Abelian varieties and curves in $W_d(C)$

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1. Introduction

The questions dealt with in this paper were originally raised by Joe Silverman in [10]. A further impetus for studying them is given by the recent results of Faltings [1].

The starting point is the following. Suppose that $C' \to C$ is a nonconstant map of smooth algebraic curves. It is a classical observation in this situation that if $C'$ is hyperelliptic then $C$ must be as well. An immediate generalization of this is the statement that if $C'$ admits a map of degree $d$ or less to $\mathbb{P}^1$, then $C$ does as well. This is elementary: if $f \in K(C')$ is a rational function of degree $d$, then either its norm is a nonconstant rational function of degree $d$ or less on $C$; or else its norm is constant, in which case the norm of some translate $f - z_0$ will not be constant.

In [10], Silverman poses a similar problem: if in the above situation the curve $C'$ is bielliptic -- that is, admits a map of degree 2 to an elliptic curve or $\mathbb{P}^1$ -- does it follow that $C$ is as well? Silverman answers this affirmatively under the additional hypothesis that the genus $g = g(C) \geq 9$.

The most general question along these lines is this. Say a curve $C$ is of type $(d, h)$ if it admits a map of degree $d$ or less to a curve of genus $h$ or less. We may then make the

STATEMENT $S(d, h)$: If $C' \to C$ is a nonconstant map of smooth curves and $C'$ is of type $(d, h)$, then $C$ is of type $(d, h)$

and ask for which $(d, h)$ this holds. We may further refine the question by specifying the genus of the curve $C$: we thus have the

STATEMENT $S(d, h, g)$: Suppose $C' \to C$ is a nonconstant map of smooth curves with $C$ of genus $g$. If $C'$ is of type $(d, h)$, then $C$ is of type $(d, h)$.

As we remarked, this is known to hold in case $h = 0$. In the next case $h = 1$, Silverman in [10] gives some positive results: he shows that $S(2, 1, g)$ holds for $g \geq 9$. We have been able to extend this: we show in Theorem 1 below that $S(d, h)$ holds in general for $h = 1$ and $d = 2$ or 3, and that $S(4, 1, g)$ holds if $g \neq 7$. On the other hand, we see that the statement $S(d, h)$ is not true in general: by way of an example we construct a family of curves of genus 5 that are images of
curves of type (3, 2) but are not of type (3, 2) themselves. It remains an interesting problem to determine for which values of $d$, $h$ and $g$ it does hold.

Our interest in these questions was greatly increased by the recent results of Faltings. Specifically, Faltings shows that if $A$ is an abelian variety defined over a number field $K$ and $X \subset A$ a subvariety not containing any translates of positive-dimensional sub-abelian varieties of $A$, then the set $X(K)$ of $K$-rational points of $X$ is finite. To apply this, suppose that $C$ is a curve that does not possess any linear series of degree $d$ or less (i.e., is not of type $(d, 0)$). Let $C^{(d)}$ be the $d$-th symmetric product of $C$, and $\text{Pic}^d(C)$ the variety of line bundles of degree $d$ on $C$ (this is isomorphic, though noncanonically, to the Jacobian $J(C)$ of $C$). It is then the case that $C^{(d)}$ embeds in $\text{Pic}^d(C)$ as the locus $W_d(C)$ of effective line bundles; applying Falting's result we see that if the subvariety $W_d(C) \subset \text{Pic}^d(C)$ contains no translates of abelian subvarieties of $\text{Pic}^d(C)$, then $C^{(d)}$ has only finitely many points defined over $K$.

We may reexpress this as follows. We consider not only the set $C(K)$ of points of $C$ rational over $K$, but the union $\Gamma_{C,d}(K)$ of all sets $C(L)$ for extensions $L$ of degree $d$ or less over $K$: that is, we set

$$\Gamma_{C,d}(K) = \{ p \in C \mid [K(p) : K] \leq d \}.$$ 

Since any point of $C$ whose field of definition has degree $d$ over $K$ gives rise to a point of $C^{(d)}$ defined over $K$ it follows in turn that under the hypotheses above – that is, if $C$ admits no map of degree $d$ or less to $\mathbb{P}^1$ and if the subvariety $W_d(C) \subset \text{Pic}^d(C)$ contains no translates of subabelian varieties of $\text{Pic}^d(C)$, then $C$ has only finitely many points defined over number fields $L$ of degree $d$ or less over $K$, i.e.,

$$\# \Gamma_{C,d}(K) < \infty.$$ 

Note that conversely if $C$ does admit a map of degree $d$ to $\mathbb{P}^1$ then there will be infinitely many points defined over extension fields of degree $d$ or less over $K$ (the inverse images of $K$-rational points of $\mathbb{P}^1$).

