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A fine limit property of functions superharmonic outside a manifold

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Abstract. Let \((X', X'')\) denote a typical point of \(\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}\), where \(n > 3\) and \(1 \leq k \leq n - 2\). Also, let \(E = \{|X''| < f(|X'|)\}\), where \(f : [0, \infty) \rightarrow [0, \infty)\) is increasing. A necessary and sufficient condition is given for \(E\) to be thin at the origin. This, in turn, is used to study the behaviour of functions \(u\) which are superharmonic on the complement of a \(C^2\) \(k\)-dimensional manifold \(S\). In particular it is shown that, if \(u^- \) does not grow too quickly near \(S\), then \(|X - Y|^{n-2} u(X)\) has a finite non-negative fine limit as \(X \rightarrow Y\), for any \(Y \in S\).

1. Main results

A set \(E\) in Euclidean space \(\mathbb{R}^n\) is said to be thin at a point \(Y\) if there is a superharmonic function \(u\) on a neighbourhood of \(Y\) such that

\[
\liminf_{X \rightarrow Y, X \in E \setminus \{Y\}} u(X) > u(Y).
\]

The classical criterion of Wiener [7, Theorem 10.21] characterizes thinness at \(Y\) in terms of the convergence of a series involving the Newtonian (outer) capacity of the sets \(E \cap \{2^{-j-1} \leq |X - Y| \leq 2^{-j}\}\), where \(j \in \mathbb{N}\). (Here \(|X|\) denotes the Euclidean norm of \(X\).) The notion of thinness is important in the study of the Dirichlet problem: a boundary point \(Y\) is regular for the Dirichlet problem on (an open set) \(\Omega\) if and only if \(\mathbb{R}^n \setminus \Omega\) is not thin at \(Y\). In this context a classical example of a set which is thin at the origin in \(\mathbb{R}^3\) is the “Lebesgue spine” defined by \(\{(x, y, z) : x > 0 \text{ and } y^2 + z^2 \leq e^{-c/x}\}\), where \(c > 0\) (see [7, p. 175]). Our first result gives a simple geometric characterization of spine-like sets which are thin at the origin \(O\). Let \(X = (X', X'')\) denote a typical point of \(\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}\), where \(n \geq 3\) and \(k \in \{1, 2, \ldots, n-2\}\).

**THEOREM 1.** Let \(E = \{X : |X'| < f(|X'|)\}\), where \(f : [0, \infty) \rightarrow [0, \infty)\) is increasing. Then \(E\) is thin at \(O\) if and only if

\[
\int_0^1 t^{-1} \left\{ \frac{f(t)}{t} \right\}^{n-2-k} \, dt < \infty \quad (k = 1, \ldots, n-3),
\]

\[
\int_0^1 \frac{dt}{t \{1 + \log^+(t/f(t))\}} < \infty \quad (k = n-2).
\]

1 \(\cdot\) 1 \(\cdot\)
The axially symmetric case \( (k=1) \) of Theorem 1 has been given by several authors under the stronger hypothesis that \( f(t)/t \) is increasing. In this form it appears in the recent book by Hayman [6, Theorem 7.15], where it is attributed to Câmera [3]. However, it can also be found in Armitage [1] and Port and Stone [8, Chap. 3, Prop. 3.5]. The case \( k=n-3 \) was recently established by Burdzy [2, Theorem 2.4] using probabilistic methods. The case \( k=n-1 \) does not appear in Theorem 1 because a set of the form \( \{(X', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < f(|X'|)\} \) is thin at \( O \) if and only if the (increasing) function \( f \) is valued 0 on \([0, \varepsilon)\) for some \( \varepsilon > 0 \). (Hyperplanes are non-thin at each constituent point.) In Section 2 we deduce Theorem 1 from Wiener's criterion and estimates of the capacity of certain ellipsoids given in Hayman [6].

