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DAN ABRAMOVICH
JOE HARRIS

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Abelian varieties and curves in $W_d(C)$

DAN ABRAMOVICH and JOE HARRIS
Department of Mathematics, Harvard University, Cambridge, MA02138, USA

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1. Introduction

The questions dealt with in this paper were originally raised by Joe Silverman in [10]. A further impetus for studying them is given by the recent results of Faltings [1].

The starting point is the following. Suppose that $C' \to C$ is a nonconstant map of smooth algebraic curves. It is a classical observation in this situation that if C' is hyperelliptic then C must be as well. An immediate generalization of this is the statement that if C' admits a map of degree d or less to $\mathbf{P}^1$, then C does as well. This is elementary: if $f \in K(C')$ is a rational function of degree d, then either its norm is a nonconstant rational function of degree d or less on C; or else its norm is constant, in which case the norm of some translate $f - z_0$ will not be constant.

In [10], Silverman poses a similar problem: if in the above situation the curve C' is bielliptic – that is, admits a map of degree 2 to an elliptic curve or $\mathbf{P}^1$ – does it follow that C is as well? Silverman answers this affirmatively under the additional hypothesis that the genus $g = g(C) \geqslant 9$.

The most general question along these lines is this. Say a curve C is of type (d, h) if it admits a map of degree d or less to a curve of genus h or less. We may then make the

STATEMENT $S(d, h)$: If $C' \to C$ is a nonconstant map of smooth curves and C' is of type (d, h), then C is of type (d, h)

and ask for which (d, h) this holds. We may further refine the question by specifying the genus of the curve C: we thus have the

STATEMENT $S(d, h, g)$: Suppose $C' \to C$ is a nonconstant map of smooth curves with C of genus g. If C' is of type (d, h), then C is of type (d, h).

As we remarked, this is known to hold in case $h = 0$. In the next case $h = 1$, Silverman in [10] gives some positive results: he shows that $S(2, 1, g)$ holds for $g \geqslant 9$. We have been able to extend this: we show in Theorem 1 below that $S(d, h)$ holds in general for $h = 1$ and $d = 2$ or 3, and that $S(4, 1, g)$ holds if $g \neq 7$. On the other hand, we see that the statement $S(d, h)$ is not true in general: by way of an example we construct a family of curves of genus 5 that are images of curves of type (3, 2) but are not of type (3, 2) themselves. It remains an interesting problem to determine for which values of d, h and g it does hold.

Our interest in these questions was greatly increased by the recent results of Faltings. Specifically, Faltings shows that if A is an abelian variety defined over a number field K and $X \subset A$ a subvariety not containing any translates of positive-dimensional sub-abelian varieties of A, then the set $X(K)$ of K-rational points of X is finite. To apply this, suppose that C is a curve that does not possess any linear series of degree d or less (i.e., is not of type $(d, 0)$). Let $C^{(d)}$ be the d-th symmetric product of C, and $Pic^d(C)$ the variety of line bundles of degree d on C (this is isomorphic, though noncanonically, to the Jacobian $J(C)$ of C). It is then the case that $C^{(d)}$ embeds in $Pic^d(C)$ as the locus $W_d(C)$ of effective line bundles; applying Falting's result we see that if the subvariety $W_d(C) \subset Pic^d(C)$ contains no translates of abelian subvarieties of $Pic^d(C)$, then $C^{(d)}$ has only finitely many points defined over K.

We may reexpress this as follows. We consider not only the set $C(K)$ of points of C rational over K, but the union $\Gamma_{C,d}(K)$ of all sets $C(L)$ for extensions L of degree d or less over K: that is, we set

$$\Gamma_{C,d}(K) = \{p \in C \mid [K(p)\colon K] \leqslant d\}.$$

Since any point of C whose field of definition has degree d over K gives rise to a point of $C^{(d)}$ defined over K it follows in turn that under the hypotheses above – that is, if C admits no map of degree d or less to $\mathbf{P}^1$ and if the subvariety $W_d(C) \subset Pic^d(C)$ contains no translates of subabelian varieties of $Pic^d(C)$, then C has only finitely many points defined over number fields L of degree d or less over K, i.e.,

$$\#\Gamma_{C,d}(K) < \infty.$$

Note that conversely if C does admit a map of degree d to $\mathbf{P}^1$ then there will be infinitely many points defined over extension fields of degree d or less over K (the inverse images of K-rational points of $\mathbf{P}^1$).

