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The cohomological dimension of the quotient field of the two dimensional complete local domain


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Let $A$ be a complete Noetherian local domain with separably closed residue field $k$ and with quotient field $K$. For a prime number $p$ which is different from $\text{char}(K)$, let $\text{cd}_p(K)$ be the $p$-cohomological dimension of the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ (cf. [13], [14], here $K^{\text{sep}}$ denotes the separable closure of $K$). In this paper, we determine $\text{cd}_p(K)$ in the case $\dim(A) = 2$, $\text{char}(k) = p$ and $\text{char}(K) = 0$.

In general, if $\text{char}(k) \neq p$, a standard conjecture (Artin [2]) is that

$$\text{cd}_p(K) = \dim(A). \quad (0.1)$$

In the more delicate case where $\text{char}(k) = p$ and $\text{char}(K) = 0$, Artin suggests in [2] that the rank of the absolute differential module $\Omega^1_k = \Omega^1_{k/Z}$ should be involved in $\text{cd}_p(K)$. The precise form of his conjecture in this case should be

$$\text{cd}_p(K) = \dim(A) + \dim_k(\Omega^1_k). \quad (0.2)$$

The aim of this paper is to prove (0.2) in the case $\dim(A) = 2$.

**THEOREM.** Let $A$ be a complete Noetherian two dimensional local domain of mixed characteristic $(0, p)$ with a separably closed residue field $k$, and $K$ be the quotient field of $A$. Then,

$$\text{cd}_p(K) = \dim_k(\Omega^1_k) + 2.$$  

The conjecture (0.1) has been proved in the case $\dim(A) \leq 2$ (the case $\dim(A) = 1$ is classical and the case $\dim(A) = 2$ is due to O. Gabber [4]). The conjecture (0.2) has been proved in the case $\dim(A) = 1$ (cf. [6]II, [13], [14]), and in the case where $\dim(A) = 2$ and $k$ is algebraically closed (then $\Omega^1_k = (0)$) by K. Kato ([12] §5).
Notation

A ring means a commutative ring with a unit.

For a local ring $A$,

$
\hat{A}:$ the completion of $A$ by the maximal ideal $m_A$,

$A_\mathfrak{p}:$ the localization of $A$ at a prime ideal $\mathfrak{p},$

$k(\mathfrak{p}):$ the residue field of $A_\mathfrak{p},$

$\Omega_A^i$: the $i$th exterior power over $A$ of the absolute differential module $\Omega_A^1$.

For a field $k$,

$K_i(k):$ the $i$th Milnor's $K$-group of $k$ ([11]).

For an abelian group $M$ and a family $S_\lambda(\lambda \in \Lambda)$ of elements of $M$, $\langle S_\lambda; \lambda \in \Lambda \rangle$ is the subgroup of $M$ generated by $S_\lambda$ for $\lambda \in \Lambda$.

Proof of theorem. Throughout this paper, let $A$ be a complete Noetherian two-dimensional local domain with a separably closed residue field $k$ of characteristic $p > 0$ and with the quotient field $K$ of characteristic 0. Without loss of generality, we assume that if $p \neq 2$ (resp. $p = 2$) $K$ contains a primitive $p$th (resp. 4th) root of unity ([15]).

First of all, we have

PROPOSITION 1.

$$\text{cd}_p(K) \geq \dim_k(\Omega_A^1) + 2.$$ 

Proof. Let $\mathfrak{p}$ be a prime ideal of $A$ such that $\text{ht}(\mathfrak{p}) = 1$ and $\text{char}(k(\mathfrak{p})) = 0$, and let $K_{\mathfrak{p}}$ be the quotient field of the henselization of the local ring of $A$ at $\mathfrak{p}$. Then we have

$$\text{cd}_p(K) \geq \text{cd}_p(K_{\mathfrak{p}}) \geq \text{cd}_p(k(\mathfrak{p})) + 1 \geq \dim_k(\Omega_A^1) + 2.$$ 

Hence it remains to prove that $\text{cd}_p(K) \leq \dim_k(\Omega_A^1) + 2$. In the rest of this paper, we assume $\dim_k(\Omega_A^1) < \infty$. Let $r = \dim_k(\Omega_A^1)$ (so $[k : k^r] = p^r$).