The only problem with this statement is that it seems a priori difficult to determine whether the subvarieties $W_d(C)$ contain abelian subvarieties of $\text{Pic}^d(C)$. Certainly one way in which it can happen that $W_d(C)$ contains an abelian subvariety of dimension $h$ is the following: if for some $n$ with $nh \leq d$ the curve $C$ admits a map of degree $n$ to a curve $B$ of genus $h$, then the Picard variety $\text{Pic}^n(B) \cong J(B)$ maps to the Picard variety $\text{Pic}^{nh}(C)$. Since any divisor class of degree $h$ on $B$ is effective, the image will be contained in the locus $W_{nh}(C)$ of effective divisor classes of degree $nh$ on $C$. (On the other hand, it will also be the case if $C$ is just the image of a curve $C'$ that admits a map of degree $n$ to a curve of genus $h$. It is for this reason that the statement $S(n, h)$ above is relevant.) This raises naturally the question of the correctness of the
STATEMENT $A(d, h, g)$: Suppose $C$ is a curve of genus $g$, and for some $d < g$ the locus $W_d(C)$ contains a sub-abelian variety of dimension $h$, then $C$ is the image of a curve $C'$ that admits a map of degree at most $d/h$ to a curve of genus $h$.

Here as in the previous question the answer is yes in some cases: it is apparent when $h = 1$, and we prove in Theorem 1 below that it holds for $h = 2$ and $d \leq 4$ if $g \geq 6$. At the same time, the answer in general is no – we have a counterexample to this below. It remains a relevant question for which values of $d, g$ and $h$ it may be true. (Note in particular that a positive answer to this question in general would imply that $W_d(C)$ could never contain an abelian subvariety of dimension strictly greater than $d/2$; we know of no counterexample to this assertion.)

The point of introducing the statements $S(d, h, g)$ and $A(d, h, g)$ is that, if true, they combine with Falting’s theorem to give a simple and powerful statement about the sets $\Gamma_{c,d}(K)$: if $W_d(C)$ does contain a subtorus, and if the relevant cases of Statements $A(d, g, h)$ and $S(d, g, h)$ hold, then it follows that $C$ is of type $(n, h)$ for some $n$ and $h$ with $nh \leq d$. It would then follow that for any curve $C$ defined over a number field $K$, $\# \Gamma_{c,d}(L)$ will be infinite for some extension $L$ of $K$ if and only if $C$ is of type $(n, h)$ for some $n$ and $h$ with $nh \leq d$. (Note that one direction is clear: if $\pi: C \to B$ is a map of degree $n$ to a curve of genus $h$ then for some extension $L$ of $K$ the rank of $\text{Pic}^h(B)$ over $L$ will be positive and $C$ will similarly have infinitely many points $p$ with $[K(p): L] \leq d$.)

Of course, as we have indicated, neither of the statements $S(d, h, g)$ or $A(d, h, g)$ hold in general. Upon closer examination, however, we see that in order to establish the simplest possible statement along these lines we do not need to worry about $S(n, h)$ for all $h$. The reason is the fact that any curve of genus $g$ admits a map of degree $[g/2] + 1$ or less to $\mathbb{P}^1$. Thus, if the Picard variety $\text{Pic}^g(C)$ of a curve $C$ contains a translate of an abelian variety coming from a map of degree $n$ to a curve $B$ of genus $h$ with $nh \leq d$, and $h \geq 2$, then $B$ will be of type $(h, 0)$, and hence $C$ will be of type $(d, 0)$. The crucial case of the general question above about images of coverings of curves of low genus is the case $h = 1$, which is still very much open. It is similarly the case that we need only look at abelian subvarieties of $W_d(C)$ for $d < [g/2] + 1$, in which range there is no counterexample to the statement $A(d, g, h)$ above that any such sub-abelian variety comes from a correspondence with a curve of genus $h$. We may thus make the