The only problem with this statement is that it seems a priori difficult to determine whether the subvarieties $W_d(C)$ contain abelian subvarieties of $Pic^d(C)$. Certainly one way in which it can happen that $W_d(C)$ contains an abelian subvariety of dimension h is the following: if for some n with $nh \leqslant d$ the curve C admits a map of degree n to a curve B of genus h, then the Picard variety $Pic^h(B) \cong J(B)$ maps to the Picard variety $Pic^{nh}(C)$. Since any divisor class of degree h on B is effective, the image will be contained in the locus $W_{nh}(C)$ of effective divisor classes of degree nh on C. (On the other hand, it will also be the case if C is just the image of a curve C' that admits a map of degree n to a curve of genus h. It is for this reason that the statement $S(n, h)$ above is relevant.) This raises naturally the question of the correctness of the

STATEMENT $A(d, h, g)$: Suppose C is a curve of genus g, and for some $d < g$ the locus $W_d(C)$ contains a sub-abelian variety of dimension h, then C is the image of a curve C' that admits a map of degree at most d/h to a curve of genus h.

Here as in the previous question the answer is yes in some cases: it is apparent when $h = 1$, and we prove in Theorem 1 below that it holds for $h = 2$ and $d \leqslant 4$ if $g \geqslant 6$. At the same time, the answer in general is no – we have a counterexample to this below. It remains a relevant question for which values of d, g and h it may be true. (Note in particular that a positive answer to this question in general would imply that $W_d(C)$ could never contain an abelian subvariety of dimension strictly greater than $d/2$; we know of no counterexample to this assertion.)

The point of introducing the statements $S(d, h, g)$ and $A(d, h, g)$ is that, if true, they combine with Falting's theorem to give a simple and powerful statement about the sets $\Gamma_{C,d}(K)$: if $W_d(C)$ does contain a subtorus, and if the relevant cases of Statements $A(d, g, h)$ and $S(d, g, h)$ hold, then it follows that C is of type (n, h) for some n and h with $nh \leqslant d$. It would then follow that for any curve C defined over a number field K, $\#\Gamma_{C,d}(L)$ will be infinite for some extension L of K if and only if C is of type (n, h) for some n and h with $nh \leqslant d$. (Note that one direction is clear: if $\pi: C \to B$ is a map of degree n to a curve of genus h then for some extension L of K the rank of $Pic^h(B)$ over L will be positive and C will similarly have infinitely many points p with $[K(p): L] \leqslant d$.)

Of course, as we have indicated, neither of the statements $S(d, h, g)$ or $A(d, h, g)$ hold in general. Upon closer examination, however, we see that in order to establish the simplest possible statement along these lines we do not need to worry about $S(n, h)$ for all h. The reason is the fact that any curve of genus g admits a map of degree $[g/2]+1$ or less to $\mathbf{P}^1$. Thus, if the Picard variety $Pic^d(C)$ of a curve C contains a translate of an abelian variety coming from a map of degree n to a curve B of genus h with $nh \leqslant d$, and $h \geqslant 2$, then B will be of type $(h, 0)$, and hence C will be of type $(d, 0)$. The crucial case of the general question above about images of coverings of curves of low genus is the case $h = 1$, which is still very much open. It is similarly the case that we need only look at abelian subvarieties of $W_d(C)$ for $d < [g/2] + 1$, in which range there is no counterexample to the statement $A(d, g, h)$ above that any such sub-abelian variety comes from a correspondence with a curve of genus h. We may thus make the

CONJECTURE. If C is a curve defined over the number field K, then

$\#\Gamma_{C,d}(L) = \infty$ for some finite extension L/K	$\Leftrightarrow$	C admits a map of degree d or less to $\mathbf{P}^1$ or an elliptic curve.