We fix an algebraic closer $\overline{K}$ of $K$, and $r$ elements $b_1, b_2, \ldots, b_r$ of $A$ such that the residue classes $\overline{b}_i$ of $b_i$ ($1 \leq i \leq r$) form a $p$-basis of $k$. Then we can pick up elements $\{b_{i,j}; \ 1 \leq i \leq r, \ j = 0, 1, 2, \ldots \}$ of $\overline{K}$ which satisfy the following conditions:
For integers \( n (n = 0, 1, 2, \ldots, \infty) \), we define extensions \( A^{(n)} \) (resp. \( K^{(n)} \)) of \( A \) (resp. \( K \)) by

\[
A^{(n)} = A[b_{i,n}; 1 \leq i \leq r]
\]

\[
A^{(\infty)} = \bigcup_{n=0}^{\infty} A^{(n)}
\]

\[
K^{(n)} = K(b_{i,n}; 1 \leq i \leq r)
\]

\[
K^{(\infty)} = \bigcup_{n=0}^{\infty} K^{(n)}.
\]

**PROPOSITION 2.**

\[
\text{cd}_p(K) \leq r + 3.
\]

**Proof.** In Lemma 1 below, we shall prove that \( A^{(\infty)} \) is an excellent henselian local domain. Since the residue field of \( A^{(\infty)} \) is algebraically closed, \( \text{cd}_p(K^{(\infty)}) = 2 \) (cf. [12] §5 Th. the excellence of \( A^{(\infty)} \) is needed here). Let \( \zeta_p^* \) be a subgroup of \( K^* \) which consists of all roots of unity of \( p \)-primary orders. Then \( \text{cd}_p(K^{(\infty)}(\zeta_p^*)) \leq 2 \) ([14]).

On the other hand, the field \( K^{(\infty)}(\zeta_p^*) \) is a Galois extension of \( K \) and the Galois group of \( K^{(\infty)}(\zeta_p^*)/K \) is isomorphic to \( \mathbb{Z}_p^{-1} \) (\( \mathbb{Z}_p \) is the ring of \( p \)-adic integers). Then we have inequalities

\[
\text{cd}_p(K) \leq \text{cd}_p(\mathbb{Z}_p^{-1}) + \text{cd}_p(K^{(\infty)}(\zeta_p^*))
\]

\[
\leq r + 3. \quad ([14])
\]

**LEMMA 1.** \( A^{(\infty)} \) is an excellent henselian two dimensional local ring.

**Proof.** By [9], \( A \) is finite over \( R[[X]] \) where \( R \) is a complete discrete valuation ring with mixed characteristic containing \( b_1, b_2, \ldots, b_r \) whose residue field is the same as that of \( A \), and \( X \) is a variable. So we may assume that \( A = R[[X]], b_1, b_2, \ldots, b_r \in R \). We define rings \( R^{(n)} = R[b_{i,n}; 1 \leq i \leq r] \) for integers \( n \geq 0 \) and \( R^{(\infty)} = \bigcup_{n=0}^{\infty} R^{(n)} \), and fix a prime element \( \pi \) of \( R \).

First, we will prove that \( A^{(\infty)} \) is Noetherian.

It is enough to show that every prime ideal \( \mathfrak{p} \) of \( A^{(\infty)} \) is finitely generated ([9]). Since \( A^{(\infty)} \) is a two dimensional ring, every prime ideal \( \neq (0) \) is either maximal or of height one. Assume \( \mathfrak{m} \) is a maximal ideal of \( A^{(\infty)} \). Then \( \mathfrak{m} \cap R^{(n)}[[X]] \) is a...
maximal ideal of $R^{(n)}[[X]]$ for all integers $n \geq 0$. Hence $\mathcal{M} \cap R^{(n)}[[X]] = (\pi, X)$ for all $n \geq 0$, and this implies $\mathcal{M} = (\pi, X)$. On the other hand, if $\mathcal{P}$ is a prime ideal of $A^{(\infty)}$ of height one, $\mathcal{P} \cap R^{(n)}[[X]]$ is $(\pi)$ or $(X^m + a_1X^{m-1} + \cdots + a_m)$ ($m$ is a positive integer and $a_i$ are elements of the maximal ideal of $R^{(n)}$ for $1 \leq i \leq r$) for all integers $n \geq 0$. When $\mathcal{P} \cap R[[X]] = (\pi)$, $\mathcal{P} \cap R^{(n)}[[X]] = (\pi)$ also. This implies $\mathcal{P} = (\pi)$. When $\mathcal{P} \cap R[[X]] \neq (\pi)$, the degree $m$ of the above polynomial becomes stable for sufficiently large integers $n$. So $\mathcal{P}$ is generated by an element which generates $\mathcal{P} \cap R^{(n)}[[X]]$ for integers $n > 0$. Thus $A^{(\infty)}$ is Noetherian.

