\mathscr{U}}$ such that*

$$\bar{T}x = (Tx)^{\Delta} \quad (x \in D),$$

$$\bar{T}x \leqslant \phi(x)^{\Delta} \quad (x \in E).$$

(See the diagram above.)

(ii) *Let, in addition, E be a Riesz space, D a Riesz subspace of E, ϕ an M-seminorm and $T: D \to F$ a Riesz homomorphism. Then I, $\mathscr{U}$ and T can be arranged such that $\bar{T}$ is a Riesz homomorphism.*

Proof. Let $\mathscr{X}$ be the set of all finite subsets of E. For $X \in \mathscr{X}$ let D_X be the linear [for (ii): Riesz] subspace of E generated by X, and $\mathscr{T}_X$ the set of all ϕ-dominated linear maps [for (ii): Riesz homomorphisms] $D_X \to F$ that coincide with T on $X \cap D$. All we have left to prove is that each $\mathscr{T}_X$ is nonempty.

In the general situation (i) this is easy to do by following the usual proof of the Hahn-Banach Theorem: Let $\{a_1, \ldots, a_N\} = X \in \mathscr{X}$. Let $D_0 := D_X \cap D$ and $D_n := D_0 + \mathbf{R}a_1 + \cdots + \mathbf{R}a_n$ $(n = 1, \ldots, N)$. Define inductively linear maps $T_n: D_n \to F$ by

$$T_0 x := Tx \quad (x \in D_0),$$

$$T_n x := T_{n-1} x \quad (x \in D_{n-1}: n = 1, \ldots, N),$$

$$T_n a_n := \inf\{T_{n-1}x + \phi(a_n - x): x \in D_{n-1}\} \quad (n = 1, \ldots, N).$$

Then $T_N \in \mathscr{T}_X$.

The proof for situation (ii), although based on the same principle, is much harder; we will treat it in a separate lemma.

The complications are partly due to the delicate behavior of Riesz homomorphisms, partly to the simple fact that the codimension of $D_X \cap D$ in D_X may be infinite. We solve the latter problem by a separability argument. This, in turn, enables us to lighten the condition on F: Without making the proof noticeably more difficult we may substitute σ-Dedekind completeness for full Dedekind completeness.

2.3. LEMMA. *Let F be a σ-Dedekind complete Riesz space, E a finitely generated Archimedean Riesz space, D a finitely generated Riesz subspace of E. Let $\phi: E \to F$ be a Riesz seminorm [an M-seminorm] and $T: D \to F$ a ϕ-dominated linear map [Riesz homomorphism]. Then T extends to a ϕ-dominated linear map [Riesz homomorphism] $E \to F$.*

Proof. (I) Let e be an order unit for E. This e determines an M-norm $\| \ \|_e: E \to \mathbf{R}$ by

$$\|x\|_e := \inf\{\lambda \in \mathbf{R}^+ : |x| \leqslant \lambda e\}.$$

If $u := \phi(e)$, then all values of T and ϕ lie in the Riesz ideal of F generated by u so we may as well assume that u is an order unit for F. Define an M-norm $\| \ \|_u: F \to \mathbf{R}$ analogous to $\| \ \|_e$. We have

$$\|\phi(x)\|_u \leqslant \|x\|_e \quad (x \in E), \tag{1}$$

$$\|Tx\|_u \leqslant \|x\|_e \quad (x \in D). \tag{2}$$

Applying 2.5 of [6] we introduce a multiplication $*$ in F, turning F into an f-algebra with u as a neutral element. Note that, if $\varepsilon \in (0, \infty)$ and $z \in F^+$, then $\varepsilon u + z$ is invertible in F. ([6], 3.7(v); or [27], 11.1).

D is a finitely generated Riesz space. Let D_0 be a countable $\mathbf{Q}$-linear subspace of D that is also a sublattice of D and contains a generating subset of D. It follows from Lemma 1.1(i) of [6] that $\| \ \|_e$-closure of D_0 is a Riesz space, hence contains D.

