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A fine limit property of functions superharmonic outside a manifold

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Abstract. Let (X', X'') denote a typical point of $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k}$, where $n \geqslant 3$ and $1 \leqslant k \leqslant n-2$. Also, let $E = \{|X''| < f(|X'|)\}$, where $f: [0, \infty) \to [0, \infty)$ is increasing. A necessary and sufficient condition is given for E to be thin at the origin. This, in turn, is used to study the behaviour of functions u which are superharmonic on the complement of a C^2 k-dimensional manifold S. In particular it is shown that, if u^- does not grow too quickly near S, then $|X - Y|^{n-2}u(X)$ has a finite non-negative fine limit as $X \to Y$, for any $Y \in S$.

1. Main results

A set E in Euclidean space $\mathbf{R}^n$ is said to be *thin* at a point Y if there is a superharmonic function u on a neighbourhood of Y such that

$$\liminf_{X \to Y, X \in E \setminus \{Y\}} u(X) > u(Y).$$

The classical criterion of Wiener [7, Theorem 10.21] characterizes thinness at Y in terms of the convergence of a series involving the Newtonian (outer) capacity of the sets $E \cap \{2^{-j-1} \leqslant |X - Y| \leqslant 2^{-j}\}$, where $j \in \mathbf{N}$. (Here $|X|$ denotes the Euclidean norm of X.) The notion of thinness is important in the study of the Dirichlet problem: a boundary point Y is regular for the Dirichlet problem on (an open set) Ω if and only if $\mathbf{R}^n \backslash \Omega$ is not thin at Y. In this context a classical example of a set which is thin at the origin in $\mathbf{R}^3$ is the "Lebesgue spine" defined by $\{(x, y, z)$: $x > 0$ and $y^2 + z^2 \leqslant e^{-c/x}\}$, where $c > 0$ (see [7, p. 175]). Our first result gives a simple geometric characterization of spine-like sets which are thin at the origin O. Let $X = (X', X'')$ denote a typical point of $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k}$, where $n \geqslant 3$ and $k \in \{1, 2, \ldots, n-2\}$.

THEOREM 1. *Let* $E = \{X: |X''| < f(|X'|)\}$, *where* $f: [0, \infty) \to [0, \infty)$ *is increasing. Then* E *is thin at* O *if and only if*

$$\begin{aligned} &\int_0^1 t^{-1} \left\{ \frac{f(t)}{t} \right\}^{n-2-k} \mathrm{d}t < \infty \quad (k = 1, \ldots, n-3), \\ &\int_0^1 \frac{\mathrm{d}t}{t\{1 + \log^+(t/f(t))\}} < \infty \quad (k = n-2). \end{aligned} \tag{1}$$

The axially symmetric case ($k=1$) of Theorem 1 has been given by several authors under the stronger hypothesis that $f(t)/t$ is increasing. In this form it appears in the recent book by Hayman [6, Theorem 7.15], where it is attributed to Cámera [3]. However, it can also be found in Armitage [1] and Port and Stone [8, Chap. 3, Prop. 3.5]. The case $k=n-3$ was recently established by Burdzy [2, Theorem 2.4] using probabilistic methods. The case $k=n-1$ does not appear in Theorem 1 because a set of the form $\{(X', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}: |x_n| < f(|X'|)\}$ is thin at O if and only if the (increasing) function f is valued 0 on $[0, \varepsilon)$ for some $\varepsilon > 0$. (Hyperplanes are non-thin at each constituent point.) In Section 2 we deduce Theorem 1 from Wiener's criterion and estimates of the capacity of certain ellipsoids given in Hayman [6].

The *fine topology* on $\mathbf{R}^n$ is the coarsest topology for which all superharmonic functions are continuous. Thus a superharmonic function u on an open set has a fine limit at every interior point. It also has the following property, which we shall label by (P): $|X-Y|^{n-2}u(X)$ *has a finite non-negative fine limit as* $X \to Y$, *for every interior point* Y. (In fact, this limit is equal to $\mu(\{Y\})$, where μ is the Riesz measure associated with u: see [5, 1.XI.4].) The connection between thin sets and the fine topology is given by the fact that a set E in $\mathbf{R}^n$ is thin at a point Y if and only if Y is not a fine limit point of E.