Secondly, recall that, a Noetherian local ring $S$ is excellent, if and only if, $S$ is a $G$-ring and universally catenary (cf. [9] Ch. 13, 34).

It is easily deduced that $A^{(\infty)}$ is universally catenary from the fact that $A^{(\infty)}$ is the union of subrings which are finite over the excellent ring $A$. And $A^{(\infty)}$ is a $G$-ring when $\hat{A}^{(\infty)} \otimes_{A^{(\infty)}} L$ is regular for any prime ideal $\mathcal{P}$ of $A^{(\infty)}$ and any finite extension $L$ of $k(\mathcal{P})$. The regularity is easy and we omit the proof.

We have shown that $\text{cd}_p(K) = r + 2$ or $r + 3$. To prove that $\text{cd}_p(K) = r + 2$, it is sufficient to show that the Galois cohomology groups $H^{r+3}(L, \mathbb{Z}/p\mathbb{Z})$ vanish for all finite extension fields $L$ over $K$.

**LEMMA 2.** The cohomology symbol map (cf. [6]II)

$$h_{K/p}^{r+3}; K_{r+3}(K)/p \rightarrow H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

is surjective.

Proof. In the first place, we consider the fact ($\ast$).

($\ast$) Let $k$ be a field, $S$ a Galois extension of $k$ of infinite degree, $p$ a prime number which is invertible in $k$, and $q \geq 0$ an integer. Suppose that $\text{cd}_p(\text{Gal}(S/k)) \leq q$ and $\text{cd}_p(S) \leq 2$, and that for any open subgroup $J$ of $\text{Gal}(S/k)$, the cup product

$$\otimes^q H^1(J, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^q(J, \mathbb{Z}/p\mathbb{Z})$$

is surjective. Then, $h_{K/p}^{r+2}$ is surjective.

Using the fact that $h_k^2$ is surjective for any field $k$ ([10]), the arguments in the proof of Proposition 3 of [6]II, §1.3 can be used to prove ($\ast$) (we replace $\text{cd}_p(S) \leq 1$ (resp. $h_k^{r+1}$) by $\text{cd}_p(S) \leq 2$ (resp. $h_{K/p}^{r+2}$)).

We apply ($\ast$) to $k = K$, $S = K^{(\infty)}(\zeta_p^e)$ and $q = r + 1$. From the proof of Proposition 2, $\text{cd}_p(K^{(\infty)}(\zeta_p^e)) \leq 2$, $\text{cd}_p(\text{Gal}(K^{(\infty)}(\zeta_p^e)/K)) = r + 1$ and $\text{Gal}(K^{(\infty)}(\zeta_p^e)/K) \cong \mathbb{Z}_p^{r+1}$. Any open subgroup of $\mathbb{Z}_p^{r+1}$ is isomorphic to $\mathbb{Z}_p^{r+1}$ and the cup product

$$\otimes^{r+1} H^1(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^{r+1}(\mathbb{Z}_p^{r+1}, \mathbb{Z}/p\mathbb{Z})$$
is surjective. Hence the assumption of (\(\ast\)) is satisfied. This lemma is proved.

We consider the condition.

\(\ast\ast\) A is regular and there exist two elements \(u\) and \(v\) of \(A\) which generate the maximal ideal of \(A\), such that \(p\) is invertible in \(A[1/uv]\).

For \(w = u\) or \(v\), let \(\mu_w = (w)\), \(\bar{A}_w = A/\mu_w\) and

\[
 f_w = \begin{cases} 
 (p/p - 1)\ord_w(p) & \text{if } \ker(\mu_w) = p \\
 0 & \text{if } \ker(\mu_w) = 0.
\end{cases}
\]

We distinguish two cases;

- case (I) \(\ker(\mu_u) = \ker(\mu_v) = p\)
- case (II) \(\ker(\mu_u) = 0, \ker(\mu_v) = p\).

**Lemma 3.** Assume \(\ast\ast\). Let

\[
\Delta = \left\langle \left\{a, b_1, \ldots, b_{r+2}; \begin{array}{l}
 b_s \in A[1/uv]^* \text{ for } 1 \leq s \leq r + 2 \\
 a \in 1 + uvA \text{ in the case (I)} \\
 (\text{resp. } a \in 1 + vA \text{ in the case (II)})
\end{array} \right\} \right\rangle \text{ in } K_{r+3}(K)/p.
\]

Then \(\Delta = 0\).

