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Kyo Nishiyama

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Decomposing oscillator representations of $\mathfrak{osp}(2n/n; \mathbb{R})$ by a super dual pair $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathfrak{so}(n)$*

KYO NISHIYAMA

Institute of Mathematics, Yoshida College, Kyoto University, Kyoto 606, Japan

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Introduction

The aim of this article is to exhibit fruitful structures of representations of orthosymplectic Lie super algebras. Of course for theoretical physicists Lie super algebras are indispensable tools, among them orthosymplectic algebras are of particular importance (for example, see [1], [4], [5], [13]). However, it seems that mathematicians are less interested in the theory of Lie super algebras. Even R. Howe would not use the notion of Lie super algebras in spite of his excellent works on invariant theory of commuting and anti-commuting variables (see, e.g., [8]).

Recently the author was aware of the importance of the role of Heisenberg super algebra in the unified theory of Weil representations (commuting variables) for symplectic algebras and spin representations (anti-commuting variables) for orthogonal algebras (cf. [15]). Analysis of Weil representations requires the use of Heisenberg algebras ([7], [9], [14], [16] and so on). To extend the theory to spin representations, we need Heisenberg super algebras. Classically this is achieved by Dirac spinors (see, e.g., [12, §1]), and Heisenberg super algebra represents Dirac spinors with multiplications or bracket product. For these topics, we refer the readers to [10].

In this article we treat an orthosymplectic Lie super algebra $\mathfrak{osp}(2n/n; \mathbb{R})$ and its super dual pair $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathfrak{so}(n)$. It seems interesting to decompose the oscillator representation of $\mathfrak{osp}(2n/n; \mathbb{R})$, which is unitary by the result of [10], when restricted to $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathfrak{so}(n)$. This clarifies the correspondence between irreducible unitary representations of $\mathfrak{osp}(2/1; \mathbb{R})$ and finite dimensional representations of $\mathfrak{so}(n)$ (Howe's correspondence).

The similar investigation is very useful for the dual pair $\mathfrak{osp}(2m/n; \mathbb{R}) \times \mathfrak{so}(N)$ in $\mathfrak{osp}(2mN/nN; \mathbb{R})$ in order to get large family of unitary representations of orthosymplectic algebras as in [11]. However, we do not succeed in decomposing oscillator representations completely by general super dual pairs yet. Therefore we hope this article, treating very special cases, becomes a beginning point of such investigations.

*Dedicated to Professor Nobuhiko Tatsuuma on his 60th birthday.

Let us explain each section briefly. After preparing some basic notions in section 1, we review the definition of oscillator representations in sections 2 and 3 (Theorem 3.2). The heart of this article consists in section 4, where a most simple example of 'super' dual pairs in $\mathfrak{osp}(2n/n; \mathbb{R})$ is given. We treat a pair $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathfrak{so}(n)$. It may seem strange that a 'super' dual pair is a pair of a Lie super algebra and a usual Lie algebra. We will give more general theory of super dual pairs in [11], in which a super dual pair is a pair of Lie super algebras. However, it seems very difficult to treat the same problem as in this article.

For the pair $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathfrak{so}(n)$, we study the decomposition of the oscillator representation and solve the problem completely (Propositions 4.2 and 4.4). The way of the decomposition produces interesting examples of analysis of commuting and anti-commuting variables. This is a very useful method to obtain various super unitary representations for $\mathfrak{osp}(m/n; \mathbb{R})$ as commented above. For $\mathfrak{osp}(2/1; \mathbb{R})$, this is a reproduction of a part of the results in [6] (Corollary 4.5).

It seems that the notion of super groups is not fully established yet. However, it is necessary to consider the super dual pair of super groups to produce Howe's duality for our pair. In the present article we only consider the situation where one of the pair, namely $\mathfrak{so}(n)$, is replaced by a Lie group $\mathrm{Spin}^{\pm}(n)$. For this pair $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathrm{Spin}^{\pm}(2m)$, we obtain a duality theorem (Theorem 4.6).

1. Lie super algebras

1.1. *Definition and basic notions*

Let $K = \mathbb{R}$ or $\mathbb{C}$ (field of real numbers or complex numbers). We put $\varepsilon(\alpha, \beta) = (-1)^{\alpha\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$.

DEFINITION 1.1. A $\mathbb{Z}_2$-graded (non-associative) algebra $L = L_0 \oplus L_1$ over K with multiplication $[\cdot, \cdot]$ is called a *Lie super algebra* (=LSA) if the bracket product $[\cdot, \cdot]$ satisfies the followings:

$$[L_\alpha, L_\beta] \subset L_{\alpha+\beta} \qquad (\alpha, \beta \in \mathbb{Z}_2), \tag{1.1}$$

$$[A, B] = -\varepsilon(\alpha, \beta)[B, A] \qquad (A \in L_\alpha,\ B \in L_\beta), \tag{1.2}$$

$$\varepsilon(\gamma, \alpha)[A, [B, C]] + \varepsilon(\alpha, \beta)[B, [C, A]] + \varepsilon(\beta, \gamma)[C, [A, B]] = 0 \tag{1.3}$$

$$(A \in L_\alpha,\ B \in L_\beta,\ C \in L_\gamma).$$

The condition (1.1) requires nothing but that L is a $\mathbb{Z}_2$-graded algebra. The conditions (1.2) and (1.3) are called *super skew symmetry* and *super Jacobi identity* respectively.