We will use Theorem 1 to establish a fine limit property of superharmonic functions defined on the complement of a k-dimensional manifold. Let E be a relatively closed polar subset of $B(1)$, where $B(X, r)=\{Y: |Y-X| < r\}$ and $B(r)=B(O, r)$. If u is a positive superharmonic function on $B(1)\backslash E$, then u has a positive superharmonic extension to $B(1)$ (see [7, Theorem 7.7]) and so property (P) holds. The positivity requirement on u can be relaxed here: if u is superharmonic on $B(1)\backslash E$ and there is a negative subharmonic function s on $B(1)\backslash E$ such that $u \geqslant s$ there, then u can be represented (outside a polar set) as the difference of two positive superharmonic functions on $B(1)$ and (P) continues to hold. Now suppose that E takes the form $\{(X', O''): X' \in \mathbf{R}^k\}$. If we write $u^- = \max\{0, -u\}$, then the above reasoning shows that any superharmonic function u on $B(1)\backslash E$ which satisfies

$$u^-(X) \leqslant |X''|^{k+2-n} \quad (k=1, \ldots, n-3), \qquad u^-(X) \leqslant \log(1/|X''|) \quad (k=n-2)$$

will have property (P). The next result shows that (P) remains true under significantly weaker assumptions on the growth of u^-, where the Riesz decomposition theorem does not apply in an obvious way. Let $S=\{X \in B(1): \Phi(X)=O''\}$, where $\Phi: B(1) \to \mathbf{R}^{n-k}$ is a C^2 function whose derivative matrix has full rank throughout $B(1)$, and let $\operatorname{dist}(X, S) = \inf\{|X-Y|: Y \in S\}$.

THEOREM 2. *Let $g: (0, 1] \to (0, \infty)$ be a decreasing continuous function such that*

$$\begin{aligned} &\int_0^1 t^{n-3-k}\{g(t)\}^{(n-2-k)/(n-2)}\,dt < \infty \quad (k=1,\dots,n-3), \\ &\int_0^{1/2} \frac{\log g(t)}{t\{\log t\}^2}\,dt < \infty \quad (k=n-2). \end{aligned} \tag{2}$$

If u is a superharmonic function on $B(1)\backslash S$ satisfying $u^-(X) \leqslant g(\mathrm{dist}(X, S))$, then $|X-Y|^{n-2}u(X)$ has a finite non-negative fine limit $u^*(Y)$ as $X \to Y$ for any $Y \in B(1)$. Further, the set $\{Y \in B(r): u^*(Y) > \varepsilon\}$ is finite for each $r \in (0, 1)$ and each $\varepsilon > 0$.

Theorem 2 is the main result of the paper. It can be regarded as an interior fine limit analogue of a result of Rippon [10, Theorem 3] on minimal fine behaviour of subharmonic functions. We will prove it in Section 3 using Theorem 1, ideas from [9, 10], and estimates of the balayage of the function $X \mapsto |X|^{2-n}$ relative to certain sets which are thin at the origin.

Proof of Theorem 1

2.1. For $a, b, c > 0$ we define the sets

$$K_k(a,b) = \{(X', X'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}: |X'| \leqslant a, |X''| \leqslant b\},$$

$$A_k(a, b; c) = \{(X', X'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}: a \leqslant |X'| \leqslant b, |X''| \leqslant c\},$$

$$E_k(a,b) = \{(X', X'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}: |X'|^2/a^2 + |X''|^2/b^2 \leqslant 1\}.$$

Let $\mathscr{C}(A)$ denote the Newtonian capacity of an arbitrary Borel (and hence capacitable) set $A \subseteq \mathbf{R}^n$. We refer to Helms [7, Chapters 7, 10] for basic results on capacity. In addition we require the following result from Hayman [6, p. 432] concerning the capacity of the ellipsoid $E_k(a, b)$.

