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p -Divisible groups with complex multiplication over $W(k)^*$

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1. Statement of the result

We refer to Waterhouse [8, §4] for p -divisible groups with complex multiplication. Let R be a complete discrete valuation ring, with residue field k , algebraically closed of characteristic $p > 0$ and fraction field K of characteristic 0. Let \bar{K} be an algebraic closure of K and $\Gamma_K = \text{Gal}(\bar{K}/K)$. For a p -divisible group G over R of height h , denote its Tate module by $T(G)$ and let $V(G) = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T(G)$. Let E be an extension of \mathbf{Q}_p with degree h . We say G has complex multiplication by E if there is given a homomorphism of E into $\mathbf{Q}_p \otimes \text{End}(G) = \text{End}_{\Gamma_K} V(G)$. Then $V(G)$ is a one-dimensional E -vector space and the action of Γ_K on $V(G)$ is given by a continuous homomorphism $\rho: \Gamma_K \rightarrow E^\times$. The action of E on $t_G(K) = K \otimes_R t_G$ (t_G is the tangent space to G) has character $\Sigma_\Phi \tau$ for some subset Φ of $\text{Hom}(E, \bar{K})$ (the set of all \mathbf{Q}_p -embeddings of E into \bar{K}) and we say G has type (E, Φ) . Then on an open subgroup of Γ_K , ρ is determined by Φ and this shows that any two p -divisible groups of prescribed type (E, Φ) are isogenous ([8, Theorem 4.1]).

Let E' be a subextension of E over \mathbf{Q}_p . For a subset Φ' of $\text{Hom}(E', \bar{K})$, write $\Phi'^E = \{\lambda \in \text{Hom}(E, \bar{K}) : \lambda|_{E'} \in \Phi'\}$. For $\Phi \subset \text{Hom}(E, \bar{K})$, we say that (E, Φ) is elementary if Φ is not of the form Φ'^E for any subextension $E' (\neq E)$ of E and for any subset Φ' of $\text{Hom}(E', \bar{K})$. A p -divisible group G of type (E, Φ) is said to be elementary if (E, Φ) is elementary. This is equivalent to saying that $E = \mathbf{Q}_p \otimes \text{End}(G)$. Any p -divisible group with complex multiplication is isogenous to a direct product of elementary groups of the same type ([8, p. 64]).

In this paper we assume that $R = W = W(k)$ the ring of Witt vectors over k and every p -divisible group G with complex multiplication is of type (E, Φ) with non-empty Φ ; this implies that G is a formal (Lie) group ([8, Corollary 4.4]). Write σ for the Frobenius automorphism of K . We denote by K_h the unique unramified extension of degree h over \mathbf{Q}_p in K and by W_h its maximal order.

Our theorem gives the complete classification of p -divisible groups with complex multiplication over W up to isomorphism.

*Dedicated to Professor Tsuneo Kanno on his 60th birthday.

THEOREM. (i) For any type (K_h, Φ) , there exists a p -divisible group over W of type (K_h, Φ) . A p -divisible group G over W with complex multiplication of height h is elementary if and only if $\text{End}(G) \simeq W_h$.

(ii) Let G be a p -divisible group over W of height h with complex multiplication. Then G is isomorphic over W to a direct product of several copies of an elementary group G_1 over W .

(iii) Any two p -divisible groups over W with complex multiplication of prescribed type (K_h, Φ) are isomorphic over W .

In case $\dim G = 1$, the assertion of our theorem follows from [4, Proposition 3.6] and [7].

REMARK. Let

$$(K_h, \{\sigma^{e_1}, \dots, \sigma^{e_n}\}) (0 \leq e_1 < \dots < e_n < h)$$

be elementary. Clearly this is the same as to say that the period of the map $i: \mathbf{Z}/h\mathbf{Z} \rightarrow \{0, 1\}$ with $i(e_k) = 1$ ($k = 1, \dots, n$) and $i(j) = 0$ ($j \neq e_1, \dots, e_n$) is h . Define χ_h to be the composite homomorphism

$$\Gamma_K \rightarrow I \xrightarrow{d} W_h^\times \xrightarrow{i} W_h^\times$$

where I is the inertia subgroup of $\text{Gal}(\bar{K}_h/K_h)$, d is the map given by classfield theory, and $i(x) = x^{-1}$ for $x \in W_h^\times$. Let G be of type $(K_h, \{\sigma^{e_1}, \dots, \sigma^{e_n}\})$. Then the p -adic representation $\rho: \Gamma_K \rightarrow K_h^\times$ attached to G is a crystalline (or B -admissible) abelian representation in the sense of Fontaine [2], [3]. As χ_h is crystalline, it follows that $\rho = \prod_{i=1}^n \sigma^{-e_i} \circ \chi_h$ on Γ_K (see [6, Chapter III Appendix] and [2, §3]).

2. A construction of p -divisible groups with complex multiplication over W

LEMMA 1. Let G be an n -dimensional p -divisible group over W of height h which has complex multiplication by E , then it has also complex multiplication by K_h (but, in general, E is not isomorphic to K_h).