Let A be a $\mathbb{Z}_2$-graded associative algebra so that A is a direct sum of homogeneous subspaces $A = A_0 \oplus A_1$ and the multiplication is compatible with the grading: $A_\alpha A_\beta \subset A_{\alpha+\beta}(\alpha, \beta \in \mathbb{Z}_2)$. If we put

$$[x, y] = xy - \varepsilon(\xi, \eta)yx \qquad (x \in A_\xi,\ y \in A_\eta),$$

then with this bracket product, A is an LSA. The only thing to check is super Jacobi identity (1.3), but this is easily verified after some calculations. We always considered a $\mathbb{Z}_2$-graded associative algebra as an LSA in this fashion.

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. Then the endomorphism algebra $\mathrm{gl}(V)$ of V has natural $\mathbb{Z}_2$-graded algebra structure:

$$\mathrm{gl}(V)_\alpha = \{A \in \mathrm{gl}(V) | AV_\gamma \subset V_{\gamma+\alpha}\ (\gamma \in \mathbb{Z}_2)\}.$$

Since $\mathrm{gl}(V)$ is a $\mathbb{Z}_2$-graded associative algebra, it is an LSA by the above manner. We denote this LSA by $\mathrm{gl}(V; \varepsilon)$ or simply by $\mathrm{gl}(V)$ if there is no confusion.

DEFINITION 1.2. Let L be an LSA and V a $\mathbb{Z}_2$-graded vector space. We call a homomorphism $\rho: L \to \mathrm{gl}(V; \varepsilon)$ a *representation* of L.

2. Orthosymplectic algebra in Clifford-Weyl algebra

2.1. *Clifford-Weyl algebra and orthosymplectic algebra*

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space and b a bilinear form on V such that

$$b(x, y) = -\varepsilon(\xi, \eta)b(y, x) \qquad (x \in V_\xi,\ y \in V_\eta),$$

where $\varepsilon(\xi, \eta) = (-1)^{\xi\eta}$. We call b a super skew symmetric bilinear form. Moreover we assume that b is homogeneous of degree zero, i.e.,

$$b(x, y) \neq 0\ (x \in V_\xi,\ y \in V_\eta) \quad \text{implies} \quad \xi + \eta = 0.$$

Let $T(V)$ be a tensor algebra of V which is $\mathbb{Z}_2$-graded. Consider a bi-ideal $I(b)$ of $T(V)$ generated by the elements $x \otimes y - \varepsilon(\xi, \eta)y \otimes x - b(x, y)$, where $x \in V_\xi$ and $y \in V_\eta$. Then $I(b)$ is a homogeneous ideal. In fact, the term $x \otimes y - \varepsilon(\xi, \eta)y \otimes x$ is of degree $\xi + \eta$ and $b(x, y)$ does not vanish if and only if $\xi + \eta = 0$ by the assumption that b is homogeneous of degree zero. We define a Clifford-Weyl algebra as in [15].

DEFINITION 2.1 ([15, §2]). We put $C = C(V; b) = T(V)/I(b)$ and call it a *Clifford-Weyl algebra* for (V, b). The projection $T(V) \to C$ is denoted by p.

Let (V, b) be as above. We say that a homogeneous linear transformation $A \in \mathfrak{gl}(V)_\alpha$ preserves b if it satisfies

$$b(Ax, y) + \varepsilon(\alpha, \xi)b(x, Ay) = 0 \qquad (x \in V_\xi,\ y \in V_\eta). \tag{2.1}$$

We denote by $\mathfrak{osp}(b)$ all the sum of homogeneous linear transformations which preserves b:

$$\mathfrak{osp}(b) = \mathfrak{osp}(b)_0 \oplus \mathfrak{osp}(b)_1,$$

$$\mathfrak{osp}(b)_\alpha = \{A \in \mathfrak{gl}(V)_\alpha \mid A \text{ satisfies (2.1)}\}.$$

By the bracket product defined in section 1, $\mathfrak{osp}(b)$ is a sub LSA of $\mathfrak{gl}(V)$ and is called an *orthosymplectic algebra* with respect to b.

2.2. *Embedding of orthosymplectic algebra into Clifford-Weyl algebra*

In this subsection, after H. Tilgner [15], we show that orthosymplectic algebra $\mathfrak{osp}(b)$ can be realized in the second order elements of $C(V; b)$ if b is non-degenerate. More general situation for ε-graded Lie algebras can be found in [10].

For homogeneous elements $x \in V_\xi$ and $y \in V_\eta$, we put $m(x, y) = x \otimes y + \varepsilon(\xi, \eta) y \otimes x \in T_2(V)$. We define $L(b) \subset C(V; b)$ by

$$L(b) = \langle p(m(x, y)) | x, y\text{: homogeneous in } V \rangle / K\text{-vector space.}$$

Then $L(b)$ becomes a sub LSA of $C(V; b)$.

