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Finite-dimensional categorical complement theorems in strong shape theory and a principle of reversing maps between open subsets of spheres

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Introduction

The well-known finite-dimensional complement theorems in shape theory (see e.g. [16], [17], [20]) assert that compacta in $\mathbb{R}^n$ (or S^n), satisfying suitable dimension and embedding conditions, have the same shape if and only if their complements are homeomorphic. This classifies the shapes of compacta by a *geometric property* of the complements, but at the same time *fails* to exhibit a classifying *homotopy property*. To compensate this deficiency, we developed in [13] a theory of finite-dimensional *categorical complement theorems*. The reader is assumed to be familiar with notation and results of [13], and should recall in particular that ($\mathfrak{ANR}_m$, (Ad_m), $\mathbf{wH}_m\mathbf{C}$) are *data of a categorical complement theorem for* $\mathbf{Sh}(\mathfrak{CM}_m)$, the Borsuk-Mardešić shape category of compacta X with fundamental dimension $FdX \leqslant m$. This result removes all geometric niceness conditions on the ambient spaces (such conditions are indispensable to establish 'ordinary' complement theorems) and specifies what embeddings are acceptable in order that the complements keep sufficient *homotopical* information for the classification of shapes. An immediate consequence is the following 'homotopical complement theorem': Compacta X_i with $FdX_i \leqslant m$, being m-admissibly embedded into m-connected ANR's M_i with a complete uniform structure, have the same shape if and only if their complements $M_i - X_i$ have the same weak complete m-homotopy type.

The present paper continues the program of [13] in the framework of *strong shape theory* (concerning 'strong shape' see [1], [3], [4], [5], [7], [10]). Our main result is the following.

THEOREM A. ($\mathfrak{ANR}_{m+1}$, (Ad_{m+2}), $\mathbf{H}_{m+1}\mathbf{C}$), *where* $\mathbf{H}_{m+1}\mathbf{C}$ *is the complete* $(m+1)$*-homotopy category* (cf. section 2), *are data of a categorical complement theorem for the strong shape category* $\mathbf{SSh}(\mathfrak{CM}_m)$ *of compacta* X *with* $FdX \leqslant m$. *In particular: Compacta* X_i *with* $FdX_i \leqslant m$, *being* $(m+2)$*-admissibly embedded into* $(m+1)$*-connected* ANR's M_i *with a complete uniform structure, have the same* *shape if and only if their complements have the same complete* $(m+1)$*-homotopy type.*

We see that the passage from shape to strong shape costs *one additional dimension* of control on the complements – $\mathbf{wH}_m\mathbf{C}$ has to be replaced by $\mathbf{H}_{m+1}\mathbf{C}$ (this 'additional dimension of control' reflects exactly the conceptual difference between shape and strong shape). The essential advantage of Theorem A is that the *complementary category* $\mathbf{H}_{m+1}\mathbf{C}$ is a 'homotopy category' and not a 'weak homotopy category' as in the case of ordinary shape. The price for such a more natural complementary category is a more restrictive embedding condition – (Ad_m) has to be replaced by (Ad_{m+2}). Results improving upon dimensions in Theorem A, however, are available for compacta of stable shape (for details see section 4).

Fixing the sphere S^n as an ambient space, Theorem A implies that the assignment $X \mapsto S^n - X$ induces a *covariant* category isomorphism from a full subcategory of $\mathbf{SSh}(\mathfrak{CM})$ to a suitable 'homotopy category of proper maps'. On the other hand, the *Strong Shape S-Duality Theorem* of Q. Haxhibeqiri and S. Nowak [6] asserts that $X \mapsto S^n - X$ induces a *contravariant* category isomorphism from the stable strong shape category to the stable homotopy category. We may therefore expect that there exists a 'principle of reversing the direction of maps between open subsets of spheres'. This general idea is made precise by the following *duality theorem.*

