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The cohomological dimension of the quotient field of the two dimensional complete local domain

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Let A be a complete Noetherian local domain with separably closed residue field k and with quotient field K. For a prime number p which is different from $\mathrm{char}(K)$, let $\mathrm{cd}_p(K)$ be the p-cohomological dimension of the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ (cf. [13], [14], here K^{sep} denotes the separable closure of K). In this paper, we determine $\mathrm{cd}_p(K)$ in the case $\dim(A) = 2$, $\mathrm{char}(k) = p$ and $\mathrm{char}(K) = 0$.

In general, if $\mathrm{char}(k) \neq p$, a standard conjecture (Artin [2]) is that

$$\mathrm{cd}_p(K) = \dim(A). \tag{0.1}$$

In the more delicate case where $\mathrm{char}(k) = p$ and $\mathrm{char}(K) = 0$, Artin suggests in [2] that the rank of the absolute differential module $\Omega_k^1 = \Omega_{k/\mathbb{Z}}^1$ should be involved in $\mathrm{cd}_p(K)$. The precise form of his conjecture in this case should be

$$\mathrm{cd}_p(K) = \dim(A) + \dim_k(\Omega_k^1). \tag{0.2}$$

The aim of this paper is to prove (0.2) in the case $\dim(A) = 2$.

THEOREM. *Let A be a complete Noetherian two dimensional local domain of mixed characteristic $(0, p)$ with a separably closed residue field k, and K be the quotient field of A. Then,*

$$\mathrm{cd}_p(K) = \dim_k(\Omega_k^1) + 2.$$

The conjecture (0.1) has been proved in the case $\dim(A) \leqq 2$ (the case $\dim(A) = 1$ is classical and the case $\dim(A) = 2$ is due to O. Gabber [4]). The conjecture (0.2) has been proved in the case $\dim(A) = 1$ (cf. [6]II, [13], [14]), and in the case where $\dim(A) = 2$ and k is algebraically closed (then $\Omega_k^1 = (0)$) by K. Kato ([12] §5).

Notation

A ring means a commutative ring with a unit.

For a local ring A,

$\hat{A}$: the completion of A by the maximal ideal m_A,
$A_{\mathfrak{p}}$: the localization of A at a prime ideal $\mathfrak{p}$,
$\kappa(\mathfrak{p})$: the residue field of $A_{\mathfrak{p}}$,
Ω^i_A: the ith exterior power over A of the absolute differential module Ω^1_A.

For a field k,

$K_i(k)$: the ith Milnor's K-group of k ([11]).

For an abelian group M and a family $S_\lambda (\lambda \in \Lambda)$ of elements of M, $\langle S_\lambda; \lambda \in \Lambda \rangle$ is the subgroup of M generated by S_λ for $\lambda \in \Lambda$.

Proof of theorem. Throughout this paper, let A be a complete Noetherian two dimensional local domain with a separably closed residue field k of characteristic $p > 0$ and with the quotient field K of characteristic 0. Without loss of generality, we assume that if $p \neq 2$ (resp. $p = 2$) K contains a primitive pth (resp. 4th) root of unity ([15]).

First of all, we have

PROPOSITION 1.

$$\mathrm{cd}_p(K) \geqq \dim_k(\Omega^1_k) + 2.$$

Proof. Let $\mathfrak{p}$ be a prime ideal of A such that $\mathrm{ht}(\mathfrak{p}) = 1$ and $\mathrm{char}(\kappa(\mathfrak{p})) = 0$, and let $K_{\mathfrak{p}}$ be the quotient field of the henselization of the local ring of A at $\mathfrak{p}$. Then we have

$$\begin{aligned} \mathrm{cd}_p(K) &\geqq \mathrm{cd}_p(K_{\mathfrak{p}}) && ([14]) \\ &= \mathrm{cd}_p(\kappa(\mathfrak{p})) + 1 && ([8]) \\ &= \dim_k(\Omega^1_k) + 2. && ([8]) \end{aligned}$$

Hence it remains to prove that $\mathrm{cd}_p(K) \leqq \dim_k(\Omega^1_k) + 2$. In the rest of this paper, we assume $\dim_k(\Omega^1_k) < \infty$. Let $r = \dim_k(\Omega^1_k)$ (so $[k : k^p] = p^r$).