Combining the results mentioned above, we have the main result of this paper: the

THEOREM 1. *The conjecture above holds when $d = 2$ or 3, and when $d = 4$ provided the genus of C is not 7.*

REMARK. (1) Results related to the above have been obtained by many people, including Gross–Rohrlich [5], Hindry [7], Mazur [8] and others. The conjecture is also related to the generalized Mordell conjectures of Lang and Vojta (for example, in the case $d = 2$ the Lang–Vojta conjectures say that a nonhyperelliptic curve possessing infinitely many points of degree 2 must admit a correspondence of bidegree $(2, m)$ with an elliptic curve, though they do not specify m).

The case $d = 2$ was proved before by Harris and Silverman [6]. We give a slightly strengthened version here (see Theorem 3). Using methods similar to theirs, one can show the following amusing result: if our C' maps with degree 2 to a *hyperelliptic* curve of genus h, then C maps with degree 2 to a hyperelliptic (or rational) curve of genus at most h, with one exception which we cannot prove: $h = 2$ and $g = 3$.

(2) Vojta tries to attack this problem from another point of view, in [11, 12]. He assumes the existence of a map $f: C \to \mathbf{P}^1$ of low degree, and deduces that all but finitely many points of low degree over K relate to this map: $K(p) \neq K(f(p))$. In general, the existence of such a map f rules out the possibility of another map of low degree to a curve of low genus, assuming the genus of C is large. In view of this, it turns out that in case of points of degree 2 and 3 his results give the same bounds as ours. In particular, on a trigonal curve over K of genus at least 8, all but finitely many point of degree 3 over K on C map to rational points on $\mathbf{P}^1$ (this is sharp simply because there are curves of genus 7 which are trigonal and trielliptic). It would be interesting to have results similar to Vojta's for maps to an elliptic curve instead of $\mathbf{P}^1$.

2. Preliminary lemmas

Let A be a complex abelian variety of dimension $a \geqslant 1$, and let $A \hookrightarrow W_d(C)$ be an embedding. Here C is a smooth complex algebraic curve, and $W_d(C)$ is the variety of effective line bundles of degree d over C.

We assume this embedding is minimal, that is, the line bundles given by A do not have a common divisor: $A \not\subset p + W_{d-1}(C)\ \forall p \in C$. We also assume that $A \not\subset \Delta$, where Δ is the image of the big diagonal of $C^{(d)}$ in $W_d(C)$.

Note that A is a coset of a subgroup in $Pic(C)$. If we write

$$A_k = \{\alpha_1 + \cdots + \alpha_k \mid \alpha_i \in A\}$$

then A_k is a coset of the same subgroup, and thus $A_k \simeq A$.

For $\alpha \in A_k$ we write L_α for the associated line bundle and D_α for any effective divisor such that $\mathcal{O}(D_\alpha) = L_\alpha$. For any $\alpha \in Pic(C)$ we write $r(\alpha) = h^0(L_\alpha) - 1$.

The ideas of the proof of the main theorem are as follows:

1. We produce families of maps to projective spaces by taking sections of L_α for $\alpha \in A_2$ (Lemma 1).
2. In case the general such map is not birational onto the image, we reduce our problem to lower genus and an appropriately lower d. In the cases of our theorems, we actually get the required maps (Lemmas 2 and 3).
3. When these maps are birational, we use an estimate similar to Castelnuovo's bound, only stronger, to show that $g(C) \leqslant O(d^2/a)$.