The fine topology on \( \mathbb{R}^n \) is the coarsest topology for which all superharmonic functions are continuous. Thus a superharmonic function \( u \) on an open set has a fine limit at every interior point. It also has the following property, which we shall label by \((P)\): \( |X - Y|^{n-2} u(X) \) has a finite non-negative fine limit as \( X \rightarrow Y \), for every interior point \( Y \). (In fact, this limit is equal to \( \mu(\{Y\}) \), where \( \mu \) is the Riesz measure associated with \( u \): see [5, 1.XI.4].) The connection between thin sets and the fine topology is given by the fact that a set \( E \) in \( \mathbb{R}^n \) is thin at a point \( Y \) if and only if \( Y \) is not a fine limit point of \( E \).

We will use Theorem 1 to establish a fine limit property of superharmonic functions defined on the complement of a \( k \)-dimensional manifold. Let \( E \) be a relatively closed polar subset of \( B(1) \), where \( B(X, r) = \{Y : |Y - X| < r\} \) and \( B(r) = B(O, r) \). If \( u \) is a positive superharmonic function on \( B(1) \setminus E \), then \( u \) has a positive superharmonic extension to \( B(1) \) (see [7, Theorem 7.7]) and so property \((P)\) holds. The positivity requirement on \( u \) can be relaxed here: if \( u \) is superharmonic on \( B(1) \setminus E \) and there is a negative subharmonic function \( s \) on \( B(1) \setminus E \) such that \( u \geq s \) there, then \( u \) can be represented (outside a polar set) as the difference of two positive superharmonic functions on \( B(1) \) and \((P)\) continues to hold. Now suppose that \( E \) takes the form \( \{(X', O^n) : X' \in \mathbb{R}^k\} \). If we write \( \bar{u} = \max\{0, -u\} \), then the above reasoning shows that any superharmonic function \( u \) on \( B(1) \setminus E \) which satisfies

\[
\bar{u}^-(X) \leq |X''|^{k+2-n} \quad (k=1, \ldots, n-3), \quad \bar{u}^-(X) \leq \log(1/|X'|) \quad (k=n-2)
\]

will have property \((P)\). The next result shows that \((P)\) remains true under significantly weaker assumptions on the growth of \( u^- \), where the Riesz decomposition theorem does not apply in an obvious way. Let \( S = \{X \in B(1) : \Phi(X) = O^n\} \), where \( \Phi : B(1) \rightarrow \mathbb{R}^{n-k} \) is a \( C^2 \) function whose derivative matrix has full rank throughout \( B(1) \), and let \( \text{dist}(X, S) = \inf\{|X - Y| : Y \in S\} \).
THEOREM 2. Let \( g: (0, 1] \rightarrow (0, \infty) \) be a decreasing continuous function such that
\[
\int_0^1 t^{n-3-k} \{g(t)^{(n-2-k)/(n-2)} dt < \infty \quad (k = 1, \ldots, n-3),
\]
\[
\int_0^{1/2} \frac{\log g(t)}{t^{(\log t)^2}} dt < \infty \quad (k = n-2).
\]

If \( u \) is a superharmonic function on \( B(1) \setminus S \) satisfying \( u^{-}(X) \leq g(\text{dist}(X, S)) \), then \( |X - Y|^{n-2}u(X) \) has a finite non-negative fine limit \( u^*(Y) \) as \( X \to Y \) for any \( Y \in B(1) \). Further, the set \( \{ Y \in B(r): u^*(Y) > \varepsilon \} \) is finite for each \( r \in (0, 1) \) and each \( \varepsilon > 0 \).

Theorem 2 is the main result of the paper. It can be regarded as an interior fine limit analogue of a result of Rippon [10, Theorem 3] on minimal fine behaviour of subharmonic functions. We will prove it in Section 3 using Theorem 1, ideas from [9, 10], and estimates of the balayage of the function \( X \mapsto |X|^2 - n \) relative to certain sets which are thin at the origin.