We fix an algebraic closer $\bar{K}$ of K, and r elements $b_1, b_2, \ldots, b_r$ of A such that the residue classes $\bar{b}_i$ of b_i $(1 \leqq i \leqq r)$ form a p-basis of k. Then we can pick up elements $\{b_{i,j};\ 1 \leqq i \leqq r,\ j = 0, 1, 2, \ldots\}$ of $\bar{K}$ which satisfy the following conditions:

$$(b_{i,j})^p = b_{i,j-1} \qquad 1 \leqq i \leqq r, j = 1, 2, \ldots$$
$$b_{i,0} = b_i \qquad 1 \leqq i \leqq r.$$

For integers n ($n = 0, 1, 2, \ldots, \infty$), we define extensions $A^{(n)}$ (resp. $K^{(n)}$) of A (resp. K) by

$$A^{(n)} = A[b_{i,n};\ 1 \leqq i \leqq r]$$
$$A^{(\infty)} = \bigcup_{n=0}^{\infty} A^{(n)}$$
$$K^{(n)} = K(b_{i,n};\ 1 \leqq i \leqq r)$$
$$K^{(\infty)} = \bigcup_{n=0}^{\infty} K^{(n)}.$$

PROPOSITION 2.

$$\mathrm{cd}_p(K) \leqq r + 3.$$

Proof. In Lemma 1 below, we shall prove that $A^{(\infty)}$ is an excellent henselian local domain. Since the residue field of $A^{(\infty)}$ is algebraically closed, $\mathrm{cd}_p(K^{(\infty)}) = 2$ (cf. [12] §5 Th. the excellence of $A^{(\infty)}$ is needed here). Let ζ_{p^∞} be a subgroup of K^* which consists of all roots of unity of p-primary orders. Then $\mathrm{cd}_p(K^{(\infty)}(\zeta_{p^\infty})) \leqq 2$ ([14]).

On the other hand, the field $K^{(\infty)}(\zeta_{p^\infty})$ is a Galois extension of K and the Galois group of $K^{(\infty)}(\zeta_{p^\infty})/K$ is isomorphic to $\mathbb{Z}_p^{r+1}$ ($\mathbb{Z}_p$ is the ring of p-adic integers). Then we have inequalities

$$\begin{aligned} \mathrm{cd}_p(K) &\leqq \mathrm{cd}_p(\mathbb{Z}_p^{r+1}) + \mathrm{cd}_p(K^{(\infty)}(\zeta_{p^\infty})) \\ &\leqq r + 3. \qquad \text{([14])} \end{aligned}$$

LEMMA 1. *$A^{(\infty)}$ is an excellent henselian two dimensional local ring.*

Proof. By [9], A is finite over $R[[X]]$ where R is a complete discrete valuation ring with mixed characteristic containing $b_1, b_2, \ldots, b_r$ whose residue field is the same as that of A, and X is a variable. So we may assume that $A = R[[X]]$, $b_1, b_2, \ldots, b_r \in R$. We define rings $R^{(n)} = R[b_{i,n};\ 1 \leqq i \leqq r]$ for integers $n \geqq 0$ and $R^{(\infty)} = \bigcup_{n=0}^{\infty} R^{(n)}$, and fix a prime element π of R.

First, we will prove that $A^{(\infty)}$ is Noetherian.