Since $V \simeq p(V) \subset C(V; b)$, we consider V as a subspace of $C(V; b)$. An element x of $C(V; b)$ acts on itself as an inner derivation: $\operatorname{ad} x = [x, \cdot]$. We call this representation the adjoint representation of $C(V; b)$. Note that the adjoint representation restricted to $L(b)$ preserves V. In fact, it holds that

$$\begin{aligned}[m(x, y), v] = 2[xy, v] &= 2(xyv - \varepsilon(\xi + \eta, v)vxy) \\ &= 2(b(y, v)x + \varepsilon(\xi, \eta)b(x, v)y),\end{aligned}$$

where $x \in V_\xi$, $y \in V_\eta$. Thus we get a representation of $L(b)$ on V.

THEOREM 2.2 ([15]). *The adjoint representation of* $L(b)$ *on* V *gives a homomorphism of* $L(b)$ *into* $\mathfrak{osp}(b)$. *Moreover, if* b *is non-degenerate, then the representation gives an isomorphism between* $L(b)$ *and* $\mathfrak{osp}(b)$.

Due to this theorem, we get a representation $\tilde{\tau}$ of $\mathfrak{osp}(b)$ from a representation τ of $C(V; b)$ by restricting τ to a sub LSA $L(b) \simeq \mathfrak{osp}(b)$.

3. Oscillator representation for orthosymplectic algebra

In this section we only treat superalgebras over $K = \mathbb{R}$. All the representations in this section are representations on complex vector spaces.

We assume that the super skew symmetric bilinear form b on $V = V_0 \oplus V_1$ is nondegenerate and there are basis $\{p_k, q_k \mid 1 \leqslant k \leqslant n\}$ of V_0 and $\{r_l, s_l \mid 1 \leqslant l \leqslant m\} \cup \{c\}$ of V_1 (c appears if and only if $\dim V_1$ is odd) such that

$$b(p_i, q_j) = -b(q_j, p_i) = \delta_{ij},\ b(p_i, p_j) = b(q_i, q_j) = 0,$$

and $\{r_l, s_l \mid 1 \leqslant l \leqslant m\} \cup \{c\}$ is an orthogonal basis in V_1 with respect to b with length $\sqrt{2}$.

Clifford-Weyl algebra $C(V; b)$ is then generated by p, q, r, s and c with relations:

$$p_i q_j - q_j p_i = \delta_{ij},\ r_i s_j + s_j r_i = 0,\ r_i r_j + r_j r_i = 2\delta_{ij},\ s_i s_j + s_j s_i = 2\delta_{ij},$$

and all the other pairs of p, q, r, s commute with each other. In addition to these, if $\dim V_1$ is odd, there are relations which contain c:

$$c^2 = 1,\ cr_i + r_i c = 0,\ cs_i + s_i c = 0,$$

c commutes with p_k and q_k.

DEFINITION 3.1. A representation (ρ, E) of an LSA L is called *super unitary* if there exists a super Hermitian form $(\cdot, \cdot)$ on E such that

$$(\rho(x)v, w) = -(-1)^{\xi\nu}(v, \rho(x)w) \qquad (x \in L_\xi,\ v \in E_\nu \text{ and } w \in E_\omega) \tag{3.1}$$

and

$$(v, w) = 0 \text{ if } \nu + \omega = 1,\ (v, v) > 0 \text{ for } v \in E_0 \backslash \{0\},$$
$$\delta\sqrt{-1}(w, w) > 0 \text{ for } w \in E_1 \backslash \{0\}. \tag{3.2}$$

Here $\delta = \pm 1$ is called an *associate constant* for (ρ, E) (see [6]).

Let $C_{\mathbb{R}}(r_l, c \mid 1 \leqslant l \leqslant m)$ (respectively $C_{\mathbb{R}}(r_l \mid 1 \leqslant l \leqslant m)$) be a subalgebra of $C_{\mathbb{R}}(V; b)$ which is generated by the elements $\{r_l \mid 1 \leqslant l \leqslant m\} \cup \{c\}$ (respectively $\{r_l \mid 1 \leqslant l \leqslant m\}$) over $\mathbb{R}$. Then $C_{\mathbb{R}}(r_l, c \mid 1 \leqslant l \leqslant m)$ is a usual Clifford algebra with usual $\mathbb{Z}_2$-grading $C^0_{\mathbb{R}}(r_l, c \mid 1 \leqslant l \leqslant m)$ and $C^1_{\mathbb{R}}(r_l, c \mid 1 \leqslant l \leqslant m)$. We denote by α_l an automorphism of $C_{\mathbb{R}}(r_l, c \mid 1 \leqslant l \leqslant m)$ such that

$$\alpha_l(r_i) = \begin{cases} r_i & \text{if } i \neq l \\ -r_l & \text{if } i = l \end{cases}, \qquad \alpha_l(c) = c.$$