LEMMA A. *As $b/a \to 0$, the following quantities tend to finite positive limits $c_{n,k}$ (depending only on n and k):*

$$a^{-k}b^{k+2-n}\mathscr{C}(E_k(a,b)) \quad (k=1,\dots,n-3), \qquad a^{2-n}\log(a/b)\mathscr{C}(E_k(a,b)) \quad (k=n-2).$$

2.2. We begin with the *if* part of Theorem 1. So let $f: [0, \infty) \to [0, \infty)$ be increasing, let $E = \{X: |X''| < f(|X'|)\}$, and assume that (1) holds. It follows that the series

$$\sum_j \left\{\frac{f(2^{-j})}{2^{-j}}\right\}^{n-2-k} \quad (k=1,\ldots,n-3),$$

$$\sum_j \left\{1+\log^+\left(\frac{2^{-j}}{f(2^{-j})}\right)\right\}^{-1} \quad (k=n-2)$$

converge. In particular, $f(2^{-j})/2^{-j} \to 0$. Since $K_k(a/\sqrt{2}, b/\sqrt{2}) \subseteq E_k(a,b)$, we have

$$E \cap \{2^{-j-1} \leqslant |X| \leqslant 2^{-j}\} \subseteq K_k(2^{-j}, f(2^{-j})) \subseteq E_k(2^{-j+1/2}, f(2^{-j})\sqrt{2}).$$

Thus, for all sufficiently large j,

$$\frac{2^{j(n-2)}\mathscr{C}(E \cap \{2^{-j-1} \leqslant |X| \leqslant 2^{-j}\})}{2c_{n,k}2^{(n-2)/2}} \leqslant \begin{cases} \{f(2^{-j})/2^{-j}\}^{n-2-k} & (k=1,\ldots,n-3) \\ \{\log(2^{-j}/f(2^{-j}))\}^{-1} & (k=n-2) \end{cases}$$

by Lemma A. It now follows from Wiener's criterion that E is thin at O.

2.3. It remains to prove the *only if* part of Theorem 1. So let the function $f:[0,\infty) \to [0,\infty)$ be increasing and assume that the set $E=\{X:|X''|<f(|X'|)\}$ is thin at O. Let $\delta \in (0,1)$ be chosen small enough so that, for $0<b/a\leqslant\delta$, the displayed quantities in Lemma A lie in the interval $[2c_{n,k}/3, 4c_{n,k}/3]$. Let $h(t)=\min\{f(t), \delta t\}$ on $[0,\infty)$ and $E_h=\{X:|X''|<h(|X'|)\}$. Since $E_h \subseteq E$, it follows that E_h is also thin at O. Hence, by Wiener's criterion, we have

$$\sum_{j=1}^{\infty} d^{j(n-2)}\mathscr{C}(E_h \cap \{d^{-j} \leqslant |X| \leqslant d^{1-j}\}) < \infty,$$

where $d=2^{2+n/2}$, and so

$$\sum_{j=1}^{\infty} d^{j(n-2)}\mathscr{C}(A_k(d^{-j}, d^{1-j}; h(d^{-j})) < \infty.$$

Using the subadditivity property of capacity and Lemma A, we obtain

$$\begin{aligned} \mathscr{C}(A_k(d^{-j}, d^{1-j}; h(d^{-j})) &\geqslant \mathscr{C}(K_k(d^{1-j}, h(d^{-j}))) - \mathscr{C}(K_k(d^{-j}, h(d^{-j}))) \\ &\geqslant \mathscr{C}(E_k(d^{1-j}, h(d^{-j}))) - \mathscr{C}(E_k(d^{-j}\sqrt{2}, h(d^{-j})\sqrt{2})) \\ &\geqslant \begin{cases} (2c_{n,k}/3)d^{-kj}\{h(d^{-j})\}^{n-2-k}\{d^k - 2^{n/2}\} & (k=1,\ldots,n-3) \\ (c_{n,n-2}/3)d^{-(n-2)j}\{\log(d^{-j}/h(d^{-j}))\}^{-1}\{d^{n-2}-2^{(n+2)/2}\} & (k=n-2) \end{cases} \end{aligned}$$