LEMMA 1. *For any $\alpha \in A_2$ we have $r(\alpha) \geqslant a$.*

Proof. Let $\pi_d\colon C^{(d)} \to W_d(C)$ be the natural map, and let $\tilde{A} \subset C^{(d)}$ be the proper transform of A under this map. Recall that the symmetrization map $C^{(d)} \times C^{(d)} \to C^{(2d)}$ is finite. Therefore $\tilde{A} \times \tilde{A} \to \tilde{A}_2$ is finite, where $\tilde{A}_2$ is the proper transform of A_2. So $\dim \tilde{A}_2 \geqslant 2a$, and the fibers of $\pi_2|_{\tilde{A}_2}\colon \tilde{A}_2 \to A_2$ have dimension at least a. Abel's theorem says that $r(\alpha) \geqslant a$ for all $\alpha \in A_2$. □

Note that the linear systems $|D_\alpha|$ obtained above are base point free. Special care is needed in case $a = 1$:

LEMMA 2. *Assume $a = 1$.*

1. *If the general point $p \in C$ belongs to exactly one D_α with $\alpha \in A$, such that $r(\alpha) = 0$, then there is a map of degree d from C to the elliptic curve A.*
2. *Assume that for the general $\alpha \in A_2$ we have $r(\alpha) = 1$. Let $\phi_\alpha\colon C \to \mathbf{P}^1$ be the map defined by the global sections of L_α. Then ϕ_α factors through a d-to-1 map to A.*

Proof. (1) is formal, and may be shown as follows: let $F\colon C \times C^{(d-1)} \to W_d(C)$ be the natural map, and let C' be the normalization of the part of $F^{-1}(A)$ dominating A. Our minimality conditions mean that C' is exactly a d-sheeted cover of A. On the other hand, the projection onto the first factor $\pi_1\colon C \times C^{(d-1)} \to C$ induces a map from C' to C, the degree of which is the number of times a general point of C belongs to a divisor D_α. If this degree is 1, then $C \simeq C'$ and therefore C admits a map to A of degree d.

For (2), note that for any $\beta \in A$ we have $\alpha - \beta \in A$. Therefore $\alpha - \beta$ is effective, and thus D_β imposes one condition on the linear system $|D_\alpha|$, so D_β lies in a fiber of ϕ_α.

If the general fiber of ϕ_α is written uniquely as a sum $D_\beta + D_{\beta'}$ where $\beta + \beta' = \alpha$, we are in case (1). Otherwise, for every α with $r(\alpha) = 1$ we have ∞^1 equations:

$$D_\beta + D_{\beta'} = D_\gamma + D_{\gamma'} \quad (\beta + \beta' = \gamma + \gamma' = \alpha). \tag{1}$$

Fixing β and changing α (and thus β') we see that since $D_\beta \cap D_\gamma \neq \varnothing$ the divisors D_γ have a common divisor. Similarly for $D_{\gamma'}$. At least one of the two moves, and so the divisors of A would have a common divisor, contradicting our assumption. □

The following lemma, together with the previous one, will establish the cases when the general ϕ_α is not birational.

LEMMA 3. *Assume $a \geqslant 1$ and $r(\alpha) > 1$ for all $\alpha \in A_2$. If $\phi_\alpha: C \to \mathbf{P}^{r(\alpha)}$ is not birational for general α, then either $A \subset W^1_{d'}(C)$ with $d' \leqslant d$, or ϕ_α factors as:*

$$C \xrightarrow{\rho} C' \xrightarrow{\bar{\phi}_\alpha} \mathbf{P}^{r(\alpha)}$$

and there is an imbedding $A \subset W_{d'}(C')$ where $d' = d/\deg \rho = \deg \bar{\phi}_\alpha$.

Proof. Recall that the set of maps from C to curves of positive genus (up to automorphisms) is discrete. If the general ϕ_α map to rational curves of degree m, then their images must be rational normal curves (the linear series in question are complete) and we get an imbedding $A \subset W^1_{d'}(C)$, where $d' = d/m$. Otherwise, there is a generic image curve for the ϕ_α, call it C'. Let $p \in C'$ and let q_1, $q_2 \in \rho^{-1}(p)$. Suppose $q_1 \in D_\beta$ for some $\beta \in A$. We claim that $q_2 \in D_\beta$, which gives the lemma. If we let α vary in A_2 and set $\beta' = \alpha - \beta$, then $D_\beta + D_{\beta'}$ is a hyperplane section of $C \subset \mathbf{P}^r$ containing q_1. Therefore, since ϕ_α factors through C', also $q_2 \in D_\beta + D_{\beta'}$. But the divisors $D_{\beta'}$ do not have a common divisor, therefore $q_2 \in D_\beta$. This means that A is a pull-back of an abelian variety from $W_{d'}(C')$. Again, since the linear series are complete, this pull-back is an isomorphism. □