THEOREM 3.2 ([10, Theorem 5.5]). *The representation* $(\tilde{\rho}, F)$ *of* $\mathfrak{osp}(b; \mathbb{R})$ *obtained from the representation* (ρ, F) *of* $C_{\mathbb{R}}(V; b)$ *given below is super unitary*:

$$F = F_0 \oplus F_1,$$
$$F_0 = \mathbb{C}[z_k | 1 \leqslant k \leqslant n] \otimes C^0_{\mathbb{C}}(r_l, c | 1 \leqslant l \leqslant m),$$
$$F_1 = \mathbb{C}[z_k | 1 \leqslant k \leqslant n] \otimes C^1_{\mathbb{C}}(r_l, c | 1 \leqslant l \leqslant m),$$

and the operators are given by

$$\begin{aligned}
\rho(p_k) &= \frac{\sqrt{-1}j}{\sqrt{2}}\left(z_k - \frac{\partial}{\partial z_k}\right) \otimes 1 && (1 \leqslant k \leqslant n)\\
\rho(q_k) &= \frac{j}{\sqrt{2}}\left(z_k + \frac{\partial}{\partial z_k}\right) \otimes 1 && (1 \leqslant k \leqslant n)\\
\rho(r_l) &= 1 \otimes r_l && (1 \leqslant l \leqslant m)\\
\rho(s_l) &= 1 \otimes \sqrt{-1}r_l\alpha_l && (1 \leqslant l \leqslant m)\\
\rho(c) &= 1 \otimes c
\end{aligned} \tag{3.3}$$

where j *is a square root of* $\sqrt{-1}$ *and* c *does not appear if* $\dim V_1$ *is even.*

We call this representation $(\tilde{\rho}, F)$ *oscillator representation.* Remark that this representation is not irreducible but has two irreducible components.

4. Super dual pair

4.1. *Explicit realization of the super dual pair*

After this section we assume that $n = 2m \geqslant 4$ and $\dim V_1 = 2m$ is even. Then we write $\mathfrak{osp}(2n/n; \mathbb{R}) = \mathfrak{osp}(b; \mathbb{R})$. Put

$$\sqrt{2}c_{2j-1} = r_j, \sqrt{2}c_{2j} = s_j \qquad (1 \leqslant j \leqslant m).$$

By Theorem 2.2, we identify $\mathfrak{osp}(2n/n; \mathbb{R})$ with a sub LSA of $C_{\mathbb{R}}(V; b)$. Hence $m(x, y)$ in section 2 is an element of $\mathfrak{osp}(2n/n; \mathbb{R})$. Put

$$P = \sum_{i=1}^{n} m(p_i, c_i); \quad Q = \sum_{i=1}^{n} m(q_i, c_i).$$

Then we have

$$[P, Q] = 2\sum_{i=1}^{n} m(p_i, q_i); \; [P, P] = 2\sum_{i=1}^{n} m(p_i, p_i); \; [Q, Q] = 2\sum_{i=1}^{n} m(q_i, q_i)$$

and

$\mathfrak{a} = \langle P, Q \rangle$/generated as an LSA
$= \langle P, Q, [P, Q], [P, P], [Q, Q] \rangle$/generated as a vector space over $\mathbb{R}$
turns to be isomorphic to $\mathfrak{osp}(2/1; \mathbb{R})$.

PROPOSITION 4.1 ([8, p. 552]). *Let $\mathfrak{a}'$ be a commutant of $\mathfrak{a} \simeq \mathfrak{osp}(2/1; \mathbb{R})$. Then $\mathfrak{a}'$ is a Lie algebra isomorphic to $\mathfrak{so}(n)$, and $\mathfrak{a}$ and $\mathfrak{a}'$ are commutants with each other.*

More explicitly, $\mathfrak{a}'$ has a basis

$$m(p_j, q_i) - m(p_i, q_j) + m(c_i, c_j) \qquad (1 \leqslant i \neq j \leqslant m).$$

In the matrix form, $\mathfrak{a}$ and $\mathfrak{a}'$ are of the following forms:

$$\mathfrak{a} = \left\{ \left[\begin{array}{cc|c} a1_n & b1_n & d1_n \\ c1_n & -a1_n & e1_n \\ \hline -e1_n & d1_n & 0_n \end{array} \right] \middle| \; a, b, c, d, e \in \mathbb{R} \right\},$$

$$\mathfrak{a}' = \left\{ \left[\begin{array}{cc|c} A & 0_n & 0_n \\ 0_n & A & 0_n \\ \hline 0_n & 0_n & A \end{array} \right] \middle| \; A \in \mathfrak{so}(n) \right\}.$$

4.2. *Decomposition of the oscillator representation and Howe's conjecture*

How does the oscillator representation break up as a representation of $\mathfrak{a} \times \mathfrak{a}'$? Lie algebra case tells us that the decomposition should be multiplicity free (for example, see [7, Theorem D]) and, in fact, that is true. We start with the decomposition of the representation of $\mathfrak{a}' \simeq \mathfrak{so}(n)$. We write oscillator representation (ρ, F) instead of $(\tilde{\rho}, F)$ for the sake of simplicity. Since $\tilde{\rho}$ is the restriction of the representation ρ of $C(V; b)$, there is no problem to do so.

