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A directional compactification of the complex Fermi surface

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1. Introduction

Let $\Gamma \subset R^3$ be a lattice of maximal rank and $L^2_R(R^3/\Gamma)$ the Hilbert space of square-integrable, real-valued functions on the torus R^3/Γ. Let q be in $L^2_R(R^3/\Gamma)$. For each $k \in R^3$ the self-adjoint boundary value problem

$$(-\Delta + q(x))\psi = \lambda\psi, \qquad \psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x) \quad \forall\, \gamma \in \Gamma,$$

has a discrete spectrum customarily denoted by

$$\varepsilon_1(k) \leqslant \varepsilon_2(k) \leqslant \varepsilon_3(k) \leqslant \cdots.$$

The eigenvalue $\varepsilon_n(k)$, $n \geqslant 1$, defines a function of k called the nth band function. It is continuous and periodic with respect to the lattice

$$\Gamma^{\#} := \{b \in R^3 \mid \langle \gamma, b\rangle \in 2\pi Z \text{ for all } \gamma \in \Gamma\}$$

dual to Γ.

The physical Fermi surface for energy λ is the set

$$F_{\text{phys},\lambda}(q) := \{k \in R^3 \mid \varepsilon_n(k) = \lambda \text{ for some } n \geqslant 1\}.$$

For example, if $q(x) = \text{const}$, then $F_{\text{phys},\lambda}(q)$ is the union of the spheres

$$\{k \in R^3 \mid (k_1 + b_1)^2 + (k_2 + b_2)^2 + (k_3 + b_3)^2 = \lambda\text{-const}\}$$

with $b = (b_1, b_2, b_3) \in \Gamma^{\#}$.

In section 3 we prove

THEOREM 1. *If q is in $L^2_R(R^3/\Gamma)$ and if for a single real λ one of the components of $F_{\text{phys},\lambda}(q)$ is a sphere (not necessarily centered at a point in the dual lattice), then q is constant.*

Actually, we conjecture that the same conclusion holds if $F_{\text{phys},\lambda}(q)$ contains an algebraic component X. In section 3 we prove this with some further assumptions on the algebraic surface X. These assumptions are fulfilled if X is a sphere or an ellipsoid.

To prove Theorem 1 we complexify the Fermi surface. The (lifted) complex

Fermi surface is defined by

$F_\lambda(q) := \{k \in C^3 \mid$ there is a nontrivial solution ψ in $H^2_{\mathrm{loc}}(R^3)$ of $(-\Delta + q(x))\psi = \lambda\psi$, satisfying $\psi(x+\gamma) = e^{i\langle k,\gamma\rangle}\psi(x)$ for all γ in $\Gamma\}$

Clearly, the dual lattice $\Gamma^\#$ acts on $F_\lambda(q)$ by $k \to k + b$, $b \in \Gamma^\#$. Furthermore, we have $F_\lambda(q) \cap R^3 = F_{\mathrm{phys},\lambda}(q)$ the physical Fermi surface.

It is easy to show, using regularized determinants (see [KT], section 1), that $F_\lambda(q)$ is a complex analytic hypersurface in C^3. The main purpose of this paper is to construct a directional compactification of $F_\lambda(q)$ in the sense of [KT]. The above statement follows from the analysis of the points added at "infinity" and a geometric interpretation of Borg's theorem [Bo].

To compactify $F_\lambda(q)$ we first embed C^3 in a quadric Q lying in P^4. For each affine line $g = \{c + tb \mid t \in R\}$ in R^3, where $b, c \in \Gamma^\#$ and b is primitive, we blow up two distinguished points of P^4 that lie on the quadric Q, to get, by using inverse limits, a space M. Denote by $E_1(g)$ and $E_2(g)$ the corresponding exceptional divisors.

THEOREM 2. *The directional closure of $F_\lambda(q)$ in the space M intersects $E_1(g)$ and $E_2(g)$ along curves both of which are isomorphic to the one-dimensional Bloch variety*[(1)]

$$B(q_g) \text{ where } q_g(x) = \sum_{n=-\infty}^{+\infty} \hat{q}(n \cdot b)\, e^{i\langle n\cdot b, x\rangle},\ x \in g$$

Recall that in [KT] the complex one-dimensional Bloch variety for $p(x) \in L^2(R/|b|\cdot Z)$ is

$B(p) = \{(k, \lambda) \in C \times C \mid$ there is a non-trivial function ψ in $H^2_{\mathrm{loc}}(R)$ satisfying $-\psi'' + p(x)\psi = \lambda\psi$ and $\psi(x + |b|\cdot n) = e^{ik\cdot(|b|\cdot n)}\psi(x)$ for all $n \in Z\}$.

To get Theorem 1 we apply Borg's theorem [Bo] in the version of [KT]: Assume p is real then B(p) contains a component that is the graph of an entire function $\lambda(k)$ if and only if p is constant. In section 3 we prove that if for example $F_{\mathrm{phys},\lambda}(q)$ is a sphere, its directional closure meets sufficiently many one-dimensional Bloch-varieties $B(q_g)$, so $B(g_g)$ is algebraic and therefore q_g is constant.

We conjecture, that for all $q \in L^2(R^3/\Gamma)$ and for each $\lambda \in C$ the complex Fermi surface $F_\lambda(q)/\Gamma^\#$ is irreducible. In other words, the conjecture is that for any two irreducible components C_1, C_2 of $F_\lambda(q)$ there is a $b \in \Gamma^\#$ such that $C_2 = b + C_1$.

At the end of section 1 we prove the conjecture for split potentials. That is, we consider potentials of the form

$$q(x) = p_1(x_1) + p_2(x_2, x_3) \tag{1}$$

[(1)] $\hat{q}(b) := \int_{R^3/\Gamma} q(x)\, e^{-i\langle b,x\rangle}\, dx$ for $b \in \Gamma^\#$. Without loss of generality we assume that R^3/Γ has volume one.

or

$$q(x) = p_1(x_1) + p_2(x_2) + p_3(x_3) \tag{2}$$

and show that $F_\lambda(q)/\Gamma^\#$ for split potentials is always irreducible. The full conjecture for the discrete periodic Schrödinger operator is proven in [Bä].

We say that the physical Fermi surface $F_{\text{phys},\lambda}(q)$ is non-degenerate, if some piece of it is a two-dimensional real surface. For example, $F_{\text{phys},0}(0) = \{0\}$ is degenerate. It follows from the conjecture that for a real λ, a non-degenerate $F_{\text{phys},\lambda}(q)$ determines the complexified Fermi surface and by Theorem 1 the isospectral classes of all averaged potentials q_g.

In particular we obtain

THEOREM 3. *If* $q \in L^2(R^3/\Gamma)$ *and the Fermi surface* $F_{\text{phys},\lambda}(q)$ *is the same as* $F_{\text{phys},\lambda}(q')$, *where* q' *is a split potential of the form* (1) *or* (2) *then* q *also splits.*

We further conjecture that, generically, the physical Fermi surface determines the potential q up to (obvious) translations and reflections.

2. The compactification

First we construct a compactification of C^3 which serves as the ambient space for the directional compactification of $F_\lambda(q)$. This compactification of C^3 will be independent of q. Its construction is motivated by considering the free Fermi surface $F_\lambda(0)$ which is the union of the quadrics

$$\{k \in C^3 \mid (k_1+b_1)^2 + (k_2+b_2)^2 + (k_3+b_3)^2 = \lambda\}, \qquad b = (b_1, b_2, b_3) \in \Gamma^\#.$$

We want to compactify C^3 in such a way that the closures of the different components of $F_\lambda(0)$ intersect as nicely as possible. If we compactified C^3 in the naïve way to P^3 or $P^1 \times P^1 \times P^1$ we would have to blow-up many times before the components of $F_\lambda(0)$ would be in general position. Instead we embed C^3 in a complex projective 3-dimensional nonsingular quadric

$$Q := \{(k_1, k_2, k_3, y, z) \in P^4 \mid yz = k_1^2 + k_2^2 + k_3^2\}$$

by mapping (k_1, k_2, k_3) to $(k_1, k_2, k_3, k_1^2 + k_2^2 + k_3^2, 1)$. The image of the embedding is the complement in Q of

$$\begin{aligned} Q_\infty &:= \{(k_1, k_2, k_3, y, z) \in Q \mid z = 0\} \\ &= \{(k_1, k_2, k_3, y, z) \in P^4 \mid z = 0,\ k_1^2 + k_2^2 + k_3^2 = 0\}. \end{aligned}$$

The closures of the components of $F_\lambda(0)$ in Q are the intersections of Q with the hyperplanes H_b in P^4 given by

$$y + 2\langle k, b\rangle + (b^2 - \lambda)z = 0, \qquad b \in \Gamma^\#.$$

If $b \neq b'$, then $H_b \cap H'_b$ is a plane in P^4. It intersects Q_∞ in the set $D_{b,b'}$, consisting of two points given by the equations

$$z=0, \quad k_1^2+k_2^2+k_3^2=0, \quad \langle k, b-b'\rangle=0, \quad y+2\langle k, b\rangle=0.$$

One checks that $D_{b,b'}$ and $D_{b'',b'''}$ are disjoint if b, b', b'', b''' do not lie on a line and that $D_{b,b'} = D_{b'',b'''}$ if these four points of $\Gamma^\#$ are on a line. Thus, a point of $D_{b,b'}$ lies precisely on the hyperplanes H_c where $c \in \Gamma^\#$ is on the line through b and b'. So let us denote this line by g and the points $D_{b,b'}$ by $D(g)$.