**CONJECTURE.** If $C$ is a curve defined over the number field $K$, then

\[ \# \Gamma_{c,d}(L) = \infty \text{ for some finite extension } L/K \]

for some finite extension $L/K$ if and only if $C$ admits a map of degree $d$ or less to $\mathbb{P}^1$ or an elliptic curve.

Combining the results mentioned above, we have the main result of this paper: the
THEOREM 1. The conjecture above holds when \( d = 2 \) or 3, and when \( d = 4 \) provided the genus of \( C \) is not 7.

REMARK. (1) Results related to the above have been obtained by many people, including Gross–Rohrlich [5], Hindry [7], Mazur [8] and others. The conjecture is also related to the generalized Mordell conjectures of Lang and Vojta (for example, in the case \( d = 2 \) the Lang–Vojta conjectures say that a nonhyperelliptic curve possessing infinitely many points of degree 2 must admit a correspondence of bidegree \((2, m)\) with an elliptic curve, though they do not specify \( m \)).

The case \( d = 2 \) was proved before by Harris and Silverman [6]. We give a slightly strengthened version here (see Theorem 3). Using methods similar to theirs, one can show the following amusing result: if our \( C' \) maps with degree 2 to a hyperelliptic curve of genus \( h \), then \( C \) maps with degree 2 to a hyperelliptic (or rational) curve of genus at most \( h \), with one exception which we cannot prove: \( h = 2 \) and \( g = 3 \).

(2) Vojta tries to attack this problem from another point of view, in [11, 12]. He assumes the existence of a map \( f: C \to \mathbf{P}^1 \) of low degree, and deduces that all but finitely many points of low degree over \( K \) relate to this map: \( K(p) \neq K(f(p)) \). In general, the existence of such a map \( f \) rules out the possibility of another map of low degree to a curve of low genus, assuming the genus of \( C \) is large. In view of this, it turns out that in case of points of degree 2 and 3 his results give the same bounds as ours. In particular, on a trigonal curve over \( K \) of genus at least 8, all but finitely many point of degree 3 over \( K \) on \( C \) map to rational points on \( \mathbf{P}^1 \) (this is sharp simply because there are curves of genus 7 which are trigonal and trielliptic). It would be interesting to have results similar to Vojta’s for maps to an elliptic curve instead of \( \mathbf{P}^1 \).

2. Preliminary lemmas

Let \( A \) be a complex abelian variety of dimension \( a \geq 1 \), and let \( A \subset W_d(C) \) be an embedding. Here \( C \) is a smooth complex algebraic curve, and \( W_d(C) \) is the variety of effective line bundles of degree \( d \) over \( C \).

We assume this embedding is minimal, that is, the line bundles given by \( A \) do not have a common divisor: \( A \not\subset p + W_{d-1}(C) \forall p \in C \). We also assume that \( A \not\subset \Delta \), where \( \Delta \) is the image of the big diagonal of \( C^{(d)} \) in \( W_d(C) \).

Note that \( A \) is a coset of a subgroup in \( Pic(C) \). If we write

\[
A_k = \{ \alpha_1 + \cdots + \alpha_k \mid \alpha_i \in A \}
\]

then \( A_k \) is a coset of the same subgroup, and thus \( A_k \simeq A \).
For $x \in A_2$ we write $L_a$ for the associated line bundle and $D_a$ for any effective divisor such that $\mathcal{O}(D_a) = L_a$. For any $x \in \text{Pic}(C)$ we write $r(x) = h^0(L_a) - 1$.

The ideas of the proof of the main theorem are as follows:

1. We produce families of maps to projective spaces by taking sections of $L_a$ for $x \in A_2$ (Lemma 1).
2. In case the general such map is not birational onto the image, we reduce our problem to lower genus and an appropriately lower $d$. In the cases of our theorems, we actually get the required maps (Lemmas 2 and 3).
3. When these maps are birational, we use an estimate similar to Castelnuovo’s bound, only stronger, to show that $g(C) \leq O(d^2/a)$.