(II) Now first we replace ϕ by a smaller seminorm ψ in the following manner. Note that

$$(\varepsilon u + \phi(x))^{-1} * Tx \leqslant (\varepsilon u + \phi(x))^{-1} * (\varepsilon u + \phi(x)) = u$$

for all $\varepsilon > 0$ and $x \in D^+$. By σ-Dedekind completeness we can define an element s of F^+ by

$$s := \sup\{(\varepsilon u + \phi(x))^{-1} * Tx : x \in D_0^+;\ \varepsilon \in \mathbf{Q} \cap (0, \infty)\}; \tag{3}$$

then $s \leqslant u$. Setting

$$\psi(x) := \phi(x) * s \quad (x \in E)$$

we obtain a seminorm $\psi: E \to F$ with $\psi(x) \leqslant \phi(x)$ $(x \in E)$. For $x \in D_0^+$ and $\varepsilon \in \mathbf{Q} \cap (0, \infty)$ we have $\varepsilon s + \psi(x) = (\varepsilon u + \phi(x)) * s \geqslant Tx$; hence, $\psi(x) \geqslant Tx$ for $x \in D_0^+$ and even, by $\| \ \|_e$-continuity, for all $x \in D^+$. Thus, T is ψ-dominated.

(III) Our next step is to extend T to a ψ-dominated linear map $\bar{T}: E \to F$. Let $E_0 = \{a_1, a_2, \dots\}$ be a countable $\mathbf{Q}$-linear subspace of E that is a sublattice, contains e and generates E as a Riesz space. Again by 1.1(i) of [6], E_0 is $\| \ \|_e$-dense in E. For $n \in \mathbf{N}$ put

$$D_n := D_0 + \mathbf{Q}a_1 + \mathbf{Q}a_2 + \cdots + \mathbf{Q}a_n.$$

Inductively, for all n we define a ψ-dominated $\mathbf{Q}$-linear map $T_n: D_n \to F$ by

$$T_0 x := Tx \text{ for all } x \in D_0,$$

$$T_n x := T_{n-1} x \text{ for all } x \in D_{n-1},$$

$$T_n a_n := \inf\{T_{n-1}x + \psi(a_n - x) : x \in D_{n-1}\}.$$

These maps T_n together form a ψ-dominated $\mathbf{Q}$-linear $T_\infty: E_0 \to \mathbf{R}$ by

$$T_\infty x := T_n x \text{ if } n \in \mathbf{N},\ x \in D_n$$

If $x \in E_0$, then $|T_\infty x| \leqslant \psi(x) \leqslant \|x\|_e \psi(e)$, so T_∞ extends uniquely to a map $\bar{T}: E \to F$ with $|\bar{T}x - \bar{T}y| \leqslant \|x - y\|_e \psi(e)$ $(x, y \in E)$. This $\bar{T}$ automatically is $\mathbf{R}$-linear and ψ-dominated, hence ϕ-dominated.

This finishes the proof of the part of the lemma that deals with a Riesz seminorm ϕ and a linear map T.

(IV) Now assume that ϕ is an M-seminorm and T is a Riesz homomorphism; we prove that $\bar{T}$ is a Riesz homomorphism, too. Basically, our argument will be that $\bar{T}$ is a (very special) extreme point in the set of all ψ-dominated extensions of T.

Observe that ψ is an M-seminorm. Let A be the smallest $\mathbf{Q}$-linear subspace of F that is also a sublattice, contains $\bar{T}(E_0) \cup \phi(E_0) \cup \{s\}$ and is closed for the multiplication $*$. In particular, A contains the order unit u of F. Let F_1 be the $\| \ \|_u$-closure of A in F; then F_1 is a Riesz subspace of F. ([6], 1.1(i).) By (1), $\phi(E) \subset F_1$ and as $|\bar{T}x| \leqslant \|x\|_e \psi(e)$ $(x \in E)$, also $\bar{T}(E) \subset F_1$. Moreover, F_1 is an f-subalgebra of F ([7]), so $\psi(E) \subset F_1$.