The group $\Gamma^\#$ acts by translation on C^3. One easily sees that this action extends to Q and that $c \in \Gamma^\#$ maps $D(g)$ to $D(c+g)$.

If b and $b' \in \Gamma^\#$ are different points on a line $g = c_1 + Rc_2$ $(c_1, c_2 \in \Gamma^\#)$ then $Q \cap H_b$ and $Q \cap H_{b'}$, have different tangent planes at the points of $D(g)$. Therefore, we can separate $Q \cap H_b$ and $Q \cap H_{b'}$, by blowing up the points of $D(g)$. Precisely, for each line $g = c_1 + Rc_2$ $(c_1, c_2 \in \Gamma^\#)$ let $M(g)$ be the space obtained from P^4 by blowing up the points of $D(g)$, $Q(g)$ the strict transform of Q in $M(g)$, $Q_\infty(g)$ the strict transform of Q_∞, and $E_1(g)$, $E_2(g)$ the two exceptional divisors over the two points of $D(g)$.

So we introduce as compactification M of C^3 the inverse limit of all the spaces $M(G)$, where G is a finite set of affine lines and $M(G)$ is obtained from P^4 by blowing up the points of $\bigcup_{g \in G} D(g)$, defined by the natural maps $M(G_1) \to M(G_2)$ for $G_2 \subset G_1$. In the following we are not going to use this inverse limit, but work directly with the manifolds $M(g)$. A precise version of Theorem 2 stated in the introduction is

THEOREM 2'. *For each affine line $g = \{c + tb \mid t \in R\}$; b, $c \in \Gamma^\#$, b primitive, there is a subset $\Sigma(g)$ of C^3 such that for any $q \in L^2_R(R^3/\Gamma)$ the closure of $F_\lambda(q) \cap \Sigma(g)$ in $Q(g)$ intersects $E_1(g)$ and $E_2(g)$ along curves isomorphic to the Bloch-variety $B(q_g)$ of the one-dimensional potential*

$$q_g(x) := \sum_{n=-\infty}^{+\infty} \hat{q}(n \cdot b)\, e^{in\langle b, x\rangle}, \quad x \in g.$$

Proof. Using the action of $\Gamma^\#$ we may assume that g passes through the origin, i.e. that $c = 0$. After rotating and scaling we further assume that $b = \tilde{b} := (1, 0, 0)$. Then

$$D(g) = \{(0, \pm i, 1, 0, 0) \in P^4\}$$

We consider the exceptional divisor $E_1 := E_1(g)$ lying above the point $(0, i, 1, 0, 0)$, the other plane is treated similarly. Near this point we take

$$\left(\frac{k_1}{k_3}, \frac{k_2}{k_3} - i, \frac{y}{k_3}, \frac{z}{k_3}\right)$$

as coordinates.

In the blown-up space $M := M(g)$ we consider the chart with the coordinates

l_1, l_2, y', z such that

$$\frac{k_1}{k_3} = zl_1, \quad \frac{k_2}{k_3} - i = zl_2, \quad \frac{y}{k_3} = zy', \quad k_3 = \frac{1}{z}.$$

For convenience we perform the change of variables

$$y' = -\mu + l_1^2 + \lambda$$

In these coordinates the blow-up map $M \to P^4$ is

$$k_1 = l_1, \quad k_2 = l_2 + \frac{i}{z}, \quad y = -\mu + l_1^2 + \lambda, \quad k_3 = \frac{1}{z}.$$

Therefore, $Q(g)$ has the equation

$$2il_2 + z(l_2^2 + \mu - \lambda) = 0.$$

In particular $Q(g)$ intersects E_1 in the plane $z = l_2 = 0$. Finally, the strict transform of the hyperplane H_b, $b \in \Gamma^{\#}$, does not meet E_1 if $b_2 \neq 0$ or $b_3 \neq 0$. Further, the strict transform of $H_{(b_1,0,0)}$ intersects E_1 in

$$(l_1 + b_1)^2 - \mu = 0.$$

Remember that the strict transform of $Q \cap H_b$ is the curve of a component of the free Fermi-surface $F_\lambda(0)$, and that the one-dimensional Bloch-variety for potential zero is

$$\bigcup_{n \in Z} \{(l, \mu) \in C \times C \mid (l + n)^2 - \mu = 0\}.$$

This shows that for $q \equiv 0$ the union of the closures of the components of $F_\lambda(0)$ meets $E_1 \cap Q(g)$ along a curve isomorphic to the one-dimensional Bloch-variety for potential zero. Observe however that the closure of $F_\lambda(0)$ in $Q(g)$ is bigger than the union of the closures of its components. This indicates that it is necessary for the general case to restrict the way one takes limits to E_1. This restriction is made precise by the introduction of the set $\Sigma(g)$.

Recall from [KT] that (without loss of generality we assume $\hat{q}(0) = 0$)

$$\det{}_2[(-\Delta_k + q - \lambda)\cdot(-\Delta_k - \lambda)^{-1}] = \det{}_2\left(\delta_{cb} + \frac{\hat{q}(c-b)}{(k+b)^2 - \lambda}\right) = 0$$

is an equation for $F_\lambda(q)$ outside of the free Fermi surface $F_\lambda(0)$, and that this determinant can be computed by taking limits of finite principal minors.

In the coordinates (l_1, l_2, μ, z) of $M(g)$ the entries of the matrix for $(-\Delta_k + q - \lambda)\cdot(-\Delta_k - \lambda)^{-1}$ considered above are

$$\delta_{cb} + \frac{\hat{q}(c-b)}{\frac{2}{z}(ib_2 + b_3) + [(l_1 + b_1)^2 - \mu + 2l_2b_2 + b_2^2 + b_3^2]}.$$

We are interested in the restriction of the determinant of this matrix to $Q(g)$, but for the moment we consider it on all $M(g)$.

We block the matrix in the form

$$\begin{array}{c} \\ c\in Z\cdot\tilde{b}\{ \\ c\notin Z\cdot\tilde{b}\{ \end{array} \begin{array}{cc} \overbrace{}^{b\in Z\cdot\tilde{b}} & \overbrace{}^{b\notin Z\cdot\tilde{b}} \\ \multicolumn{2}{c}{\begin{pmatrix} A(l_1, l_2, \mu, z) & B(l_1, l_2, \mu, z) \\ C(l_1, l_2, \mu, z) & D(l_1, l_2, \mu, z) \end{pmatrix}} \end{array} =: F(l_1, l_2, \mu, z)$$

With this notation

$$A(l_1, l_2, \mu, z) = \left(\delta_{c_1 b_1} + \frac{\hat{q}(c_1 - b_1, 0, 0)}{(l_1 + b_1)^2 - \mu}\right) c_1, b_1 \in Z.$$

This is the matrix whose determinant describes the Bloch-variety of the averaged potential q_g outside $B(0)$. Furthermore on $Q(g) \cap E_1 = \{z = l_2 = 0\}$ the matrix $B = 0$ and $D = 1$.

We will define $\Sigma(g)$ in such a way that the matrix $F(l_1, l_2, \mu, z)$ restricted to $\Sigma(g)$ converges in Hilbert–Schmidt norm to

$$F(l_1, l_2, \mu, 0) = \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} \text{ as } z \to 0.$$

The square of the Hilbert–Schmidt norm of

$$F(l_1, l_2, \mu, z) - F(l_1, 0, \mu, 0)$$

is bounded by

$$\|q\|_2^2 \cdot \sum_{\substack{b\in\Gamma^\# \\ b\notin Z\cdot\tilde{b}}} \frac{1}{\left|\frac{2}{z}(ib_2 + b_3) + [(l_1 + b_1)^2 - \mu + 2l_2 b_2 + b_2^2 + b_3^2]\right|^2}.$$

We define

$$\Sigma(g) := \left\{ (l_1, l_2, \mu, z) \in C^4 \,\middle|\, \sum_{\substack{b\in\Gamma^\# \\ b\notin Z\cdot\tilde{b}}} \frac{1}{\left|\frac{2}{z}(ib_2 + b_3) + [(l_1 + b_1)^2 - \mu + 2l_2 b_2 + b_2^2 + b_3^2]\right|^2} \right.$$

$$\left. + \sum_{\substack{b\in\Gamma^\# \\ b\notin Z\cdot\tilde{b}}} \frac{|l_1 + b_1|^2 + b_2^2}{\left|\frac{2}{z}(ib_2 + b_3) + ((l_1 + b_1)^2 - \mu + 2l_2 b_2 + b_2^2 + b_3^2]\right|^4} < |z|^{1/5} \right\}$$

REMARK. The second term in the definition of $\Sigma(g)$ is needed not only to control the Hilbert–Schmidt norm of $F(l_1, l_2, \mu, z) - F(l_1, 0, \mu, 0)$ but also that of its derivatives $\partial_{l_1} F$, $\partial_{l_2} F$, $\partial_\mu F$.

Clearly, the restriction of $\det_2 F$ to $\Sigma(g)$ is continuous at $z=0$:

$$\|F(l_1, l_2, \mu, z) - F(l_1, 0, \mu, 0)\|^2_{\text{Hilbert-Schmidt}} = 0(\|q\|_2^2 \cdot |z|^{1/5}).$$

Therefore the intersection of $\overline{F_\lambda(q) \cap \Sigma(g)}$ with $Q(g) \cap E_1$ is contained in $B(q_g)$.