**LEMMA 1.** For any $x \in A_2$ we have $r(x) \geq a$.

**Proof.** Let $\pi_d: C^{(d)} \rightarrow W_d(C)$ be the natural map, and let $\tilde{A} \subset C^{(d)}$ be the proper transform of $A$ under this map. Recall that the symmetrization map $C^{(d)} \times C^{(d)} \rightarrow C^{(2d)}$ is finite. Therefore $\tilde{A} \times \tilde{A} \rightarrow \tilde{A}_2$ is finite, where $\tilde{A}_2$ is the proper transform of $A_2$. So $\dim \tilde{A}_2 \geq 2a$, and the fibers of $\pi_2|_{\tilde{A}_2}: \tilde{A}_2 \rightarrow A_2$ have dimension at least $a$. Abel’s theorem says that $r(a) > a$ for all $x \in A_2$. □

Note that the linear systems $|D_a|$ obtained above are base point free. Special care is needed in case $a = 1$:

**LEMMA 2.** Assume $a = 1$.

1. If the general point $p \in C$ belongs to exactly one $D_a$ with $x \in A$, such that $r(x) = 0$, then there is a map of degree $d$ from $C$ to the elliptic curve $A$.

2. Assume that for the general $x \in A_2$ we have $r(x) = 1$. Let $\phi_a: C \rightarrow \mathbb{P}^1$ be the map defined by the global sections of $L_a$. Then $\phi_a$ factors through a $d$-to-$1$ map to $A$.

**Proof.** (1) is formal, and may be shown as follows: let $F: C \times C^{(d-1)} \rightarrow W_d(C)$ be the natural map, and let $C'$ be the normalization of the part of $F^{-1}(A)$ dominating $A$. Our minimality conditions mean that $C'$ is exactly a $d$-sheeted cover of $A$. On the other hand, the projection onto the first factor $\pi_1: C \times C^{(d-1)} \rightarrow C$ induces a map from $C'$ to $C$, the degree of which is the number of times a general point of $C$ belongs to a divisor $D_a$. If this degree is 1, then $C \cong C'$ and therefore $C$ admits a map to $A$ of degree $d$.

For (2), note that for any $\beta \in A$ we have $x - \beta \in A$. Therefore $x - \beta$ is effective, and thus $D_\beta$ imposes one condition on the linear system $|D_a|$, so $D_\beta$ lies in a fiber of $\phi_a$.

If the general fiber of $\phi_a$ is written uniquely as a sum $D_\beta + D_\gamma$ where $\beta + \gamma = x$, we are in case (1). Otherwise, for every $x$ with $r(x) = 1$ we have $\infty^1$ equations:

$$D_\beta + D_\gamma = D_\gamma + D_\gamma' \quad (\beta + \gamma' = \gamma + \gamma' = x).$$  (1)
Fixing $\beta$ and changing $\alpha$ (and thus $\beta'$) we see that since $D_{\beta} \cap D_{\beta'} \neq \emptyset$ the divisors $D_{\beta}$ have a common divisor. Similarly for $D_{\beta'}$. At least one of the two moves, and so the divisors of $A$ would have a common divisor, contradicting our assumption.

The following lemma, together with the previous one, will establish the cases when the general $\phi_\alpha$ is not birational.

**LEMMA 3.** Assume $a \geq 1$ and $r(\alpha) > 1$ for all $\alpha \in A_2$. If $\phi_\alpha : C \to P^{r(\alpha)}$ is not birational for general $\alpha$, then either $A \subset W_{d'}(C)$ with $d' \leq d$, or $\phi_\alpha$ factors as:

$$C \xrightarrow{\rho} C' \xrightarrow{\tilde{\phi}_\alpha} P^{r(\alpha)}$$

and there is an imbedding $A \subset W_{d'}(C')$ where $d' = d/\deg \rho = \deg \tilde{\phi}_\alpha$.