THEOREM B. *Let* $d(n) = \max\{k \mid 2k+2 \leqslant n\}$ *and* $c(n) = \max\{k \mid 4k \leqslant n\}$. *Let* $\mathbf{T}_n$ *be the full subcategory of the homotopy category* **HTop** *whose objects are all complements* $S^n - X$ *of* $c(n)$*-shape-connected compacta* $X \subset S^n$ *having* $FdX \leqslant d(n) - 1$ *and satisfying the inessential loops condition ILC* [20], *and let* $\mathbf{H}_{d(n)}\mathbf{P}$ *be the proper* $d(n)$*-homotopy category* (cf. section 2). *There exists a contravariant full embedding*

$$\Delta\colon \mathbf{T}_n \to \mathbf{H}_{d(n)}\mathbf{P}$$

such that $\Delta(U) = U$ *for each object* U.

Roughly speaking, Theorem B says that each map $f\colon U \to V$ between suitable open subsets U, V of S^n can be 'canonically reversed' to obtain a proper map $f^*\colon V \to U$, and vice versa. In certain special cases we can say even more about this reversing process: $f\colon U \to V$ and $f^*\colon V \to U$ can be chosen in such manner that they are 'inverse to each other' in a very weak sense; for details see Theorem 5.5.

1. Filtered spaces

Let $\tilde{X} = (X; X_n)$ be a filtered space (consisting of an underlying space X and a sequence of closed subspaces $X_n \subset X$ such that $X_0 = X$ and $X_{n+1} \subset \operatorname{int} X_n$ for each n). We call $\tilde{X}$ *functionally filtered* if there is a map $h\colon X \to [0, \infty)$ with $X_n = h^{-1}([n, \infty))$ for each n. The symbol $\mathbf{F}$ will denote the category of functionally filtered spaces and filtered maps. Here are some properties of functionally filtered spaces $\tilde{X}$.

1.1. PROPOSITION. (a) *$\tilde{X} \times I = (X \times I;\, X_n \times I)$ is a functionally filtered space, and each $i_t\colon X \to X \times I$, $i_t(x) = (x, t)$, is a filtered map $i_t\colon \tilde{X} \to \tilde{X} \times I$.*

(b) *For each $A \subset X$, $\tilde{X} \cap A = (A;\, X_n \cap A)$ is a functionally filtered space, and the inclusion $i\colon A \to X$ is a filtered map $i\colon \tilde{X} \cap A \to \tilde{X}$.*

A filtered subspace $\tilde{A} \subset \tilde{X}$ (i.e. $\tilde{A} = \tilde{X} \cap A$ for some $A \subset X$) is called *cofinal* if A contains some X_n. In this case also the inclusion $i\colon \tilde{A} \to \tilde{X}$ is said to be cofinal. Let $\Sigma_{\mathbf{F}}$ denote the class of all cofinal inclusions in $\mathbf{F}$, and let $\mathbf{F}_\infty$ denote the quotient category $\mathbf{F}\backslash\Sigma_{\mathbf{F}}$. It is easy to show that $\mathbf{F}_\infty$ admits a calculus of right fractions (cf. [5] §6.2).

Next, let $\mathbf{C}$ be the category of σ-complete uniform spaces and complete maps (introduced in [13]). Recall that a uniform space X is σ-complete iff it has a *filtration function* h_X, i.e. a map $h_X\colon X \to [0, \infty)$ with

(C1) For each complete $S \subset X$, the closure $\mathrm{cl}(h_X(S))$ is compact.
(C2) For each n, $h_X^{-1}([0, n])$ is complete.

By a *filtration functor* on $\mathbf{C}$ we mean a functor $V\colon \mathbf{C} \to \mathbf{F}$ such that

(V1) For each $X \in \mathrm{Ob}\,\mathbf{C}$, $V(X)$ is a *filtered model* of X (i.e. $V(X) = (X;\, h_X^{-1}([n, \infty))$ for some filtration function h_X).
(V2) For each $f \in \mathbf{C}(X, Y)$, $V(f) = f$.