We now want to prove the converse. For this we need information about the structure of $\Sigma(g)$ in the neighbourhood of any point of $Q(g) \cap E_1$.

PROPOSITION. *For every point* $p=(l_1^*, l_2^*, \mu^*, 0)$ *of* $E_1(g)$ *and for all* $A>0$ *there is a neighbourhood* U *of* p *in* $M(g)$ *and an open set* $Z \subset C$ *having* 0 *as a cluster point such that*

$$\{(l_1, l_2, \mu, z) \in U \mid z \in Z, |l_2 - l_2^*| \leqslant A|z|\} \subset \sum(g).$$

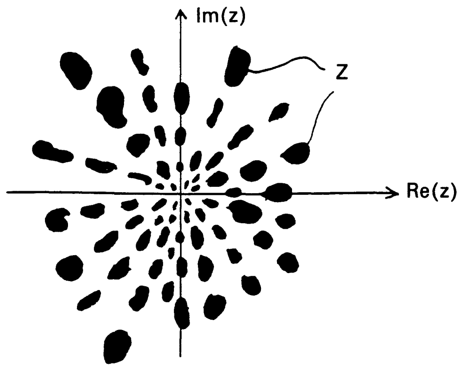

Fig. 1

Proof. We have to estimate the function $S := \tilde{S}_1 + \tilde{S}_2$, where

$$\tilde{S}_1(l_1, l_2, \mu, z) := \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0)}} \frac{1}{\left|\frac{2}{z}(ib_2 + b_3) + [(l_1+b_1)^2 - \mu + 2l_2 b_2 + b_2^2 + b_3^2]\right|^2}$$

$$\tilde{S}_2(l_1, l_2, \mu, z) := \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0)}} \frac{|l_1+b_1|^2 + b_2^2}{\left|\frac{2}{z}(ib_2 + b_3) + [(l_1+b_1)^2 - \mu + 2l_2 b_2 + b_2^2 + b_3^2]\right|^4}$$

near $z=0$ in a neighbourhood of $(l_1^*, l_2^*, \mu^*, 0)$. We first substitute $\frac{2}{z} =: -iw$. Then

$$\tilde{S}_1(l_1, l_2, \mu, w) = \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0)}} \frac{1}{(b_2^2+b_3^2)\left|w - \left(1 + \frac{(l_1+b_1)^2 - \mu + 2l_2 b_2}{b_2^2 + b_3^2}\right)(-b_2 + ib_3)\right|^2}$$

and

$$\tilde{S}_2(l_1, l_2, \mu, w) = \sum_{\substack{b\in\Gamma^{\#} \\ b\notin Z\cdot(1,0,0)}} \frac{|l_1+b_1|^2+b_2^2}{(b_2^2+b_3^2)^2\left|w-\left(1+\frac{(l_1+b_1)^2-\mu+2l_2b_2}{b_2^2+b_3^2}\right)(-b_2+ib_3)\right|^4}.$$

For $b\in\Gamma^{\#}\backslash Z\cdot(1,0,0)$, set

$$f_b(l_1, \mu, w) := \sup_{|l_2|\leqslant A/|w|} \frac{1}{\left|w-w_b(l_1,\mu)+2l_2\frac{b_2}{b_2^2+b_3^2}(-b_2+ib_3)\right|^2}$$

where

$$w_b(l_1, \mu) = \left(1 + \frac{(l_1+b_1)^2-\mu}{b_2^2+b_3^2}\right)(-b_2+ib_3).$$

Observe that

$$f_b(l_1, \mu, w) \geqslant \begin{cases} \infty, & \text{if } |w-w_b| \leqslant \frac{2}{|w|} A \frac{|b_2|}{(b_2^2+b_3^2)^{1/2}} \\ \dfrac{1}{\left(|w-w_b|-2\frac{A}{|w|}\frac{|b_2|}{(b_2^2+b_3^2)^{1/2}}\right)^2} & \text{otherwise.} \end{cases}$$

So, using the estimate $|w_b(l_1,\mu)| \geqslant \frac{1}{2}|b_2|$ for all $b\in\Gamma^{\#}\backslash Z\cdot(1,0,0)$ outside a finite set $S(l_1,\mu) \subset \Gamma^{\#}$ and the fact that the union

$$\mathscr{S} = \bigcup_{\substack{(l_1,\mu)\in \text{Neighbourhood} \\ U_1 \text{ of } (l_1^*,\mu^*)}} S(l_1, \mu)$$

is finite we have:

$$\forall w\in C, \quad (l_1,\mu)\in U_1 \text{ with } |l_2| \leqslant \frac{A}{|w|} \text{ and } |w-w_b| \geqslant \frac{4A}{(b_2^2+b_3^2)^{1/2}}$$

$$\tilde{S}_1(l_1, l_2, \mu, w) \leqslant S_1(l_1, \mu, w) := \tfrac{1}{4} \sum_{\substack{b\in\Gamma^{\#} \\ b\notin Z\cdot(1,0,0) \\ b\notin\mathscr{S}}} \frac{1}{(b_2^2+b_3^2)|w-w_b(l_1,\mu)|^2} + 0\left(\frac{1}{|w|^2}\right)$$

$$\tilde{S}_2(l_1, l_2, \mu, w) \leqslant S_2(l_1, \mu, w) := \tfrac{1}{16} \sum_{\substack{b\in\Gamma^{\#} \\ b\notin Z\cdot(1,0,0) \\ b\notin\mathscr{S}}} \frac{|l_1+b_1|^2+b_2^2}{(b_2^2+b_3^2)^2|w-w_b(l_1,\mu)|^4} + 0\left(\frac{1}{|w|^4}\right).$$

Furthermore, for each w in a disc of radius $\left(\dfrac{b_1}{b_2^2+b_3^2}\right)^{1/2}$ centered at w_b

$$S_2(l_1, \mu, w) \geqslant \frac{b_1^2+b_2^2}{b_1^2} \geqslant 1.$$

Thus, these regions must be excluded since we want S to go to 0 as $|w| \to \infty$.

LEMMA 1. *Let $D_b(l_1, \mu)$ be the disc*

$$\left\{ w \in C \,\middle|\, |w - w_b(l_1, \mu)| \leqslant \frac{|b_1| + 4A}{(b_2^2 + b_3^2)^{2/5}} \right\}.$$

Then, there exists a neighbourhood U_2 of (l_1^*, μ^*) such that for all $(l_1, \mu) \in U_2$ the open set

$$G := \overline{C \setminus \bigcup_{(l_1,\mu)\in U_2} \bigcup_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0)}} D_b(l_1, \mu)}$$

is not bounded.

Proof. Let S_R be the shell $\{w \in C | R \leqslant |w| \leqslant 2R\}$. We show that $G \cap S_R$ has positive Lebesque-measure for all R big enough, even

$$\lim_{R \to \infty} [\mathrm{meas}(G \cap S_R)/\mathrm{meas}(S_R)] = 1.$$

Since,

$$|w_b(l_1, \mu) - w_b(l_1^*, \mu^*)| \leqslant \frac{|b_1|}{(b_2^2 + b_3^2)^{1/2}} \left| \frac{(l_1^2 - l_1^{*2}) + 2(l_1 - l_1^*)b_1 - (\mu - \mu^*)}{b_1} \right|$$

there exists a neighbourhood U_2 of (l_1^*, μ^*) such that for all $(l_1, \mu) \in U_2$

$$D_b(l_1, \mu) \subset D_b'(l_1^*, \mu^*),$$

where

$$D_b'(l_1^*, \mu^*) = \left\{ w \in C \,|\, |w - w_b(l_1^*, \mu^*)| \leqslant 2\frac{|b_1| + 4A}{(b_2^2 + b_3^2)^{2/5}} \right\}.$$

Therefore, the Lebesque-measure

$$\mathrm{meas}\left(\left(\bigcup_{(l_1,\mu)\in U_2} \bigcup_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0)}} D_b(l_1, \mu) \right) \cap S_R \right) \leqslant \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0) \\ R \leqslant |w_b| \leqslant 2R}} \mathrm{meas}(D_b'(l_1^*, \mu^*)).$$

Let R be big enough, such that $\mathscr{S} \cap S_R = \emptyset$. Since $|w_b(l_1^*, \mu^*)| \leqslant 2R$ and $b \notin \mathscr{S}$, we have $|(l_1^* + b_1)^2 - \mu| = O(R)$, i.e. $b_1^2 \leqslant \text{Const}\, R$. So

$$\sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0) \\ R \leqslant |w_b| \leqslant 2R}} \mathrm{meas}(D_b'(l_1^*, \mu^*)) = O\left(R^{1/2} \sum_{\substack{(b_2,b_3) \\ 1 \leqslant |-ib_2 + b_3| \leqslant 2R}} \left[\frac{(\text{Const}\, R)^{1/2} + 4A}{(b_2^2 + b_3^2)^{2/5}} \right]^2 \right)$$

$$= O\left(R^{3/2} \sum_{\substack{(b_2,b_3) \neq (0,0) \\ |b_2 + ib_3| \leqslant 2R}} \frac{1}{(b_2^2 + b_3^2)^{4/5}} \right) = O(R^{3/2} \cdot R^{2/5}) = O(R^{19/10})$$

On the other hand $\mathrm{meas}(S_R) = O(R^2)$. This implies Lemma 1.