Proof of Theorem 1

2.1. For \( a, b, c > 0 \) we define the sets
\[
K_k(a, b) = \{(X', X'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}: |X'| \leq a, |X''| \leq b\},
\]
\[
A_k(a, b; c) = \{(X', X'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}: a \leq |X'| \leq b, |X''| \leq c\},
\]
\[
E_k(a, b) = \{(X', X'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}: |X'|^2/a^2 + |X''|^2/b^2 \leq 1\}.
\]

Let \( \mathcal{C}(A) \) denote the Newtonian capacity of an arbitrary Borel (and hence capacitable) set \( A \subseteq \mathbb{R}^n \). We refer to Helms [7, Chapters 7, 10] for basic results on capacity. In addition we require the following result from Hayman [6, p. 432] concerning the capacity of the ellipsoid \( E_k(a, b) \).

LEMMA A. As \( b/a \to 0 \), the following quantities tend to finite positive limits \( c_{n,k} \) (depending only on \( n \) and \( k \)):
\[
a^{-k}b^{k+2-n}6(E_k(a, b)) \quad (k = 1, \ldots, n-3), \quad a^{2-n} \log(a/b)6(E_k(a, b)) \quad (k = n-2).
\]

2.2. We begin with the \( if \) part of Theorem 1. So let \( f: [0, \infty) \to [0, \infty) \) be increasing, let \( E = \{ X: |X''| < f(|X'|) \} \), and assume that (1) holds. It follows that
the series

$$\sum_j \left\{ \frac{f(2^{-j})}{2^{-j}} \right\}^{n-2-k} (k = 1, \ldots, n-3),$$

$$\sum_j \left\{ 1 + \log^+ \left( \frac{2^{-j}}{f(2^{-j})} \right) \right\}^{-1} (k = n-2)$$

converge. In particular, \( f(2^{-j})/2^{-j} \to 0 \). Since \( K_k(a/\sqrt{2}, b/\sqrt{2}) \subseteq E_k(a, b) \), we have

$$E \cap \{ 2^{-j-1} \leq |X| \leq 2^{-j} \} \subseteq K_k(2^{-j}, f(2^{-j})) \subseteq E_k(2^{-j+1/2}, f(2^{-j})\sqrt{2}).$$

Thus, for all sufficiently large \( j \),

$$\frac{2^{j(n-2)}c(E \cap \{ 2^{-j-1} \leq |X| \leq 2^{-j} \})}{2c_{n,k}2^{(n-2)/2}} \leq \left\{ \frac{f(2^{-j})}{2^{-j}} \right\}^{n-2-k} (k = 1, \ldots, n-3)$$

$$\left\{ \log(2^{-j}/f(2^{-j})) \right\}^{-1} (k = n-2)$$

by Lemma A. It now follows from Wiener's criterion that \( E \) is thin at \( O \).

2.3. It remains to prove the only if part of Theorem 1. So let the function \( f : [0, \infty) \to [0, \infty) \) be increasing and assume that the set \( E = \{ X : |X| \leq f(|X'|) \} \) is thin at \( O \). Let \( \delta \in (0, 1) \) be chosen small enough so that, for \( 0 < b/a \leq \delta \), the displayed quantities in Lemma A lie in the interval \([2c_{n,k}/3, 4c_{n,k}/3]\). Let \( h(t) = \min\{ f(t), \delta t \} \) on \([0, \infty)\) and \( E_h = \{ X : |X| < h(|X'|) \} \). Since \( E_h \subseteq E \), it follows that \( E_h \) is also thin at \( O \). Hence, by Wiener's criterion, we have

$$\sum_{j=1}^\infty d^{j(n-2)}c(E_h \cap \{ d^{-j} \leq |X| \leq d^{1-j} \}) < \infty,$$

where \( d = 2^{2+n/2} \), and so

$$\sum_{j=1}^\infty d^{j(n-2)}c(A_k(d^{-j}, d^{1-j}; h(d^{-j}))) < \infty.$$

Using the subadditivity property of capacity and Lemma A, we obtain

$$c(A_k(d^{-j}, d^{1-j}; h(d^{-j}))) \geq c(K_k(d^{1-j}, h(d^{-j}))) - c(K_k(d^{-j}, h(d^{-j})))$$