**Proof.** Recall that the set of maps from $C$ to curves of positive genus (up to automorphisms) is discrete. If the general $\phi_\alpha$ map to rational curves of degree $m$, then their images must be rational normal curves (the linear series in question are complete) and we get an imbedding $A \subset W_{d'}^1(C)$, where $d' = d/m$. Otherwise, there is a generic image curve for the $\phi_\alpha$, call it $C'$. Let $p \in C'$ and let $q_1, q_2 \in \rho^{-1}(p)$. Suppose $q_1 \in D_\beta$ for some $\beta \in A$. We claim that $q_2 \in D_\beta'$, which gives the lemma. If we let $\alpha$ vary in $A_2$ and set $\beta' = \alpha - \beta$, then $D_\beta + D_{\beta'}$ is a hyperplane section of $C \subset P^r$ containing $q_1$. Therefore, since $\phi_\alpha$ factors through $C'$, also $q_2 \in D_\beta + D_{\beta'}$. But the divisors $D_{\beta'}$ do not have a common divisor, therefore $q_2 \in D_\beta$. This means that $A$ is a pull-back of an abelian variety from $W_{d'}(C')$. Again, since the linear series are complete, this pull-back is an isomorphism.

We use the following classical lemma:

**LEMMA 4.** Let $C \to P^r$ be birational onto its image. Then for every $s \leq r$ there do not exist $\infty$ divisors of degree $s + 1$, each spanning an $s-1$-plane.

**Proof.** By a projection from a generic secant we reduce to the fact that a plane curve has finitely many singularities.

Let $r_k = \min\{r(\alpha) | \alpha \in A_k\}$, that is, the general dimension of the complete linear series $|D_\alpha|$, $\alpha \in A_k$.

**LEMMA 5.** Suppose $r_2 = a$. Then $\phi_\alpha$ is not birational.

**Proof.** In case $a = 1$ this is trivial. Otherwise, the fibers of $\pi_2$ as in Lemma 1 are in general projective spaces of dimension $a$, which are surjected by the quotient of $A$ by an involution. In dimension $a > 1$ these are never rational. Therefore each divisor of $D_\alpha$ is represented in at least two ways as the sum of two divisors from $A$, and we get $\infty^a$ equations as in (1). Fixing $D_\beta$ again and letting $D_{\beta'}$ vary, and vice versa, we see that $A$ is generated by two subvarieties
$X_1 \subset W_d(C)$ and $X_2 \subset W_d(C)$, where $\dim(X_1) = a_1 > 0$ and $\dim(X_2) = a_2 > 0$ and $a_1 + a_2 \geq a$. In the target space of $\phi_\alpha$ we see that we get $\infty^{a_1}$ divisors, given by $X_1$, each spanning only an $a_1 - 1$-plane. By Lemma 4 the map is not birational onto the image.

The following lemma uses the same kind of information for the next possible dimension:

**Lemma 6.** Suppose $r_2 = a + 1$, and suppose the general $\phi_\alpha$ is birational. Then for general points $p_1, \ldots, p_a$ in $C$, and any $D_\beta$ with $\beta \in A$ such that $p_1 < D_\beta$ there is another divisor $D_{\beta'}$ with $\beta' \in A$ so that $\gcd(D_\beta, D_{\beta'}) = p_1 + \cdots + p_a$.

**Proof.** Now the fibre of $\pi_2$ is in general a projective space of dimension $a + 1$, and the quotient of $A$ by an involution maps to it by a finite map. If the image is a linear space, we have a linear series to which we may apply the previous lemma. Otherwise, the image is of higher degree, in which case the line defined by general $a$ points of $C$ intersects this image several times. This means that the divisor $p_1 + \cdots + p_a$ lies on several of the hyperplanes defined by $A$, and therefore is in general contained in several divisors of $A$. If they all contain an extra point, we get $\infty^a$ intersections of hyperplanes containing $a + 1$ points, contradicting Lemma 4.