From [13] 2.3(2) we infer

1.2 PROPOSITION. (a) *There exist filtration functors on $\mathbf{C}$; in fact, each choice of filtration functions h_X, $X \in \mathrm{Ob}\,\mathbf{C}$, determines a unique filtration functor on $\mathbf{C}$.*

(b) *Any two filtration functors on $\mathbf{C}$ are naturally isomorphic.*

(c) *Each filtration functor on $\mathbf{C}$ is a full embedding.*

An easy consequence of the definition of the quotient categories $\mathbf{C}_\infty$ (cf. [13]) and $\mathbf{F}_\infty$ is

1.3 PROPOSITION. *Each filtration functor $V\colon \mathbf{C} \to \mathbf{F}$ induces a unique functor $V_\infty\colon \mathbf{C}_\infty \to \mathbf{F}_\infty$ with $U_{\mathbf{F}}V = V_\infty U_{\mathbf{C}}$ (where $U_{\mathbf{C}}\colon \mathbf{C} \to \mathbf{C}_\infty$ and $U_{\mathbf{F}}\colon \mathbf{F} \to \mathbf{F}_\infty$ are the canonical quotient functors); V_∞ is a full embedding.*

2. Homotopy relations

The concept of *homotopy* was defined in [5] for the categories $\mathbf{P}$ (=proper category = category of σ-compact spaces and proper maps) and $\mathbf{P}_\infty$ (=proper category at ∞ = **P**\(cofinal inclusions in **P**)), and in [12] for the category **C**. More generally, in each of the categories $\mathbf{D} = \mathbf{P}, \mathbf{P}_\infty, \mathbf{C}, \mathbf{C}_\infty, \mathbf{F}, \mathbf{F}_\infty$ the cylinder functor '$\times I$' can be used to define the relation of homotopy: $\alpha_0, \alpha_1 \in \mathbf{D}(A, B)$ are called *homotopic* in **D**, $\alpha_0 \cong \alpha_1$, if there exists a *homotopy* $H \in \mathbf{D}(A \times I, B)$ such that $Hi_k = \alpha_k$, $k = 0, 1$ (where $i_k: A \to A \times I$ are the obvious morphisms in **D**).

We shall also need the following 'relative' version of homotopy: Given any class of objects $\mathfrak{M} \subset \mathrm{Ob}\,\mathbf{D}$, $\alpha_0, \alpha_1 \in \mathbf{D}(A, B)$ are called *$\mathfrak{M}$-homotopic* in **D**, $\alpha_0 \cong_{\mathfrak{M}} \alpha_1$, if $\alpha_0\varphi \cong \alpha_1\varphi$ for each $\varphi \in \mathbf{D}(A', A)$ with $A' \in \mathfrak{M}$. It is fairly obvious that '$\mathfrak{M}$-homotopy in **D**' is an equivalence relation for morphisms in **D** which is compatible with composition. The *$\mathfrak{M}$-homotopy category* $\mathbf{H}_{\mathfrak{M}}\mathbf{D}$ is obtained from **D** by identifying $\mathfrak{M}$-homotopic morphisms. For $\mathfrak{M} = \mathrm{Ob}\,\mathbf{D}$ we get the *homotopy category* $\mathbf{HD} = \mathbf{H}_{\mathrm{Ob}\mathbf{D}}\mathbf{D}$ (note that $\alpha_0 \cong_{\mathrm{Ob}\mathbf{D}} \alpha_1$ iff $\alpha_0 \cong \alpha_1$).

2.1 EXAMPLE. Let $\mathfrak{P}(m)$ be the class of all σ-compact polyhedra P with $\dim P \leqslant m$ (cf. [13] 1.4). Then we obtain the *proper m-homotopy category*, $\mathbf{H}_m\mathbf{P} = \mathbf{H}_{\mathfrak{P}(m)}\mathbf{P}$, the *proper m-homotopy category at* ∞, $\mathbf{H}_m\mathbf{P}_\infty = \mathbf{H}_{\mathfrak{P}(m)}\mathbf{P}_\infty$, the *complete m-homotopy category*, $\mathbf{H}_m\mathbf{C} = \mathbf{H}_{\mathfrak{P}(m)}\mathbf{C}$, and the *complete m-homotopy category at* ∞, $\mathbf{H}_m\mathbf{C}_\infty = \mathbf{H}_{\mathfrak{P}(m)}\mathbf{C}_\infty$.