We use the following classical lemma:

LEMMA 4. *Let $C \to \mathbf{P}^r$ be birational onto its image. Then for every $s < r$ there do not exist ∞^s divisors of degree $s+1$, each spanning an $s-1$-plane.*

Proof. By a projection from a generic secant we reduce to the fact that a plane curve has finitely many singularities. □

Let $r_k = \min\{r(\alpha) \mid \alpha \in A_k\}$, that is, the general dimension of the complete linear series $|D_\alpha|$, $\alpha \in A_k$.

LEMMA 5. *Suppose $r_2 = a$. Then ϕ_α is not birational.*

Proof. In case $a = 1$ this is trivial. Otherwise, the fibers of π_2 as in Lemma 1 are in general projective spaces of dimension a, which are surjected by the quotient of A by an involution. In dimension $a > 1$ these are never rational. Therefore each divisor of D_α is represented in at least two ways as the sum of two divisors from A, and we get ∞^a equations as in (1). Fixing D_β again and letting $D_{\beta'}$ vary, and vice versa, we see that A is generated by two subvarieties

$X_1 \subset W_{d_1}(C)$ and $X_2 \subset W_{d_2}(C)$, where $\dim(X_1) = a_1 > 0$ and $\dim(X_2) = a_2 > 0$ and $a_1 + a_2 \geqslant a$. In the target space of ϕ_α we see that we get ∞^{a_1} divisors, given by X_1, each spanning only an $a_1 - 1$-plane. By Lemma 4 the map is not birational onto the image. □

The following lemma uses the same kind of information for the next possible dimension:

LEMMA 6. *Suppose $r_2 = a + 1$, and suppose the general ϕ_α is birational. Then for general points $p_1, \dots, p_a$ in C, and any D_β with $\beta \in A$ such that $p_i < D_\beta$ there is another divisor $D_{\beta'}$ with $\beta' \in A$ so that* $\gcd(D_\beta, D_{\beta'}) = p_1 + \cdots + p_a$.

Proof. Now the fibre of π_2 is in general a projective space of dimension $a+1$, and the quotient of A by an involution maps to it by a finite map. If the image is a linear space, we have a linear series to which we may apply the previous lemma. Otherwise, the image is of higher degree, in which case the line defined by general a points of C intersects this image several times. This means that the divisor $p_1 + \cdots + p_a$ lies on several of the hyperplanes defined by A, and therefore is in general contained in several divisors of A. If they all contain an extra point, we get ∞^a intersections of hyperplanes containing $a+1$ points, contradicting Lemma 4. □

3. Number of conditions

We are left with the cases when $r(\alpha) > 1$ for all $\alpha \in A_2$ and ϕ_α birational for the general α. We continue and derive a strengthened Castelnuovo type bound on the genus of C. The argument is similar to the original argument of Castelnuovo's bound (see [2], Chapt. 3) and the generalized one by Accola [3]. The idea is to estimate the number of conditions a divisor D_β for $\beta \in A$ imposes on the sections of a general k-fold sum $\alpha \in A_k$. The fact that we are working with cosets of subgroups plays an important role.

First, some observations. Since $\{D_\beta \mid \beta \in A\}$ have no common divisor, for all $p \in C$ the general D_β does not contain p. As an immediate result we get:

LEMMA 7. $r_{k+1} - r_k \geqslant r_{k-1}$.