**Proof.** For integers \(i \geq 0\) and \(j > 0\), we define

\[
\Delta_{i,j} = \left\langle \left\{1 + au^i v^j, b_1, \ldots, b_r, c, d; \begin{array}{l}
 a \in A, b_s \in A^* \text{ for } 1 \leq s \leq r \text{ and } c, d \in A[1/uv]^*
\end{array} \right\} \right\rangle \text{ in } K_{r+3}(K)/p.
\]

In the case (I) (resp. (II)), we deduce \(\Delta = 0\) from the following facts,

1. \(\Delta = \Delta_{1,1}\) (resp. \(\Delta = \Delta_{0,1}\))
2. if \(0 \leq i \leq f_u, 0 < j < f_v\) and either \(p \nmid i\) or \(p \mid j\), we have \(\Delta_{i,j} = \Delta_{i,j+1}\)
3. \(\Delta_{f_u, f_v} = 0\).

**Proof of (1).** This is easy and we omit the proof.

**Proof of (2).** We can define the homomorphism

\[
\chi_1; \Omega_{\lambda}^{r+2} \rightarrow \Delta_{i,j}/\Delta_{i,j+1}
\]

by

\[
\frac{\alpha \cdot d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{r+2}}{\alpha_{r+1}} \mapsto \{1 + au^i v^j, a_1, \ldots, a_{r+2}\}.
\]
If \( p \nmid j \), by the following calculation in \( K_2(K) \), the map \( \chi_1 \) is surjective.

\[
j \{ 1 + au^h v^j, v \} = \{ 1 + au^h v^j, -au^h \}
\]
\[
\{ 1 + au^h v^j, 1 + bv \} = \{ 1 + au^h v^j, 1 - abu^h v^{j+1} \} - \{ 1 + bv, 1 - abu^h v^{j+1} \}
\]
for \( a \in A^* \), \( b \in A \) and \( h \geq i \).

In the case \( p \mid j \) and \( p \mid i \), we can define the homomorphism

\[
\chi_2; \Omega_{A_v}^{r+1} \to \Delta_{i,j}/\Delta_{i,j+1}
\]

by

\[
\tilde{a} \frac{d \tilde{a}_1}{\tilde{a}_1} \wedge \cdots \wedge \frac{d \tilde{a}_{r+1}}{\tilde{a}_{r+1}} \mapsto \{ 1 + au^j v, a_1, \ldots, a_{r+1}, v \}.
\]

Then every element of \( \Delta_{i,j}/\Delta_{i,j+1} \) is a sum of elements of the images of \( \chi_1 \) and \( \chi_2 \).

On the other hand, the equalities \( \text{char}(K(\overline{tv})) = p \) and \( [\kappa(\overline{tv}); \kappa(\overline{tv})^p] = p^{r+1} \) imply \( \Omega_{A_v}^{r+2} = 0 \).

\[
\Omega_{A_v}^{r+1} = \left\langle \tilde{a}^p, \frac{d \tilde{a}_1}{\tilde{a}_1} \wedge \cdots \wedge \frac{d \tilde{a}_{r+1}}{\tilde{a}_{r+1}} ; \tilde{a} \in \overline{A}_v, \tilde{a}_j \in \overline{A}_v^*, 1 \leq j \leq r + 1 \right\rangle + d\Omega_{A_v}^r
\]

([7] II). Then we have \( \Delta_{i,j} = \Delta_{i,j+1} \).

Proof of (3). Let \( c \) be the element of \( A^* \) such that \( p = cu^{f(v)(p-1)}/pv^{f(v)(p-1)/p} \). We can take a solution \( x \in A \) of the equation \( X^p + cX - a = 0 \) for any \( a \in A \). Then,

\[
1 + au^{f(u)/p} = (1 + xu^{f(u)/p})^p.
\]

Thus the proof of Lemma 3 is complete.

LEMMA 4. Assume (**). Let

\[
\Delta' = \langle \{ a, b_1, \ldots, b_{r+2} \} ; a \in 1 + m_A \text{ and } b_j \in A[1/uw]^* \text{ for } 1 \leq j \leq r + 2 \rangle
\]

in \( K_{r+3}(K)/p \).

where \( m_A \) is the maximal ideal of \( A \).

Then \( \Delta' = 0 \).

