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On certain degenerate eigenvalues of Hecke operators

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Introduction

The purpose of this paper is to give a purely local criterion in terms of degeneration of eigenvalues of Hecke operators, which characterizes, whether a holomorphic Siegel modular form of genus two is a lift of Saito-Kurokawa type or not. In contrast, all known criteria involving eigenvalues of Hecke operators are of a global nature, involving either poles of L-series or the notion of CAP-representations (cuspidal associated to parabolic). The method extends to certain nonholomorphic automorphic forms, provided their γ-functions related to the archimedian place are well behaved. This is formulated as the archimedian assumption (2). The validity of this assumption in the nonholomorphic case seems to be intimately related to an indefinite version of the Koecher effect. The results of this paper considerably simplify earlier results obtained in [W2] mainly through the use of a more flexible version of the converse theorem. In [W2] a special nonholomorphic case was considered, which lead to a complete characterisation of the group of cycle classes attached to Siegel modular threefolds in terms of the Saito-Kurokawa lift.

The Saito-Kurokawa lift

In [PS2] Piatetski-Shapiro defines a lift from the metaplectic group $\overline{\mathrm{Sl}}(2,\mathbb{A})$ to the group $\mathrm{GSp}(4,\mathbb{A})$ of symplectic similitudes. This lift attaches to each irreducible cuspidal automorphic representation σ of $\overline{\mathrm{Sl}}(2,\mathbb{A})$ and each non-trivial character ψ of $\mathbb{A}/k$ an automorphic representation $\theta(\sigma,\psi)$ of $\mathrm{GSp}(4,\mathbb{A})$ with trivial central character ω_Π. Here $k=\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{A}$ its adele ring. If $\theta(\sigma,\psi)$ contains a cuspidal form, it is automatically irreducible. An irreducible cuspidal automorphic representation Π with trivial central character on $\mathrm{GSp}(4,\mathbb{A})$ is in the image of the lift iff it is strongly associated either to a Borel group B or to a maximal parabolic group P of GSp(4) with abelian radical. (See [PS2] Thm. 2.2). Π is strongly associated to

P or B means, that Π_p is isomorphic to a subquotient of some representation $\Pi(\tau, z)_p$ for almost all primes p, where

$$\Pi(\tau, z) = \mathrm{Ind}_{P_A}^{\mathrm{GSp}(4,\mathbb{A})}(\tau \otimes \lambda^z),$$

is (unitarily normalized) induced from an irreducible automorphic representation τ of $\mathrm{Gl}(2, \mathbb{A})$ and a character

$$\lambda(p) = \prod_p \lambda_p(p) = |\det(A)/x|, \qquad P \ni p = \begin{pmatrix} A & * \\ 0 & x^t A^{-1} \end{pmatrix} \; A \in \mathrm{Gl}(2, \mathbb{A}),\ x \in \mathbb{A}^*.$$

Here the group $\mathrm{GSp}(4, \mathbb{A})$ is realized as the group of all matrices $g \in M_{4,4}(\mathbb{A})$ with the property ${}^t gJg = x(g)J$, where $x(g) \in \mathbb{A}^*$ and where

$$J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

One can show ([PS2], thm. 2.5 and 2.6), that the only values for the exponent z, that actually occur, are $z = -\frac{1}{2}$ and dually $z = \frac{1}{2}$.

Generalized Whittaker models

Let N be the unipotent radical of P. It is isomorphic to the vector space $\mathrm{Symm}^2(\mathbb{Q}^2)$ of symmetric 2 by 2 matrices. Fix such an isomorphism as in [PSS], p. 506. For a fixed nontrivial character ψ of $\mathbb{A}$ a symmetric matrix T from $\mathrm{Symm}^2(\mathbb{Q}^2)$ defines a character $\psi_T(n) = \psi(\mathrm{Tr}(T \cdot n))$ of $N(\mathbb{A})$. If $\det(T) \neq 0$ the character is called nondegenerate. The stabilizer in $P(\mathbb{A})$ of a nondegenerate character contains a group $D(\mathbb{A})$, where D is the group of units in the semisimple algebra $\mathbb{Q}[X]/(X^2 + \det(T))$ over $\mathbb{Q}$ of rank two. Let denote D^1 the kernel of the idele norm map $D(\mathbb{A}) \to \mathbb{R}^*$.

For a cusp form f on $\mathrm{GSp}(\mathbb{A})$, a nondegenerate character ψ_T of $N(\mathbb{A})$ and a character ν of D^1 the integral

$$f_{T,\nu}(g) = \int_{N(k)\backslash N(\mathbb{A})} \int_{D(k)\backslash D^1} f(nrg)\psi_T(n)^{-1}\nu(r)^{-1}\,\mathrm{d}n\,\mathrm{d}r$$

is well defined. This uses reduction theory for $\mathrm{GSp}(4, \mathbb{A})$ in case D splits.

Suppose Π is an irreducible cuspidal representation of $\mathrm{GSp}(4, \mathbb{A})$. A choice (T, ν, ψ), such that $f_{T,\nu} \neq 0$ for some $f \in \Pi$, will be called a global Whittaker datum for Π. The theory of singular forms shows, that global Whittaker data always exist. The character ν then has a unique extension to $D(\mathbb{A})$, whose restriction to $\mathbb{A}^*$ equals the central character ω_Π. This follows immediately from the explicit formulas given in [PSS], p. 507.

For local irreducible preunitary admissable representations Π_p of $\mathrm{GSp}(4, k_p)$ a local Whittaker datum is a pair (T_p, v_p, ψ_p), such that there exists a nontrivial linear map $l: \Pi_p \to \mathbb{C}$ with the property

$$l(\Pi_p(rn)v) = \psi_p(\mathrm{Tr}(T_p \cdot n))v_p(r)l(v) \qquad r \in D(k_p),\ n \in N(k_p),$$

where T_p is a symmetric 2 by 2 matrix with entries in K_p and $\det(T_p) \neq 0$, ψ_p is a nontrivial character of k_p and v_p is a character of $D(k_p)$. In the archimedean case one assumes l to be defined and continuous on the C^∞ vectors. For fixed Whittaker datum the map l is unique up to a constant. ([PSS], thm. 1.1). The space of Whittaker functions $w(g) = l(\Pi_p(g)v)$ for $v \in \Pi_p$ defines the (generalized) Whittaker model of the representation Π_p attached to the Whittaker data (v_p, T_p, ψ_p).