$$\geq c(E_k(d^{1-j}, h(d^{-j}))) - c(E_k(d^{-j}\sqrt{2}, h(d^{-j})\sqrt{2}))$$

$$\geq \left\{ \frac{(2c_{n,k}/3)d^{-kj}}{d^{-k/2}} \{ h(d^{-j}) \}^{n-2-k} \{ d^k - 2^{k/2} \} \right\} (k = 1, \ldots, n-3)$$

$$\left\{ (c_{n,k}/2)2^{-(n-2)}d^{-(n-2)} \right\} \{ \log(d^{-j}/h(d^{-j})) \}^{-1} \{ d^{n-2} - 2^{(n+2)/2} \} (k = n-2)$$
provided $\delta \leq 1/d$. Hence the series
\[ \sum_{j} \left\{ \frac{h(d^{-j})}{d^{-j}} \right\}^{n-2-k} (k = 1, \ldots, n-3), \quad \sum_{j} \left\{ \log \left( \frac{d^{-j}}{h(d^{-j})} \right) \right\}^{-1} (k = n-2) \]
converge, and it follows that
\[ \int_{0}^{1} t^{-1} \left\{ \frac{h(t)}{t} \right\}^{n-2-k} dt < \infty \quad (k = 1, \ldots, n-3), \]
\[ \int_{0}^{1} \frac{dt}{t \left\{ \log(t/h(t)) \right\}} < \infty \quad (k = n-2). \]

The convergence of these integrals and the monotonicity of $h$ imply that $h(t)/t \to 0$ as $t \to 0^+$. Hence $h(t) = f(t)$ for all sufficiently small $t$, establishing (1). The proof of Theorem 1 is now complete.

3. Proof of Theorem 2

3.1. Let $g$ be as in the statement of Theorem 2. By adding a suitable function if necessary, we can assume that $g$ is strictly decreasing and unbounded on $(0, 1]$. There is also no loss of generality in assuming that $g(1) = 1$. Let $f$ denote the inverse of the increasing function $t \mapsto \{g(t)\}^{1/(2-n)}$. We are going to show that (1) follows from (2).

Let $\delta, \varepsilon \in (0, 1)$ and $k \in \{1, \ldots, n-3\}$. Using the decreasing property of $g$ and (2) we have
\[ \delta^{n-2-k} \{g(\delta)\}^{(n-2-k)/(n-2)} \leq (n-2-k) \int_{0}^{\delta} t^{n-3-k} \{g(t)\}^{(n-2-k)/(n-2)} dt \rightarrow 0 \quad (\delta \to 0^+). \]

Hence
\[ \int_{\varepsilon}^{1} t^{-1} \left\{ \frac{f(t)}{t} \right\}^{n-2-k} dt = \frac{-1}{n-2-k} \int_{\varepsilon}^{1} \left\{ f(t) \right\}^{n-2-k} d(t^{k+2-n}) \]
\[ = \frac{-1}{n-2-k} \int_{f(\varepsilon)}^{1} x^{n-2-k} d((g(x))^{(n-2-k)/(n-2)}) \]
\[ = \int_{f(\varepsilon)}^{1} x^{n-3-k} \{g(x)\}^{(n-2-k)/(n-2)} dx - \frac{1}{n-2-k} \left[ x^{n-2-k} \{g(x)\}^{(n-2-k)/(n-2)} \right]_{f(\varepsilon)}^{1} \]
\[ \rightarrow \int_{0}^{1} x^{n-3-k} \{g(x)\}^{(n-2-k)/(n-2)} dx - \frac{1}{n-2-k} \]
as $\varepsilon \to 0^+$. If $k = n - 2$, then

$$\log \frac{g(\delta)}{\log(1/\delta)} \leq \int_0^\delta \frac{\log g(t)}{t \log t}^2 \, dt \to 0 \quad (\delta \to 0^+).$$