Proof. As is easily seen, \( 1 + m_A \) is generated by elements of the form \( 1 - aw \) (\( a \in A^* \)) for \( w = u \) or \( v \). From this, we see that \( \Delta' \) is generated by elements of the forms \( \{ 1 - aw, b_1, \ldots, b_r, c, d \} \) with \( a, b_1, \ldots, b_r \) and \( c \in A^* \), \( d \in A[1/uw]^* \) and \( w = u \) or \( v \) such that \( b_1, \ldots, b_{r-1} \) and \( b_r \) form a \( p \)-basis of \( k \).
Hence it suffices to prove
\[ \{ 1 - au, b_1, \ldots, b_r, c, d \} \in pK_{r+3}(K) \]
for \( a, b_1, \ldots, b_r, c \) and \( d \) as above. Let
\[ B = A[(au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}] \subset \bar{K} \]
and \( L \) be the quotient field of \( B \). Since \( B/vB = (A/vA)^{1/p} \), there exist elements \( c' \in B^* \) and \( c'' \in B \) such that \( c = (c')^p(1 + c''v) \). We apply Lemma 3 to \( B \),
\[ \{ 1 - (au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}, 1 + c''v, d \} \in pK_{r+3}(L). \]

With \( N_{L/K} \) denoting the norm map, we have
\[ \{ 1 - au, b_1, \ldots, b_r, c, d \} = N_{L/K}(\{ 1 - (au)^{1/p}, b_1^{1/p}, \ldots, b_r^{1/p}, c, d \}) \in N_{L/K}(pK_{r+3}(L)) \subset pK_{r+3}(K). \]

**Lemma 5.** Assume (**). Let
\[ \Delta'' = \langle \{ a_1, a_2, \ldots, a_{r+3} \}; a_j \in A[1/uv]^* \text{ for } 1 \leq j \leq r + 3 \rangle \]
in \( K_{r+3}(K)/p \).

Then \( \Delta'' = 0 \).

**Proof.**
\[ \Delta'' = \langle \{ a_1, \ldots, a_{r+1}, b, c \}; a_j \in A^* \text{ for } 1 \leq j \leq r + 1, b, c \in A[1/uv]^* \rangle \]

Hence it suffices to prove
\[ \{ a_1, \ldots, a_{r+1}, b, c \} \in pK_{r+3}(K) \]
for \( a_1, \ldots, a_{r+1}, b \) and \( c \) as above. Since \( \bar{a}_i = a_i \mod m_A (1 \leq i \leq r + 1) \) cannot be \( p \)-independent, there exists \( s \) such that \( 1 \leq s \leq r \) and \( \bar{a}_{s+1} \in k^p(a_1, \ldots, a_s) \). Let
\[ B = A[a_1^{1/p}, \ldots, a_{s+1}^{1/p}] \subset \bar{K} \]
and \( L \) be the quotient field of \( B \). Since the residue field of \( B \) contains \( \bar{a}_{s+1}^{1/p} \), there exist elements \( a' \in B^* \) and \( a'' \in m_B \) (\( m_B \) is the maximal ideal of \( B \)) such that
$a_{s+1} = (a')r(1 + a')$. By applying Lemma 4 to $B$, we have

$$\{a_1, \ldots, a_{r+1}, b, c\} = N_{L/K}(\{a_1^{1/p}, \ldots, a_s^{1/p}, a_{s+1}, \ldots, a_{r+1}, b, c\}) \in N_{L/K}(pK_{r+3}(L)) \subset pK_{r+3}(K).$$

We follow the method of [12] §5.

**Lemma 6.** Let $\mathfrak{X} \to \text{Spec}(A)$ be a proper birational morphism with regular such that $Y = \mathfrak{X} \otimes_A A/m_A$ is a reduced divisor with normal crossing on $\mathfrak{X}$ ([1], [5]). Then,

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{x \in Y_0} H^{r+3}(K_x, \mathbb{Z}/p\mathbb{Z})$$

where $Y_0$ denotes the set of closed points of $Y$ and for each $x \in Y_0$, $K_x$ denotes the quotient field of the henselization of $\mathcal{O}_{X,x}$.