A choice of a local Whittaker datum for Π_p determines local L and ε functions

$$L_p(\Pi_p \otimes \mu_p, s), \qquad \varepsilon_p(\Pi_p \otimes \mu_p, s).$$

See [PSS], p. 509. The γ-function $\gamma_p(\Pi_p \otimes \mu_p, s)$ used later is defined as

$$\gamma_p(\Pi_p \otimes \mu_p, s) = \varepsilon_p(\Pi_p \otimes \mu_p, s)L_p(\tilde{\Pi}_p \otimes \mu_p^{-1}, 1 - s)/L_p(\Pi_p \otimes \mu_p, s).$$

Here $\tilde{\Pi}$ denotes the contragredient representation of Π.

For global cuspidal Π each choice of a global Whittaker datum (T, v, ψ) induces local Whittaker data of Π_p for all places p. This defines functions $L(\Pi \otimes \mu, s) = \prod L_p(\Pi_p \otimes \mu_p, s)$ and $\varepsilon(\Pi \otimes \mu, s) = \prod \varepsilon_p(\Pi_p \otimes \mu_p, s)$, which allow meromorphic continuation to the complex plane and have functional equations

$$L(\Pi \otimes \mu, s) = \varepsilon(\Pi \otimes \mu, s)L(\tilde{\Pi} \otimes \mu^{-1}, 1 - s).$$

For nonarchimedean p and $\Pi_p \otimes \mu_p$ unramified $L_p(\Pi_p \otimes \mu_p, s)$ is the local L-function attached to the four dimensional standard representation of the L-group ${}^L G = \mathrm{GSp}(4, \mathbb{C})$. This determines almost all factors of the global L-series $L(\Pi \otimes \mu, s)$ in terms of Π and μ alone. The remaining finitely many factors may depend on the Whittaker datum (T, v, ψ). For details see [PS1], [PSS], [PS3], p. 148ff and [PS2], p. 317ff.

For the unramified constituent Π_p of $\Pi(\tau_p, z)$ as above and unramified characters μ_p this implies

$$L_p(\Pi_p \otimes \mu_p, s) = L_p(\omega_{\Pi_p} \otimes \mu_p, s + z)L_p(\tau_p, s)L_p(\mu_p, s - z)$$

for the local L-factor. Here

$$\omega_{\Pi_p} = \omega_{\tau_p}$$

is the central character of Π_p or τ_p and the L-factors on the right are the usual standard L-factors for the groups Gl(1) and Gl(2) in the sense of Tate and Jaquet-Langlands. This is easily deduced from [PS2], page 318.

A lifting criterion

Let Π be an irreducible cuspidal automorphic representation with unitary central character ω_Π for the group $GSp(4, \mathbb{A})$. We assume, that the infinite component Π_∞ is fixed in the following sense

Archimedian assumptions:

Assume existence of a irreducible cuspidal automorphic representation $\theta(\sigma, \psi)$ such that

(1) For some character ψ and some holomorphic or antiholomorphic irreducible cuspidal automorphic representation σ of $\overline{Sl}(2, \mathbb{A})$ of weight $r + \frac{1}{2}$ with $r \geqslant 1$ in $\mathbb{N}$

$$\Pi_\infty \cong \theta(\sigma, \psi)_\infty$$

holds.

(2) For some global Whittaker data (T', v', ψ') resp. (T, v, ψ) of Π resp. $\theta(\sigma, \psi)$ the local γ-functions should coincide

$$\gamma_\infty(\Pi_\infty \otimes \mu_\infty, s) = c \cdot \gamma_\infty(\theta(\sigma, \psi)_\infty \otimes \mu_\infty, s)$$

for the two characters $\mu_\infty = 1$ and $\mu_\infty = \text{sign}$ up to some fixed constant c.

Condition (1) is purely local, whereas (2) might be not. It is expected, that the γ-functions do not depend on the choice of Whittaker data, but only on the local representations. This of course would imply that (2) is a consequence of (1) In the holomorphic cases, this is well known. This is related to the Koecher effect and explained in the last section.

Theorem. Let Π *be an irreducible cuspidal automorphic representation of* $GSp(4, \mathbb{A})$ *with central character* ω_Π. *Assume there exists an integer r such that* $\Pi_\infty \cong \theta(\sigma, \psi)_\infty$ *is fixed as in the assumptions* (1) *and* (2) *above. Then the following properties* (*a*) *and* (*b*) *are equivalent:*

(a) Π *is of Saito-Kurokawa type:* $\Pi \cong \theta(\tilde{\sigma}, \tilde{\psi})$. *This holds for some pair* $(\tilde{\sigma}, \tilde{\psi})$, *where* $\tilde{\sigma}$ *is a holomorphic or antiholomorphic irreducible cuspidal automorphic representation of* $\overline{\mathrm{Sl}}(2, \mathbb{A})$ *of the same weight* $r + \frac{1}{2}$ *as* σ.

(b) *There exists a finite set S of places such that for all p not in S the representation* Π_p *is a subquotient of a locally induced representation*

$$\Pi(\tau_p, -1/2) = \mathrm{Ind}_{P_p}^{\mathrm{GSp}(4,k_p)}(\tau_p \otimes \lambda_p^{-1/2})$$

for some irreducible admissable representations τ_p *of* $\mathrm{Gl}(2, k_p)$. *Furthermore for every* $p \notin S$ *either the central character* $\omega_{\Pi_p} = 1$ *is trivial or the representation* τ_p *is (unitarily normalized) induced from a pair of characters* ν_p, μ_p, *such that* $\nu_p = \omega_{\Pi_p}|\cdot|_p^{1/2}$ *and* $\mu_p = |\cdot|_p^{-1/2}$.

Remark. Note that, though $\omega_\Pi = 1$ is not a priori assumed in assertion (b), implication (b) $\Rightarrow$ (a) of the theorem implies $\omega_\Pi = 1$. One might ask, whether the first part of (b) without any further assumption on the central character already implies (a), and especially $\omega_\Pi = 1$. In any case the formulation above is sufficiently general for the applications made in [W2].

Proof of theorem. The implication (a) $\Rightarrow$ (b) of the theorem is obvious from what was explained above. If Π is a Saito-Kurokawa lift, it is strongly associated to either the parabolic P or B with central character $\omega_\pi = 1$. The induction parameter z is either $1/2$ or $-1/2$. In the unramified cases it is easy to show (using the action of the Weyl group) that $\Pi(\tau_p, z)$ has the same constituents as $\Pi(\tau_p \otimes \omega_{\Pi_p}^{-1}, -z) \otimes \omega_{\Pi_p}$. By $\omega_\Pi = 1$, we can therefore always assume

$$z = -1/2.$$

This shows (b).