provided $\delta \leqslant 1/d$. Hence the series

$$\sum_j \left\{\frac{h(d^{-j})}{d^{-j}}\right\}^{n-2-k} \quad (k=1,\dots,n-3), \qquad \sum_j \left\{\log\left(\frac{d^{-j}}{h(d^{-j})}\right)\right\}^{-1} \quad (k=n-2)$$

converge, and it follows that

$$\int_0^1 t^{-1}\left\{\frac{h(t)}{t}\right\}^{n-2-k} \mathrm{d}t < \infty \quad (k=1,\dots,n-3),$$

$$\int_0^1 \frac{\mathrm{d}t}{t\{\log(t/h(t))\}} < \infty \quad (k=n-2).$$

The convergence of these integrals and the monotonicity of h imply that $h(t)/t \to 0$ as $t \to 0+$. Hence $h(t) = f(t)$ for all sufficiently small t, establishing (1). The proof of Theorem 1 is now complete.

3. Proof of Theorem 2

3.1. Let g be as in the statement of Theorem 2. By adding a suitable function if necessary, we can assume that g is strictly decreasing and unbounded on $(0, 1]$. There is also no loss of generality in assuming that $g(1) = 1$. Let f denote the inverse of the increasing function $t \mapsto \{g(t)\}^{1/(2-n)}$. We are going to show that (1) follows from (2).

Let $\delta, \varepsilon \in (0, 1)$ and $k \in \{1, \dots, n-3\}$. Using the decreasing property of g and (2) we have

$$\delta^{n-2-k}\{g(\delta)\}^{(n-2-k)/(n-2)} \leqslant (n-2-k)\int_0^\delta t^{n-3-k}\{g(t)\}^{(n-2-k)/(n-2)}\,\mathrm{d}t$$
$$\to 0 \quad (\delta \to 0+).$$

Hence

$$\int_\varepsilon^1 t^{-1}\left\{\frac{f(t)}{t}\right\}^{n-2-k}\mathrm{d}t = \frac{-1}{n-2-k}\int_\varepsilon^1 \{f(t)\}^{n-2-k}\,\mathrm{d}(t^{k+2-n})$$
$$= \frac{-1}{n-2-k}\int_{f(\varepsilon)}^1 x^{n-2-k}\,\mathrm{d}(\{g(x)\}^{(n-2-k)/(n-2)})$$
$$= \int_{f(\varepsilon)}^1 x^{n-3-k}\{g(x)\}^{(n-2-k)/(n-2)}\,\mathrm{d}x - \frac{1}{n-2-k}\left[x^{n-2-k}\{g(x)\}^{(n-2-k)/(n-2)}\right]_{f(\varepsilon)}^1$$
$$\to \int_0^1 x^{n-3-k}\{g(x)\}^{(n-2-k)/(n-2)}\,\mathrm{d}x - \frac{1}{n-2-k}$$

as $\varepsilon \to 0+$.

If $k = n-2$, then

$$\frac{\log g(\delta)}{\log(1/\delta)} \leqslant \int_0^\delta \frac{\log g(t)}{t\{\log t\}^2}\,\mathrm{d}t \to 0 \quad (\delta \to 0+).$$