It follows that $\log(1/t) = o(\log(1/f(t))$ as $t \to 0^+$. Thus, for suitably small $a > 0$ and $\varepsilon \in (0, a)$, we have

$$\int_\varepsilon^a \frac{dt}{t \{1 + \log^+(t/f(t))\}} \leq 2 \int_\varepsilon^a \frac{dt}{t \log(1/f(t))}$$

$$= 2 \int_{f(a)}^{f(\varepsilon)} \frac{d\log\{g(x)^{1/(2-n)}\}}{\log(1/x)}$$

$$= 2 \left\{ \frac{\int_{f(a)}^{f(\varepsilon)} \log g(x)}{x \log x}^2 \, dx - \left[ \frac{\log g(x)}{\log(1/x)} \right]_{f(a)} \right\}$$

$$\to 2 \left\{ \frac{\int_0^{f(a)} \log g(x)}{x \log x}^2 \, dx - \frac{\log g(a)}{\log(1/f(a))} \right\}$$

as $\varepsilon \to 0^+$, using (2).

It follows from (2) that (1) holds for all $k \in \{1, \ldots, n - 2\}$. Hence, by Theorem 1, the set $E = \{X: |X'| < 2f(|X'|)\}$ is thin at $O$.

3.2. Let $\Phi, S$ be as in the paragraph preceding Theorem 2, let $r \in (0, 1)$, and let $Z \in S \cap B(r)$. From the implicit function theorem we can (using a suitable new coordinate system centered at $Z$) find a $C^2$ function $\psi: \mathbb{R}^k \to \mathbb{R}^{n-k}$ and numbers $a_r > 1$ and $\rho_r > 0$ (depending on $r$ and $\Phi$ but not on $Z$) such that

$$\{(X', X'') \in S: |X'| < \rho_r, |X''| < \rho_r\} = \{(X', \psi(X')): |X'| < \rho_r\},$$

$$|\psi(X')| \leq a_r |X'|^2 \quad (|X'| < \rho_r),$$

and

$$\text{dist}(X, S) \geq |X'' - \psi(X')|/2 \quad (|X'| < \rho_r, |X''| < \rho_r). \quad (3)$$

It can be arranged that $\rho_r \in (0, 1/(4a_r))$. Further, since $f(t)/t \to 0$ as $t \to 0^+$ (where $f$ is as defined in Section 3.1), we can choose $\rho_r$ to be sufficiently small so that $f(t)/t \leq 1/4$ for $t \in (0, 2\rho_r)$. Thus $\rho_r$ now depends also on $g$. We can also find a number $b_r > 0$ (depending on $r$ and $\Phi$ but not on $Z$) such that

$$|\psi(X') - \psi(Y')| \leq b_r |X' - Y'| \quad (|X'| < \rho_r, |Y'| < \rho_r). \quad (4)$$
This new coordinate system will remain in force in what follows.

3.3. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be defined by \( F(X', X'') = (X', X'' + \psi(X')) \), let \( E \) be as defined at the end of Section 3.1, let \( E_1 = \{ X \in E : |X'| < \rho_r \} \) and \( E_2 = F(E_1) \). Further, let \( v_1, v_2 \) denote the balayage of the fundamental function \( X \mapsto |X|^{2-n} \) relative to the sets \( E_1, E_2 \) respectively.

**Lemma 1.** (i) The set \( E_2 \) is thin at \( O \).

(ii) If \( |X''| > 2|X'| \), then \( v_2(X) \leq \{16(1+b_r/7)^n-2v_1(X) \}

To prove the lemma, let \( \mu \) be the measure associated with the Newtonian potential \( v_1 \), and let \( w \) be the potential corresponding to the measure \( v \) defined on Borel sets \( A \) by \( v(A) = \mu(\{ X : F(X) \in A \}) \). Since \( E_1 \) is thin at \( O \) (by Section 3.1) we have \( \mu(\{O\}) = 0 \), and hence \( v(\{O\}) = 0 \) also.