What needs a proof is the implication (b) $\Rightarrow$ (a). For that we have to show, that the local representations τ_p of $\mathrm{Gl}(2, k_p)$, $p \notin S$, which occur in (b), patch together to a global automorphic representation of the group $\mathrm{Gl}(2, \mathbb{A})$. Then Π is strongly associated to P or B in the sense above and therefore of Saito-Kurokawa type, as explained above. Observe

$$\omega_{\tau_p} = \omega_{\Pi_p}.$$

Hence at least the central characters of the τ_p patch together to the global character ω_Π. So we are in a situation, where we should try to apply the Hecke-Weil converse theorem. We have to show that certain L-series attached to twists of the local representations τ_p have functional equations and nice analytic behaviour in the complex plane. This will be shown in the next sections. Actually we do not consider the representations τ_p itself, but rather some substitutes $\tilde{\tau}_p$

with central characters $\omega_{\Pi_p}^2$. This is due to the fact, that we do not a priori assume $\omega_\Pi = 1$. In the following we will assume that we have given a cuspidal representation Π of GSp(4, $\mathbb{A}$) satisfying condition (b). We always assume S to be a finite set of places including all ramified places, the infinite place, the prime 2, the places where $k(\sqrt{-\det(T)})/k$ is ramified and the places excluded in assumption (b).

The functional equation

We remark for later use the following two facts

LEMMA 1 (ramified case). *Let k_p be nonarchimedian and Π_p be an irreducible admissable preunitary representation of* GSp(4, k_p) *of dimension* $\dim(\Pi_p) > 1$. *Fix some local Whittaker data for Π_p to define L and ε-factors. Then there exists a constant C depending only on Π_p, such that for all unramified characters μ_p of k_p^* and for all characters ν_p of k_p^*, for which ν_p^2 has conductor* cond(ν_p^2) *at least C, the following holds*

(1) *The L-factor $L_p(\Pi_p \otimes \mu_p \nu_p, s) = 1$ is trivial.*
(2) *For constants $c_p \in \mathbb{C}^*$ and $f_p \in \mathbb{Z}$ depending only on Π_p and ν_p, but not on s and μ_p*

$$\varepsilon_p(\Pi_p \otimes \mu_p \nu_p, s) = c_p(\mu_p(p)p^{-s})^{f_p}.$$

Proof. [PPS], thm. 2.2 and a comparison of central characters excludes so called exceptional poles of the local L-factors unless $\omega_{\Pi_p}\mu_p^2\nu_p^2 = |\ |_p^{-1-2s_0}$ for some exponent s_0. Hence by our assumption there are no exceptional poles for $L_p(\Pi_p \otimes \mu_p \nu_p, s)$ for C large enough. The first claim therefore follows from [PSS], thm. 2.1, giving the regular poles.

From the first claim (1) and [PSS], thm. 1.2 follows that the function $\varepsilon_p(\Pi_p \otimes \chi_p, s)$ is entire without zeros and can be written for $\chi_p = \mu_p \nu_p$ as the quotient

$$\varepsilon_p(\Pi_p \otimes \chi_p, s) = L_p(\tilde{w}, \hat{\phi}, \omega_{\Pi_p}^{-1}\chi_p^{-1}, 1 - s)L_p(w, \phi, \chi_p, s)^{-1}$$

of two zeta integrals

$$L_p(w, \phi, s_p, s) = \int_{N_p \backslash \mathrm{Gl}(2, D_p)'} w(g)\phi[(0,\ 1)g]\chi_p(\det(g))|\det(g)|_p^{s+1/2}\, dg.$$

Here $\mathrm{Gl}(2, D_p)'$ is some subgroup of $\mathrm{Gl}(2, D_p)$ suitably embedded into GSp(4, k_p), w a Whittaker function and ϕ a Schwartz-Bruhat function on D_p^2. For precise definitions see [PSS], especially formula (1.4) loc. cit. The quotient is well defined for a suitable choice of $w(g)$ and ϕ, making the denominator not vanish identically. On the other hand one can show, that all $L(w, \phi, \chi_p, s)$ are rational functions in the variable p^{-s}. This follows from a direct inspection of the integrals and uses the asymptotic expansion of Whittaker functions ([PSS], formula (2.1)) for

$$w\left(\begin{pmatrix} tE & 0 \\ 0 & E \end{pmatrix}\right), \qquad t \in k_p^*$$

and small $|t|_p$ and the fact, that this function vanishes for large $|t|_p$. For the details see [W2]. Therefore $\varepsilon_p(\Pi_p \otimes \nu_p, s)$ is a rational function in p^{-s} as a quotient of two rational functions. On the other hand it is entire without zeros. See [PSS]. thm. 1.2. This proves our assertion for fixed μ, namely $\varepsilon_p(\Pi_p \otimes \nu_p, s) = \text{const} \cdot p^{-f_p s}$ for some integer f.

From its expression as the quotient of integrals as above also follows

$$\varepsilon_p(\Pi \otimes \chi\mu, s) = \varepsilon_p(\Pi_p \otimes \chi_p, s + \log_p(\mu_p(p))$$

for unramified characters μ_p. This proves the claim.

LEMMA 2 (unramified case). *Let the situation be as in lemma* 1. *In addition assume Π_p to be unramified and $p \neq 2$. Furthermore assume, that Π_p admits a local Whittaker model for a pair (T_p, ν_p, ψ_p), such that $k_p(\sqrt{-\det(T_p)})/k_p$ is not a ramified field extension. Let μ_p be an arbitrary character of k_p^*. Let ω_{Π_p} be the central character of Π_p, then*

$$\varepsilon_p(\Pi_p \otimes \mu_p, s) = \omega_{\Pi_p}(p^{\text{cond}(\mu_p)})^2 \varepsilon_p(\mu_p, s)^4,$$

where $\varepsilon_p(\mu_p, s)$ *is Tate's factor* [T]. *Furthermore if μ_p is ramified, then*

$$L_p(\Pi_p \otimes \mu_p, s) = 1.$$

Proof. Consider the Whittaker data (ν_p, T_p, ψ_p) of Π_p from the assumptions of the lemma. Then one can show, that the unramified irreducible admissable representation Π_p of $\mathrm{GSp}(4, k_p)$ can be obtained as a quotient of the representation

$$\Pi(\sigma_p) = \Pi(\sigma_p, \nu_p, T_p, \psi_p)$$

as defined in [PSS], p. 515. This holds for some unramified representation σ_p of the connected component H of the group $\mathrm{GO}(2, 2)$ of similitudes of a split rank 4 quadratic form over k_p. This is proved in [W2]. The idea of proof is to reduce the statement to a similar statement on the Weil correspondence of the dual reductive pair $\mathrm{Sp}(4, k_p)$ and $O(2,2)$ proved in [R], page 500ff. This involves a comparison of the representations of the groups SO(2, 2), $O(2,2)$ and H. Furthermore unramified representations as in assumption (b) with trivial central character play an exceptional role. At one stage of the argument one has to show, that for fixed T_p these special representations allow Whittaker models at most for the trivial character $v_p = 1$. For the details we refer to [W2].