2.2 REMARK. Recall from [13] that we regard $\mathbf{P}, \mathbf{P}_\infty$ as full subcategories $\mathbf{P} \subset \mathbf{C}$, $\mathbf{P}_\infty \subset \mathbf{C}_\infty$. This induces the following inclusions as full subcategories: $\mathbf{HP} \subset \mathbf{HC}$, $\mathbf{HP}_\infty \subset \mathbf{HC}_\infty$, $\mathbf{H}_m\mathbf{P} \subset \mathbf{H}_m\mathbf{C}$, $\mathbf{H}_m\mathbf{P}_\infty \subset \mathbf{H}_m\mathbf{C}_\infty$.

From 1.3 we infer

2.3 PROPOSITION. *Each filtration functor* $V: \mathbf{C} \to \mathbf{F}$ *induces a unique* $HV_\infty: \mathbf{HC}_\infty \to \mathbf{HF}_\infty$; $\mathbf{HV}_\infty$ *is a full embedding.*

One also readily checks that $\mathbf{U}_C: \mathbf{C} \to \mathbf{C}_\infty$ induces a quotient functor $H_mU_{\mathbf{C}}: \mathbf{H}_m\mathbf{C} \to \mathbf{H}_m\mathbf{C}_\infty$.

3. The end functor

Let **H(pro-Top)** denote the homotopy category of **pro-Top** with respect to its closed model structure as defined in [5] §2.3. Recall that the natural functor π:**H(pro-Top)** → **pro-HTop** has the following property (see [5] 5.2.9):

(3.1) Let $\underline{X}, \underline{Y}$ be inverse systems in **pro-Top** which are isomorphic in **H(pro-Top)** to towers. Then $\underline{X}$ and $\underline{Y}$ are isomorphic in **H(pro-Top)** iff $\pi\underline{X}$ and $\pi\underline{Y}$ are isomorphic in **pro-HTop**.

We now study the end functor $\varepsilon: \mathbf{C}_\infty \to$ **pro-Top** constructed in [13].

3.2 THEOREM. ε *induces a full embedding*

ε: $\mathbf{HC}_\infty \to$ **H(pro-Top)**.

Proof. The proof follows the lines of [5] §3.7 and §6.3. Let us first observe that $\varepsilon\colon \mathbf{C}_\infty \to \textbf{pro-Top}$ carries homotopies in $\mathbf{C}_\infty$ to *left homotopies* in **pro-Top** (cf. [15]), and therefore actually induces a functor $\varepsilon\colon \mathbf{HC}_\infty \to \mathbf{H(pro\text{-}Top)}$.

We now construct an end functor $\varepsilon'\colon \mathbf{F} \to \textbf{tow-Top} \subset \textbf{pro-Top}$ ($\textbf{tow-Top} \subset \textbf{pro-Top}$ is the full subcategory of towers). For each object $\tilde{X} = (X; X_n)$ of **F**, let $\varepsilon'(\tilde{X})$ be the tower $\{X_n\}$ in pro-Top (bonded by inclusions). For each morphism $f\colon \tilde{X} = (X; X_n) \to \tilde{Y} = (Y; Y_n)$ in **F**, define $f^*\colon \mathbb{N} \to \mathbb{N}$ by $f^*(n) = \min\{r \mid f(X_r) \subset Y_n\}$. Let $f_n\colon X_{f^*(n)} \to Y_n$ be the restriction of f. Then (f_n, f^*) is a map of inverse systems which represents a unique morphism $\varepsilon'(f) \in \textbf{tow-Top}(\varepsilon'(\tilde{X}), \varepsilon'(\tilde{Y}))$.