If \( X \in E_1 \), then

\[
|X - F(X)| = |\psi(X')| \leq a_r |X'|^2 \leq a_r \rho_r |X'| \leq |X|/4,
\]

and so \( |F(X)| \geq 3|X|/4 \). Using (4) and (5) we have

\[
w(F(X)) = \int_{E_1} |F(X) - F(Y)|^{2-n} \, d\mu(Y)
\]

\[
\geq \int_{E_1} \{|X - Y| + |\psi(X') - \psi(Y')|\}^{2-n} \, d\mu(Y)
\]

\[
\geq (1+b_r)^2 v_1(X)
\]

\[
\geq \{4(1+b_r/3)^2 |F(X)|^{2-n} \} \quad (X \in E_1).
\]

It follows that \( E_2 \) is thin at \( O \), and also that

\[
w(X) \geq \{4(1+b_r/3)^2 v_2(X) \} \quad (X \in \mathbb{R}^n).
\]

It remains to prove (ii). If \( Y \in E_1 \) (so that \( |Y''| \leq 2f(|Y'|) \leq |Y'|/2 \) by our choice of \( \rho_r \) in Section 3.2) and \( |X''| > 2|X'| \), then \( |X - Y| \geq 3|Y|/5 \). Using (5) we have

\[
|X - F(Y)| \geq |X - Y| - |Y - F(Y)| \geq 7|X - Y|/12,
\]

and so

\[
w(X) = \int_{E_1} |X - F(Y)|^{2-n} \, d\mu(Y) \leq (7/12)^2 v_1(X).
\]

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Combining (6) and (7) we obtain (ii). The lemma is now proved.

3.4. Let $E_2$ be as above and let $U = \{X: |X'| < \rho_r, |X''| < \rho_r\} \setminus E_2$. Since

$$E_2 = \{X: |X'| < \rho_r \text{ and } |X'' - \psi(X')| \leq 2 f(|X'|)\},$$

we have from (3) that

$$\text{dist}(X, S) \geq f(|X'|) \quad (X \in U). \quad (8)$$

Now let $u$ be as in the statement of Theorem 2. For $X \in U$ we have $\text{dist}(X, S) < \rho_r \sqrt{2} < 2 \rho_r$ and so $f(\text{dist}(X, S)) \leq \text{dist}(X, S)$. Hence

$$u^-(X) \leq g(\text{dist}(X, S))$$

$$\leq g(\text{dist}(X, S)) \left\{ \frac{2}{1 + [\text{dist}(X, S)/f^{-1}(\text{dist}(X, S))]^2} \right\}^{(n-2)/2}$$

$$= \left\{ \frac{2}{[f^{-1}(\text{dist}(X, S))]^2 + [\text{dist}(X, S)]^2} \right\}^{(n-2)/2}$$

$$\leq \left\{ \frac{8}{4|X'|^2 + |X'' - \psi(X')|^2} \right\}^{(n-2)/2}, \quad (9)$$

using (3) and (8). If $|X'' - \psi(X')| \geq |X''|/2$, then (9) shows that $u^-(X) \leq (32/|X|^2)^{(n-2)/2}$. Otherwise we have $|X'' - \psi(X')| < |X''|/2$, whence

$$|X''| < 2|\psi(X')| \leq 2a_r |X'|^2 < 2a_r \rho_r |X'| < |X'|/2,$$

and (9) now shows that $u^-(X) \leq (2/|X|)^{n-2}$. In either case we thus have

$$u^-(X) \leq (8/|X|)^{n-2} \quad (X \in U). \quad (10)$$

We now know that $|X|^n-2 u(X)$ is bounded below on $U$. Also, $O$ is an irregular boundary point of the open set $U$, by Lemma 1(i). It follows (see Doob [5, 1.XI.21]) that $|X|^n-2 u(X)$ has a finite fine limit $l$ as $X \to O$. In particular (see [4]), $r^n-2 u(rY) \to l$ as $r \to 0+$ for all $Y \in \partial B(1) \setminus A$, where $A$ is some polar set. So, if $l < 0$ and we choose $Y \in \partial B(1) \setminus A$ such that $Y'' \neq O''$, then $u(rY) < (l/2)r^{2-n}$ for all sufficiently small $r > 0$. Combining this with our hypothesis on $u^-$, it follows that $g(t)$ dominates a positive multiple of $t^{2-n}$ on some interval of the form $(0, \eta)$, where $\eta > 0$. This, in turn, contradicts (2). Hence $l \geq 0$.