The representation

$$\sigma_p = (\sigma_{1,p}, \sigma_{2,p})$$

of the connected component $H \cong \mathrm{Gl}(2, k_p) \times \mathrm{Gl}(2, k_p)/\{(tE, t^{-1}E)\}$ of the group GO(2, 2) mentioned above corresponds to a pair of unramified representations of the group $\mathrm{Gl}(2, k_p)$ with central characters

$$\omega_{\sigma_{1,p}} = \omega_{\sigma_{2,p}} = \omega_\Pi.$$

Observe, that the Whittaker model of the quotient Π_p of $\Pi(\sigma_p)$ attached to the Whittaker data (v_p, T_p, ψ_p) induces the unique Whittaker model of $\Pi(\sigma_p)$ attached to these data ([PSS], thm. 1.6) via the quotient map. It follows that both representations have the same L- and ε-functions, because they are defined through the same Whittaker functions. Hence by [PSS], thm. 2.4 and thm. 3.1 one obtains

$$L_p(\Pi_p \otimes \mu_p, s) = L_p(\sigma_{1,p} \otimes \mu_p, s) L_p(\sigma_{2,p} \otimes \mu_p, s)$$

and then

$$\varepsilon_p(\Pi_p \otimes \mu_p, s) = \varepsilon_p(\sigma_{1,p} \otimes \mu_p, s) \varepsilon_p(\sigma_{2,p} \otimes \mu_p, s).$$

This also shows the independence of these functions from the Whittaker model chosen. This being said, the claim follows from the well known formulas for the right sides provided by Jaquet-Langlands theory in the GL(2) case. For further details see [W2].

From now on let v and μ always be global characters, such that v^2 is highly ramified in S respectively μ is unramified in S. Thus lemma 1 can be applied in the following. Let $L^S(\Pi \otimes \mu v, s)$ denote the partial L-series defined by the product over all local factors for primes $p \notin S$. Similar notation is used for other kind of L-series. Observe that the partial L-series $L^S(\Pi \otimes \mu v, s)$ depends only on Π and not on the choice of a Whittaker datum for Π. This follows from lemma 2.

Let $L(\chi, s)$ be the classical Dirichlet series for a character χ and similarly let $L^S(\chi, s)$ be the partial L-series corresponding to $L(\chi, s)$. We define new Euler products $\Lambda(\chi, s)$ by

$$\Lambda(\chi, s) = \frac{L^S(\Pi \otimes \chi, s)}{L^S(\chi, s + \frac{1}{2})L^s(\chi, s - \frac{1}{2})}.$$

From our assumption on S (and using lemma 2 for characters χ ramified outside S) we get equivalently

$$\Lambda(\chi, s) = L^S(\tau \otimes \chi, s)\frac{L^S(\omega_\Pi \chi, s - \frac{1}{2})}{L^S(\chi, s - \frac{1}{2})}.$$

To these functions the converse theorem will be applied. Observe that each local p-factor of $\Lambda(\chi, s)$ has exactly two Euler factors

$$\Delta_p(\chi_p, s) = (1 - a_p p^{-s})^{-1}(1 - b_p p^{-s})^{-1}, \qquad (p \notin S),$$

hence can be associated formally to some unramified representation of the group $Gl(2, k_p)$. This is important, when we want to apply [W]. The assertion is clear for the primes, where $\omega_{\Pi_p} = 1$. For the remaining primes the assertion follows from the second part of assumption (b). It gives $a_p = \omega_{\Pi_p}(p)\chi_p(p)p^{1/2}$ and $b_p = \omega_{\Pi_p}(p)\chi_p(p)p^{-1/2}$ in that case. For all $p \notin S$ we therefore have

$$a_p b_p = \omega_{\Pi_p}(\mathrm{p})^2 \chi_p(p)^2$$

for the coefficient at p^{-2s}.

We will consider the series $\Lambda(\chi, s)$ only for characters $\chi = \mu\nu$ with fixed ν highly ramified in S (hopefully not to be confused with the character occuring in the notion of Whittaker data) and varying characters μ unramified in S! If ν^2 is ramified highly enough, the same holds true when ν is replaced by $\nu\omega_\Pi^n$ for any integer n. The functional equations for $L(\Pi \otimes \chi, s)$ and $L(\chi, s)$ together with lemma 1 and lemma 2 imply

$$\Lambda(\chi, s) = \varepsilon_\Lambda(\chi, s)\overline{\Lambda(\chi, 1 - \bar{s})}$$

for all such characters χ. It is this functional equation, which leads us to consider $\Lambda(\chi, s)$ instead of $L(\tau \otimes \chi, s)$.

Remark. For the admissable (complex) representation Π we can consider the complex conjugate $\bar{\Pi}$. It is defined on the same representation space as Π by

$\bar{\Pi}(g)(v) = c(\Pi(g)(c(v)))$ for some arbitrarily chosen complex structure c on the representation space V. Its isomorphism class does not depend on the choice of c. If Π is preunitary, then it is easy to show that $\bar{\Pi}$ is isomorphic to the contragradient $\tilde{\Pi}$ of Π.

This remark allows to replace the local terms of the right hand L-series of the functional equation of $L(\Pi \otimes \chi, s)$ by $L_p(\bar{\Pi}_p \otimes \chi_p^{-1}, 1 - s) = \bar{L}_p(\Pi_p \otimes \chi_p, 1 - \bar{s})$ for all $p \notin S$. Namely if Π_p is the spherical constituent of some $\mathrm{Ind}_B^G(\rho_p)$ for some character ρ_p of the Borelgroup, then $\bar{\Pi}_p$ is the spherical constituents of $\mathrm{Ind}_B^G(\bar{\rho}_p)$.