Now assume that for all R big enough and for all $w \in G$ with $|w| \geqslant R$, $(l_1, \mu) \in U_1 \cap U_2 =: U$

$$S_1(l_1, \mu, w) + S_2(l_1, \mu, w) \geqslant \frac{1}{|w|^{1/5}}. \tag{*}$$

If we can show that this claim leads to a contradiction the proposition is proved.

So let us assume (*). Then

$$\int_{S_R \cap G} (S_1(l_1, \mu, w) + S_2(l_1, \mu, w)) |\mathrm{d}w \wedge \mathrm{d}\bar{w}| \geqslant \int_{S_R \cap G} \frac{1}{|w|^{1/5}} |\mathrm{d}w \wedge \mathrm{d}\bar{w}|$$

Since $\lim_{R \to \infty} (\mathrm{meas}(G \cap S_R)/\mathrm{meas}(S_R)) = 1$ (see proof of Lemma 1) we have

$$\int_{S_R \cap G} \frac{1}{|w|^{1/5}} |\mathrm{d}w \wedge \mathrm{d}\bar{w}| = \mathrm{Const}\ R^{2-1/5} = \mathrm{Const}\ R^{9/5} \tag{1}$$

for R big enough.

We now calculate the left-hand side of the above inequality. For simplicity we do it for $l_1 = \mu = 0$ ($\mathscr{S} = \varnothing$). We have

$$\int_{S_R \cap G} S_2(0,0,w) |\mathrm{d}w \wedge \mathrm{d}\bar{w}| \leqslant \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0)}} \frac{b_1^2 + b_2^2}{(b_2^2 + b_3^2)^2} \int_{\substack{w \in G \\ 1 \leqslant |w| \leqslant 2R}} \frac{|\mathrm{d}w \wedge \mathrm{d}\bar{w}|}{|w - w_b(0,0)|^4}$$

If $|w_b| \leqslant 3R$ then

$$\int_{\substack{1 \leqslant |w| \leqslant 2R \\ w \in G}} \frac{|\mathrm{d}w \wedge \mathrm{d}\bar{w}|}{|w - w_b(0,0)|^4} \leqslant 2\pi \int_{\frac{4A + |b_1|}{(b_2^2 + b_3^2)^{2/5}}}^{5R} \frac{r\,\mathrm{d}r}{r^4} \leqslant \pi \frac{(b_2^2 + b_3^2)^{4/5}}{(4A + |b_1|)^2}$$

If $|w_b| \geqslant 3R$ then we can bound this integral by $\pi \cdot \dfrac{1}{(|w_b| - R)^2}$. Therefore,

$$\int_{S_R \cap G} S_2(0,0,w) |\mathrm{d}w \wedge \mathrm{d}\bar{w}| \leqslant \pi \cdot \left\{ \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0) \\ |w_b| \leqslant 3R}} \frac{b_1^2 + b_2^2}{(4A + |b_1|)^2 (b_2^2 + b_3^2)^{6/5}} \right.$$
$$\left. + \sum_{\substack{b \in \Gamma^\# \\ b \notin Z\cdot(1,0,0) \\ |w_b| \geqslant 3R}} \frac{b_1^2 + b_2^2}{(|w_b| - R)^2 (b_2^2 + b_3^2)^2} \right\}$$

The first sum is bounded by (since $|w_b| \leqslant 3R$ and therefore $b_1^2 \leqslant 3R$)

$$\sum_{\substack{(b_2, b_3) \neq 0 \\ |-b_2 + ib_3| \leqslant 3R}} \frac{1}{(b_2^2 + b_3^2)^{6/5}} \cdot \sum_{b_1^2 \leqslant 3R} \frac{b_1^2}{(4A + |b_1|)^2}$$

$$+ \sum_{\substack{(b_2,b_3)\neq 0 \\ |-b_2+ib_3|\leqslant 3R}} \frac{1}{(b_2^2+b_3^2)^{1/5}} \sum_{b_1} \frac{1}{(4A+|b_1|)^2} = O(1\cdot R^{1/2}+R^{8/5}\cdot 1) = O(R^{8/5}).$$

For the second sum we first sum over b_1. The coefficient in the sum is monotonically decreasing in $|b_1|$ (if $|b_1|\neq 0$), so comparable with

$$\sum_{|-b_2+ib_3|\geqslant 3R} \frac{2(b_2^2+b_3^2)}{(b_2^2+b_3^2)^2(|-b_2+ib_3|-R)^2}\cdot(b_2^2+b_3^2)^{1/2}$$

$$+ \sum_{|-b_2+ib_3|\geqslant 3R} \frac{1}{(b_2^2+b_3^2)^2} \int_{\substack{x\geqslant|-b_2+ib_3| \\ |w_b|\geqslant 3R}} \frac{x^2+b_2^2}{(|w_b|-R)^2}\,\mathrm{d}x$$

For the integral put $y^2 := |w_b|-R = \dfrac{x^2}{|-b_2+ib_3|} + |-b_2+ib_3|-R$, then this expression is bounded by

$$\sum_{|-b_2+ib_3|\geqslant 3R} \frac{2(b_2^2+b_3^2)^{3/2}}{(b_2^2+b_3^2)^3(\frac{2}{3})^2} + \sum_{|-b_2+ib_3|\geqslant 3R} \frac{1}{(b_2^2+b_3^2)^2}$$

$$\times \int_{\substack{y^2\geqslant|b_2+ib_3|-R \\ y^2\geqslant R}} \frac{|-b_2+ib_3|(y^2+R)+b_3^2}{y^4}(|-b_2+ib_3|^{1/2}\,\mathrm{d}y)$$

This is equal to

$$O\left(\frac{1}{R}\right) + O\left(\sum_{|-b_2+ib_3|\geqslant 3R} \frac{1}{(b_2^2+b_3^2)^{5/4}}\left(\frac{1}{R^{1/2}}+\frac{R}{R^{3/2}}\right)\right.$$

$$\left. + \sum_{|-b_2+ib_3|\geqslant 3R} \frac{b_3^2}{(b_2^2+b_3^2)^{7/4}} \frac{1}{(|-b_2+ib_3|-R)^{3/2}}\right)$$

$$= O\left(\frac{1}{R}\right) + O\left(\frac{R^{-1/2}}{R^{1/2}} + \sum_{|-b_2+ib_3|\geqslant 3R} \frac{1}{(b_2^2+b_3^2)^{3/4}} \frac{1}{(b_2^2+b_3^2)^{3/4}} \frac{1}{(2/3)^{3/2}}\right.$$

$$= O\left(R^{-1} + \frac{1}{R}\right) = O(R^{-1})$$

So we get

$$\int_{S_R\cap G} S_2(0,0,w)|\mathrm{d}w\wedge \mathrm{d}\bar{w}| = O(R^{8/5}).$$

With the same methods one calculates

$$\int_{S_R\cap G} S_1(0,0,w)|\,\mathrm{d}w\wedge \mathrm{d}\bar{w}| = O(R^{3/2}).$$

Comparing both expressions with (1) leads to a contradiction of (*). This proves the proposition.

REMARK. The method of proof does not give much information about the form of $\Sigma(g)$, i.e. of Z.

However, by estimating the function $S(l_1, l_2, \mu, z)$, which defines $\Sigma(g)$, explicitly, the following can be shown: Let

$$g = \{t \cdot b \mid t \in R\} \quad \text{with } b = (1,0,0)$$

and consider a direction $\gamma = (b_2, b_3)$, where γ is a primitive vector of

$$\{b = (b_1, b_2, b_3) \in \Gamma^{\#} \mid b_1 = 0\}.$$

Exclude all points in

$$D := \bigcup_{\substack{b \in \Gamma^{\#} \\ b \notin Z \cdot (1,0,0)}} D'_b(l_1^*, \mu^*)$$

on the line

$$R \cdot \left(1 + \frac{l_1^{*2} - \mu^*}{|\gamma|^2}\right) \gamma.$$

Then on the segment $L_\gamma := \{t \cdot \gamma \mid [|\gamma|] - 1 \leqslant t \leqslant [|\gamma|]\}$ there are points which do not lie on D and for these points $w \in L_\gamma \backslash D$ one has

$$S(l_1, l_2, \mu, w) \leqslant \frac{1}{(t|\gamma|)^{1/5}} \leqslant \frac{1}{|w|^{1/5}}$$

for (l_1, l_2, μ) in a neighbourhood of (l_1^*, μ^*), $|l_2| \leqslant A/|w|$.

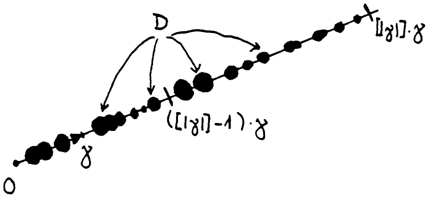

Fig. 2

We therefore can reach $w = \infty$ on $\Sigma(g)$ by hopping from a segment L_γ to another segment $L_{\gamma'}$ with $|\gamma'| > |\gamma|$.

We now prove Theorem 2′. We already observed that

$$\overline{(F_\lambda(q) \cap \sum(g))} \cap (Q(g) \cap E_1) \subset B(q_g).$$

So we have to prove the opposite inclusion. Let us fix a smooth point $p = (l_1^*, 0, \mu^*, 0)$ of $Q(g) \cap E_1 \cap B(q_g)$. For simplicity we assume that p does not lie on the free Bloch-variety $B(0)$ in $Q(g) \cap E_1$. By the proposition there is a neighbourhood U of p in $M(g)$ and an open subset Z of C having 0 as a cluster point, such that

$$T := \{(l_1, l_2, \mu, z) \in U \mid z \in Z, |l_2| \leqslant A \cdot |z|\}$$

is contained in $\Sigma(g)$.