Proof. First assume that G is elementary of type (E, Φ) . Let E' be the maximal unramified subextension of E . By the operation of $\text{End}(G)$ on the tangent space of G we obtain a homomorphism $j: E \rightarrow M_n(K)$ (the full matrix ring of order n over K). Then the character of j is $\sum_{\Phi} \lambda$. If the restriction of j to E' has a character $\sum_{\Phi} \tau$, then j is equivalent to the direct product of the regular representations of E over $\tau(E')$ for $\tau \in \Phi'$. This implies $\Phi = \Phi'^E$ and since G is elementary, we have $E = E'$ and $E = K_h$. If G is not elementary, then G is isogenous to a direct product of elementary groups of the same type (cf. [8, p. 64]). One can now verify at once that $\mathbf{Q}_p \otimes \text{End}(G)$ contains a subfield isomorphic to K_h . This completes the proof.

We will now construct an *n*-dimensional *p*-divisible group G_0 over W of type (K_h, Φ) where $\Phi = \{\sigma^{e_1}, \dots, \sigma^{e_n}\}$, $0 \leq e_1 < \dots < e_n < h$. We use a result on a classification of commutative formal groups over W by systems of Honda (cf. [1, Chapter IV and V §2] and [4]). Let $D = W_\sigma[[F]]$ be the non-commutative power series ring on F with the multiplication rule; $Fa = a^\sigma F$ for $a \in W$. Let $A_{n/h} = K_h[\theta]$ denote the associative K_h -algebra with unit generated by θ such that $\theta^h = p^n$, $\theta a = a^\sigma \theta$ ($a \in K_h$). It is the central simple algebra of rank h^2 over \mathbf{Q}_p and invariant n/h . Consider the left K -space

$$M_{n/h} = K \otimes_{K_h} A_{n/h}.$$

It is a K -space with basis $\theta^i = 1 \otimes \theta^i$ ($i = 0, \dots, h - 1$) and a right $A_{n/h}$ -space. We define a D -module structure on $M_{n/h}$ by putting $F\theta^i = \theta^{i+1}$. The D -endomorphisms of $M_{n/h}$ are the right multiplications by elements of $A_{n/h}$ (cf. [5, Chapter III §4]). Now we put

$$\xi_i = p^{n-i} \theta^{e_i} \quad (i = 1, \dots, n).$$

Let L_0 (respectively M_0) be the W -submodule (respectively D -submodule) of $M_{n/h}$ generated by ξ_1, \dots, ξ_n . Then we can easily check that (L_0, M_0) is a system of Honda. Let G_0 be the *p*-divisible group over W associated to (L_0, M_0) . Put $g(0) = h + e_1 - e_n$ and $g(i) = e_{i+1} - e_i$ ($1 \leq i \leq n - 1$), then G_0 corresponds to a special element

$$u = pI - \begin{pmatrix} 0 & \dots & 0 & F^{g(0)} \\ F^{g(1)} & \dots & & 0 \\ \vdots & \dots & & \vdots \\ 0 & \dots & F^{g(n-1)} & 0 \end{pmatrix}$$

Let $D(a) = \text{diag}(a^{\sigma^{e_1}}, \dots, a^{\sigma^{e_n}})$ for $a \in W_h$. Then we have $D(a)u = uD(a)$. Therefore G_0 is of type (K_h, Φ) and $\text{End}(G_0) \supset W_h$ (see [4, Theorem 3]).

Now let $f = (h, n)$ and $h = fh_1, n = fn_1$. We extend g to a function on $\mathbf{Z}/n\mathbf{Z}$ by

$$g(i + n\mathbf{Z}) = g(i) \quad \text{for } i = 0, \dots, n - 1.$$

LEMMA 2. *Let r be the least positive divisor of n such that g is a function on $\mathbf{Z}/r\mathbf{Z}$. Put $\Phi_1 = \{\sigma^{e_1}, \dots, \sigma^{e_r}\}$.*

(i) *Then r is a multiple of n_1 and if we put $r = f_0 n_1$, we have*

$$e_{k+r} = e_k + f_0 h_1 \quad (1 \leq k \leq n - r).$$

(ii) Let G_1 be the group which is constructed from $(K_{f_0h_1}, \Phi_1)$ as above. Then G_1 is elementary and $\text{End}(G_1) \simeq W_{f_0h_1}$.

(iii) G_0 is isomorphic over W to $(G_1)^{f/f_0}$.

Proof. Since $\sum_{i=0}^{n-1} g(i) = h$, we have for $1 \leq k \leq n - r$

$$g(k) + \dots + g(k + r - 1) (= e_{k+r} - e_k) = hr/n = h_1r/n_1.$$

This shows that r is a multiple of n_1 and (i) follows. If $(K_{f_0h_1}, \Phi_1)$ is not elementary, there exists a divisor $f' (\neq f_0)$ of f_0 such that $\{e_1, \dots, e_{f'n_1}\} (e_{f'n_1} < f'h_1)$ is a complete system of representatives of $\{e_1, \dots, e_r\} \bmod f'h_1$ and it is also that of $\{e_1, \dots, e_n\} \bmod f'h_1$. This gives $e_{k+f'n_1} - e_k = f'h_1$. Then