A similar argument applies to the Dirichlet L-series involved in the definition of $\Lambda(\chi, s)$. By our assumption on χ none of the L-series involved in the functional equations has nontrivial terms for places in S except the infinite place. A computation of the factor $\varepsilon_\Lambda(\chi, s)$ for $\chi = \mu\nu$ therefore gives

$$c(S, \nu)\mu\left(\frac{N_1}{N_2}\right)\left(\frac{N_1}{N_2}\right)^{-s} \gamma_\infty(\Pi \otimes \mu\nu, s)\gamma_\infty(\mu\nu, s - \tfrac{1}{2})^{-1}$$

$$\times\gamma_\infty(\mu\nu, s + \tfrac{1}{2})^{-1} \prod_{p \notin S} (\omega_\Pi^2 \nu^2)_p(p^{\mathrm{cond}(\mu_p)})\varepsilon_p(\mu_p, s)^2$$

for a constant $c(S, \nu)$ and positive integers N_1, N_2 not depending on μ. Here by some abuse of notation we do not distinguish between idele class character and their associated Dirichlet characters. The first three terms result from lemma 1, the next three terms arise from the infinite places, whereas the last term arises from the unramified places $p \notin S$. The factors of the product are determined by lemma 2 and the local factors $\varepsilon_p(\chi_p, s + \frac{1}{2})\varepsilon_p(\chi_p, s - \frac{1}{2}) = \varepsilon_p(\chi_p, s)^2$. One uses the formula $\varepsilon_p(\chi_p, s) = \nu_p(p^{\mathrm{cond}(\mu_p)})\varepsilon_p(\mu_p, s)$ for $\chi_p = \nu_p\mu_p$ and unramified ν_p $(p \notin S)$.

It remains to simplify the archimedian contribution. This does the next lemma. It shows, that the Euler product $\Lambda(\nu, s)$ and its μ-twists have functional equations like that of an elliptic holomorphic modular form of weight $2r$ and nebentype character $\omega_\Pi^2\nu^2$, in the form required in [W] for the converse theorem.

LEMMA 3 (archimedian case). *Assume* Π *satisfies the archimedian assumptions* (1) *and* (2). *Then there exists a constant c independent of* μ *and a holomorphic discrete series representation* τ_∞ *of* $\mathrm{GL}(2, \mathbb{R})$ *of weight* $2r$, *such that*

$$\gamma_\infty(\tau_\infty \otimes \mu\nu, s) = c \cdot \gamma_\infty(\Pi \otimes \mu\nu, s)\gamma_\infty(\mu\nu, s - \tfrac{1}{2})^{-1}\gamma_\infty(\mu\nu, s + \tfrac{1}{2})^{-1}.$$

Proof. Assumption (2) gives $\gamma_\infty(\Pi \otimes \chi, s) = c \cdot \gamma_\infty(\theta(\sigma, \psi) \otimes \chi, s)$, for some representation $\theta(\sigma, \psi)$. For the proof of lemma 3 we can therefore assume $\Pi = \theta(\sigma, \psi)$. According to [PS2], thm. 4 then

$$\Lambda(\chi, s) = L^S(\tau \otimes \chi, s)$$

holds for a holomorphic cuspidal automorphic representation τ of $\mathrm{Gl}(2, \mathbb{A})$ of weight $2r$. Furthermore $\omega_\Pi = \omega_\tau = 1$ holds for the central characters involved.

We now compare the automorphic functional equations for $L(\tau, s)$ with the above functional equations defined via the associated $\Lambda(\chi, s)$ function. A comparison of ε-factors shows that the quotients of the two sides in the equality of lemma 3 is

$$c \cdot \mu\left(\frac{M_1}{M_2}\right)\left(\frac{M_1}{M_2}\right)^{-s}$$

for all characters μ unramified in S with c, M_1, M_2 independent from μ. Here again v^2 is assumed to be highly ramified in the exceptional set of primes. On the other hand the quotient depends on the character μ only via its infinite component μ_∞, hence is independent of μ for all even characters. Apply this to all even characters μ of sufficiently large prime conductor m, e.g. $m > M_2 + M_1$ and $m \notin S$. This forces $M_1 = M_2 = 1$. Thus the quotient is equal to the constant c. This was the claim.

PROOF OF THEOREM. In this section we discuss the analytic behaviour of the functions $\Lambda(\chi, s)$ in the complex plane. For this we maintain our assumptions on the Dirichlet character $\chi = \mu v$.

The functions $L(\Pi \otimes \chi, s)$ are holomorphic for $\mathrm{Re}(s) \geqslant \frac{3}{2}$ except a possible pole at $s = \frac{3}{2}$ and absolutely convergent in some right half plane. See [PS2], Theorem 1 or [PS3], Lemma 3.1. The same holds true for the partial L-series $L^S(\Pi \otimes \chi, s)$ by the nonvanishing of local factors ([PS2], page 318). $\Lambda(\chi, s)$ inherits these properties in the domain $\mathrm{Re}(s) \geqslant \frac{3}{2}$ by definition, because the Dirichlet L-series $L^S(\chi, s + \frac{1}{2})$ and $L^S(\chi, s - \frac{1}{2})$ in the denominator do not vanish in this region.

On the other hand, the functional equation of $\Lambda(\chi, s)$ relates the points s and $1 - \bar{s}$. This reduces us to consider the analytic continuation of the functions $\Lambda(\chi, s)$ into the strip $\frac{1}{2} \leqslant \mathrm{Re}(s) \leqslant \frac{3}{2}$. Here the zeros of the Dirichlet L-series in the denominator of the formula defining $\Lambda(\chi, s)$ might produce some poles.

Let me sketch the further argument: The first important observation concerns the points s with nonvanishing imaginary part. At these points the functions $\Lambda(\chi, s)$ turn out to be analytic. This follows from a unitary property of scattering operators attached to the group $\mathrm{Sp}(6, \mathbb{A})$ and will be discussed in the next sections. The same kind of reasonings also shows, that the functions $\Lambda(\chi, s)$ under considerations are bounded in vertical strips outside of suitable compact sets.

Let us assume this for the moment. The argument is then completed as follows: Each of the functions $\Lambda(\chi, s)$ for $\chi = v\mu$ has finitely many real poles in the strip $-\frac{1}{2} \leqslant \mathrm{Re}(s) \leqslant \frac{3}{2}$ and is bounded in vertical strips outside suitable compact sets. Viewing v fixed and μ varying the functions $\Lambda(v\mu, s)$ have functional equations like that of twists of an elliptic modular form of weight $2r$ and virtual conductor N_1/N_2 and character $\omega_\Pi^2 \nu^2$. Furthermore $\Lambda(\nu, s)$ has an Euler product expansion with two local Euler factors for each prime $p \notin S$, and coefficient $\omega_{\Pi_p}^2(p)\nu_p^2(p)$ at p^{-2s}. We are now in a situation, where we can apply a version of the converse theorem, which allows each of the functions $\Lambda(\chi)$ to have finitely many poles. See [W]. It follows that the Dirichlet series $\Lambda(\nu, s)$ is the L-series attached to an elliptic holomorphic modular eigenform of weight $k = 2r$ or to the nonholomorphic Eisenstein series $E_2(z)$ of weight 2 in case $r = 1$. This follows from [W] and the existence of an Euler product for $\Lambda(\nu, s)$. To be honest, this is true only up to a rescaling of the parameter s by the value $(k - 1)/2$ and some trivial changes due to a more classical notation in [W] and the additional reparametrisation of the conductor, used in [W]. Let $\tilde{\tau}$ denote the corresponding automorphic representation of $\mathrm{Gl}(2, \mathbb{A})$.