3. Number of conditions

We are left with the cases when $r(\alpha) > 1$ for all $\alpha \in A_2$ and $\phi_\alpha$ birational for the general $\alpha$. We continue and derive a strengthened Castelnuovo type bound on the genus of $C$. The argument is similar to the original argument of Castelnuovo’s bound (see [2], Chapt. 3) and the generalized one by Accola [3]. The idea is to estimate the number of conditions a divisor $D_\beta$ for $\beta \in A$ imposes on the sections of a general $k$-fold sum $\alpha \in A_k$. The fact that we are working with cosets of subgroups plays an important role.

First, some observations. Since $\{D_\beta | \beta \in A\}$ have no common divisor, for all $p \in C$ the general $D_\beta$ does not contain $p$. As an immediate result we get:

**Lemma 7.** $r_{k+1} - r_k \geq r_{k-1}$.

**Proof.** Let $\alpha \in A_{k+1}$ be a general point, and let $D$ be a general divisor coming from $A$. Let $D_\gamma, \gamma \in A$ be a general divisor, such that $\gcd(D, D_\gamma) = 0$. By the generality assumption, there are $r_k - r_{k-1}$ points in $D$ which impose independent conditions on sections of $L_{\alpha - \gamma}$. Multiplying by the canonical section of $\ell(D_\gamma)$, which does not vanish on any point of $D$, certainly keeps this property. Hence the lemma.

**Lemma 8.**

1. If for general $\alpha \in A_2$ the map $\phi_\alpha$ is birational onto its image then $r_3 \geq 2r_2$ and
\[ r_{k+2} - r_k \geq \min(r_k - r_{k-2} + r_2, 2d) \text{ for any } k \geq 2. \]

2. If \( r_2 = a + 1 \) then \( r_{k+1} - r_k \geq \min(ka + 1, d) \).

**Proof.** The fact that \( r_3 \geq 2r_2 \) follows immediately from Lemma 7.

Let \( D_a = p_1 + \cdots + p_{2d} \) be a general divisor. Now, by the uniform position lemma (see [2]) if we take a general \( \alpha \in A_2 \) then there is a divisor \( D_\alpha \) such that the common divisor with \( D_a \) is \( p_1 + \cdots + p_{r_2} \).

Also, for a general \( \gamma \in A_k \) we have a divisor \( D_\gamma \) so that \( \gcd(D_\gamma, D_\alpha) = p_{r_2+1} + \cdots + p_{r_2+k-r_2-2} \). But the order of the chosen points is unimportant. Therefore, for the general \( \delta = \gamma + \alpha \in A_{k+2} \) we have that \( \alpha \) imposes at least \( r_2 + r_k - r_{k-2} \) conditions on \( |D_\delta| \).

For the second claim, notice that by Lemma 6, if \( D_\beta = q_1 + \cdots + q_d \) is a general divisor corresponding to points of \( A \), then there are divisors \( D_\beta_i \) so that \( \gcd(D_\beta, D_\beta_i) = q(i-1)a_i + \cdots + qia_i \). Summing up \( k \) of these, and using Lemma 7 for an extra divisor, we get the inequality in (2).

**REMARK.** Notice that we didn’t make use, in the proof of first part of the lemma, of the fact that we have \( \infty^a \) ways to choose \( \alpha \). For our results this turns out to be sufficient.

**COROLLARY 1.** In the case of the lemma, we have \( r_3 \geq \min(2a+3, a+1+d) \), and \( r_4 \geq \min(3a+6, r_3+d) \).

As a byproduct we get a theorem:

**THEOREM 2.** Let \( A \subset W_1(C) \) be an abelian variety of positive dimension \( a \). Assume that for the general \( \alpha \in A_2 \) the map \( \phi_\alpha \) is birational onto its image. Then

\[ g(C) \leq \binom{d}{2} + 1. \]

**Proof.** We know that \( r_2 \geq 2 \). If equality holds, we have \( r_d \geq 2 + 3 + \cdots + d = \binom{d+1}{2} - 1 \). But for \( \alpha \in A_d \), \( \deg \alpha = d^2 \), so \( 2r(\alpha) > \deg \alpha \) for all \( \alpha \), and by Clifford’s theorem (see [2]) \( \alpha \) is non-special, that is, \( g(C) = \deg \alpha - r(\alpha) \leq \binom{d}{2} + 1 \). Similarly, if \( r_2 \geq 3 \) one proves by induction, using Lemmas 8 and 7, that \( r_k \geq \binom{k+1}{2} - 1 \), and continues as before.