It follows that $\log(1/t) = o(\log(1/f(t))$ as $t \to 0+$. Thus, for suitably small $a > 0$ and $\varepsilon \in (0, a)$, we have

$$\begin{aligned}
\int_\varepsilon^a \frac{\mathrm{d}t}{t\{1+\log^+(t/f(t))\}} &\leqslant 2\int_\varepsilon^a \frac{\mathrm{d}t}{t\log(1/f(t))} \\
&= 2\int_{f(\varepsilon)}^{f(a)} \frac{\mathrm{d}(\log\{g(x)^{1/(2-n)}\})}{\log(1/x)} \\
&= \frac{2}{n-2}\left\{\int_{f(\varepsilon)}^{f(a)} \frac{\log g(x)}{x\{\log x\}^2}\,\mathrm{d}x - \left[\frac{\log g(x)}{\log(1/x)}\right]_{f(\varepsilon)}^{f(a)}\right\} \\
&\to \frac{2}{n-2}\left\{\int_0^{f(a)} \frac{\log g(x)}{x\{\log x\}^2}\,\mathrm{d}x - \frac{\log g(f(a))}{\log(1/f(a))}\right\}
\end{aligned}$$

as $\varepsilon \to 0+$, using (2).

It follows from (2) that (1) holds for all $k \in \{1, \ldots, n-2\}$. Hence, by Theorem 1, the set $E = \{X : |X''| < 2f(|X'|)\}$ is thin at O.

3.2. Let Φ, S be as in the paragraph preceding Theorem 2, let $r \in (0, 1)$, and let $Z \in S \cap \overline{B(r)}$. From the implicit function theorem we can (using a suitable new coordinate system centered at Z) find a C^2 function $\psi: \mathbf{R}^k \to \mathbf{R}^{n-k}$ and numbers $a_r > 1$ and $\rho_r > 0$ (depending on r and Φ but not on Z) such that

$$\{(X', X'') \in S : |X'| < \rho_r, |X''| < \rho_r\} = \{(X', \psi(X')) : |X'| < \rho_r\},$$

$$|\psi(X')| \leqslant a_r|X'|^2 \quad (|X'| < \rho_r),$$

and

$$\operatorname{dist}(X, S) \geqslant |X'' - \psi(X')|/2 \quad (|X'| < \rho_r, |X''| < \rho_r). \tag{3}$$

It can be arranged that $\rho_r \in (0, 1/(4a_r))$. Further, since $f(t)/t \to 0$ as $t \to 0+$ (where f is as defined in Section 3.1), we can choose ρ_r to be sufficiently small so that $f(t)/t \leqslant 1/4$ for $t \in (0, 2\rho_r)$. Thus ρ_r now depends also on g. We can also find a number $b_r > 0$ (depending on r and Φ but not on Z) such that

$$|\psi(X') - \psi(Y')| \leqslant b_r|X' - Y'| \quad (|X'| < \rho_r, |Y'| < \rho_r). \tag{4}$$

This new coordinate system will remain in force in what follows.

3.3. Let $F: \mathbf{R}^n \to \mathbf{R}^n$ be defined by $F(X', X'') = (X', X'' + \psi(X'))$, let E be as defined at the end of Section 3.1, let $E_1 = \{X \in E : |X'| < \rho_r\}$ and $E_2 = F(E_1)$. Further, let v_1, v_2 denote the balayage of the fundamental function $X \mapsto |X|^{2-n}$ relative to the sets E_1, E_2 respectively.

LEMMA 1. (i) *The set E_2 is thin at O.*
(ii) *If $|X''| > 2|X'|$, then $v_2(X) \leqslant \{16(1+b_r)/7\}^{n-2} v_1(X)$.*

To prove the lemma, let μ be the measure associated with the Newtonian potential v_1, and let w be the potential corresponding to the measure ν defined on Borel sets A by $\nu(A) = \mu(\{X : F(X) \in A\})$. Since E_1 is thin at O (by Section 3.1) we have $\mu(\{O\}) = 0$, and hence $\nu(\{O\}) = 0$ also.