It is enough to show that every prime ideal $\mathscr{P}$ of $A^{(\infty)}$ is finitely generated ([9]). Since $A^{(\infty)}$ is a two dimensional ring, every prime ideal $\neq (0)$ is either maximal or of height one. Assume $\mathscr{M}$ is a maximal ideal of $A^{(\infty)}$. Then $\mathscr{M} \cap R^{(n)}[[X]]$ is a maximal ideal of $R^{(n)}[[X]]$ for all integers $n \geqq 0$. Hence $\mathscr{M} \cap R^{(n)}[[X]] = (\pi, X)$ for all $n \geqq 0$, and this implies $\mathscr{M} = (\pi, X)$. On the other hand, if $\mathscr{P}$ is a prime ideal of $A^{(\infty)}$ of height one, $\mathscr{P} \cap R^{(n)}[[X]]$ is (π) or $(X^m + a_1X^{m-1} + \cdots + a_m)$ (m is a positive integer and a_i are elements of the maximal ideal of $R^{(n)}$ for $1 \leqq i \leqq r$) for all integers $n \geqq 0$. When $\mathscr{P} \cap R[[X]] = (\pi)$, $\mathscr{P} \cap R^{(n)}[[X]] = (\pi)$ also. This implies $\mathscr{P} = (\pi)$. When $\mathscr{P} \cap R[[X]] \neq (\pi)$, the degree m of the above polynomial becomes stable for sufficiently large integers n. So $\mathscr{P}$ is generated by an element which generates $\mathscr{P} \cap R^{(n)}[[X]]$ for integers $n \gg 0$. Thus $A^{(\infty)}$ is Noetherian.

Secondly, recall that, a Noetherian local ring S is excellent, if and only if, S is a G-ring and universally catenary (cf. [9] Ch. 13, 34).

It is easily deduced that $A^{(\infty)}$ is universally catenary from the fact that $A^{(\infty)}$ is the union of subrings which are finite over the excellent ring A. And $A^{(\infty)}$ is a G-ring when $\hat{A}^{(\infty)} \otimes_{A^{(\infty)}} L$ is regular for any prime ideal $\mathscr{P}$ of $A^{(\infty)}$ and any finite extension L of $\kappa(\mathscr{P})$. The regularity is easy and we omit the proof.

We have shown that $\mathrm{cd}_p(K)$ is $r + 2$ or $r + 3$. To prove that $\mathrm{cd}_p(K) = r + 2$, it is sufficient to show that the Galois cohomology groups $H^{r+3}(L, \mathbb{Z}/p\mathbb{Z})$ vanish for all finite extension fields L over K.

LEMMA 2. *The cohomology symbol map* (*cf.* [6]II)

$$h_K^{r+3};\ K_{r+3}(K)/p \to H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

is surjective.

Proof. In the first place, we consider the fact (*).

(*) *Let k be a field, S a Galois extension of k of infinite degree, p a prime number which is invertible in k, and $q \geqq 0$ an integer. Suppose that $\mathrm{cd}_p(\mathrm{Gal}(S/k)) \leqq q$ and $\mathrm{cd}_p(S) \leqq 2$, and that for any open subgroup J of $\mathrm{Gal}(S/k)$, the cup product*

$$\otimes^q H^1(J, \mathbb{Z}/p\mathbb{Z}) \to H^q(J, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Then, h_k^{q+2} is surjective.

Using the fact that h_k^2 is surjective for any field k ([10]), the arguments in the proof of Proposition 3 of [6]II, §1.3 can be used to prove (*) (we replace $\mathrm{cd}_p(S) \leqq 1$ (resp. h_k^{q+1}) by $\mathrm{cd}_p(S) \leqq 2$ (resp. h_k^{q+2})).

We apply (*) to $k = K$, $S = K^{(\infty)}(\zeta_{p^\infty})$ and $q = r + 1$. From the proof of Proposition 2, $\mathrm{cd}_p(K^{(\infty)}(\zeta_{p^\infty})) \leqq 2$, $\mathrm{cd}_p(\mathrm{Gal}(K^{(\infty)}(\zeta_{p^\infty})/K)) = r + 1$ and $\mathrm{Gal}(K^{(\infty)}(\zeta_{p^\infty})/K) \cong \mathbb{Z}_p^{r+1}$. Any open subgroup of $\mathbb{Z}_p^{r+1}$ is isomorphic to $\mathbb{Z}_p^{r+1}$ and the cup product

$$\otimes^{r+1} H^1(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z}) \to H^{r+1}(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Hence the assumption of (*) is satisfied. This lemma is proved.

We consider the condition.