A is countable. Hence, in the terminology of [6], F_1 is a slender Riesz space. It has u as an order unit. It follows from Cor. 2.3 in [6] that the points of F_1 are separated by the Riesz homomorphisms $\omega: F_1 \to \mathbf{R}$ that satisfy

$$\omega(s) = \sup\{\omega((\varepsilon u + \phi(x))^{-1} * Tx) : x \in D_0^+;\ \varepsilon \in \mathbf{Q} \cap (0, \infty)\}, \tag{4}$$

$$\omega(\bar{T}a_n) = \inf\{\omega(\bar{T}x + \psi(a_n - x)) : x \in D_{n-1}\} \quad (n \in \mathbf{N}), \tag{5}$$

$$\omega(u) = 1. \tag{6}$$

Hence, if for every such ω the function $\omega \circ \bar{T}: E \to \mathbf{R}$ is a Riesz homomorphism, then $\bar{T}$ will be a Riesz homomorphism $E \to F_1$.

Thus, let ω be a Riesz homomorphism $F_1 \to \mathbf{R}$ for which (4), (5) and (6) are true. Put $\tau := \omega \circ T$, $\bar{\phi} := \omega \circ \phi$, $\bar{\psi} := \omega \circ \psi$. As $\omega(u) = 1$, ω is multiplicative (14.5 in [27]). We see that

$\bar{\phi}$ and $\bar{\psi}$ are M-seminorms $E \to \mathbf{R}$,

$\bar{\psi} = \omega(s)\bar{\phi}$,

τ is $\bar{\psi}$-dominated and $\tau \in E^{\sim +}$.

We wish to prove that τ is a Riesz homomorphism. If $\omega(s) = 0$ that is trivial because then $\bar{\psi} = 0$. We assume $\omega(s) \neq 0$. From (4) we obtain

$$\begin{aligned}\omega(s) &= \sup\left\{\frac{\tau(x)}{\varepsilon + \bar{\phi}(x)} : x \in D_0^+,\ \varepsilon \in \mathbf{Q} \cap (0, \infty)\right\} \\ &= \sup\left\{\frac{\tau(x)}{\bar{\phi}(x)} : x \in D_0^+,\ \bar{\phi}(x) \neq 0\right\}\end{aligned}$$

and thereby

$$\begin{aligned}1 &= \sup\left\{\frac{\tau(x)}{\omega(s)\bar{\phi}(x)} : x \in D_0^+,\ \bar{\phi}(x) \neq 0\right\} \\ &= \sup\{\tau(x) : x \in D_0^+,\ \bar{\psi}(x) = 1\}. \qquad (7)\end{aligned}$$

Furthermore, (5) yields:

$$\tau(a_n) = \inf\{\bar{\tau}(x) + \bar{\psi}(a_n - x) : x \in D_{n-1}\} \quad (n \in \mathbf{N}). \qquad (8)$$

Let $\sigma \in E^{\sim +}$, $\sigma(x) \leqslant \tau(x)$ $(x \in E^+)$. According to exercise 5, page 105 of [1], τ will be a Riesz homomorphism if and only if it follows that σ is a scalar multiple of τ. By the same token, $\tau|_D$ being a Riesz homomorphism, we know that there exists an α in $[0, 1]$ for which

$$\sigma(x) = \alpha\tau(x) \quad (x \in D). \qquad (9)$$

Defining

$\lambda := \sup\{\sigma(x) : x \in E^+,\ \bar{\psi}(x) = 1\}$

$\mu := \sup\{(\bar{\tau} - \sigma)(y) : y \in E^+,\ \bar{\psi}(y) = 1\}$,

we see from (7) and (9) that $\alpha \leqslant \lambda$ and $1 - \alpha \leqslant \mu$. On the other hand,

$$\begin{aligned}\lambda + \mu &= \sup\{\sigma(x) + (\tau - \sigma)(y) : x, y \in E^+, \bar{\psi}(x) = \bar{\psi}(y) = 1\} \\ &\leqslant \sup\{\tau(x \vee y) : x, y \in E^+, \bar{\psi}(x) = \bar{\psi}(y) = 1\} \leqslant 1\end{aligned}$$