PROPOSITION 4.2. *Let (ρ, F) be the oscillator representation of $\mathfrak{osp}(2n/n; \mathbb{R})$ considered above. Put*

$$PF(k) = (\textit{homogeneous polynomials of degree } k) \otimes C_{\mathbb{C}}(r_l \,|\, 1 \leqslant l \leqslant m) \subset F.$$

(1) *The space $PF(k)$ is invariant under the action of $\mathfrak{a}'$ and it breaks up as $\mathfrak{a}'$-module like this; for $k \geqslant 1$,*

$$PF(k) \simeq PF(k-2) \oplus \tau_1^+(k) \oplus \tau_1^-(k) \oplus \tau_2^+(k) \oplus \tau_2^-(k), \tag{4.1}$$

where $\tau_j^{\pm}(k)$ is an irreducible finite dimensional representation of $\mathfrak{so}(n)$ with highest weight listed below in (2). Intertwining map $PF(k-2) \to PF(k)$ is given by the multiplication by $(z_1^2 + z_2^2 + \cdots + z_n^2) \otimes 1$.

(1′) $PF(0) = C_{\mathbb{C}}(r_l \mid 1 \leqslant l \leqslant m) \oplus \tau_1^+(0) \oplus \tau_1^-(0)$.

(2) *Heighest weights and heighest weight vectors for $\tau_j^{\pm}(k)$ are explicitly given in Table I*

Table I

	highest weight	highest weight vector
τ_1^+	$(k+\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$	$v_k^+ = (z_1 + \sqrt{-1}z_2)^k \otimes 1$
τ_1^-	$(k+\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2})$	$v_k^- = (z_1 + \sqrt{-1}z_2)^k \otimes r_m$
τ_2^+	$(k-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$	$w_k^+ = (z_1 + \sqrt{-1}z_2)^{k-1}\{\Sigma_{j=1}^{m-1}(z_{2j-1} + \sqrt{-1}z_{2j}) \otimes r_j r_m + (z_{n-1} - \sqrt{-1}z_n) \otimes 1\}$
τ_2^-	$(k-\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{1}{2})$	$w_k^- = (z_1 + \sqrt{-1}z_2)^{k-1}\{\Sigma_{j=1}^{m}(z_{2j-1} + \sqrt{-1}z_{2j}) \otimes r_j\}$

Sketch of Proof. Since operators $(z_1^2 + z_2^2 + \cdots + z_n^2) \otimes 1$ and $\rho(\mathfrak{a}')$ commute, the map $f \otimes v \to (z_1^2 + z_2^2 + \cdots + z_n^2) f \otimes v$ gives an embedding $PF(k-2) \hookrightarrow PF(k)$. Let us see that the vectors listed in Table I are really highest weight vectors. Making use of the explicit formula (3.3) for ρ, we see positive simple root vectors of $\mathfrak{a}'$ act on F as

$$\begin{aligned}&\Big\{(z_{2j-1} + \sqrt{-1}z_{2j})\left(\frac{\partial}{\partial z_{2j+1}} \pm \sqrt{-1}\frac{\partial}{\partial z_{2j+2}}\right)\\ &\quad -(z_{2j+1} \pm \sqrt{-1}z_{2j+2})\left(\frac{\partial}{\partial z_{2j-1}} + \sqrt{-1}\frac{\partial}{\partial z_{2j}}\right)\Big\} \otimes 1\\ &\quad +1 \otimes r_j r_{j+1}(1-\alpha_j)(1 \mp \alpha_{j+1}) \qquad (1 \leqslant j \leqslant m-1)\end{aligned}$$

Now one can check that the vectors in Table I are killed by the operators above. Similarly their weights can be calculated and we conclude that they are highest weight vectors. Since dimensions of the both hand sides of (4.1) are equal by Weyl's dimension formula, we obtain desired decomposition. □

As in the arguments in [8], full orthogonal group $O(m)$ must have taken a major role in the above proposition. However, oscillator representation contains spin representations of $\mathfrak{so}(n)$ which cannot be integrated up to $SO(n)$ but the double cover Spin(n). So we first give a brief review on a spinor group.

Let G be a Clifford group in $C_{\mathbb{R}}(r_l, s_l | 1 \leqslant l \leqslant m)$:

$$G = \{x \in C_{\mathbb{R}}(r_l,\ s_l | 1 \leqslant l \leqslant m) | x \text{ is invertible and } x^{-1}Wx = W\},$$

where W is the real linear span of $\{r_l, s_l | 1 \leqslant l \leqslant m\}$. For $g \in G$ we put $\psi(g) = \mathrm{Ad}\, g|_W$. Then ψ is a surjective group homomorphism from G onto $O(W) \simeq O(n)$. Let β be an anti-automorphism of $C_{\mathbb{R}}(r_l, s_l | 1 \leqslant l \leqslant m)$ which is identity on W. Let $G^{\pm}$ be the set of all the homogeneous elements in G. Then $G^{\pm}$ becomes a subgroup in G and

$$N: G^{\pm} \ni g \mapsto \beta(g)g \in \mathbb{R}$$

is a group homomorphism. The morphism N is called spinorial norm. Put

$$G^{+} = G \cap C^{0}_{\mathbb{R}}(r_l,\ s_l | 1 \leqslant l \leqslant m) \text{ and } G^{-} = G \cap C^{1}_{\mathbb{R}}(r_l,\ s_l | 1 \leqslant l \leqslant m),$$