We have shown that $|X - Z|^n-2 u(X)$ has a finite non-negative fine limit as $X \to Z$ for any $Z \in S \cap \overline{B(r)}$. Since $r \in (0, 1)$ was arbitrary, and since property (P)
holds automatically on $B(1) \setminus S$, the first assertion of Theorem 2 is now established.

Before proving the final sentence of Theorem 2, we make some further observations. We claim that

$$u(X) + (l + 8^{n-2})[v_2(X) + \rho_l^{-2-n}] - l|X|^{2-n} \geq 0 \quad (X \in U).$$

(11)

To see this, we denote the left-hand side of (11) by $-s$, so that $s$ is subharmonic on $U$. Further,

$$\limsup_{X \to Y, Y \in U} s(X) \leq 0 \quad (Y \in \partial U \setminus \{O\}).$$

Hence the function $s^+$ is subharmonic on $\mathbb{R}^n \setminus \{O\}$, if we assign it the value 0 outside $U$. Also, the thinness of $E_2$ at $O$ implies (see [5, 1.XI.3]) that $|X|^{n-2} \nu_2(X)$, and hence $|X|^{n-2}s^+(X)$, has fine limit 0 at $O$. Thus there is an open set $E_3 \subset U$, thin at $O$, such that

$$|X|^{n-2}s^+(X) \to 0 \quad (X \to O, X \notin E_3).$$

Since $s(X) \leq (8^{n-2} + l)|X|^{2-n}$ on $E_3$, and since the surface area measure of $E_3 \cap \partial B(R)$ tends to $0$ as $R \to 0$, we now have $R^{n-2} L(s^+, R) \to 0$ as $R \to 0$, where $L(s^+, R)$ denotes the mean of $s^+$ over $\partial B(O, R)$. It follows from easy estimates of the Poisson kernel for $\mathbb{R}^n \setminus B(R)$ that $s^+ \equiv 0$ on $\mathbb{R}^n \setminus \{O\}$, proving (11).

From Lemma 1(ii) we now have

$$u(X) + (l + 8^{n-2})[\{1 + b_l)/7\}^{n-2} \nu_1(X) + \rho_l^{-2-n}] \geq l|X|^{2-n}$$

for $X \in U$ satisfying $|X''| > 2|X'|$. By the thinness of $E_1$ at $O$ it follows that

$$u(X) \geq (l/2)|X|^{2-n} \quad (|X| < \delta_r, |X''| > 2|X'|),$$

(12)

for some suitably small $\delta_r > 0$ (depending on $r$, but not on $Z$).

3.5. We are now in a position to establish the final assertion of Theorem 2. Suppose that, for given $r \in (0, 1)$ and $\varepsilon > 0$, the set $\{Y \in B(r): \nu(Y) > \varepsilon\}$ is infinite. Then we can find a convergent sequence $(Y_j)$ of points in this set with some limit $Z$. Because the Riesz measure associated with $u$ is locally finite in $B(1) \setminus S$, we can conclude that $Z \in S \cap \overline{B(r)}$. We choose a new coordinate system centered at $Z$ as in Section 3.2 for the following discussion.