because $\bar{\psi}$ is an M-seminorm. Thus, $\lambda = \alpha$ and $\mu = 1 - \alpha$. Therefore, $\sigma \leqslant \alpha\bar{\psi}$ and $\tau - \sigma \leqslant (1 - \alpha)\bar{\psi}$ relative to the natural ordering of the space of all functions $E \to \mathbf{R}$. In this sense we find that

$$\begin{aligned}\tau + (\sigma - \alpha\tau) &= (1 - \alpha)\tau + \sigma \leqslant (1 - \alpha)\bar{\psi} + \alpha\bar{\psi} = \bar{\psi}, \\ \tau - (\sigma - \alpha\tau) &= (\tau - \sigma) + \alpha\tau \leqslant (1 - \alpha)\bar{\psi} + \alpha\bar{\psi} = \bar{\psi},\end{aligned}$$

so that both $\tau + (\sigma - \alpha\tau)$ and $\tau - (\sigma - \alpha\tau)$ are $\bar{\psi}$-dominated. However, it is not difficult to derive from (8) that τ is an extreme point in the set of all $\bar{\psi}$-dominated elements of $E^{\sim +}$. Thus, $\sigma = \alpha\tau$. □□

2.4. Comments on the proof of Theorem 2.2

(1) Once the nonemptiness of the sets $\mathscr{T}_X$ is known, the proof is an application of 2.1. There, $\mathscr{U}$ is any filter in I that contains $\{I_x : x \in E\}$. Consequently, *if the Ultrafilter Theorem is assumed, in Theorem 2.2. one may replace "filter" by "ultrafilter"*. This observation will be of use later on.

(2) It follows from Lemma 2.3 that for 2.2(ii) σ-Dedekind completeness of F is enough. It is not hard to see that the same is true for 2.2(i) and that even the σ-interpolation property will do, but such refinements of 2.2 are pointless. (It may, however, be worth observing that the σ-interpolation property is not strong enough to replace σ-Dedekind completeness in Lemma 2.3.)

3. Some applications

3.1. THEOREM. *Let F be an Archimedean Riesz space. Then* $(\alpha) \Leftrightarrow (\beta) \Leftrightarrow (\gamma)$ *and* $(\alpha') \Leftrightarrow (\beta')$.

(α) *If D is a Riesz subspace of an [Archimedean] Riesz space E and $\phi : E \to F$ is a Riesz seminorm, then every ϕ-dominated (hence positive) linear map $D \to F$ extends to a ϕ-dominated linear map $E \to F$.*

(β) *If F is a majorizing Riesz subspace of an [Archimedean] Riesz space G, then there is an S in $L^+(G, F)$ with $S = I$ on F.*

(γ) *If D is a linear subspace of a vector space E and $\phi : D \to F$ is sublinear, then*

every ϕ-dominated linear map $D \to F$ extends to a ϕ-dominated linear map $E \to F$.

(α') *If D is a Riesz subspace of an [Archimedean] Riesz space E and $\phi: E \to F$ is an M-seminorm, then every ϕ-dominated Riesz homomorphism $D \to F$ extends to a ϕ-dominated Riesz homomorphism $E \to F$.*

(β') *If F is a majorizing Riesz subspace of an [Archimedean] Riesz space G, then there is a Riesz homomorphism $S: G \to F$ with $S = I$ on F.*

Furthermore, each of these conditions implies Dedekind completeness of F.

Before we get into the proof of this theorem something must be said about the interpretation of the clause "[Archimedean]". The item (α), for instance, is to be taken as shorthand for *two* statements, (α_{arch}) and (α_{gen}), say, in which E runs through the class of all Archimedean Riesz spaces and through the class of *all* Riesz spaces, respectively. Trivially, (α_{gen}) implies (α_{arch}), but the converse is not hard to see either: Just factor out the infinitesimal kernel of E.

(β), (α') and (β') are to be treated similarly. Thus, the formula $(\alpha) \Leftrightarrow (\beta) \Leftrightarrow (\gamma)$ really indicates the equivalence of five statements.