LEMMA 2. *The restriction of the function*

$$f(l_1, l_2, \mu, z) := \det_2 F(l_1, l_2, \mu, z)$$

to $\bar{T}$ has the following properties

(i) $f(p) = 0$.
(ii) *There is a constant C, such that*

$$|f(l_1, l_2, \mu, z) - f(l_1, l_2, \mu, 0)| \leqslant C \cdot |z|^{1/5}$$

for all $z \in Z$, $(l_1, l_2, \mu, z) \in U$.
(iii) *For any $z \in \bar{Z}$ the mapping $f(\cdot, z)$ is differentiable and*

$$(l_1, l_2, \mu, z) \rightarrow (\nabla_{(l_1,l_2,\mu)} f)(l_1, l_2, \mu, z) \text{ is continuous on } \bar{T}.$$

(iv) $\dfrac{\partial f}{\partial l_1}(p)$ *and* $\dfrac{\partial f}{\partial \mu}(p)$ *are not both equal to zero.*

Lemma 2 implies the theorem as follows: Since $Q(g)$ intersects E_1 transversally, we can choose (l_1, μ, z) as local coordinates on $Q(g) \cap U =: V$ near p (observe that there exists an $A > 0$ such that $|l_2| \leqslant A \cdot |z|$ for all points near p on $Q(g)$). Assume $\dfrac{\partial f}{\partial l_1}(p) \neq 0$ (the other case is treated similarly, using $\dfrac{\partial l_2(\mu, z)}{\partial \mu}(p) = 0$) and consider the continuous mapping

$$\Phi\colon V \subset R^4 \times \bar{Z} \rightarrow R^4$$

defined by

$$\Phi(l_1, \mu, z) := (f(l_1, l_2(\mu, z), \mu, z), \mu - \mu^*).$$

This mapping fulfills the assumptions of the modification of the implicit function theorem described in Appendix 1. Therefore p lies in the closure of the zero-set of f in $(Q(g) - Q_\infty(g)) \cap T$, hence in the closure of $F_\lambda(q) \cap \Sigma(g)$. One knows that the equation defining the one-dimensional Bloch-variety $B(q_g)$ is reduced. So the smooth points are dense in the zero-set of $f(l_1, 0, \mu, 0)$ and the proof of the theorem is complete.

We now prove Lemma 2: Statement (i) is obvious, (iv) follows from the fact that p is a smooth point of $B(q_g)$. (ii) Is a consequence of the definition of $\Sigma(g)$

and the fact that $\det_2$ is continuous in the Hilbert–Schmidt norm. Similarly we have

$$\begin{aligned}&\|\partial_{l_1}F(l_1, l_2, \mu, z) - \partial_{l_1}F(l_1, l_2, \mu, 0)\|^2_{\mathrm{H.S.}} + \|\partial_{l_2}F(l_1, l_2, \mu, z)\\ &\quad - \partial_{l_2}F(l_1, l_2, \mu, 0)\|^2_{\mathrm{H.S.}} + \|\partial_{\mu}F(l_1, l_2, \mu, z) - \partial_{\mu}F(l_1, l_2, \mu, 0)\|^2_{\mathrm{H.S.}}\\ &\leqslant \|q\|_2^2 \cdot \sum_{\substack{b\in\Gamma^{\#}\\ b\notin Z\cdot\bar{b}}} \frac{4\cdot|l_1+b_1|^2+4b_2^2+1}{\left|\frac{2}{z}(ib_2+b_3)+[(l_1+b_1)^2-\mu+2l_2b_2+b_2^2+b_3^2]\right|^4}\end{aligned}$$

so (iii) again follows from the definition of $\Sigma(g)$.

COROLLARY. *Assume that q is a real potential and that $F_\lambda(q)$ contains an algebraic component X. If the closure $\bar{X}$ of X in Q contains one of the curves*

$$\{(k, y, 0)\in Q_\infty \mid \langle k, c\rangle + y = 0\}$$

with $c\in\Gamma^{\#}$ then q is constant.

Proof. For $b\in\Gamma^{\#}-\{0\}$ let g_b be the line $\{c+tb \mid t\in R\}$. Then $\bar{X}$ contains all the sets $D(g_b)$, $b\in\Gamma^{\#}$. By the proposition above for each $b\in\Gamma^{\#}$ the closure of $X\cap\Sigma(g_b)$ in $Q(g_b)$ meets $E_1(g_b)$ and $E_2(g_b)$ along a (non-empty) algebraic curve, namely the intersection of the strict transform of $\bar{X}$ with $E_1(g_b)$ resp. $E_2(g_b)$. Hence by Theorem 2′ the Bloch varieties of all the averaged potentials q_b each contains an algebraic component. As each q_b is real, Borg's theorem [Bo] implies that q_b is constant. Therefore q is constant.

The assumption of the corollary is fulfilled if $F_\lambda(q)$ contains a sphere around a point of $\Gamma^{\#}$.

In the introduction we conjectured that $F_\lambda(q)/\Gamma^{\#}$ is always irreducible. Assuming this conjecture one could finish the proof of Theorem 1 stated in the introduction immediately: If X were any algebraic component of $F_\lambda(q)$ (e.g. the complexification of a sphere) then by Theorem 2′ there would be an affine line g in each direction, such that $\overline{X\cap\Sigma(g)}$ intersects $E_i(g)$ $(i=1,2)$ along a curve. Then one would deduce the fact that q is constant as above. In the next section we will see that, under further assumptions on X, one does not need the irreducibility of $F_\lambda(q)/\Gamma^{\#}$ to complete the proof of Theorem 1.

We can prove the conjecture for split potentials.

PROPOSITION. *Let $\Gamma\subset R^3$ be a lattice generated by the vectors $a_1, a_2, a_3\in R^3$ with $\langle a_1, a_i\rangle=0$ $(i=2,3)$. Then for all potentials*

$$q(x_1, x_2, x_3) = p_1(x_1) + p_2(x_2, x_3)$$

with $p_1\in L^2(R/a_1\cdot Z)$ and $p_2\in L^2(R^2/a_2Z+a_3Z)$ the Fermi surface $F_\lambda(p_1+p_2)/\Gamma^{\#}$ is irreducible.

Proof. Let $B(p_1) \subset C^2$ resp. $B(p_2) \subset C^3$ be the one-, resp. two-dimensional Bloch variety for the potential p_1 resp. p_2. Suppose that $(k_1, \lambda_1) \in B(p_1)$ with corresponding eigenfunction $\psi_1(x_1)$ and $(k_2, k_3, \lambda) \in B(p_2)$ with eigenfunction $\psi_2(x_2, x_3)$, then $\psi(x) := \psi_1(x_1) \cdot \psi_2(x_2, x_3)$ is an eigenfunction for

$$(k_1, k_2, k_3, \lambda_1 + \lambda_2) \in F_{\lambda_1 + \lambda_2}(q).$$

Therefore the image of the map

$$j \colon B(p_1) \times B(p_2) \to C^3 \times C \quad ((k_1, \lambda_1), (k_2, k_3, \lambda_2)) \to (k_1, k_2, k_3, \lambda_1 + \lambda_2)$$

is contained in the three dimensional Bloch variety. Since the operator defining $B(p_1 + p_2)$ has a compact resolvent it follows from the method of separation of variables that each $(k_1, k_2, k_3, \lambda) \in B(p_1 + p_2)$ corresponds to an eigenfunction of the form $\psi(x_1) \cdot \psi(x_2, x_3)$, so j is surjective.

Next consider the map $\pi \colon B(p_1) \times B(p_2) \to C$ defined by $\pi((k_1, \lambda_1), (k_2, k_3, \lambda_2)) = \lambda_1 + \lambda_2$. The restriction of j to $\pi^{-1}(\lambda)$ is a surjective map from $\pi^{-1}(\lambda)$ to F_λ. In order to show that $F_\lambda / \Gamma^\#$ is irreducible it thus suffices to show that $\pi^{-1}(\lambda)/\Gamma^\#$ is irreducible. By [KT], Theorem 1 of section 3, the varieties

$$B(p_1)/\{\gamma \in \Gamma^\# \mid \gamma_2 = \gamma_3 = 0\} \quad \text{and} \quad B(p_2)/\{\gamma \in \Gamma^\# \mid \gamma_1 = 0\}$$

are irreducible. We can view $\pi^{-1}(\lambda)$ as the fibered product of $B(p_1)$ and $B(p_2)$ with respect to the maps

$$\pi_1 \colon B(p_1) \to C, \qquad (k_1, \lambda_1) \to \lambda_1$$

and

$$\pi_2 \colon B(p_2) \to C, \qquad (k_2, k_3, \lambda_2) \to \lambda - \lambda_2.$$

This construction is compatible with the action of $\Gamma^\#$, and therefore $\pi^{-1}(\lambda)/\Gamma^\#$ is a fiber product of two irreducible varieties with $\dim B(p_2) > 1$, hence irreducible.

The proposition above also applies to potentials of the form

$$p_1(x_1) + p_2(x_2) + p_3(x_3)$$

as a special case. As described in the introduction one deduces Theorem 3 from this proposition.