**Proof.** Let $\lambda : \text{Spec}(K) \to \mathfrak{X}$ be the inclusion map and put $\mathfrak{Y} = R\lambda_*(\mathbb{Z}/p\mathbb{Z})$. By the proper base change theorem, we have

$$H^q(Y, i^*\mathfrak{Y}) = H^q(\mathfrak{X}, \mathfrak{Y}) = H^q(K, \mathbb{Z}/p\mathbb{Z})$$

where $i : Y \to \mathfrak{X}$ is the inclusion map. From this, we obtain an exact sequence

$$\cdots \to \bigoplus_{x \in Y_0} H^q(Y, i^*\mathfrak{Y}) \to H^q(K, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{\eta} H^q(K_{\eta}, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{x \in Y_0} H^{q+1}(Y, i^*\mathfrak{Y}) \to \cdots$$

where $\eta$ ranges over all generic points of $Y$ and $K_{\eta}$ denotes the quotient field of the henselization of $\mathcal{O}_{X,\eta}$. For each $x \in Y_0$, we have an exact sequence

$$\cdots \to H^q(Y, i^*\mathfrak{Y}) \to H^q(K_x, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{\nu} H^q(K_{\nu}, \mathbb{Z}/p\mathbb{Z}) \to H^{q+1}(Y, i^*\mathfrak{Y}) \to \cdots$$

where $\nu$ ranges over all generic points of the henselization of $\text{Spec}(\mathcal{O}_{X,x})$, and $K_{\nu}$ denotes the quotient field of the henselization of the discrete valuation ring of $K_x$ corresponding to $\nu$.

Since $\text{cd}_p(K_{\eta}) = \text{cd}_p(K_{\nu}) = r + 2$ ([8]), we have

$$H^{r+3}(K_{\eta}, \mathbb{Z}/p\mathbb{Z}) = H^{r+3}(K_{\nu}, \mathbb{Z}/p\mathbb{Z}) = 0.$$
On the other hand, from the classical approximation theorem for a finite family of discrete valuation on $K^*$ and $K^*_x$, the maps

$$K_{r+2}(K) \to \bigoplus_{\eta} K_{r+2}(K\eta)/p \quad \text{and} \quad K_{r+2}(Kx) \to \bigoplus_{v} K_{r+2}(Kv)/p$$

are subjective. By [3] §5, the cohomological symbol maps

$$K_{r+2}(K\eta)/p \to H^{r+2}(K\eta, \mathbb{Z}/p\mathbb{Z})$$

and

$$K_{r+2}(Kx)/p \to H^{r+2}(Kx, \mathbb{Z}/p\mathbb{Z})$$

are subjective. Hence the maps

$$H^{r+2}(K, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{\eta} H^{r+2}(K\eta, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^{r+2}(Kx, \mathbb{Z}/p\mathbb{Z}) \to \bigoplus_{v} H^{r+2}(Kv, \mathbb{Z}/p\mathbb{Z})$$

are also subjective.

Putting these things together,

$$\bigoplus_{x\in Y_0} H^{r+3}_x(Y, i^*\mathcal{G}) \xrightarrow{\sim} H^{r+3}(K, \mathbb{Z}/p\mathbb{Z})$$

and

$$H^{r+3}_x(Y, i^*\mathcal{G}) \xrightarrow{\sim} H^{r+3}(Kx, \mathbb{Z}/p\mathbb{Z}).$$

These isomorphisms induce the isomorphism of Lemma 6.

**PROPOSITION 3.** For $r = \text{dom}_k(\Omega^1_k)$,

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

**Proof.** By Lemma 2 and 6, we have

$$\bigoplus_{x\in Y_0} K_{r+3}(Kx)/p \to \bigoplus_{x\in Y_0} H^{r+3}(Kx, \mathbb{Z}/p\mathbb{Z}) \uparrow$$

$$K_{r+3}(K)/p \to H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}).$$
For any family of fixed elements $a_1, a_2, \ldots, a_{r+2} \in K^*$, we can take $X$ such that the union $Z$ of $Y$ with the supports of the divisor of $a_1, \ldots, a_{r+2}$ and $a_{r+3}$ on $X$ is normally crossing divisor ([5]). Then, by Lemma 5,

$$\{a_1, a_2, \ldots, a_{r+3}\} \in pK_{r+3}(K_X)$$

for any $x \in Y_0$. This shows that

$$H^{r+3}(K, \mathbb{Z}/p\mathbb{Z}) = 0.$$ 

We are now in the position to complete the proof of our theorem. By Proposition 1 and 2, $\text{cd}_p(K) = r + 2$ or $r + 3$. For any finite extension field $K'$ over $K$,

$$H^{r+3}(K', \mathbb{Z}/p\mathbb{Z}) = 0$$

by Proposition 3. Hence $\text{cd}_p(K) = r + 2$ ([14]).

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**References**