Proof of Theorem 3.1. We first prove the final statement of the theorem. To see that (β) and (β') imply Dedekind completeness, one simply lets G be the Dedekind completion of F. Further, (γ) trivially implies (α). Thus, we are done if (α) and (α') each imply that all principal ideas of F are Dedekind complete. Assume (α) or (α'). Let $a \in F^+$. Let D be the principal ideal of F generated by a and E the Dedekind completion of D. We have an M-seminorm $\| \ \|_a: E \to \mathbf{R}$ defined by

$$\|x\|_a := \inf\{\lambda \in \mathbf{R}^+ : |x| \leqslant \lambda a\} \quad (x \in E).$$

$x \mapsto \|x\|_a a$ $(x \in E)$ is an M-seminorm $E \to F$. It follows from (α) or (α') that the identity map $D \to F$ extends to an element of $L^+(E, F)$, whose values necessarily lie in D. Then D is Dedekind complete. Thus, all principal ideals of F are Dedekind complete. Then so is F itself.

Next, we prove $(\alpha) \Rightarrow (\beta)$ and $(\beta) \Rightarrow (\gamma)$. (As the implication $(\gamma) \Rightarrow (\alpha)$ is clear, we then have $(\alpha) \Leftrightarrow (\beta) \Leftrightarrow (\gamma)$. The proofs of $(\alpha') \Rightarrow (\beta')$ and $(\beta') \Rightarrow (\alpha')$ closely resemble the ones of $(\alpha) \Rightarrow (\beta)$ and $(\beta) \Rightarrow (\gamma)$.)

$(\alpha) \Rightarrow (\beta)$. F being Dedekind complete, we can define a Riesz seminorm $\phi: G \to F$ by

$$\phi(x) := \inf\{y \in F^+ : y \geqslant |x|\} \quad (x \in G).$$

Now take $D := F$, $E := G$ and extend the identity map $D \to F$.

$(\beta) \Rightarrow (\gamma)$. Let D, E, ϕ be as in (γ) and let T be a ϕ-dominated linear map $D \to F$.

In the language of 2.1 and 2.2, T induces a linear $\bar{T}: E \to F_{\mathscr{U}}$ with $\bar{T}x = (Tx)^{\Delta}$ for all $x \in D$ and $\bar{T}x \leqslant \phi(x)^{\Delta}$ for all $x \in E$. Now (β) yields an $S \in L^{+}(F_{\mathscr{U}}, F)$ such that $S(y^{\Delta}) = y$ ($y \in F$). Then $S\bar{T}$ is the desired extension of T. □

Comments. (i) Obviously, (β') implies (β). Therefore, we have the nontrivial implication (α') $\Rightarrow$ (α). (It would be nice to have a direct proof for this, at least for the case $F = \mathbf{R}$.)

(ii) In general, (α) and (α') are not equivalent. For $F = \mathbf{R}$, (γ) is the Hahn-Banach Theorem, but we shall see in section 4 that (α') and (β') are equivalent to the Prime Ideal Theorem. It follows that (for $F = \mathbf{R}$) (α') is strictly stronger than (α).

(iii) Assuming the Axiom of Choice, all five of the conditions of the theorem are equivalent to Dedekind completeness of F. Luxemburg proved **AC** $\Rightarrow$ (β') (see Cor. 1.9, page 215 of [18]), Kantorovic showed **AC** $\Rightarrow$ (γ) (2.8 in [1]), Buskes and Van Rooij showed **AC** $\Rightarrow$ (α') ([5]) and for **AC** $\Rightarrow$ (α) we refer to 2.3 in [1].

It follows that the Kantorovic Extension Theorem (2.8 in [1]) is equivalent to the Hahn-Banach Theorem for (positive) operators with values in a Dedekind complete Riesz space and that the Lipecki-Luxemburg-Schep Extension Theorem ([16], [22]) is equivalent to the Hahn-Banach Theorem for Riesz homomorphisms [5]. A similar result in the setting of Boolean Algebras was obtained in [2] where it is shown that Sikorski's Extension Theorem is effectively equivalent to Monteiro's Hahn-Banach-like theorem ([26]).