$$g(i + f'n_1) - g(i) = (e_{i+1+f'n_1} - e_{i+1}) - (e_{i+f'n_1} - e_i) = 0.$$

This contradicts to the choice of r . Since $\text{End}(G_1) \supset W_{f_0h_1}$, (ii) is clear. Let us prove (iii). Put $s = f/f_0$ and $\eta_i = \sum_{k=0}^{s-1} \xi_{i+kr} (i = 1, \dots, r)$. Let L_1 (respectively M_1) be the W -submodule (respectively D -submodule) of M_0 generated by η_1, \dots, η_r . Then (L_1, M_1) is isomorphic to the system of Honda associated to G_1 and

$$(L_0, M_0) = (L_1, M_1)\omega_1 \oplus \dots \oplus (L_1, M_1)\omega_s$$

for a basis $\{\omega_1, \dots, \omega_s\}$ of $W_h/W_{f_0h_1}$. This proves (iii).

LEMMA 3. Let (L, M) be a system of Honda such that

$$(L_0, M_0) \supset (L, M) \supset p(L_0, M_0).$$

Then we have $\text{End}(L, M) \supset W_{f_0h_1}$ where f_0h_1 is as in Lemma 2.

Proof. We may suppose that $L \neq pL_0$. Let $x \in L - pL_0$. As x can be uniquely expressed in the form

$$x = \sum_{i=1}^n a_i \xi_i \quad (a_i \in W)$$

we write

$$S(x) = \{i: 1 \leq i \leq n, a_i \not\equiv 0 \pmod p\}.$$

Put

$$d = \text{Max}\{g(i): i \in S(x)\}, \quad A = \{i \in S(x): d = g(i)\} \quad \text{and} \quad y = \sum_{i \in A} a_i \xi_i.$$

Since

$$F^d \xi_i = F^{d-g(i)} F^{g(i)} \xi_i = p F^{d-g(i)} \xi_{i+1},$$

we have

$$F^d y = p \sum_{i \in A} a_i^{\sigma^d} \xi_{i+1} = F^d x - F^d(x - y) \in pL_0 \cap FM.$$

Here we put $\xi_{n+1} = \xi_1$. As $L \cap FM = pL$, we obtain an element $\delta(x) = \sum_{i \in A} a_i^{\sigma^d} \xi_{i+1}$ of L . The n th iteration of the operation δ gives an element x' of $L - pL_0$ which satisfies

$$g(i + k) = g(j + k) \quad \text{for } i, j \in S(x') \quad \text{and } k \in \mathbf{Z}. \tag{*}$$

Clearly L is generated by $\{x' : x \in L - pL_0\}$ and pL_0 over W . Now (*) implies that g is a function on $\mathbf{Z}/(j-i)\mathbf{Z}$. Then $j - i$ is a multiple of r and by Lemma 2(i) we have $e_i \equiv e_j \pmod{f_0 h_1}$. Therefore for $i \in S(x')$, $x'a = a^{\sigma^i} x'$ ($a \in W_{f_0 h_1}$). This shows that $\text{End}(L, M) \supset W_{f_0 h_1}$.

3. Proof of the theorem

Let G be a p -divisible group with complex multiplication over W of height h . By Lemma 1 G is of type (K_h, Φ) . Let G_0 be the group of type (K_h, Φ) constructed in Section 2. We claim that G and G_0 are isomorphic. By [8, Theorem 4.1] there exists an isogeny $\alpha: G \rightarrow G_0$ over W and α defines an injection $T(G) \rightarrow T(G_0)$. Hence we may assume that $T(G) \subset T(G_0)$. There is an integer m such that $p^m T(G_0) \subset T(G)$. Let $T_i = T(G) + p^i T(G_0)$ ($i = 0, 1, \dots, m$). Then T_i is a Γ_K -sublattice of $T(G_0)$. Hence $T_i = T(H_i)$ for some group H_i over W ([8, Theorem 1.3]). Since $T_0 \supset T_1 \supset pT_0$, the system of Honda of H_1 satisfies the condition of Lemma 3. Therefore $\text{End}(H_1) \supset W_{f_0 h_1}$. By Lemma 2, $T_0 = T(G_0) \simeq \bigoplus_s T(G_1)$ is a free $W_{f_0 h_1}$ -module of rank s and T_1 is a $W_{f_0 h_1}$ -sublattice of T_0 . Then T_0 and T_1 are $W_{f_0 h_1}$ -isomorphic, and also Γ_K -isomorphic, since the operation of Γ_K is given by the p -adic representation

$$\Gamma_K \rightarrow W_{f_0 h_1}^\times = \text{Aut}_R T(G_1) \subset \text{Aut}(\mathbf{Q}_p \otimes T_0) = \text{Aut}(\mathbf{Q}_p \otimes T_1)$$

where $R = \text{End}(G_1) \simeq W_{f_0 h_1}$. Proceeding inductively, we see that $T_m = T(G)$ is Γ_K -isomorphic to $T(G_0)$. This implies that G and G_0 are isomorphic over W . Our theorem now follows immediately from Lemma 2 (ii), (iii).

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