3. Algebraic components of the Fermi surface

If q is a real potential and the physical Fermi surface contains an ellipsoid then the complexification of this ellipsoid is an algebraic component of $F_\lambda(q)$. In this case one verifies that the closure of this component is transversal to Q_∞ in almost all points of the intersection. So, Theorem 1 stated in the introduction is a special case of

THEOREM 1'. *Let q be a real potential. Assume that $F_\lambda(q)$ contains an algebraic component X whose closure $\bar{X}$ is transversal to Q_∞ at almost every point of $\bar{X} \cap Q_\infty$. Then q is constant.*

For the proof it suffices to show that

$$\bar{X} \cap Q_\infty \subset \bigcup_{b \in \Gamma^\#} \{(k, y, 0) \in Q_\infty \mid \langle k, b\rangle + y = 0\}. \qquad (*)$$

In this case $\bar{X} \cap Q_\infty$ contains one of the curves

$$\{(k, y, 0) \in Q_\infty \mid \langle k, b\rangle + y = 0\}, \quad b \in \Gamma^\#,$$

and one can apply the corollary of the previous section. A first step towards proving the inclusion (*) is to show that points of $\bar{X} \cap Q_\infty$ are all well approximable by the curves $\{(k, y, 0) \in Q_\infty \mid \langle k, b\rangle + y = 0\}$. More precisely, set

$$D := \left\{(\kappa_1, \kappa_2, \kappa_3, 1, 0) \in Q_\infty \,|\, \text{there are } M, \tau \geqslant 0 \text{ such that for } b \in \Gamma^\# - \{0\} \text{ one} \right.$$
$$\left. \text{has } |\langle \kappa, b\rangle| \geqslant \frac{M}{|b|^\tau} \text{ and } |\langle \kappa, b\rangle + 1| \geqslant \frac{M}{|b|^\tau} \right\}$$

PROPOSITION. *Let $q \in L^2$ be any potential, and let $p = (\kappa, 1, 0)$ be a point of D. Then there is no algebraic component of $F_\lambda(q)$ whose closure passes through p and is transversal to Q_∞ at this point.*

Its proof is similar to that of the Proposition and Lemma 2 of the previous chapter: one blows up the point $p \in D$ in P^4 and shows that $\det_2 F$, whose zero set is the Fermi surface, is different from zero near the exceptional divisor. The diophantine conditions we imposed on p are used to get an upper bound for the Hilbert Schmidt norm of $F-1$. The complete proof is given in Appendix 2.

Now let C be a component of $\bar{X} \cap Q_\infty$, and assume that C is not contained in $\bigcup_{b\in\Gamma^\#} \{(k, y, 0) \in Q_\infty \mid \langle k, b\rangle + y = 0\}$. Then, C meets $\{(k, y, 0) \in Q_\infty \mid y = 0\}$ in only finitely many points, i.e.

$$C' := \{(k, 1, 0) \in Q_\infty \mid (k, 1, 0) \in C\}$$

is an affine curve. By the proposition above $C' \cap D$ consists of only finitely many points. We want to show that this leads to a contradiction.

Let D_0 be the set of all points $[y] = [y_1, y_2, y_3]$ in $P_2(R)$ for which one – and then all – representative y fulfills a diaphantine estimate

$$|\langle y, b\rangle| \geqslant \frac{K}{|b|^\tau}, \quad \forall\, b \in \Gamma^\# - \{0\}$$

with some $K, \tau \geqslant 0$. Clearly a point $(k, 1, 0) \in Q_\infty$ with $k \neq 0$ lies in D if its imaginary part $Im k$ represents a point of D_0. So let $\pi_0: C' - \{(0; 1, 0)\} \to P_2(R)$ be the projection $(k, 1, 0) \to [Im k]$. From what we said above it follows that the image of π_0 intersects D_0 in only finitely many points. On the other hand one easily verifies that $P_2(R) - D_0$ has measure zero. Hence by Sard's theorem π_0 does not have maximal rank anywhere. A first step towards reaching the contradiction is

LEMMA. *C' is contained in a plane.*

Proof. Let $p = (\kappa; 1, 0) = (\kappa_1, \kappa_2, \kappa_3, 1, 0)$ be a smooth point of C' where the torque of C' is non-zero. Since π_0 has rank $\leqslant 1$ at p any tangent vector $v = (v_1, v_2, v_3)$ to C' at p fulfills

$$\det(\operatorname{Im} k, \operatorname{Re} v, \operatorname{Im} v) = 0.$$

Therefore, we can find a linear change of variables

$$k' = Ak$$

where A is a real, invertible 3×3 – matrix such that $k'_1 = v'_1 = 0$ at p. Without loss of generality we may assume that $\operatorname{Im} k'_2$ and $\operatorname{Im} k'_3$ are different from zero. In these new variables C' has the parametrization

$$k'_1 = \beta \cdot t^2 + O(t^3)$$

$$k'_2 = k'_2(p) + v'_2(p)t + O(t^2)$$

$$k'_3 = k'_3(p) + v'_3(p)t + O(t^2).$$

Since the torque of C' at p is non-zero we have $\beta \neq 0$.

The image $\pi_0(C')$ has the parametrization (in the coordinate $y' = Ay$)

$$y'_1 = \frac{1}{\operatorname{Im} k'_3} \cdot \operatorname{Im}(\beta t^2) + O(t^3)$$

$$y'_2 = \frac{\operatorname{Im} k'_2}{\operatorname{Im} k'_3}\left(1 + \frac{1}{\operatorname{Im} k'_2} \cdot \operatorname{Im}(v'_2 t)\right)\left(1 - \frac{1}{\operatorname{Im} k'_3} \operatorname{Im} v'_3 \cdot t\right) + O(t^2)$$

$$y'_3 = 1$$

where the components of k' on the right-hand side are evaluated at p. For this map to have rank $\leqslant 1$ for all points $t = t_1 + it_2$ in a neighbourhood of 0 in $C = R + iR$ one must have

$$\frac{v'_2}{\operatorname{Im} k'_2(p)} = \frac{v'_3}{\operatorname{Im} k'_3(p)}.$$

In other words v' is proportional to $\operatorname{Im} k'$, and hence v is proportional to k. This shows that π_0 has rank zero at every point of C' where the torque does not vanish.

If the torque of C' is zero everywhere then C' is contained in a plane, and the lemma is proven. Otherwise, $\pi_0(C')$ is a point $[y_0]$ and the tangent vector of C' at each of its smooth points is proportional to y_0. But then C' is a line, and again the lemma is proven.

Returning to the proof of Theorem 1′, if $C \subset \{(k, y, 0) \in Q_\infty \mid y = 0\}$, we are finished. Otherwise there is $\gamma \in C^3$ such that

$$C \subset \{(k, y, 0) \in Q_\infty \mid \langle k, \gamma \rangle + y = 0\}.$$

From the fact that π_0 has rank $\leqslant 1$ one concludes that γ is either purely real or purely imaginary. We discuss the case that γ is real, the other case being similar. Thus, we may now assume that

$$C' = \{(k, 1, 0) \in Q_\infty \mid \langle k, \gamma \rangle + 1 = 0\}.$$

We want to show that $\gamma \in \Gamma^\#$. So, assume that $\gamma \notin \Gamma^\#$. By the proposition above it suffices to show that $C' \cap D$ consists of infinitely many points.

First we show that almost all points of C' fulfill the first diophantine condition in the definition of D. For this purpose we introduce the following notation: If

$$k = (k_1, k_2, k_3) \in C^3 - \{0\} \quad \text{with } k_1^2 + k_2^2 + k_3^2 = 0$$

(i.e. $|\mathrm{Re}\, k| = |\mathrm{Im}\, k|$ and $\langle \mathrm{Re}\, k, \mathrm{Im}\, k \rangle = 0$) let $v(k)$ be the unit vector in R^3 such that $\mathrm{Re}\, k$, $\mathrm{Im}\, k$, $v(k)$ form an oriented orthogonal basis. Also put

$$D_1 = \left\{ v \in R^3 \,\middle|\, |v| = 1,\ v \neq \frac{b}{|b|} \text{ for all } b \in \Gamma^\# - \{0\} \text{ and there are only finitely many } b \in \Gamma^\# \text{ such that } \left| v - \frac{b}{|b|} \right| < \frac{1}{|b|^2} \right\}.$$

A standard argument (as in [SM] §25, p. 191) shows that the complement of D_1 on the unit sphere S^2 has Lebesque measure zero.

LEMMA. *For any $k \in C^3 - \{0\}$ with $k_1^2 + k_2^2 + k_3^2 = 0$ and $v(k) \in D_1$ there is $K' > 0$ such that for all $b \in \Gamma^\# - \{0\}$*

$$|\langle k, b \rangle| \geqslant \frac{K'}{|b|^2}.$$

Proof. Let E be the plane in R^3 passing through by 0, $\mathrm{Re}\, k$, $\mathrm{Im}\, k$. For any $b \in \Gamma^\#$, $K > 0$ the set

$$\left\{ \xi \in E \,\middle|\, |\langle \xi, b \rangle| \leqslant \frac{K}{|b|^2} \right\}$$

is a strip centered at $\{\xi \in E \mid \langle \xi, b \rangle = 0\}$ of width at most

$$4\pi K \cdot \frac{1}{\left| v(k) - \frac{b}{|b|} \right| \cdot |b|^3}.$$

For all $b \in \Gamma^{\#} - \{0\}$ outside a finite set S this width is at most $\dfrac{4\pi k}{|b|}$. If K is chosen sufficiently small then none of these strips contains both $\operatorname{Re} k$ and $\operatorname{Im} k$

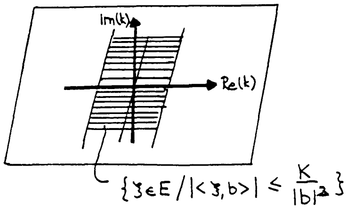

Fig. 3

Since $\langle b, k\rangle \neq 0$ for all $b \in \Gamma^{\#} - \{0\}$ we can shrink K such that $|\langle k, b\rangle| \geqslant \dfrac{K}{|b|^2}$ also for all $b \in S$.