If $X \in E_1$, then

$$|X - F(X)| = |\psi(X')| \leqslant a_r |X'|^2 \leqslant a_r \rho_r |X'| \leqslant |X|/4, \tag{5}$$

and so $|F(X)| \geqslant 3|X|/4$. Using (4) and (5) we have

$$\begin{aligned} w(F(X)) &= \int_{E_1} |F(X) - F(Y)|^{2-n} \, \mathrm{d}\mu(Y) \\ &\geqslant \int_{E_1} \{|X - Y| + |\psi(X') - \psi(Y')|\}^{2-n} \, \mathrm{d}\mu(Y) \\ &\geqslant (1 + b_r)^{2-n} v_1(X) \\ &\geqslant \{4(1+b_r)/3\}^{2-n} |F(X)|^{2-n} \quad (X \in E_1). \end{aligned}$$

It follows that E_2 is thin at O, and also that

$$w(X) \geqslant \{4(1+b_r)/3\}^{2-n} v_2(X) \quad (X \in \mathbf{R}^n). \tag{6}$$

It remains to prove (ii). If $Y \in E_1$ (so that $|Y''| \leqslant 2f(|Y'|) \leqslant |Y'|/2$ by our choice of ρ_r in Section 3.2) and $|X''| > 2|X'|$, then $|X - Y| \geqslant 3|Y|/5$. Using (5) we have

$$|X - F(Y)| \geqslant |X - Y| - |Y - F(Y)| \geqslant 7|X - Y|/12,$$

and so

$$w(X) = \int_{E_1} |X - F(Y)|^{2-n} \, \mathrm{d}\mu(Y) \leqslant (7/12)^{2-n} v_1(X). \tag{7}$$

Combining (6) and (7) we obtain (ii). The lemma is now proved.

3.4. Let E_2 be as above and let $U = \{X : |X'| < \rho_r, |X''| < \rho_r\} \backslash E_2$. Since

$$E_2 = \{X : |X'| < \rho_r \text{ and } |X'' - \psi(X')| \leqslant 2f(|X'|)\},$$

we have from (3) that

$$\operatorname{dist}(X, S) \geqslant f(|X'|) \quad (X \in U). \tag{8}$$

Now let u be as in the statement of Theorem 2. For $X \in U$ we have $\operatorname{dist}(X, S) < \rho_r\sqrt{2} < 2\rho_r$ and so $f(\operatorname{dist}(X, S)) \leqslant \operatorname{dist}(X, S)$. Hence

$$\begin{aligned} u^-(X) &\leqslant g(\operatorname{dist}(X,S)) \\ &\leqslant g(\operatorname{dist}(X,S))\left\{\frac{2}{1+[\operatorname{dist}(X,S)/f^{-1}(\operatorname{dist}(X,S))]^2}\right\}^{(n-2)/2} \\ &= \left\{\frac{2}{[f^{-1}(\operatorname{dist}(X,S))]^2+[\operatorname{dist}(X,S)]^2}\right\}^{(n-2)/2} \\ &\leqslant \left\{\frac{8}{4|X'|^2+|X''-\psi(X')|^2}\right\}^{(n-2)/2}, \end{aligned} \tag{9}$$

using (3) and (8). If $|X'' - \psi(X')| \geqslant |X''|/2$, then (9) shows that $u^-(X) \leqslant (32/|X|^2)^{(n-2)/2}$. Otherwise we have $|X'' - \psi(X')| < |X''|/2$, whence

$$|X''| < 2|\psi(X')| \leqslant 2a_r|X'|^2 < 2a_r\rho_r|X'| < |X'|/2,$$

and (9) now shows that $u^-(X) \leqslant (2/|X|)^{n-2}$. In either case we thus have

$$u^-(X) \leqslant (8/|X|)^{n-2} \quad (X \in U). \tag{10}$$

We now know that $|X|^{n-2}u(X)$ is bounded below on U. Also, O is an irregular boundary point of the open set U, by Lemma 1(i). It follows (see Doob [5, 1.XI.21]) that $|X|^{n-2}u(X)$ has a finite fine limit l as $X \to O$. In particular (see [4]), $r^{n-2}u(rY) \to l$ as $r \to 0+$ for all $Y \in \partial B(1)\backslash A$, where A is some polar set. So, if $l < 0$ and we choose $Y \in \partial B(1)\backslash A$ such that $Y'' \neq O''$, then $u(rY) < (l/2)r^{2-n}$ for all sufficiently small $r > 0$. Combining this with our hypothesis on u^-, it follows that $g(t)$ dominates a positive multiple of t^{2-n} on some interval of the form $(0, \eta)$, where $\eta > 0$. This, in turn, contradicts (2). Hence $l \geqslant 0$.