**4. Statement and proof of main theorems**

**THEOREM 3.** If \( A \subset W_2(C) \), then either \( C \) has genus at most 2, or \( C \) is bielliptic. If \( g(C) > 3 \) then \( C \) is not hyperelliptic.

**Proof.** Lemma 2 settles the theorem when \( r(\alpha) = 1 \) for general \( \alpha \in A_2 \). If
r(\alpha) > 1$, we have a family of $g_a^2$, which does not exist unless $g(C) \leq 2$, because for genus 3 a $g_a^2$ is the canonical series, and for higher genera it has to be twice a unique hyperelliptic series. For the last statement, a bi-elliptic hyperelliptic curve is of type $(2,4)$ on a smooth quadric, and therefore of genus at most 3.

\begin{proof}
Lemma 2 and 3 settle the theorem for $\phi_2$ not birational for general $\alpha \in A_2$. Corollary 1 shows that any other case has a $g_3^2$, and by Clifford's theorem ([2]) has genus at most 4, but these have a $g_3^1$ [2]. Similarly, if we take $a = 2$ we see that $g \leq 3$.
\end{proof}

\begin{proof}
Again we may assume $\phi_2$ is birational for general $\alpha \in A_2$. Corollary 1 and Clifford's theorem show that $g(C) \leq 7$.

Curves of genus at most 6 have a $g_4^1$. For the last statement, we see that if $a > 1$ then in fact $g \leq 5$. In the next section we show that there is a counterexample with $g = 5$.
\end{proof}

5. An example

We construct a 6 dimensional family of curves of genus 5, all having a curve of genus 2 in $W_3$, and none of them admits a map of degree 2 or 3 to curves of genus 0, 1 or 2. As a byproduct, we explain how a curve of genus 5 can possess an abelian surface in $W_4$ without being a double cover of a curve of genus 2. The construction is a special case of the tetragonal construction for Prym varieties, as in [4].

Let $f: C_2 \to \mathbb{P}^1$ be a map of degree 4, from a curve $C_2$ of genus 2 to $\mathbb{P}^1$. Assume $f$ has only simple ramifications.

Let $C' = C_2 \times_{\mathbb{P}^1, C_2 - \Delta} C_2 - \Delta$ be the curve of pairs of distinct points in the fibres over $\mathbb{P}^1$. Let $C_5$ be the quotient of $C'$ by the symmetrization involution: $C_5 = (C_2)^{[2]} - \Delta$. $C_5$ is our curve of genus 5. Note that $C_5$ admits an involution that assigns to an unordered pair of distinct elements in a fiber, the residual pair in that fiber. The quotient is a curve of genus 3, $C_3$. We have the following commutative diagram:

\begin{center}
\begin{tikzcd}
C_2 \arrow{r}{f} \arrow{d}{\phi} & \mathbb{P}^1 \\
C_5 \arrow{r}{\pi} & C_3
\end{tikzcd}
\end{center}
where $\pi$ is unramified of degree 2, $\rho$ of degree 3 with 10 ramifications, and $f$ of degree 4 with 10 ramifications.

The corresponding ramification behavior of $C_2$, $C_5$ and $C_3$ over $\mathbb{P}^1$ is sketched below:

\begin{itemize}
  \item ramification in $C_2$
  \item ramification in $C_5$
  \item ramification in $C_3$
\end{itemize}

From this construction we see that the curves $C_5$ vary in at most 6 parameters: in fact the curves $C_3$ have only so many moduli. We show that the construction may be reversed, and that we really get 6 parameters.

Let $C_3$ be a nonhyperelliptic curve of genus 3. Let $\rho$ be any $g_1^3$ on the curve, with simple ramifications. Let $\pi: C_5 \to C_3$ be any connected unramified double cover. One checks that the monodromy of $C_5$ over $\mathbb{P}^1$ via the map $\rho \circ \pi$ is $S_4$.

Now take the subvariety $D$ of the triple relative symmetric power $(C_3)^{(3)}$ that does not map into a diagonal of $(C_3)^{(3)}$. This subvariety is composed of two isomorphic components of genus 2, called $C_2$.