One sees that the map $C' \to S^2$, $(k; 1, 0) \to v(k)$ has maximal rank almost everywhere. Therefore, for all points $(k, 1, 0)$ outside a set of Lebesque measure zero in C' there is $K > 0$ such that $|\langle k, b\rangle| \geqslant \dfrac{K}{|b|^2}$ for all $b \in \Gamma^{\#} - \{0\}$.

Now let E' be the plane in R^3

$$E' := \{x \in R^3 | \langle x, \gamma\rangle + 1 = 0\}.$$

One easily checks that the map $C' \to E'$, $(k, 1, 0) \to \operatorname{Re} k$ is surjective and submersive. Thus, the Theorem 1′ will be proven once we have shown

LEMMA. *The set of points x in E' for which there is $K, \tau > 0$ such that*

$$|\langle x, b\rangle + 1| \geqslant \frac{1}{|b|^{\tau}}$$

has positive (in fact infinite) Lebesgue measure.

Proof. For each $b \in \Gamma^{\#} - \{0\}$ the set

$$\left\{x \in E' \,\middle|\, |\langle x, b\rangle + 1| \leqslant \frac{K}{|b|^{\tau}}\right\}$$

is a strip around the line

$$\{x \in E' \,|\, \langle x, b\rangle + 1 = 0\}$$

of width at most

$$\frac{4\pi K}{\left|\dfrac{b}{|b|} - \dfrac{\gamma}{|\gamma|}\right| \cdot |b|^{\tau+1}}.$$

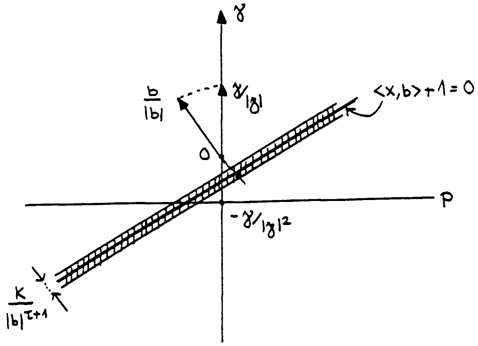

Fig. 4

The distance of this line $\{x \in E' \mid \langle x, b\rangle + 1 = 0\}$ from the point $\frac{\gamma}{|\gamma|^2}$ on E' is at most

$$\frac{1}{2|\gamma| \cdot \left|\frac{b}{|b|} + \frac{\gamma}{|\gamma|}\right|}.$$

Therefore, for $R > 0$,

$$\text{area}\left\{x \in E' \,\middle|\, \left|x + \frac{\gamma}{|\gamma|^2}\right| < R \text{ and } |\langle x, b\rangle + 1| \leqslant \frac{K}{|b|^\tau} \text{ for some } b \in \Gamma^\# - \{0\}\right\}$$

$$\leqslant \sum_{\substack{b \in \Gamma^\# - \{0\} \\ 2|\gamma|(b/|b| + \gamma/|\gamma|) \geqslant R^{-1}}} 2R \cdot \frac{4\pi K}{\left|\frac{b}{|b|} + \frac{\gamma}{|\gamma|}\right| \cdot |b|^{\tau+1}}$$

$$\leqslant \sum_{b \in \Gamma^\# - \{0\}} 16|\gamma| K \frac{R^2}{|b|^{\tau+1}} = \left(16|\gamma| K \cdot \sum_{b \in \Gamma^\#} \frac{1}{|b|^{\tau+1}}\right) \cdot R^2.$$

If $\tau > 2$ and K is sufficiently small this area is smaller than $2\pi R^2$, the area of the whole disc of radius R around $\frac{\gamma}{|\gamma|^2}$.

4. Appendix 1

Let Z be an open subset of R^m, that has zero as a cluster point, and let V be a neighbourhood of zero in R^n. Consider a continuous mapping

$F\colon V \times \bar{Z} \subset R^n \times R^m \to R^n$

with the following properties:

(i) $F(0, 0) = 0$.
(ii) For $z \in \bar{Z}$ the mapping $F(\cdot, z)$: $V \to R^n$ is differentiable and $(x, z) \to F_x(x, z)$ is continuous on $V \times \bar{Z}$.
(iii) $|F(x, z) - F(x, 0)| \leqslant |z|^\alpha$ for all $x \in V$, $z \in \bar{Z}$ (where α is a real number > 0).
(iv) $F_x(0, 0)$ is invertible.

Then there exists a sequence $(x_k, z_k)_{k \in N}$ in $V \times Z$ with $z_k \neq 0$ converging to $(0, 0)$ such that $F(x_k, z_k) = 0$.

Proof. Using properties (ii) and (iv) we can apply, for each fixed $z \in \bar{Z}$ near 0, the inverse mapping theorem to $F(\cdot, z)$. So there exists an open neighbourhood $W_1 \times W_2 \subset V \times R^m$ of 0 such that for each $z \in W_2 \cap \bar{Z}$ the mapping

$$F(\cdot, z): W_1 \to F(\cdot, z)(W_1) \subset R^n$$

is bijective, F^{-1} continuous and $F(\cdot, z)(W_1)$ contains an open ball. Then the mapping

$$G: W_1 \times W_2 \cap V \times \bar{Z} \to G(W_1 \times W_2 \cap V \times \bar{Z})$$

defined by

$$G(x, z) := (F(x, z), z)$$

is bijective and G^{-1} is continuous.

We have to show: $(0, z) \in G(W_1 \times W_2 \cap V \times \bar{Z})$ for z sufficiently close to 0.

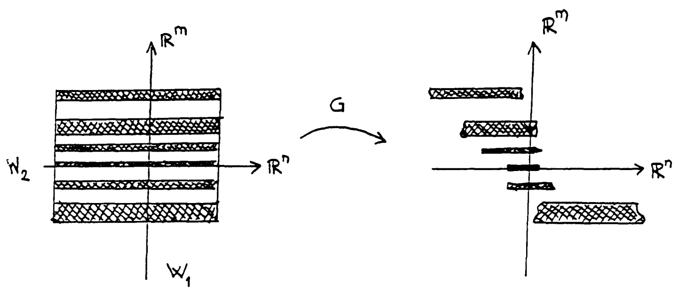

Fig. 5

Observe that by (i), $G(W_1 \times \{0\})$ is an open neighbourhood of zero in $R^n \times \{0\}$. Using (iii) we see that for $|z|$ small enough an open ball in $G(W_1 \times \{z\})$ of $R^n \times \{z\}$ contains $\{0\} \times \{z\}$. This proves the modification of the implicit function theorem.

5. Appendix 2: Proof of the proposition of Chapter 3

We blow up p in P^4. In the blown up space there are coordinates l_1, l_2, l_3, z such that

$$2 \cdot k_i = \frac{\kappa_i}{z} + l_i,\ i = 1, 2, 3 \quad \text{and} \quad y = \frac{1}{z}.$$

The exceptional plane E lying over p is $\{z = 0\}$. Suppose that X is an algebraic component of $F_\lambda(q)$ whose closure $\bar{X}$ passes transversally to Q_∞ through p. Then the strict transform X' of $\bar{X}$ would pass transversally through a point $(\hat{l}_1, \hat{l}_2, \hat{l}_3, 0)$ of E. Then for all points (l_1, l_2, l_3, z) in X near to this point the matrix

$$\left(\delta_{cb} + \frac{\hat{q}(c-b)}{\frac{1}{z}(\langle \kappa, b\rangle + 1) + (\langle l, b\rangle + b^2 - \lambda)}\right)_{c,b}$$

whose determinant describes the Fermi surface, is non-invertible, since the zero set of this determinant contains X. In particular, at these points the Hilbert–Schmidt norm of

$$\left(\frac{\hat{q}(c-b)}{\frac{1}{2}(\langle \kappa, b\rangle + 1) + \langle l, b\rangle + b^2 - \lambda}\right)_{c,b}$$

is at least one. The square of this Hilbert–Schmidt norm is bounded above by

$$\|q\|_2^2 \cdot \sum_{b \in \Gamma^\#} \frac{1}{|\langle \kappa, b\rangle + 1|^2 \left|\frac{1}{z} + \frac{b^2 + \langle l, b\rangle - \lambda}{\langle \kappa, b\rangle + 1}\right|^2}.$$

As X' is transversal to E at $(\hat{l}, 0)$ there is a constant $A > 0$ such that for all sufficiently small $z \in C$ there exists $l \in C^3$ with $|l - \hat{l}| < A \cdot |z|$ and $(l, z) \in X$. In particular the sum above is bigger than one. This contradicts the