Proof. Let $\alpha \in A_{k+1}$ be a general point, and let D be a general divisor coming from A. Let D_γ, $\gamma \in A$ be a general divisor, such that $\gcd(D, D_\gamma) = 0$. By the generality assumption, there are $r_k - r_{k-1}$ points in D which impose independent conditions on sections of $L_{\alpha-\gamma}$. Multiplying by the canonical section of $\mathcal{O}(D_\gamma)$, which does not vanish on any point of D, certainly keeps this property. Hence the lemma. □

LEMMA 8.

1. *If for general $\alpha \in A_2$ the map ϕ_α is birational onto its image then $r_3 \geqslant 2r_2$ and*

$r_{k+2} - r_k \geqslant \min(r_k - r_{k-2} + r_2, 2d)$ *for any* $k \geqslant 2$.
2. *If* $r_2 = a + 1$ *then* $r_{k+1} - r_k \geqslant \min(ka + 1, d)$.

Proof. The fact that $r_3 \geqslant 2r_2$ follows immediately from Lemma 7.

Let $D_\alpha = p_1 + \cdots + p_{2d}$ be a general divisor. Now, by the uniform position lemma (see [2]) if we take a general $\alpha' \in A_2$ then there is a divisor $D_{\alpha'}$ such that the common divisor with D_α is $p_1 + \cdots + p_{r_2}$.

Also, for a general $\gamma \in A_k$ we have a divisor D_γ so that $\gcd(D_\gamma, D_\alpha) = p_{r_2+1} + \cdots + p_{r_2+r_k-r_{k-2}-1}$. But the order of the chosen points is unimportant. Therefore, for the general $\delta = \gamma + \alpha' \in A_{k+2}$ we have that α imposes at least $r_2 + r_k - r_{k-2}$ conditions on $|D_\delta|$.

For the second claim, notice that by Lemma 6, if $D_\beta = q_1 + \cdots + q_d$ is a general divisor corresponding to points of A, then there are divisors D_{β_i} so that $gcd(D_\beta, D_{\beta_i}) = q_{(i-1)a+1} + \cdots + q_{ia}$. Summing up k of these, and using Lemma 7 for an extra divisor, we get the inequality in (2). □

REMARK. Notice that we didn't make use, in the proof of first part of the lemma, of the fact that we have ∞^a ways to choose α'. For our results this turns out to be sufficient.

COROLLARY 1. *In the case of the lemma, we have* $r_3 \geqslant \min(2a+3, a+1+d)$, *and* $r_4 \geqslant \min(3a+6, r_3+d)$.

As a byproduct we get a theorem:

THEOREM 2. *Let* $A \subset W_d(C)$ *be an abelian variety of positive dimension* a. *Assume that for the general* $\alpha \in A_2$ *the map* ϕ_α *is birational onto its image. Then*

$$g(C) \leqslant \binom{d}{2} + 1.$$

Proof. We know that $r_2 \geqslant 2$. If equality holds, we have $r_d \geqslant 2 + 3 + \cdots + d = \binom{d+1}{2} - 1$. But for $\alpha \in A_d$, $\deg\, \alpha = d^2$, so $2r(\alpha) > \deg \alpha$ for all α, and by Clifford's theorem (see [2]) α is non-special, that is, $g(C) = \deg \alpha - r(\alpha) \leqslant \binom{d}{2} + 1$. Similarly, if $r_2 \geqslant 3$ one proves by induction, using Lemmas 8 and 7, that $r_k \geqslant \binom{k+1}{2} - 1$, and continues as before. □

4. Statement and proof of main theorems

THEOREM 3. *If* $A \subset W_2(C)$, *then either* C *has genus at most* 2, *or* C *is bielliptic. If* $g(C) > 3$ *then* C *is not hyperelliptic.*

Proof. Lemma 2 settles the theorem when $r(\alpha) = 1$ for general $\alpha \in A_2$. If

$r(\alpha) > 1$, we have a family of g_4^2, which does not exist unless $g(C) \leqslant 2$, because for genus 3 a g_4^2 is the canonical series, and for higher genera it has to be twice a unique hyperelliptic series. For the last statement, a bi-elliptic hyperelliptic curve is of type $(2, 4)$ on a smooth quadric, and therefore of genus at most 3. $\square$