We will need the following theorem in the sequel.

3.2. THEOREM. *Assume the Ultrafilter Theorem. Let D be a Riesz subspace of a Riesz space E, $\phi: E \to \mathbf{R}$ an M-seminorm and $T: D \to \mathbf{R}$ a ϕ-dominated Riesz homomorphism. Then T extends to a ϕ-dominated Riesz homomorphism $E \to \mathbf{R}$.*

Proof. Since every filter is contained in an ultrafilter, we can take $\mathscr{U}$ of 2.2 to be an ultrafilter as remarked in 2.4. Furthermore, $\mathbf{R}_{\mathscr{U}} = \mathbf{R}^{\Delta}$. □

3.3. In Boolean algebras the notions of maximal ideal and prime ideal coincide. Therefore, the (Boolean) Prime Ideal Theorem is equivalent to the statement:

Every Boolean algebra contains a maximal ideal.

In distributive lattices there may be prime ideals which are not maximal. In fact, the statements "*Every distributive lattice with* 1 *contains a prime ideal*" and "*Every distributive lattice with* 1 *contains a maximal ideal*" are not equivalent. The latter implies the Axiom of Choice while the former does not. For other contexts consult [4], [8], [10] and [25].

In Riesz spaces, as in distributive lattices, there may be prime ideals which are not maximal. In 3.5 we answer some of the naturally arising questions.

3.4. It is known (see 2.21, page 261 of [15], Theorems 4.1 and 4.2 of [17] and page 282 of [28]) that *the following are equivalent*:

(i) *The Ultrafilter Theorem.*
(ii) *The Prime Ideal Theorem.*
(iii) *The Tychonoff Theorem for compact Hausdorff spaces.*
(iv) *The Alaoglu Theorem.*
(v) *Every Boolean algebra admits a* $\{0, 1\}$*-valued homomorphism.*

3.5. THEOREM. *Each of the following statements is equivalent to each of the above statements* (i)–(v).

(α) *Let D be a Riesz subspace of an [Archimedean] Riesz space E, $\phi: E \to \mathbf{R}$ an M-seminorm and $T: D \to \mathbf{R}$ a ϕ-dominated Riesz homomorphism. Then T can be extended to a ϕ-dominated Riesz homomorphism $E \to \mathbf{R}$.*
(β) *Let E be an [Archimedean] Riesz space with order unit e. Then there exists a Riesz homomorphism $T: E \to \mathbf{R}$ with $T(e) = 1$.*
(γ) *Let E be an [Archimedean] Riesz space with order unit. Then E contains a maximal Riesz ideal.*
(δ) *Let E be an [Archimedean] Riesz space with order unit. Then E contains a proper prime Riesz ideal.*
(ε) *Let E be a nonzero (Archimedean] Riesz space. Then E contains a proper prime Riesz ideal.*

The clause "[Archimedean]" is subject to the same convention here as it was in Theorem 3.1. Further, an "order unit" in E is, by definition, positive and unequal to 0. Thus, the spaces E in (β), (γ) and (δ) are automatically nonzero.

Proof. By "(i)" we denote (i) of 3.4, the Ultrafilter Theorem. We will prove (i) $\Rightarrow$ (α) $\Rightarrow$ (β) $\Rightarrow$ (γ) $\Rightarrow$ (δ) $\Rightarrow$ (ε) $\Rightarrow$ (i).

(i) $\Rightarrow$ (α). This is the content of Theorem 3.2.

(α) $\Rightarrow$ (β). Define $\phi: E \to \mathbf{R}$ by $\phi(x) := \inf\{\lambda : |x| \leqslant \lambda e\}$. Take $D = \mathbf{R}e$ and $Te = 1$. Then apply (α).

(β) $\Rightarrow$ (γ). The kernel of a nonzero Riesz homomorphism $E \to \mathbf{R}$ is a maximal ideal.