(**) *A is regular and there exist two elements u and v of A which generate the maximal ideal of A, such that p is invertible in $A[1/uv]$.*

For $w = u$ or v, let $\mathfrak{p}_w = (w)$, $\bar{A}_w = A/\mathfrak{p}_w$ and

$$f_w = \begin{cases} (p/p-1)\mathrm{ord}_w(p) & \text{if } \mathrm{char}\,\kappa(\mathfrak{p}_w) = p \\ 0 & \text{if } \mathrm{char}\,\kappa(\mathfrak{p}_w) = 0. \end{cases}$$

We distinguish two cases;

case (I) $\quad \mathrm{char}(\kappa(\mathfrak{p}_u)) = \mathrm{char}(\kappa(\mathfrak{p}_v)) = p$

case (II) $\quad \mathrm{char}(\kappa(\mathfrak{p}_u)) = 0, \quad \mathrm{char}(\kappa(\mathfrak{p}_v)) = p.$

LEMMA 3. *Assume* (**). *Let*

$$\Delta = \left\langle \begin{matrix} \{a, b_1, \ldots, b_{r+2}\}; & b_s \in A[1/uv]^* \text{ for } 1 \leqq s \leqq r+2 \text{ and} \\ & a \in 1 + uvA \text{ in the case (I)} \\ & (\text{resp. } a \in 1 + vA \text{ in the case (II)}) \end{matrix} \right\rangle$$

in $K_{r+3}(K)/p$.

Then $\Delta = 0$.

Proof. For integers $i \geqq 0$ and $j > 0$, we define

$$\Delta_{i,j} = \left\langle \begin{matrix} \{1 + au^iv^j, b_1, \ldots, b_r, c, d\}; \\ a \in A,\ b_s \in A^* \text{ for } 1 \leqq s \leqq r \text{ and } c, d \in A[1/uv]^* \end{matrix} \right\rangle \quad \text{in } K_{r+3}(K)/p.$$

In the case (I) (resp. (II)), we deduce $\Delta = 0$ from the following facts,

(1) $\Delta = \Delta_{1,1}$ (resp. $\Delta = \Delta_{0,1}$)
(2) if $0 \leqq i \leqq f_u$, $0 < j < f_v$ and either $p \nmid i$ or $p \nmid j$, we have $\Delta_{i,j} = \Delta_{i,j+1}$
(3) $\Delta_{f_u, f_v} = 0$.

Proof of (1). This is easy and we omit the proof.

Proof of (2). We can define the homomorphism

$$\chi_1;\ \Omega^{r+2}_{\bar{A}_v} \to \Delta_{i,j}/\Delta_{i,j+1}$$

by

$$\bar{a}\cdot\frac{d\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{d\bar{a}_{r+2}}{\bar{a}_{r+1}} \mapsto \{1 + au^iv^j, a_1, \ldots, a_{r+2}\}.$$

If $p \nmid j$, by the following calculation in $K_2(K)$, the map χ_1 is surjective.

$$j\{1 + au^h v^j, v\} = \{1 + au^h v^j, -au^h\}$$
$$\{1 + au^h v^j, 1 + bv\} = \{1 + au^h v^j, 1 - abu^h v^{j+1}\} - \{1 + bv, 1 - abu^h v^{j+1}\}$$
$$\text{for } a \in A^*, b \in A \text{ and } h \geqq i.$$

In the case $p \mid j$ and $p \mid i$, we can define the homomorphism

$$\chi_2;\ \Omega^{r+1}_{\bar{A}_v} \to \Delta_{i,j}/\Delta_{i,j+1}$$

by

$$\bar{a}\frac{d\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{d\bar{a}_{r+1}}{\bar{a}_{r+1}} \mapsto \{1 + au^i v^j, a_1, \ldots, a_{r+1}, v\}.$$

Then every element of $\Delta_{i,j}/\Delta_{i,j+1}$ is a sum of elements of the images of χ_1 and χ_2.