Then we have $G^{\pm} = G^{+} \cup G^{-}$. With these notations $G_0^{+} = \mathrm{Ker}\, N \cap G^{+}$ is isomorphic to $\mathrm{Spin}(n)$. In fact, the map ψ gives a double cover

$$\psi: G_0^{+} \simeq \mathrm{Spin}(n) \to SO(n) \subset O(W).$$

On the other hand, if we put $G_0^{\pm} = \mathrm{Ker}\, N \cap G^{\pm}$, then ψ gives a surjective double cover

$$\psi: G_0^{\pm} \to O(n).$$

Any non-zero $v \in W$ belongs to G^{-} and it belongs to G_0^{-} if and only if its length is $\sqrt{2}$, i.e., $v^2 = 1$. Note that $-\psi(v)$ represents a reflection with respect to the hyperplane which is orthogonal to v.

More informations on spinor groups can be found in [2, §9, no. 5] or [3, Chap. II, §XI].

Let us return to our subject. We keep the notations $G_0^{\pm}$ and ψ in the following. We define a representation $\mathscr{R}$ of $\mathfrak{so}(n)$ on $C_{\mathbb{C}}(r_l | 1 \leqslant l \leqslant m)$ by the similar formulas as in (3.3): for $v \in C_{\mathbb{C}}(r_l | 1 \leqslant l \leqslant m)$

$$\mathscr{R}(r_i)v = r_i v \quad (1 \leqslant i \leqslant m),$$

$$\mathscr{R}(s_i)v = \sqrt{-1}\, r_i \alpha_i v \quad (1 \leqslant i \leqslant m).$$

Of course, in this case, $\mathfrak{so}(n)$ is considered as a Lie subalgebra of second order elements in $C_{\mathbb{R}}(r_l, s_l | 1 \leqslant l \leqslant m)$. This representation $\mathscr{R}$ is one of the spin representations and can be integrated up to $\mathrm{Spin}(n) \simeq G_0^+$. Moreover a representation of $G_0^\pm$ can be naturally defined as $\mathscr{R}(g)$ ($g \in G_0^\pm \subset C_{\mathbb{R}}(r_l, s_l \mid 1 \leqslant l \leqslant m)$) if $\mathscr{R}$ is extended to $C_{\mathbb{R}}(r_l, s_l \mid 1 \leqslant l \leqslant m)$ as a representation of a Clifford algebra. Of course this representation agrees with the integrated one on the identity component G_0^+.

Next we consider an action $\mathscr{S}$ of $G_0^\pm$ on $\mathbb{C}[z_k \mid 1 \leqslant k \leqslant n]$. A matrix $A \in O(n)$ acts on the space $\langle z_k \mid 1 \leqslant k \leqslant n \rangle / \mathbb{C}$ by a matrix multiplication of A by ${}^t(z_1, z_2, \ldots, z_n)$ (natural representation of $O(n)$). Then $O(n)$ acts naturally on $\mathbb{C}[z_k \mid 1 \leqslant k \leqslant n]$ which is a symmetric tensor product of $\langle z_k \mid 1 \leqslant k \leqslant n \rangle / \mathbb{C}$. Now $G_0^\pm$ acts on $\mathbb{C}[z_k \mid 1 \leqslant k \leqslant n]$ by

$$G_0^\pm \xrightarrow{\psi} O(n) \xrightarrow[\text{natural representation}]{\text{symmetric tensor of}} GL(\mathbb{C}[z_k \mid 1 \leqslant k \leqslant n]).$$

We write this representation by $\mathscr{S}: G_0^\pm \to GL(\mathbb{C}[z_k \mid 1 \leqslant k \leqslant n])$.

Now the representation $\mathscr{S} \otimes \mathscr{R}$ of $G_0^\pm$ on $F = \mathbb{C}[z_k \mid 1 \leqslant k \leqslant n] \otimes C_{\mathbb{C}}(r_l \mid 1 \leqslant l \leqslant m)$ can be obtained. Clearly the differentiation of $\mathscr{S} \otimes \mathscr{R}$ agrees with the respresentation ρ for $\mathfrak{a}' \simeq \mathfrak{so}(n)$ obtained by considering super dual pair.

PROPOSITION 4.3. *Operators $\mathscr{S} \otimes \mathscr{R}(G_0^\pm)$ commute with $\rho(\mathfrak{a})$.*

Proof. Since operators $\mathscr{S} \otimes \mathscr{R}(G_0^+)$ are obtained by the integration of the operators $\rho(\mathfrak{a}')$, they commute with $\rho(\mathfrak{a})$. Note that $G_0^\pm$ is generated as a group by G_0^+ and $s_m \in G_0^-$. So only thing to check is if $\mathscr{S} \otimes \mathscr{R}(s_m)$ commutes with $\rho(\mathfrak{a})$. In the matrix form (with appropriate choice of orderings of basis $\{r_l, s_l \mid 1 \leqslant l \leqslant m\}$), we can see

$$\psi(s_m) = \begin{bmatrix} -1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \in O(n).$$