(γ) $\Rightarrow$ (δ). Every maximal ideal is prime.

(δ) $\Rightarrow$ (ε). Take $e \in E$, $e > 0$. By (δ), the principal ideal E_e of E generated by e has a maximal ideal J. This J is a prime Riesz ideal in E_e, and $\{x \in E : |x| \wedge \lambda e \in J$ for all $\lambda \in (0, \infty)\}$ is a proper prime ideal in E.

(ε) $\Rightarrow$ (i). Let X be a set and $\mathscr{F}$ a proper filter of subsets of X. We extend $\mathscr{F}$ to an ultrafilter as follows. $l^\infty(X)$ is the Riesz space of all bounded functions $X \to \mathbf{R}$, N the closed Riesz ideal of $l^\infty(X)$ consisting of all elements f that have the property $\{x \in X : |f(x)| \leqslant \varepsilon\} \in \mathscr{F}$ for all $\varepsilon > 0$. Let E be the quotient space $l^\infty(X)/N$ and $P: l^\infty(X) \to E$ the quotient map. By (ε), E has a proper prime Riesz

ideal J. Then $P^{-1}(J)$ is a prime Riesz ideal in $l^\infty(X)$ that contains N. Set

$$\mathscr{U} := \{Y : Y \subset X,\ \mathbf{1}_{X\setminus Y} \in P^{-1}(J)\}.$$

Then $\mathscr{U}$ is a filter of subsets of X that contains $\mathscr{F}$. If $Y, Z \subset X$ and $Y \cup Z \in \mathscr{U}$, then $Y \in \mathscr{U}$ or $Z \in \mathscr{U}$ (since $P^{-1}(J)$ is prime). Hence, $\mathscr{U}$ is actually an ultrafilter. □

3.6. Some remarks about Theorem 3.5 are in order. Schmidt in [30] proved that Alaoglu's Theorem (see 3.4) together with Krein-Milman Theorem implies (α). The same combination is also known to imply the Axiom of Choice ([28]).

Similar statements for Riesz homomorphisms on majorizing Riesz subspaces have drawn the attention of many authors ([5], [9], [16], [22]), all of whom use the Axiom of Choice.

4. Representation Theorems

We are now going to study representation theorems that are connected with the existence of maximal ideals in certain Riesz spaces. Similarly, Luxemburg in [24] has considered representation theorems requiring the existence of prime Riesz ideals and he refers to an unpublished manuscript by Fremlin in which the same results are obtained.

Assume the Ultrafilter Theorem. Let E be an Archimedean Riesz space with order unit. It follows from Theorem 3.5 that every proper ideal of E is contained in a maximal ideal. Let $\mathscr{M}(E)$ be the set of all maximal ideals in E with the hull-kernel topology. (Page 91 in [13].) $\mathscr{M}(E)$ is a compact Hausdorff space. (Compare the arguments on pages 94, 95 and Lemma 13.6 in [13].) In analogy to the reasoning in 13.10 of [13] we arrive at a Riesz isomorphism $f \mapsto \hat{f}$ from E onto a Riesz subspace $\hat{E}$ of $C(\mathscr{M}(E))$. Using the **AC**-free Stone-Weierstrass (1.5 in [7]) we find that $\hat{E}$ is order dense in $C(\mathscr{M}(E))$. If, in addition, E is uniformly complete, it follows *) that E is Riesz isomorphic to $C(\mathscr{M}(E))$. All in all we have:

Assume **BPI** *and* **CAC**. *If E is a uniformly complete Archimedean Riesz space with order unit, then E is Riesz isomorphic to $C(\mathscr{M}(E))$.*

We arrive at the following theorem.

THEOREM 4.1. *Assume* **CAC**. *The following are equivalent.*

(i) *The Boolean Prime Ideal Theorem.*
(ii) *The Stone Representation Theorem.*

*It is convenient, but not necessary (see [6]), to use here the Countable Axiom of Choice, **CAC**.