**THEOREM 6.** The general $C_5$ in this family does not admit a map of degree at most 3 to a curve of genus at most 2.

Let $\Lambda$ be the variety inside $\mathcal{M}_5$ described by our curves of genus 5, and let $D_{d,h}$ be the subset of $\Lambda$ of those curves that admit a map of degree $d$ to a curve of genus $h$. We need to show: $\Lambda \neq \bigcup_{d \leq 3, h \leq 2} D_{d,h}$.

**LEMMA 9.** $\dim D_{3,2} \leq 5$.

**Proof.** The dimension of the variety of curves of genus 5 admitting a map of degree 3 to a curve of genus 2 is 5.

**LEMMA 10.** $\dim D_{2,h} \leq 5$.

**Proof.** If an involution of $C_5$ commutes with $\pi$ then $C_3$ has automorphisms, and the dimension of such $C_3$ is 5. If they do not commute, the composition of the two involutions is of some order bigger than 2, and the dimension of the variety of curves of genus 5 admitting such an automorphism is again not more than 5 [2].

**LEMMA 11.** $\dim D_{3,0} \leq 5$.

**REMARK.** In fact, one can show that $D_{3,0}$ is empty.

**Proof.** We prove by specialization. Let $C_3$ be a nonhyperelliptic, bielliptic curve of genus 3, and $p: C_3 \to E$ the bielliptic map. Let $E' \to E$ be a two sheeted map of elliptic curves. Then $E' \times_E C_3$ is a bielliptic curve of genus 5 in our family.
Now, a bielliptic curve of genus 5 does not admit a $g_3^3$. In fact, if $C_5$ has a map of degree 2 or 3 to $P^1$, then as a cycle in $E' \times P^1$ we have

$$[C_5]^2 = \text{degree of ramification} = 12 \text{ or } 14.$$  

If $H = \pi_1^{-1}(p) + \pi_2^{-1}(q)$ is an ample divisor formed by fibers both ways, we have $H^2 = 2$ and $H \cdot [C_5] = 4 \text{ or } 5$. We get

$$(H \cdot [C_5])^2 \leq (H \cdot H)([C_5] \cdot [C_5])$$

which is a contradiction to the Hodge index theorem. 

LEMMA 12. $h \ast g \ast \text{Jac}(C_2) \cap \pi^* \text{Jac}(C_3)$ is finite, and their sum is the whole $\text{Jac}(C_5)$, for general $C_5$. In other words, the two jacobians give subabelian varieties which are complementary up to isogeny.

Proof. If $q \in C_2$ one checks explicitly that $\pi \ast h \ast g \ast (q) = \rho \ast f \ast (q)$ (in fact, the big square in the commutative diagram is the normalization of a fiber square). This does not depend on $q$ because $f_\ast (q) \sim f_\ast (q_1)$ on $P^1$.

If $C_5$ is general from $A$, then $\dim h \ast g \ast \text{Jac}(C_2) > 0$, otherwise $C_5$ has a $g_3^3$ (see Lemma 2).

If $C_2$ is of general moduli, it does not map to an elliptic curve, in which case $\dim h \ast g \ast \text{Jac}(C_2) = 1$. By semicontinuity, the dimension is 2 for general $C_2$. 

LEMMA 13. $\dim D_{d,1} \leq 5$.

Proof. In fact, if $C_5$ admits a map to an elliptic curve, then this elliptic curve maps to $\text{Jac}(C_5)$ by a nonconstant map. Projecting to $\text{Jac}(C_3)$ and to $\text{Jac}(C_2)$ we see that at least one of these jacobians is nonsimple. This again bounds the dimension of either $C_2$ or $C_3$.

This finishes the verification of our theorem.

COROLLARY 2. There are curves $C_5$ of genus 5 such that $W_4(C_5)$ contains an abelian surface, but the curve $C_5$ does not map to any curve of genus 2.

Proof. The Prym variety of the map $C_5 \rightarrow C_3$ has a translate which lies in $W_4$, namely the odd component of the inverse image of $K_{C_3}$ (see [9]).

References