We have shown that $|X - Z|^{n-2}u(X)$ has a finite non-negative fine limit as $X \to Z$ for any $Z \in S \cap \overline{B(r)}$. Since $r \in (0, 1)$ was arbitrary, and since property (P) holds automatically on $B(1)\backslash S$, the first assertion of Theorem 2 is now established.

Before proving the final sentence of Theorem 2, we make some further observations. We claim that

$$u(X)+(l+8^{n-2})\{v_2(X)+\rho_r^{2-n}\}-l|X|^{2-n}\geqslant 0 \quad (X\in U). \tag{11}$$

To see this, we denote the left-hand side of (11) by $-s$, so that s is subharmonic on U. Further,

$$\limsup_{X\to Y, X\in U} s(X)\leqslant 0 \quad (Y\in\partial U\backslash\{O\}).$$

Hence the function s^+ is subharmonic on $\mathbf{R}^n\backslash\{O\}$, if we assign it the value 0 outside U. Also, the thinness of E_2 at O implies (see [5, 1.XI.3]) that $|X|^{n-2}v_2(X)$, and hence $|X|^{n-2}s^+(X)$, has fine limit 0 at O. Thus there is an open set $E_3\subset U$, thin at O, such that

$$|X|^{n-2}s^+(X)\to 0 \quad (X\to O,\ X\notin E_3).$$

Since $s(X)\leqslant(8^{n-2}+l)|X|^{2-n}$ on E_3, and since the surface area measure of $E_3\cap\partial B(R)$ tends to 0 as $R\to 0$, we now have $R^{n-2}L(s^+,R)\to 0$ as $R\to 0$, where $L(s^+,R)$ denotes the mean of s^+ over $\partial B(O,R)$. It follows from easy estimates of the Poisson kernel for $\mathbf{R}^n\backslash\overline{B(R)}$ that $s^+\equiv 0$ on $\mathbf{R}^n\backslash\{O\}$, proving (11).

From Lemma 1(ii) we now have

$$u(X)+(l+8^{n-2})[\{16(1+b_r)/7\}^{n-2}v_1(X)+\rho_r^{2-n}]\geqslant l|X|^{2-n}$$

for $X\in U$ satisfying $|X''|>2|X'|$. By the thinness of E_1 at O it follows that

$$u(X)\geqslant(l/2)|X|^{2-n} \quad (|X|<\delta_r,\ |X''|>2|X'|), \tag{12}$$

for some suitably small $\delta_r>0$ (depending on r, but not on Z).

3.5. We are now in a position to establish the final assertion of Theorem 2. Suppose that, for given $r\in(0,1)$ and $\varepsilon>0$, the set $\{Y\in B(r): u^*(Y)>\varepsilon\}$ is infinite. Then we can find a convergent sequence (Y_j) of points in this set with some limit Z. Because the Riesz measure associated with u is locally finite in $B(1)\backslash S$, we can conclude that $Z\in S\cap\overline{B(r)}$. We choose a new coordinate system centered at Z as in Section 3.2 for the following discussion.