THEOREM 4. *If $A \subset W_3(C)$ and $g(C) \geqslant 5$ then C admits a map of degree at most 3 to a curve of genus 1. If $\dim A \geqslant 2$ then the genus of C is at most 3. If $g(C) \geqslant 8$ then C does not admit a g_3^1.*

Proof. Lemmas 2 and 3 settle the theorem for ϕ_α not birational for general $\alpha \in A_2$. Corollary 1 shows that any other case has a g_9^5, and by Clifford's theorem ([2]) has genus at most 4, but these have a g_3^1 [2]. Similarly, if we take $a = 2$ we see that $g \leqslant 3$. $\square$

THEOREM 5. *If $A \subset W_4(C)$ and $g(C) \geqslant 8$ then either C admits a map of degree at most 4 to a curve of genus 1, or a map of degree 2 to a curve of genus 2. If $\dim A \geqslant 2$ and $g(C) \geqslant 6$ then C is a double cover of a curve of genus 2.*

Proof. Again we may assume ϕ_α is birational for general $\alpha \in A_2$. Corollary 1 and Clifford's theorem show that $g(C) \leqslant 7$.

Curves of genus at most 6 have a g_4^1. For the last statement, we see that if $a > 1$ then in fact $g \leqslant 5$. In the next section we show that there is a counterexample with $g = 5$. $\square$

5. An example

We construct a 6 dimensional family of curves of genus 5, all having a curve of genus 2 in W_3, and none of them admits a map of degree 2 or 3 to curves of genus 0, 1 or 2. As a byproduct, we explain how a curve of genus 5 can possess an abelian surface in W_4 without being a double cover of a curve of genus 2. The construction is a special case of the tetragonal construction for Prym varieties, as in [4].

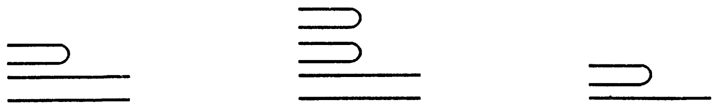
Let $f: C_2 \to \mathbf{P}^1$ be a map of degree 4, from a curve C_2 of genus 2 to $\mathbf{P}^1$. Assume f has only simple ramifications.

Let $C' = \overline{C_2 \times_{\mathbf{P}^1} C_2 - \Delta}$ be the curve of pairs of distinct points in the fibres over $\mathbf{P}^1$. Let C_5 be the quotient of C' by the symmetrization involution: $C_5 = \overline{(C_2)^{(2)}_{\mathbf{P}^1} - \Delta}$. C_5 is our curve of genus 5. Note that C_5 admits an involution that assigns to an unordered pair of distinct elements in a fiber, the residual pair in that fiber. The quotient is a curve of genus 3, C_3. We have the following commutative diagram:

$$\begin{array}{ccccc} C' & \xrightarrow{h} & C_5 & \xrightarrow{\pi} & C_3 \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \rho} \\ C_2 & & \xrightarrow{\quad f \quad} & & \mathbf{P}^1 \end{array}$$

where π is unramified of degree 2, ρ of degree 3 with 10 ramifications, and f of degree 4 with 10 ramifications.

The corresponding ramification behavior of C_2, C_5 and C_3 over $\mathbf{P}^1$ is sketched below:



ramification in C_2 ramification in C_5 ramification in C_3

From this construction we see that the curves C_5 vary in at most 6 parameters: in fact the curves C_3 have only so many moduli. We show that the construction may be reversed, and that we really get 6 parameters.

Let C_3 be a nonhyperelliptic curve of genus 3. Let ρ be any g_3^1 on the curve, with simple ramifications. Let π: $C_5 \to C_3$ be any connected unramified double cover. One checks that the monodromy of C_5 over $\mathbf{P}^1$ via the map $\rho \circ \pi$ is S_4. Now take the subvariety D of the triple relative symmetric power $(C_5)^{(3)}_{\mathbf{P}^1}$ that does not map into a diagonal of $(C_3)^{(3)}_{\mathbf{P}^1}$. This subvariety is composed of two isomorphic components of genus 2, called C_2.