Let ζ_i be an autormophism of $\mathbb{C}[z_k \mid 1 \leqslant k \leqslant n]$ which sends z_j to $(-1)^{\delta_{ij}} z_j$ and put $\zeta^i = \zeta_i \Pi_{j=1}^m \zeta_j$. Then clearly we get

$$\mathscr{S} \otimes \mathscr{R}(s_m)(f \otimes v) = \zeta^n(f) \otimes \sqrt{-1}\, r_m \alpha_m v$$
$$(f \otimes v \in \mathbb{C}[z_k \mid 1 \leqslant k \leqslant n] \otimes C_{\mathbb{C}}(r_l \mid 1 \leqslant l \leqslant m)).$$

Note that operators $\rho(\mathfrak{a})$ is generated by

$$X = -\frac{j}{2}\rho(P) - \frac{\sqrt{-1}}{2}\rho(Q) = \sum_{k=1}^m (z_{2k-1} \otimes r_k + \sqrt{-1}\, z_{2k} \otimes r_k \alpha_k)$$

and

$$Y = -\frac{j}{2}\rho(P) + \frac{\sqrt{-1}}{2}\rho(Q) = \sum_{k=1}^{m}\left(\frac{\partial}{\partial z_{2k-1}} \otimes r_k + \sqrt{-1}\frac{\partial}{\partial z_{2k}} \otimes r_k\alpha_k\right).$$

For each summand $z_{2k-1} \otimes r_k + \sqrt{-1}z_{2k} \otimes r_k\alpha_k$ $(k \neq m)$ of X, we have

$$\begin{aligned}
&\mathscr{S} \otimes \mathscr{R}(s_m^{-1})(z_{2k-1} \otimes r_k + \sqrt{-1}z_{2k} \otimes r_k\alpha_k)\mathscr{S} \otimes \mathscr{R}(s_m) \\
&= (-\sqrt{-1}\zeta^n\alpha_m r_m)(z_{2k-1} \otimes r_k + \sqrt{-1}z_{2k} \otimes r_k\alpha_k)(\sqrt{-1}\zeta^n r_m\alpha_m) \\
&= (\zeta^n\alpha_m r_m)(\zeta^n r_m\alpha_m)(z_{2k-1} \otimes r_k) + \sqrt{-1}(\zeta^n\alpha_m r_m)(\zeta^n r_m\alpha_m)(z_{2k} \otimes r_k\alpha_k) \\
&= z_{2k-1} \otimes r_k + \sqrt{-1}z_{2k} \otimes r_k\alpha_k.
\end{aligned}$$

Note that $z_j\zeta^n = -\zeta^n z_j$ if $j \neq n$ and $r_k r_m = -r_m r_k$ for $k \neq m$.

For the summand $z_{n-1} \otimes r_m + \sqrt{-1}z_n \otimes r_m\alpha_m$, we have

$$\begin{aligned}
&\mathscr{S} \otimes \mathscr{R}(s_m^{-1})(z_{n-1} \otimes r_m + \sqrt{-1}z_n \otimes r_m\alpha_m)\mathscr{S} \otimes \mathscr{R}(s_m) \\
&= (-\sqrt{-1}\zeta^n\alpha_m r_m)(z_{n-1} \otimes r_m + \sqrt{-1}z_n \otimes r_m\alpha_m)(\sqrt{-1}\zeta^n r_m\alpha_m) \\
&= (\zeta^n\alpha_m r_m)(\zeta^n r_m\alpha_m)(z_{n-1} \otimes r_m) + \sqrt{-1}(\zeta^n\alpha_m r_m)(\zeta^n r_m\alpha_m)(z_n \otimes r_m\alpha_m) \\
&= z_{n-1} \otimes r_m + \sqrt{-1}z_n \otimes r_m\alpha_m,
\end{aligned}$$

using $\zeta^n z_n = z_n\zeta^n$ and $\alpha_m r_m = -r_m\alpha_m$. Now we see X is fixed by $\mathrm{Ad}\{\mathscr{S} \otimes \mathscr{R}(s_m)\}$.

For the operator Y, we can proceed in similar way. □

For the highest weight vectors $v_k^{\pm}$ and $w_k^{\pm}$ in Proposition 4.2, it is easily calculated out that

$$\mathscr{S} \otimes \mathscr{R}(s_m)v_k^+ = (-1)^k v_k^-, \qquad \mathscr{S} \otimes \mathscr{R}(s_m)w_k^+ = (-1)^k w_k^-,$$

where $s_m \in G_0^-$. Now it is easy to see that, as a representation of $G_0^{\pm}$, $PF(k)$ decomposes as follows:

$$PF(k) \simeq PF(k-2) \oplus \tau_1(k) \oplus \tau_2(k).$$

Here $\tau_1(k)$ (respectively $\tau_2(k)$) is a representation generated by v_k^+ (respectively w_k^+).