On the other hand, the equalities $\mathrm{char}(\kappa(\not{p}_v)) = p$ and $[\kappa(\not{p}_v): \kappa(\not{p}_v)^p] = p^{r+1}$ imply $\Omega^{r+2}_{\bar{A}_v} = 0$.

$$\Omega^{r+1}_{\bar{A}_v} = \left\langle \bar{a}^p \cdot \frac{d\bar{a}_1}{\bar{a}_1} \wedge \cdots \wedge \frac{d\bar{a}_{r+1}}{\bar{a}_{r+1}};\ \bar{a} \in \bar{A}_v,\ \bar{a}_j \in \bar{A}^*_v,\ 1 \leqq j \leqq r+1 \right\rangle + d\Omega^r_{\bar{A}_v}$$

([7] II). Then we have $\Delta_{i,j} = \Delta_{i,j+1}$.

Proof of (3). Let c be the element of A^* such that $p = cu^{f_u(p-1)/p}v^{f_v(p-1)/p}$. We can take a solution $x \in A$ of the equation $X^p + cX - a = 0$ for any $a \in A$. Then,

$$1 + au^{f_u}v^{f_v} = (1 = xu^{f_u/p}v^{f_v/p})^p.$$

Thus the proof of Lemma 3 is complete.

LEMMA 4. *Assume* (**). *Let*

$$\Delta' = \langle \{a, b_1, \ldots, b_{r+2}\};\ a \in 1 + m_A \text{ and } b_j \in A[1/uv]^* \quad \text{for } 1 \leqq j \leqq r+2 \rangle$$
$$\text{in } K_{r+3}(K)/p.$$

where m_A is the maximal ideal of A.

Then $\Delta' = 0$.

Proof. As is easily seen, $1 + m_A$ is generated by elements of the form $1 - aw$ ($a \in A^*$) for $w = u$ or v. From this, we see that Δ' is generated by elements of the forms $\{1 - aw, b_1, \ldots, b_r, c, d\}$ with $a, b_1, \ldots, b_r$ and $c \in A^*$, $d \in A[1/uv]^*$ and $w = u$ or v such that $b_1, \ldots, b_{r-1}$ and b_r form a p-basis of k.

Hence it suffices to prove

$$\{1 - au, b_1, \ldots, b_r, c, d\} \in pK_{r+3}(K)$$

for $a, b_1, \ldots, b_r, c$ and d as above. Let

$$B = A[(au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}] \subset \bar{K}$$

and L be the quotient field of B. Since $B/vB = (A/vA)^{1/p}$, there exist elements $c' \in B^*$ and $c'' \in B$ such that $c = (c')^p(1 + c''v)$. We apply Lemma 3 to B,

$$\{1 - (au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}, 1 + c''v, d\} \in pK_{r+3}(L).$$

With $N_{L/K}$ denoting the norm map, we have

$$\begin{aligned}\{1 - au, b_1, \ldots, b_r, c, d\} &= N_{L/K}(\{1 - (au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}, c, d\}) \\ &\in N_{L/K}(pK_{r+3}(L)) \subset pK_{r+3}(K).\end{aligned}$$

LEMMA 5. *Assume* (**). *Let*

$$\Delta'' = \langle\{a_1, a_2, \ldots, a_{r+3}\};\ a_j \in A[1/uv]^* \ \textit{for}\ 1 \leqq j \leqq r+3\rangle \quad \textit{in}\ K_{r+3}(K)/p.$$

Then $\Delta'' = 0$.

Proof.

$$\Delta'' = \langle\{a_1, \ldots, a_{r+1}, b, c\};\ a_j \in A^* \text{ for } 1 \leqq j \leqq r+1,\ b, c \in A[1/uv]^*\rangle$$

Hence it suffices to prove

$$\{a_1, \ldots, a_{r+1}, b, c\} \in pK_{r+3}(K)$$

for $a_1, \ldots, a_{r+1}, b$ and c as above. Since $\bar{a}_i = a_i \bmod \mathfrak{m}_A\ (1 \leqq i \leqq r+1)$ cannot be p-independent, there exists s such that $1 \leqq s \leqq r$ and $\bar{a}_{s+1} \in k^p(a_1, \ldots, a_s)$. Let

$$B = A[a_1^{1/p}, \ldots, a_s^{1/p}] \subset \bar{K}$$

and L be the quotient field of B. Since the residue field of B contains $\bar{a}_{s+1}^{1/p}$, there exist elements $a' \in B^*$ and $a'' \in \mathfrak{m}_B$ ($\mathfrak{m}_B$ is the maximal ideal of B) such that