Suppose $\tilde{\tau}$ were not cuspidal. Then

$$L(\tilde{\tau}, s) = L\left(\chi_1, s - \frac{k-1}{2}\right) L\left(\chi_2, s + \frac{k-1}{2}\right)$$

holds for some unitary characters χ_1 and χ_2. If $r \geqslant 2$ then this implies, that $L(\Pi \otimes \nu\chi_1^{-1}, s)$ has a pole at the point $s = (k + 1/2) = (2r + 1/2)$. This is impossible, because the L-series under consideration is known to be holomorphic in the domain $\mathrm{Re}(s) > \frac{3}{2}$, as explained above. Furthermore if $r = 1$, then the same argument shows existence of a pole at $s = \frac{3}{2}$. Existence of a pole then implies property (a) by [PS2], thm. 2.2.

If on the other hand $\tilde{\tau}$ is cuspidal, then the L-series $L(\tilde{\tau}, s)$ does not have zeros for $\mathrm{Re}(s) > 1$. Therefore $L(\Pi, s)$ has a pole at $s = \frac{3}{2}$. Again (a) follows.

Analytic continuation

Let Q be the maximal parabolic subgroup in $\mathrm{Sp}(6, k)$, which is the stabilizer of a line and whose Levi component is $\mathrm{Sp}(4, k) \times \mathrm{Gl}(1, k)$. We put $A(g) = \prod_p |a_p(g_p)|_p$, where $a_p(g_p)$ is the split torus component a_p in the Iwasawa decomposition $g_p = m_p a_p n_p k_p$ in $\mathrm{Sp}(6, k_p)$ with respect to the parabolic group $Q(k_p)$. View the restricted Π as a representation on $\mathrm{Sp}(4, \mathbb{A})$ and $\chi|\ |^s$ as a character on $\mathrm{Gl}(1, \mathbb{A})$. Consider

$$H_{\Pi\otimes\chi}(s) = \mathrm{Ind}_{Q(\mathbb{A})}^{\mathrm{Sp}(6,\mathbb{A})}(\Pi \otimes \chi|\ |^s)$$

and similarily the local analogs $H_{p,\Pi\otimes\chi}(s)$. Identify $H_{\Pi\otimes\chi}(s)$ and $H_{\Pi\otimes\chi}(0)$ via the map

$$\varphi(g) \mapsto \varphi_s(g) = |A(g)|^{\rho+s}\varphi(g).$$

Then the scattering operator $M(\chi, s)$, defined by the integral

$$(M(\chi, s)\varphi)(g) = |A(g)|^{s-\rho} \int_{\bar{N}(\mathbb{A})} \varphi(\bar{n}wg)|A(\bar{n}wg)|^{\rho+s}\, d\bar{n},$$

is well defined for all s with $\mathrm{Re}(s)$ large enough, if $\varphi(g)$ is in $H_\mu(0)$ and K-finite under right multiplication with respect to a suitable maximal compact subgroup K of $\mathrm{Sp}(6, \mathbb{A})$. (See [A], p. 255 in the case $P = P_1 = Q_1$ and $w_s = w^{-1}$ or [L], page 277). $M(\chi, s)$ defines therefore a map from a dense subspace of the fixed Hilbert space $H_\mu(0)$ to itself.

For products $\varphi = \Pi_p \varphi_p$ in $H_\chi(0)$ the operator $M(\chi, s)$ decomposes into a product of local operators $M_p(\chi_p, s)$.

$$M(\chi, s)\varphi = \prod_p M_p(\chi_p, s)\varphi_p.$$

The local operators are given by the corresponding local integrals. It is known, that both global and the local operators admit meromorphic continuation to the complex plane.

Furthermore it is known, that for nonarchimedian places p there exists for every p a function φ_p and a point g_p, such that $(M_p(\chi_p, s)\varphi_p)(g_p) \neq 0$ is independent from s. Choose such functions for all nonarchimedian p in $S(\chi)$, the union of S and the places, where χ is ramified.

For $p \notin S(\chi)$ choose φ_p spherical in the Hilbert space $H_{\mu_v}(0)$. Then

$$M_p(\chi_p, s)\varphi_p = \zeta_p(\Pi_p, \chi_p, s)/\zeta_p(\Pi_p, \chi_p, s+1)\cdot \varphi_p$$

can be computed in terms of the standard L-series

$$\zeta(\Pi, \chi, s)$$

of the group $\mathrm{Sp}(4) \times \mathrm{Gl}(1)$, attached to the 5 dimensional standard representation of the L-group $\mathrm{SO}(5, \mathbb{C}) \times \mathrm{Gl}(1, \mathbb{C})$. For $p \notin S(\chi)$ and the unramified subquotient Π_p of $\Pi(\tau_p, z)$ the local term $\zeta_p(\Pi_p, \chi_p, s)$ is given by

$$\zeta_p(\Pi_p, \chi_p, s) = L_p(\chi_p, s)L_p(\tau_p \otimes \chi_p, s+z)L_p(\tau_p \otimes \chi_p\omega_{\Pi_p}^{-1}, s-z).$$

Under the condition (b) using especially $z = -\frac{1}{2}$, a short but clumsy calculation of the product $M^{S(\chi)}(\chi, s) = \Pi_{p \notin S(\chi)} M_p(\chi_p, s)$ over all p not in $S(\chi)$, thereby distinguishing the two cases $\omega_{\Pi_p} = 1$ and $\omega_{\Pi_p} \neq 1$, gives

$$\Lambda(\chi, s - \tfrac{1}{2}) = \Lambda(\chi, s + \tfrac{3}{2})M^{S(\chi)}(\chi\omega_\Pi, s)\,\frac{L(\chi\omega_\Pi, s)(L(\chi, s+1)L(\chi\omega_\Pi^2, s+1)}{L(\chi, s)L(\chi\omega_\Pi^2, s)L(\chi\omega_\Pi, s+1)}.$$

This identity is valid for all characters χ such that χ^2 is ramified highly enough at all primes in S. Also by abuse of notation $M^{S(\chi)}(\chi\omega_\Pi, s)$ is understood to be the spherical eigenvalue of the corresponding operator.

COROLLARY. *Analytic continuation for* $\Lambda(\chi, s)$ *into the region* $\mathrm{Re}(s) \geqslant \frac{1}{2}$ *is a consequence of analyticity of* $M(\chi\omega_\Pi, s)M_\infty(\chi_\infty\omega_{\Pi\infty}, s)^{-1}$ *in the region* $\mathrm{Re}(s) \geqslant 1$.