There are three cases to consider. The first is where

$$\limsup_{j \to \infty} \text{dist}(Y_j, S)/|Y_j| > 0.$$
By selecting a suitable subsequence of \((Y_j)\) we can find \(\eta \in (0, 1)\) such that 
\[B(Y_j, 3\eta|Y_j|)\] is disjoint from \(S\) for all \(j \in \mathbb{N}\). Applying the minimum principle on
the set \(B(Y_j, 2\eta|Y_j|)\), it follows that
\[u(X) + g(\eta|Y_j|) > \varepsilon\{(|X - Y_j|^2 - n - (2\eta|Y_j|)^2 - n\} (X \in B(Y_j, 2\eta|Y_j|)).\]

Since \((1 - \eta)|Y_j| \leq |X| \leq (1 + \eta)|Y_j|\) in \(B_j = B(Y_j, \eta|Y_j|)\), we now have
\[|X|^n - 2u(X) > (1/\eta - 1)^n - 2\varepsilon(1 - 2^n) - (1/\eta + 1)^n - 2(\eta|Y_j|)^n/2g(\eta|Y_j|)\]
for \(X \in B_j\). Since \(x^n - 2g(x) \to 0\) as \(x \to 0^+\) by (2) (cf. §3.1), we have
\[\lim_{X \to 0} \inf_{X \in \cup_j B_j} |X|^n - 2u(X) \geq (1/\eta - 1)^n - 2\varepsilon(1 - 2^n).
\]

Since \(\cup_j B_j\) is clearly non-thin at \(O\) and \(\eta \in (0, 1)\) can be arbitrarily small, we obtain a contradiction to the fact that \(u^*(O)\) is finite.

The second case is where infinitely many of the \((Y_j)\) are in \(S\). By taking a suitable subsequence, we can assume that \(Y_j \in S\) for all \(j\). Let \(\eta \in (0, 1/4)\) and let
\[Z_j = Y_j + (O', (3\eta|Y_j|, 0, \ldots, 0)), \quad B_j = B(Z_j, \eta|Y_j|) \quad (j \in \mathbb{N}).\]

It follows from (12) that, for all sufficiently large \(j\),
\[u(X) \geq (\varepsilon'/2)|X - Y_j|^2 - n \geq (\varepsilon/2)(4\eta|Y_j|)^2 - n \quad (X \in B_j).
\]

Since \(|X| \geq (1 - 4\eta)|Y_j|\) for \(X \in B_j\), we now have
\[|X|^n - 2u(X) \geq \left(\frac{\varepsilon}{2}\right)\left(\frac{1}{4\eta} - 1\right)^n - 2 \quad (X \in B_j).
\]

The set \(\cup_j B_j\) is non-thin at \(O\), so the right-hand side of (13) is a lower bound for \(u^*(O)\). But \(\eta > 0\) can be arbitrarily small. Hence \(u^*(O) = \infty\), which is a contradiction.

The final case is where dist\((Y_j, S)/|Y_j| \to 0\) as \(j \to \infty\), but only finitely many of
the points \(Y_j\) are in \(S\). These few points can be ignored. We require a suitable
modification of (11), as follows. Let \(Y\) be such that \(|Y| < \rho_r/2\) and \(|Y''| > 2|Y'|\), and let \(l\) be the fine limit of \(|X - Y|^n - 2u(X)\) as \(X \to Y\). Reasoning as in the proof of (11), it can be seen that
\[u(X) + 8^n - 2\{v_2(X) + \rho_r^2 - n\} + l\{v_r(X) + (\rho_r/2)^2 - n\} - l|X - Y|^2 - n \geq 0\]
for \( X \in U \), where \( v_Y \) denotes the balayage of the function \( X \mapsto |X - Y|^{2-n} \) relative to the set \( E_2 \). It follows from estimates in Section 3.3 that \( |X - Y| \geq \frac{7|X|}{25} \) for \( X \in E_2 \). Hence \( v_Y \leq (7/25)^{2-n}v_2 \) on \( \mathbb{R}^n \). Also, \( |X - Y|^{2-n} \geq (2|X|^2)^{2-n} \) for \( |X| > |Y| \). Combining these observations and using Lemma 1(i), we obtain

\[
    u(X) \geq 2^{1-n}|X|^{2-n} \quad (|Y| < |X| < d_r, \ |X''| > 2|X'|)
\]

for some suitably small \( d_r > 0 \) (independent of \( Y \) and \( Z \)). The remaining argument is now similar to that for the second case, with (14) replacing (12).

References