Next we consider the decomposition of the oscillator representation ρ as a representation of $\mathfrak{a} \simeq \mathfrak{osp}(2/1; \mathbb{R})$. Since $\mathfrak{a}$ and $\mathfrak{a}'$ are commutants with each other, $\rho(\mathfrak{a})$ preserves a space of highest weight vectors of $\mathfrak{a}'$ which have a fixed highest weight. Let $v_k^\pm$ and $w_k^\pm$ be highest weight vectors as in Table I. Then the space

$$V^+(k)=\{f(z_1^2+z_2^2+\cdots+z_n^2)v_k^+,\quad f(z_1^2+z_2^2+\cdots+z_n^2)w_{k+1}^+ | f(x)\in\mathbb{C}[x]\}$$

consists of all the highest weight vectors with weight $(k+\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ and

$$V^-(k)=\{f(z_1^2+z_2^2+\cdots+z_n^2)v_k^-,\quad f(z_1^2+z_2^2+\cdots+z_n^2)w_{k+1}^- | f(x)\in\mathbb{C}[x]\}$$

is the space of all the highest weight vectors with weight $(k+\frac{1}{2},\frac{1}{2},\ldots,-\frac{1}{2})$.

PROPOSITION 4.4. (1) *$V^\pm(k)$ is an irreducible lowest weight module of $\mathfrak{a}\simeq\mathfrak{osp}(2/1;\mathbb{R})$ with lowest weight $k+m$. As a representation of $\mathfrak{a}_0\simeq\mathfrak{sl}(2;\mathbb{R})$ (even part of $\mathfrak{a}$), $V^\pm(k)$ is a direct sum of two holomorphic discrete series representations with lowest weights $k+m$ and $k+m+1$.*

(2) *A lowest weight vector for $V^\pm(k)$ is $v_k^\pm$. For $\mathfrak{a}_0\simeq\mathfrak{sl}(2;\mathbb{R})$, lowest weight vectors are $v_k^\pm$ and $w_k^\pm$.*

COROLLARY 4.5 (see [6]). *An irreducible lowest weight module of $\mathfrak{osp}(2/1;\mathbb{R})$ with lowest weight λ ($\lambda\in\mathbb{Z}_{\geqslant 0}$) is unitarizable.*

REMARK. All the unitary representations of $\mathfrak{osp}(2/1;\mathbb{R})$ are classified by H. Furutsu and T. Hirai ([6, Theorem 5.11]). As a result they are either lowest weight modules or highest weight modules extended from (anti-)holomorphic discrete series representations of $\mathfrak{osp}(2/1;\mathbb{R})_0\simeq\mathfrak{sl}(2;\mathbb{R})$.

Proof of Proposition 4.4. Negative simple (odd) root vector of $\mathfrak{a}\simeq\mathfrak{osp}(2/1;\mathbb{R})$ acts on $V^\pm(k)$ as an operator

$$Y=\sum_{j=1}^{m}\left\{\frac{\partial}{\partial z_{2j-1}}\otimes r_j+\sqrt{-1}\,\frac{\partial}{\partial z_{2j}}\otimes r_j\alpha_j\right\}.$$

Since $Yv_k^\pm=0$ and $Yw_k^\pm=(2k+n-2)v_{k-1}^\pm$, we conclude that $v_k^\pm$ are only lowest weight vectors in $V^\pm(k)$. The rest of the assertions are clear. □

We summarize above results into

THEOREM 4.6. *The oscillator representation (ρ,F) is multiplicity free as a representation of $\mathfrak{a}\times G_0^\pm\simeq\mathfrak{osp}(2/1;\mathbb{R})\times\mathrm{Spin}^\pm(n)$, where $n=2m\geqslant 4$.*

(1) *The decomposition of ρ is given as follows:*

$$(\rho,F)\simeq\sum_{k=0}^{\infty}{}^{\oplus}D_{k+m}\otimes\tau_1(k),$$

where D_{k+m} is an irreducible lowest weight module of $\mathfrak{osp}(2/1;\mathbb{R})$ with lowest weight $k+m$ and $\tau_1(k)$ is a representation of $\mathrm{Spin}^\pm(n)$ which is a direct sum of

$\tau_1^+(k)$ *and* $\tau_1^-(k)$ *as a representation of* $\mathfrak{a}' \simeq \mathfrak{so}(n)$.

(2) *For a generic vector for* $D_{k+m} \otimes \tau_1(k)$, *we can take* $v_k^+ = (z_1 + \sqrt{-1}z_2)^k \otimes 1$.

Let $\Pi_{\mathfrak{osp}}$ (respectively $\Pi_{\mathrm{Spin}^\pm}$) be the set of irreducible unitary representations of $\mathfrak{osp}(2/1; \mathbb{R})$ (respectively $\mathrm{Spin}^\pm(n)$) which appear in the oscillator representation. Then the above theorem tells us that there exists a natural bijective correspondence

$$\pi\colon \Pi_{\mathfrak{osp}} \to \Pi_{\mathrm{Spin}^\pm}$$

such that the oscillator representation decomposes as a representation of $\mathfrak{osp}(2/1; \mathbb{R}) \times \mathrm{Spin}^\pm(n)$ in a multiplicity free manner:

$$(\rho, F) \simeq \sum_{D \in \Pi_{\mathfrak{osp}}} {}^{\oplus} D \otimes \pi(D).$$

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