$a_{s+1} = (a')^p(1 + a'')$. By applying Lemma 4 to B, we have

$$\begin{aligned}&\{a_1, \ldots, a_{r+1}, b, c\}\\ &\quad = N_{L/K}(\{a_1^{1/p}, \ldots, a_s^{1/p}, a_{s+1}, \ldots, a_{r+1}, b, c\})\\ &\quad \in N_{L/K}(pK_{r+3}(L)) \subset pK_{r+3}(K).\end{aligned}$$

We follow the method of [12] §5.

LEMMA 6. *Let $\mathfrak{X} \to \mathrm{Spec}(A)$ be a proper birational morphism with regular such that $Y = \mathfrak{X} \otimes_A A/\mathfrak{m}_A$ is a reduced divisor with normal crossing on $\mathfrak{X}$* ([1], [5]). *Then,*

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) \overset{\cong}{\to} \bigoplus_{x \in Y_0} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z})$$

where Y_0 denotes the set of closed points of Y and for each $x \in Y_0$, K_x denotes the quotient field of the henselization of $\mathfrak{O}_{\mathfrak{X},x}$.

Proof. Let $\lambda: \mathrm{Spec}(K) \to \mathfrak{X}$ be the inclusion map and put $\mathfrak{F} = R\lambda_*(\mathbb{Z}/p\mathbb{Z})$. By the proper base change theorem, we have

$$H^q_x(Y, i^*\mathfrak{F}) = H^q(\mathfrak{X}, \mathfrak{F}) = H^q(K, \mathbb{Z}/p\mathbb{Z})$$

where $i: Y \to \mathfrak{X}$ is the inclusion map. From this, we obtain an exact sequence

$$\to \bigoplus_{x \in Y_0} H^q_x(Y, i^*\mathfrak{F}) \to H^q(K, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{\eta} H^q(K_\eta, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{x \in Y_0} H^{q+1}_x(Y, i^*\mathfrak{F}) \to$$

where η ranges over all generic points of Y and K_η denotes the quotient field of the henselization of $\mathfrak{O}_{\mathfrak{X},\eta}$. For each $x \in Y_0$, we have an exact sequence

$$\to H^q_x(Y, i^*\mathfrak{F}) \to H^q(K_x, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v} H^q(K_v, \mathbb{Z}/p\mathbb{Z}) \to H^{q+1}_x(Y, i^*\mathfrak{F}) \to$$

where v ranges over all generic points of the henselization of $\mathrm{Spec}(\mathfrak{O}_{Y,x})$, and K_v denotes the quotient field of the henselization of the discrete valuation ring of K_x corresponding to v.

Since $\mathrm{cd}_p(K_\eta) = \mathrm{cd}_p(K_v) = r + 2$ ([8]), we have

$$H^{r+3}(K_\eta, \mathbb{Z}/p\mathbb{Z}) = H^{r+3}(K_v, \mathbb{Z}/p\mathbb{Z}) = 0.$$

On the other hand, from the classical approximation theorem for a finite family of discrete valuation on K^* and K_x^*, the maps

$$K_{r+2}(K) \to \bigoplus_{\eta} K_{r+2}(K_\eta)/p \quad \text{and} \quad K_{r+2}(K_x) \to \bigoplus_{v} K_{r+2}(K_v)/p$$

are subjective. By [3] §5, the cohomological symbol maps

$$K_{r+2}(K_\eta)/p \to H^{r+2}(K_\eta, \mathbb{Z}/p\mathbb{Z})$$

and

$$K_{r+2}(K_v)/p \to H^{r+2}(K_v, \mathbb{Z}/p\mathbb{Z})$$

are subjective. Hence the maps

$$H^{r+2}(K, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{\eta} H^{r+2}(K_\eta, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^{r+2}(K_x, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v} H^{r+2}(K_v, \mathbb{Z}/p\mathbb{Z})$$

are also subjective.