LEMMA. *Assume that for all $b \in \Gamma^\#$ one has $|\langle \kappa, b\rangle + 1| \geqslant \frac{M}{|b|^\tau}$. Let $\varepsilon, A > 0$ and $\hat{l}$ be a point in C^3. Then there is a subset Z of $C - \{0\}$ that has 0 as cluster point such that for all $z \in Z$ and all $l \in C^3$ with $|l - \hat{l}| < A \cdot |z|$*

$$\sum_{b \in \Gamma^\#} \frac{1}{\left|\frac{1}{z}(\langle \kappa, b\rangle + 1) + (\langle l, b\rangle + b^2 - \lambda)\right|^2} < \varepsilon^2.$$

Proof. We consider the special case $|\kappa| = 1, \hat{l} = 0, \lambda = 0$, the general case is just somewhat more notationally awkward. Put $w := -\frac{1}{z}$ and define for $b \in \Gamma^{\#}$

$$f_b(w) := \sup_{|l| < A/|w|} \frac{1}{|\langle \kappa, b\rangle + 1|^2 \cdot \left| w - \frac{b^2 + \langle l, b\rangle}{\langle \kappa, b\rangle + 1} \right|^2}.$$

We are going to show that there are arbitrary large $w \in C$ for which $\Sigma_{b\in\Gamma^{\#}} f_b(w) < \varepsilon^2$. This clearly implies the lemma. Observe that

$$f_b(w) \geqslant \begin{cases} \infty, \quad \text{if } \left| w - \frac{b^2}{\langle \kappa, b\rangle + 1} \right| \leqslant \frac{A}{|w|} \cdot \frac{|b|}{|\langle \kappa, b\rangle + 1|} \\ \frac{1}{|\langle \kappa, b\rangle + 1|^2 \left(\left| w - \frac{b^2}{\langle \kappa, b\rangle + 1} \right| - \frac{A}{|w|} \frac{|b|}{|\langle \kappa, b\rangle + 1|} \right)^2}, \quad \text{otherwise.} \end{cases}$$

There is a finite set $S \subset \Gamma^{\#}$ such that for all $b \in \Gamma^{\#} - S$ and all $w \in C$ with

$$|w| > A, \quad \left| w - \frac{b^2}{\langle \kappa, b\rangle + 1} \right| > \frac{4A}{|\langle \kappa, b\rangle + 1|}$$

one has

$$f_b(w) \leqslant \frac{4}{|\langle \kappa, b\rangle + 1|^2 \left| w - \frac{b^2}{\langle \kappa, b\rangle + 1} \right|^2}.$$

Put $\tilde{A} := \max(4A, 2 \cdot \varepsilon^{-1})$ and denote by $D(b)$ the disc of radius $\frac{\tilde{A}}{|\langle \kappa, b\rangle + 1|}$ around the point $\frac{b^2}{\langle \kappa, b\rangle + 1}$. Then for all $b \in \Gamma^{\#} - S$

(1) $f_b(w) \leqslant \varepsilon^2$ for all $w \in C$ with $|w| > A$, $w \notin D(b)$,

(2) $\int_{\substack{A < |w| < R \\ w \notin D(b)}} f_b(w) |dw \wedge d\bar{w}| \leqslant \frac{8 \cdot \pi}{|\langle \kappa, b\rangle + 1|^2} \ln(2R) + \ln \varepsilon + \ln \frac{|\langle \kappa, b\rangle + 1|}{2}.$

The second inequality follows, because one has to integrate a function smaller than

$$\frac{4}{|\langle \kappa, b\rangle + 1|^2 \left| w - \frac{b^2}{\langle \kappa, b\rangle + 1} \right|^2}$$

over a region, where this latter expression is smaller than ε^2. Therefore, one increases this integral, if one integrates this latter function over a ball of radius

$2R$ around $\dfrac{b^2}{\langle\kappa, b\rangle + 1}$. Thus,

$$\int_{\substack{A<|w|<R\\ w\notin D(b)}} f_b(w)|dw \wedge d\bar{w}| \leqslant \frac{4\cdot 2\pi}{|\langle\kappa, b\rangle + 1|^2}\cdot \int_{2\varepsilon^{-1}|\langle\kappa,b\rangle+1|^{-1}}^{2R} \frac{r\,dr}{r^2}$$

We now decompose $\Gamma^{\#}$ into sets of the form

$$\Gamma^{\#}(\alpha, \beta) := \{b \in \Gamma^{\#} \mid |b|^{\beta} < |1 + \langle\kappa, b\rangle| \leqslant |b|^{\alpha}\}.$$

By the diophantine estimate for κ the set $\Gamma^{\#}(\alpha, \beta)$ has finite complement in $\Gamma^{\#}$ whenever $\beta < -\tau$, $\alpha > 1$.

For $r > 0$, denote by B_r the ball of radius r in R^3. Then

$$\#\Gamma^{\#}(\alpha, \beta) \cap B_r = \begin{cases} O(r^{1+2\alpha}), & \text{if } \alpha \geqslant 0;\\ O(r^{(\tau+\alpha)/\tau}), & \text{if } \alpha < 0 \text{ and } \tau \neq 0;\\ O(1), & \text{if } \tau = 0.\end{cases}$$

The first O-estimate is the growth of the volume in the region

$$\{x \in R^3 \mid |1 + \langle\kappa, x\rangle| \leqslant |x|^{\alpha}\}.$$

(Observe that $\{x \in R^3 \mid 1 + \langle\kappa, x\rangle = 0\}$ is a line in R^3, since $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 0$.) The second estimate follows from the fact that for $b_1, b_2 \in \Gamma^{\#}(\alpha, \beta)$ with $|b_1| \leqslant |b_2|$

$$\begin{aligned} M|b_2 - b_1|^{-\tau} &\leqslant |\langle\kappa, b_2 - b_1\rangle| \leqslant |1 + \langle\kappa, b_1\rangle| + |1 + \langle\kappa, b_2\rangle|\\ &\leqslant |b_1|^{\alpha} + |b_2|^{\alpha} \leqslant 2|b_2|^{\alpha}.\end{aligned}$$

CLAIM. For any $\alpha > \beta$ with $\beta \neq 3/2$ and $\alpha - \beta \neq 1/2$

(i) area $(\{w \in C \mid A < |w| < R,\ w \in D(b) \text{ for some } b \in \Gamma^{\#}(\alpha, \beta)\})$

$$= \begin{cases} O(R^{(1+2(\alpha-\beta))/(2-\alpha)}), & \text{if } \alpha \geqslant 0;\\ O(R^{((\tau+\alpha)/\tau - 2\beta)/(2-\alpha)}), & \text{if } \alpha \leqslant 0.\end{cases}$$

(ii) $$\int_{\substack{A<|w|<R\\ w\notin \bigcup_{b\in\Gamma^{\#}(\alpha,\beta)} D(b)}} \left(\sum_{b\in\Gamma^{\#}(\alpha,b)-S} f_b(w)\right)|dw \wedge d\bar{w}|$$

$$= \begin{cases} O(R^{(1+2(\alpha-\beta))/(2-\alpha)}\ln R), & \text{if } \alpha \geqslant 0;\\ O(R^{((\tau+\alpha)/\tau - 2\beta)/(2-\alpha)}\ln R), & \text{if } \alpha \leqslant 0.\end{cases}$$

In fact, for $b \in \Gamma^{\#}(\alpha, \beta)$

$$\text{area}(D(b)) = 2\pi\left|\frac{\tilde{A}}{\langle\kappa, b\rangle + 1}\right|^2 \leqslant \frac{2\pi\tilde{A}^2}{|b|^{2\beta}}.$$

The center of $D(b)$ is $\dfrac{b^2}{\langle\kappa, b\rangle + 1}$, which has distance at least $|b|^{2-\alpha}$ from the origin.

So if R is big enough $D(b)$ does not meet the shell $A < |w| < R$ unless $|b|^{2-\alpha} \leqslant 2R$. Therefore

$$\sum_{b\in\Gamma^{\#}(\alpha,\beta)} \operatorname{area}(D(b) \cap \{w \in C | A < |w| < R\}) \leqslant \sum_{\substack{b\in\Gamma^{\#}(\alpha,\beta)\\ |b|\leqslant(2R)^{1/(2-\alpha)}}} \frac{2\pi\tilde{A}^2}{|b|^{2\beta}}.$$

This, together with the growth estimates for $\Gamma^{\#}(\alpha, \beta)$, gives part (i) of the claim. Part (ii) is proven using the argument at the end of the proof of the proposition in section 2.

We now return to the proof of the lemma. It follows from the claim that one can find $\rho > 0$ and finitely many pairs $(\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)$ such that the union of the open intervals (β_n, α_n) covers $[-\tau, 1]$ and such that for $n = 1, \ldots, N$

$$\operatorname{area}\left(\left\{w \in C \,\middle|\, |w| < R \text{ and } \sum_{b\in\Gamma^{\#}(\alpha_n,\beta_n)\backslash S} f_b(w) \geqslant \frac{\varepsilon}{N}\right\}\right) = O(R^{2-\rho}).$$

Since the complement of $\bigcup_{n=1}^{N} \Gamma^{\#}(\alpha_n, \beta_n)$ in $\Gamma^{\#}$ is finite, this implies that

$$\operatorname{area}\left(\left\{w \in C \,|\, |w| < R \text{ and } \sum_{b\in\Gamma^{\#}} f_b(w) \geqslant \varepsilon\right\}\right) = O(R^{2-\rho}).$$

As we remarked above this implies the lemma, and so the proof of the proposition is completed.

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