The same formula above also implies, that $\Lambda(\chi, s)$ has at most polynomial growth in $\mathrm{Im}(s)$ in vertical strips outside of certain compact sets. This follows from well known estimates for the local and global scattering operators. We postpone the proof to first gather some necessary information on the archimedian place.

The archimedian place

Let B denote the Borel subgroup in $\mathrm{Sp}(4, k)$ given by all matrices $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, where A is a lower triangular matrix. The operators $M_\infty(\chi, s)$ are controlled by the next

LEMMA 4. *Suppose there exists an embedding*

$$\Pi \hookrightarrow \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{Sp}(4,\mathbb{R})}(\tau),$$

where τ *is a character of* $B(\mathbb{R})$, *trivial on the unipotent radical, such that*

$$\tau(\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-2})) = a_1^n a_2^m, \qquad a_1, a_2 > 0$$

for integers $n, m \in \mathbb{Z}$. *Then* $M_\infty(\chi_\infty, s)$ *and* $M_\infty(\chi_\infty, s)^{-1}$ *are holomorphic on* $\mathbb{C}\backslash\mathbb{Z}$ *and of polynomial growth in vertical strips.*

Proof. The technique explained in [KZ], page 95 and an explicit computation, which will be skipped, shows that the operator $M_\infty(\chi, s)$ can be factorized into a product

$$M_\infty(\chi_\infty, s) = A_1(\chi_\infty, s+1) \circ A_2(\chi_\infty, s-2) \circ A_3(\chi_\infty, s) \circ A_2(\chi_\infty, s+2) \circ A_1(\chi_\infty, s-1)$$

of five operators, where each of these operators can be computed using embedded groups $\mathrm{SL}(2, \mathbb{R})$. Each of the five operators $A_i(\chi_\infty, s)$, when restricted to K_∞ types with respect to a maximal compact subgroup K_∞, can be diagonalised with entries of the form $a_k(s - s_j)$. The coefficients k are integers, which describe the character decomposition of the given K_∞-type, under restriction to certain embedded subgroups $SO(2, \mathbb{R})$. The factors $a_k(s)$ are essentially the eigenvalues of the scattering operators for the group $Sl(2, \mathbb{R})$. Because the character τ, which determines the induced representation under consideration, has integral exponents in the diagonal coordinates, all the values s_j turn out to be integral. But the functions $a_k(s)$ have zeros and poles only for integral values of s. This follows from explicit formulas for the functions $a_k(s)$ expressing them in terms of the Γ-functions. See [Wa], Lemma 7.17. This proves our claim. In addition it follows from Stirling's formula, that $a_k(s)$ and $a_k(s)$ are at most of polynomial growth in vertical strips of bounded width. The same then follows for the entries of the operator valued functions $M_\infty(\chi_\infty, s)$ and $M_\infty(\chi_\infty, s)^{-1}$. The lemma is proved.

We will show now, that the assumptions of the lemma are fullfilled in our situation. This together with the last corollary immediately implies analytic continuation of $\Lambda(\chi, s)$ to all nonreal point of the plane. This uses the well known fact, that $M(\chi, s)$ does not have poles in the region $\mathrm{Re}(s) \geqslant 0$ outside the real axis. See [L], p. 136 and also [L], appendix II.

By the archimedian assumption (1) the infinite component Π_∞ of Π is fixed. In terms of the invariant r it is either the holomorphic highest weight representation

$$\Pi_\infty^{\mathrm{hol}}(r + 1)$$

of $GSp(4, \mathbb{R})$ of weight $r + 1$ or a certain nonholomorphic representation

$$\Pi_\infty^{\mathrm{skew}}(r + 1)$$

having the same infinitesimal character as the holomorphic representation $\Pi_\infty^{\mathrm{hol}}(r + 1)$. With respect to the identification $PGSp(4, \mathbb{R}) = SO(3, 2)$ this follows from theorem 2.1a) and (b), remark 2.1 and theorem 4.3 of [R2]. The parameter $s > 1$ congruent $\frac{5}{2}$ mod 1 in loc. cit. corresponds to our $r + \frac{1}{2}, r \geqslant 1$. This follows from theorem 2.1(c) loc. cit. and an easy evaluation of the eigenvalue of the Casimir operator for the group $\overline{Sl(2, \mathbb{R})}$. On the other hand the integrality assumption on the character τ in lemma 4 is not changed under the action of the Weyl group. Hence this assumption depends only on the infinitesimal character of the representation Π_∞, by the Casselman subrepresentation theorem. So it is enough to check the holomorphic case. In the holomorphic case a valid choice of character τ is given by the exponents $(n, m) = (r, r - 1)$. This verifies the assumption.

Boundedness in vertical strips

Let us finally show that $\Lambda(\chi, s)$ is of polynomial growth in $\mathrm{Im}(z)$ in vertical strips, provided one stays away from the finitely many real poles. It is enough to consider regions $|\mathrm{Re}(s)| \leqslant C$, $|\mathrm{Im}(s)| \geqslant t_0$ for arbitary large numbers C and t. By the maximum modulus theorem it is enough to consider values on the boundary of boxes $\mathrm{Im}(s) = t$, $|\mathrm{Re}(s)| \leqslant C$ and $\mathrm{Re}(s) = \pm C$, where t runs over a suitable strictly increasing sequence of positive (and negative) real numbers. If C is in the domain of absolute convergence of $\Lambda(\chi, s)$, $\Lambda(\chi, s)$ is bounded on the axes $\mathrm{Re}(s) = C$. By Stirling's formula and the functional equation it is then of polynomial growth on the axis $\mathrm{Re}(s) = 1 - C$.

Again by the functional equation we can now restrict outselves to discuss the function on the lines $\mathrm{Im}(s) = t$ and $\frac{1}{2} \leqslant \mathrm{Re}(s) \leqslant C$ for a strictly increasing sequence of positive (and negative) real numbers t. For that we use the formula proving the last corollary. Iteration gives

$$\Lambda(\chi, s - \tfrac{1}{2}) = \prod_j L(\mu_j, s + m_j)^{\varepsilon_j} \prod_{i=0}^{v} M^{S(\chi)}(\chi\omega_\Pi, s + 2i)\Lambda(\chi, s + 2i + \tfrac{3}{2})$$

for certain numbers $m_j \geqslant 0$, $\varepsilon_j \in \{+1, -1\}$ and certain Dirichlet characters μ_j. We can assume $2i + 2$ to be in the domain of absolute convergence of $\Lambda(\chi, s)$. One gets

$$|\Lambda(\chi, s)| \leqslant \prod_{i=0}^{v} |M^{S(\chi)}(\chi\omega_\Pi, s + 2i + \tfrac{1}{2})| \cdot |P(t)|$$

for all s with $\frac{1}{2} \leqslant \mathrm{Re}(s) \leqslant C$ and $\mathrm{Im}(s) = t$, where $P(t)$ can be chosen to be a polynomial in t. This follows from well known estimates (see e.g. [Sch], page 162, 96, 92, 151) for Dirichlet series $|L(\mu, s)|$ respectively $|L(\mu, s)|^{-1}$ in the region $\mathrm{Re}(s) \geqslant 1$ and $|\mathrm{Im}(s)| \geqslant t_0$, where t_0 is a fixed positive number.