Putting these things together,

$$\bigoplus_{x \in Y_0} H_x^{r+3}(Y, i^*\mathfrak{F}) \xrightarrow{\cong} H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

and

$$H_x^{r+3}(Y, i^*\mathfrak{F}) \xrightarrow{\cong} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z}).$$

These isomorphisms induce the isomorphism of Lemma 6.

PROPOSITION 3. *For* $r = \mathrm{dom}_k(\Omega_k^1)$,

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

Proof. By Lemma 2 and 6, we have

$$\begin{array}{ccc} \bigoplus_{x \in Y_0} K_{r+3}(K_x)/p & \twoheadrightarrow & \bigoplus_{x \in Y_0} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z}) \\ & & \uparrow \wr\| \\ K_{r+3}(K)/p & \twoheadrightarrow & H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}). \end{array}$$

For any family of fixed elements $a_1, a_2, \ldots, a_{r+3} \in K^*$, we can take $\mathfrak{X}$ such that the union Z of Y with the supports of the divisor of $a_1, \ldots, a_{r+2}$ and a_{r+3} on $\mathfrak{X}$ is normally crossing divisor ([5]). Then, by Lemma 5,

$$\{a_1, a_2, \ldots, a_{r+3}\} \in pK_{r+3}(K_x)$$

for any $x \in Y_0$. This shows that

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

We are now in the position to complete the proof of our theorem. By Proposition 1 and 2, $\mathrm{cd}_p(K) = r + 2$ or $r + 3$. For any finite extension field K' over K,

$$H^{r+3}(K', \mathbb{Z}/p\mathbb{Z}) = 0$$

by Proposition 3. Hence $\mathrm{cd}_p(K) = r + 2$ ([14]).

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References

[1] Abhyankar, S.; Resolution of singularities for arithmetical surfaces. In *Arithmetical Algebraic Geometry*, New York: Harper and Row, (1963), 111–152.
[2] Artin, M.; Dimension cohomologique: Premiers résultats, in *SGA* 4, *Tome* 3, *Lecture Notes in Mathematics* 305, (1973), 43–63.
[3] Bloch, S. and Kato, K.; p-adic étale coholology. *Publ. Math.* I.H.E.S. 63 (1986) 107–152.
[4] Gabber, O.; A lecture at *I.H.E.S.* on March in 1981.
[5] Hironaka, H.; Desingularization of excellent surface. *Lecture at Advanced Seminar in Algebraic Geometry*, Bowdoin College, Summer 1967, notes by Bruce Bennett.
[6] Kato, K.; A generalization of local class field theory by using K-groups, *I. J. fac. Sci. Univ. Tokyo Sec.* IA 26 (1979) 303–376: II *Ibid* 27 (1980) 602–683: III *ibid* 29 (1982) 31–34.
[7] Kato, K.; Galois cohomology of complete discrete valuation fields. *Lecture Notes in Mathematics* 967 (1982) 215–238.
[8] Kato, K. and Kuzumaki, T.; The dimension of fields and algebraic K-theory. *Journal of Number Theory* Vol. 24 No. 2 (1986) 229–244.
[9] Matumura, H.; Commutative Algebra. W. A. Benjamin Co., New York 2nd ed. (1980).
[10] Merkuriev, A. S. and Suslin, A. A.; K-cohomology of Severi-Brauer variety and norm residue homomorphism. *Math. USSA-Izv.* 21 (1984) 307–340.

[11] Milnor, J.; Introduction to algebraic K-theory. *Ann. Math. Stud.* 72 (1971).
[12] Saito, S.; Arithmetic on two dimensional local rings. *Invent. Math.* 85 (1986) 379–414.
[13] Satz, S. S.; Profinite groups, arithmetic and geometry. *Ann. Math. Stud.* 67 (1972).
[14] Serre, J.-P.; Cohomologie Galoisienne. *Lecture Notes in Mathematics* 5 (1965).
[15] Serre, J.-P.; Sur la dimension cohomologique des groupes profinis. *Topology* 3 (1965) 413–420.