There are three cases to consider. The first is where

$$\limsup_{j\to\infty}\operatorname{dist}(Y_j,S)/|Y_j|>0.$$

By selecting a suitable subsequence of (Y_j) we can find $\eta \in (0,1)$ such that $B(Y_j, 3\eta|Y_j|)$ is disjoint from S for all $j \in \mathbf{N}$. Applying the minimum principle on the set $B(Y_j, 2\eta|Y_j|)$, it follows that

$$u(X)+g(\eta|Y_j|) > \varepsilon\{|X-Y_j|^{2-n}-(2\eta|Y_j|)^{2-n}\} \quad (X \in B(Y_j, 2\eta|Y_j|)).$$

Since $(1-\eta)|Y_j| \leqslant |X| \leqslant (1+\eta)|Y_j|$ in $B_j = B(Y_j, \eta|Y_j|)$, we now have

$$|X|^{n-2}u(X) > (1/\eta-1)^{n-2}\varepsilon(1-2^{2-n})-(1/\eta+1)^{n-2}(\eta|Y_j|)^{n-2}g(\eta|Y_j|)$$

for $X \in B_j$. Since $x^{n-2}g(x) \to 0$ as $x \to 0+$ by (2) (cf. §3.1), we have

$$\liminf_{\substack{X \to O \\ X \in \cup_j B_j}} |X|^{n-2}u(X) \geqslant (1/\eta-1)^{n-2}\varepsilon(1-2^{2-n}).$$

Since $\bigcup_j B_j$ is clearly non-thin at O and $\eta \in (0,1)$ can be arbitrarily small, we obtain a contradiction to the fact that $u^*(O)$ is finite.

The second case is where infinitely many of the (Y_j) are in S. By taking a suitable subsequence, we can assume that $Y_j \in S$ for all j. Let $\eta \in (0, 1/4)$ and let

$$Z_j = Y_j + (O', (3\eta|Y_j|, 0, \ldots, 0)), \qquad B_j = B(Z_j, \eta|Y_j|) \quad (j \in \mathbf{N}).$$

It follows from (12) that, for all sufficiently large j,

$$u(X) \geqslant (\varepsilon'/2)|X - Y_j|^{2-n} \geqslant (\varepsilon/2)(4\eta|Y_j|)^{2-n} \quad (X \in B_j).$$

Since $|X| \geqslant (1-4\eta)|Y_j|$ for $X \in B_j$, we now have

$$|X|^{n-2}u(X) \geqslant \left(\frac{\varepsilon}{2}\right)\left(\frac{1}{4\eta}-1\right)^{n-2} \quad (X \in B_j). \tag{13}$$

The set $\bigcup_j B_j$ is non-thin at O, so the right-hand side of (13) is a lower bound for $u^*(O)$. But $\eta > 0$ can be arbitrarily small. Hence $u^*(O) = \infty$, which is a contradiction.

The final case is where $\mathrm{dist}(Y_j, S)/|Y_j| \to 0$ as $j \to \infty$, but only finitely many of the points Y_j are in S. These few points can be ignored. We require a suitable modification of (11), as follows. Let Y be such that $|Y| < \rho_r/2$ and $|Y''| > 2|Y'|$, and let l be the fine limit of $|X-Y|^{n-2}u(X)$ as $X \to Y$. Reasoning as in the proof of (11), it can be seen that

$$u(X)+8^{n-2}\{v_2(X)+\rho_r^{2-n}\}+l\{v_Y(X)+(\rho_r/2)^{2-n}\}-l|X-Y|^{2-n} \geqslant 0$$

for $X \in U$, where v_Y denotes the balayage of the function $X \mapsto |X - Y|^{2-n}$ relative to the set E_2. It follows from estimates in Section 3.3 that $|X - Y| \geqslant 7|X|/25$ for $X \in E_2$. Hence $v_Y \leqslant (7/25)^{2-n} v_2$ on $\mathbf{R}^n$. Also, $|X - Y|^{2-n} \geqslant (2|X|)^{2-n}$ for $|X| > |Y|$. Combining these observations and using Lemma 1(i), we obtain

$$u(X) \geqslant 2^{1-n} l |X|^{2-n} \quad (|Y| < |X| < d_r,\ |X''| > 2|X'|) \tag{14}$$

for some suitably small $d_r > 0$ (independent of Y and Z). The remaining argument is now similar to that for the second case, with (14) replacing (12).

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