In order to estimate the partial scattering operator on a suitable finite dimensional vectorspace of a fixed K type use

$$|M^{S(\chi)}(\chi\omega_\Pi, s)| \leqslant \|M(\chi\omega_\Pi, s)\| \cdot \prod_{p \in S(\chi)} \|M_p^{-1}(\chi_p \omega_{\Pi_p}, s)\|.$$

From [L], page 136 we know the boundedness $\|M(\chi\omega_\Pi, s)\| \leqslant B$ for some constant B in regions of the type $0 \leqslant \mathrm{Re}(s) \leqslant C$ and $|\mathrm{Im}(s)| \geqslant t_0$, where t_0 is a fixed positive number. This together with lemma 4 shows, that it is enough to estimate the inverse of the finitely many remaining nonarchimedian operators $M_p(\chi_p \omega_{\Pi_p}, s)^{-1}$.

But the finitely many local operators $M_p(\chi_p\omega_{\Pi_p}, s)$ for the nonarchimedean $p \in S(\chi)$ are periodic with the periods $\frac{2\pi i}{\log(p)}n$, $n \in \mathbb{Z}$ respectively. We can find numbers x_p, such that each $M_p(\chi_p\omega_{\Pi_p}, s)$ and $M_p(\chi_p\omega_{\Pi_p}, s)^{-1}$ is holomorphic in the strip

$$-\varepsilon \leqslant \mathrm{Im}(s) - \frac{2\pi i x_p}{\log(p)} \leqslant \varepsilon \quad \text{and} \quad \tfrac{1}{2} \leqslant |\mathrm{Re}(s)| \leqslant C.$$

According to an elementary approximation argument we can find (a sequence of) arbitrarily large positive or negative real numbers t, and for each t integers n_p such that $\left| t - \frac{2\pi i}{\log(p)}(x_p + n_p) \right| < \varepsilon$ holds for all nonarchimedean primes $p \in S(\chi)$. Hence the finite product of operators $M_p(\chi_p\omega_{\Pi_p}, s)^{-1}$ for the primes $p \in S(\chi)$ has bounded norm for all values of s in the strip $\frac{1}{2} \leqslant |\mathrm{Re}(s)| \leqslant C$ and $\mathrm{Im}(s) = t$. The bound is independent from the particular t. If we substitute this information in our estimate this completes the proof: $\Lambda(\chi, s)$ has polynomial growth in vertical strips. After multiplying $\Lambda(\chi, s)$ with an additional Γ-factor, as in [W], we get a function, which is bounded in vertical strips. As already explained the converse theorem of [W] then proves the implication (b)$\Rightarrow$(a).

The Koecher effect

Suppose now Π is holomorphic, i.e.

$$\Pi_\infty \cong \Pi_\infty^{\mathrm{hol}}(r+1)$$

in our previous notation. Then the only type of global Whittaker datum (and also locally at ∞) is of the form (T, v, ψ) with

$$v_\infty = 1 \quad \text{and} \quad T \text{ definite.}$$

Whether T is positive or negative definite depends on the choice of ψ. This follows from the classical Koecher effect for holomorphic modular forms and shows, that the archimedean assumption (2) is a consequence of assumption (1) in this case.

Let us consider the nonholomorphic case now. Fix some

$$\Pi_\infty \cong \Pi_\infty^{\mathrm{skew}}(r+1).$$

An explicit computation of Fourier coefficients (see [PS2], p. 322ff) shows, that any global Whittaker datum (T, v, ψ) for a Saito-Kurokawa lift $\Pi = \theta(\sigma, \psi)$, as in assertion (1) of the theorem, which is of this nonholomorphic type Π_∞ at infinity, has the property $v_\infty = 1$ and furthermore T turns out to be indefinite! By comparison we find, that any nonholomorphic automorphic cuspidal representation Π with this Π_∞ and property (b) and (1), which admits a global Whittaker datum (v, T, ψ) with

$$v_\infty = 1 \quad \text{and } T \text{ indefinite}$$

fulfills assumption (2). Therefore by the theorem it fulfills condition (a), or in other words is a Saito-Kurokawa lift. Of course Π then can have only global Whittaker models (T, v) with T_∞ indefinite and $v_\infty = 1$. This leads to the following

CONJECTURE. Every cuspidal irreducible automorphic representation Π on $GSp(4, \mathbb{A})$ with Π_∞ as in assumption (1) of the theorem has only global Whittaker data (T, v, ψ), such that $v_\infty = 1$ and such that T is definite or indefinite according to whether Π_∞ is holomorphic or not.

The truth should be, that the same holds already locally at the infinite place: Any local representation Π_∞ as in assumption (1) should only have one type of archimedian Whittaker model (T_∞, v_∞), namely the one given in the conjecture above. This also should lead to a proof of the conjecture.

Evidently if the conjecture were true, then condition (2) is a consequence of (1). Furthermore (1) and (b) then provide a criterion for detecting Saito-Kukokawa liftings in purely local terms, depending only on the local representations Π_v for almost all places including the archimedian one. However, as long as the conjecture is not proven, this is true only if Π_∞ is assumed to be holomorphic. Because the conjecture then is a consequence of the well known Koecher effect. We get as a

COROLLARY. *Let Π be a cuspidal, irreducible automorphic representation of* $GSp(4, \mathbb{A})$ *with central character ω_Π and holomorphic infinite component* $\Pi_\infty \cong \Pi_\infty^{\mathrm{hol}}(k)$ *of weight $k \geqslant 2$. Then Π is a Saito-Kukokawa lift iff $\omega_\Pi = 1$ and for almost all primes p the representation Π_p is a subquotient of some locally induced representation*

$$\Pi(\tau_p, -\tfrac{1}{2}) = \mathrm{Ind}_{P_p}^{GSp(4,k_p)}(\tau_p \otimes \lambda_p^{-1/2}).$$

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