THEOREM 6. *The general C_5 in this family does not admit a map of degree at most* 3 *to a curve of genus at most* 2.

Let Λ be the variety inside $\mathcal{M}_5$ described by our curves of genus 5, and let $D_{d,h}$ be the subset of Λ of those curves that admit a map of degree d to a curve of genus h. We need to show: $\Lambda \neq \bigcup_{d \leqslant 3, h \leqslant 2} D_{d,h}$.

LEMMA 9. $\dim D_{3,2} \leqslant 5$.

Proof. The dimension of the variety of curves of genus 5 admitting a map of degree 3 to a curve of genus 2 is 5. □

LEMMA 10. $\dim D_{2,h} \leqslant 5$.

Proof. If an involution of C_5 commutes with π then C_3 has automorphisms, and the dimension of such C_3 is 5. If they do not commute, the composition of the two involutions is of some order bigger than 2, and the dimension of the variety of curves of genus 5 admitting such an automorphism is again not more than 5 [2]. □

LEMMA 11. $\dim D_{3,0} \leqslant 5$.

REMARK. In fact, one can show that $D_{3,0}$ is empty.

Proof. We prove by specialization. Let C_3 be a nonhyperelliptic, bielliptic curve of genus 3, and p: $C_3 \to E$ the bielliptic map. Let $E' \to E$ be a two sheeted map of elliptic curves. Then $E' \times_E C_3$ is a bielliptic curve of genus 5 in our family. □

Now, a bielliptic curve of genus 5 does not admit a g_3^1. In fact, if C_5 has a map of degree 2 or 3 to $\mathbf{P}^1$, then as a cycle in $E' \times \mathbf{P}^1$ we have

$$[C_5]^2 = \text{degree of ramification} = 12 \text{ or } 14.$$

If $H = \pi_1^{-1}(p) + \pi_2^{-1}(q)$ is an ample divisor formed by fibers both ways, we have $H^2 = 2$ and $H.[C_5] = 4$ or 5. We get

$$(H.[C_5])^2 \leqslant (H.H)([C_5].[C_5])$$

which is a contradiction to the Hodge index theorem. □

LEMMA 12. *$h_* g^* Jac(C_2) \cap \pi^* Jac(C_3)$ is finite, and their sum is the whole $Jac(C_5)$, for general C_5. In other words, the two jacobians give subabelian varieties which are complementary up to isogeny.*

Proof. If $q \in C_2$ one checks explicitly that $\pi_* h_* g^*(q) = \rho^* f_*(q)$ (in fact, the big square in the commutative diagram is the normalization of a fiber square). This does not depend on q because $f_*(q) \sim f^*(q_1)$ on $\mathbf{P}^1$.

If C_5 is general from Λ, then $\dim h_* g^* Jac(C_2) > 0$, otherwise C_5 has a g_3^1 (see Lemma 2).

If C_2 is of general moduli, it does not map to an elliptic curve, in which case $\dim h_* g^* Jac(C_2) \neq 1$. By semicontinuity, the dimension is 2 for general C_2. □

LEMMA 13. $\dim D_{d,1} \leqslant 5$.

Proof. In fact, if C_5 admits a map to an elliptic curve, then this elliptic curve maps to $Jac(C_5)$ by a nonconstant map. Projecting to $Jac(C_3)$ and to $Jac(C_2)$ we see that at least one of these jacobians is nonsimple. This again bounds the dimension of either C_2 or C_3. □

This finishes the verification of our theorem. □

COROLLARY 2. *There are curves C_5 of genus 5 such that $W_4(C_5)$ contains an abelian surface, but the curve C_5 does not map to any curve of genus 2.*

Proof. The Prym variety of the map $C_5 \to C_3$ has a translate which lies in W_4, namely the odd component of the inverse image of K_{C_3} (see [9]). □

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