(iii) *The Kakutani Representation Theorem.*
(iv) *The Gelfand Representation Theorem for Gelfand-algebras.*

Indeed, the equivalence of (i) and (ii) is known, (i) $\Rightarrow$ (iii) is the content of the above arguments and (iii) $\Rightarrow$ (i) follows from Theorem 3.5 and **CAC**. For (i) $\Leftrightarrow$ (iv), see [4]. (For other contexts the reader may want to consult [8] and [25].)

□

The Gelfand-Naimark-Segal Theorem for representation of C^*-algebras can be seen to be equivalent to each of the statements in Theorem 4.1, using similar methods.

Several other representation theorems fit in our framework. We discuss two.

The Prime Ideal Theorem implies the Maeda-Ogasawara Representation Theorem for an Archimedean Riesz space with a weak unit:

4.2. THEOREM. *Assume* **BPI**. *Let E be an Archimedean Riesz space with a weak order unit e. Let X be the Stone space of the Boolean algebra of all bands of E. Then there exists a Riesz isomorphism S of E onto an order dense Riesz subspace of $C^\infty(X)$ such that $S(e)$ is the constant function* 1.

Proof. Copy the proof of the general Maeda-Orgasawara Theorem 15.5 in [13], choosing $\{e\}$ for the maximal disjoint subset of $L\backslash\{0\}$ occurring there.

□

To obtain the complete Maeda-Ogasawara Theorem we need a maximal disjoint subset of $E\backslash\{0\}$. We do not know if the existence of such a set can be derived from **BPI**.

From **BPI** we do get a nice representation theorem for f-algebras. (See [27] for the relevant terminology.)

4.3. THEOREM. *Assume* **BPI**. *Let E be a semisimple Archimedean f-algebra. Let X be the Stone space of the Boolean algebra of all bands of E. Then there exists an f-algebra isomorphism of E onto an order dense f-subalgebra of $C^\infty(X)$.**

Proof. For $a \in E$ let $M_a \in \mathrm{Orth}(E)$ be the map $x \mapsto xa$. Since E is semisimple, $a \mapsto M_a$ is an injective and multiplicative Riesz homomorphism of E into $\mathrm{Orth}(E)$. ([27], 12.1.) The identity map of E is a (weak order) unit in $\mathrm{Orth}(E)$. ([27], 9.4.) By Theorem 4.2, for some extremally disconnected compact Hausdorff space X we have a Riesz isomorphism S of $\mathrm{Orth}(E)$ onto an order dense Riesz subspace of $C^\infty(X)$. This S is multiplicative ([27], 14.5.) Then $T: a \mapsto S(M_a)$ is an f-algebra isomorphism of E onto an f-subalgebra of $C^\infty(X)$. If we can prove

*(Added in proof.) As the referee pointed out to us, in the setting of normal Archimedean lattice rings and using the Axion of Choice, this theorem appears in [4a].

that $T(E)$ is order dense in the range of S, then it will also be order dense in $C^\infty(X)$, and X will be the announced Stone space.

Thus, take $A \in \mathrm{Orth}(E)$, $A > 0$ and set $f := S(A)$; we make an $a \in E$ with $0 < T(a) \leqslant f$. Observe that for all $x \in E$ we have $M_{Ax} = AM_x$ ([27], 12.1(ii)) and, consequently,

$$T(Ax) = fT(x). \tag{$*$}$$

Choose a b in E^+ with $g := Ab > 0$ and an ε in $(0, \infty)$ such that $\varepsilon g < 1$ on a nonempty open subset of X. Put $a := \varepsilon A(b - \varepsilon b^2)^+$. Then by ($*$),

$$T(a) = \varepsilon f(g - \varepsilon g^2)^+ = f\varepsilon g(1 - \varepsilon g)^+,$$

so that $0 < T(a) \leqslant f$. □

By the last part of this proof it follows from **BPI** that for any semisimple Archimedean f-algebra E, the space $\{M_a : a \in E\}$ is order dense in $\mathrm{Orth}(E)$, a fact for which we have not found a reference. The same result can be proved without **BPI** (and without semisimplicity) by the methods of [6].

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