If $j\colon \tilde{A} \to \tilde{X}$ is a cofinal inclusion in **F**, then $\varepsilon'(j)$ is an isomorphism in **tow-Top**, and therefore $\varepsilon'\colon \mathbf{F} \to \textbf{tow-Top}$ induces a unique functor $\varepsilon'\colon \mathbf{F}_\infty \to \textbf{tow-Top}$ such that $\varepsilon' U_{\mathbf{F}} = \varepsilon'$.

Since ε' carries homotopies in $\mathbf{F}_\infty$ to left homotopies in **tow-Top**, it induces a functor $H\varepsilon'\colon \mathbf{HF}_\infty \to \mathbf{H(two\text{-}Top)} \subset \mathbf{H(pro\text{-}Top)}$.

Let us fix an arbitrary filtration functor $V\colon \mathbf{C} \to \mathbf{F}$. It is easy to verify

(3.3) The two functors ε, $\varepsilon' \circ V\colon \mathbf{C} \to \textbf{pro-Top}$ are naturally isomorphic.

From this one readily infers

(3.4) The two functors ε, $H\varepsilon' \circ HV_\infty\colon \mathbf{HC}_\infty \to \mathbf{H(pro\text{-}Top)}$ are naturally isomorphic.

The telescope construction discussed in [5] §3.7 is readily seen to furnish a functor Tel: $\mathbf{H(tow\text{-}Top)} \to \mathbf{HF}_\infty$. Using the techniques of [5] §3.7 and §6.3 one can verify

(3.5) There are natural isomorphisms $\mathrm{Tel} \circ H\varepsilon' \cong 1$ and $H\varepsilon' \circ \mathrm{Tel} \cong 1$. In particular, $H\varepsilon'$ is an equivalence of categories and a fortiori a full embedding.

This completes the proof, since HV_∞ is a full embedding.

3.6. REMARK. The above proof shows that $\mathbf{HF}_\infty$ is a 'category of geometric models' for **H(tow-Top)**. Note that it also follows from [5] 3.7.20 that **H(tow-Top)** *embeds* as a full subcategory into $\mathbf{HF}_\infty$.

Now, let θ_m^c resp. θ_m^* be the full subcategories of **H(pro-Top)** whose objects are all inverse systems which are isomorphic in **H(pro-Top)** to some tower $\underline{P} = \{P_n, p_n\}$ such that the P_n are compact resp. arbitrary polyhedra with $\dim P_n \leqslant m$.

Let $[\mathfrak{P}(m)]$ be the full subcategory of $\mathbf{HC}_\infty$ such that $\mathrm{Ob}[\mathfrak{P}(m)] = \mathfrak{P}(m)$. Then $\mathbf{H}_m\mathbf{C}_\infty = \mathbf{HC}_\infty/[\mathfrak{P}(m)]$ (cf. [13] §2). We set

$$\theta_m = \mathrm{Sat}(\varepsilon([\mathfrak{P}(m)]) \cap \theta_m^*$$

and obtain (cf. [13] 2.10)

3.7. PROPOSITION. ε: $\mathbf{HC}_\infty \to \mathbf{H(pro\text{-}Top)}$ induces a full embedding ε_m: $\mathbf{H}_m\mathbf{C}_\infty \to \mathbf{H(pro\text{-}Top)}/\theta_m$.

It is a routine exercise (using simplicial approximation, (3.1) and the telescope construction) to verify

3.8. PROPOSITION. $\theta^c_m \subset \theta_{m+1}$.

It is unfortunately not true that $\theta^c_m \subset \theta_m$. This can easily be seen for $m = 0$. The only non-compact 0-dimensional σ-compact polyhedron is a discrete space Δ with infinitely many points. Given any tower $\underline{X} = \{X_n\}$ consisting of pathwise connected spaces X_n, there is only *one* morphism $\varepsilon(\Delta) \to \underline{X}$ in **H(pro-Top)**; i.e. *any* two morphisms φ_0, φ_1: $\underline{Y} \to \underline{X}$ ($\underline{Y}$ arbitrary!) are $\varepsilon([\mathfrak{P}(0)])$-equal although they need not be θ^c_0-equal.

The following is a partial substitute for the failure of '$\theta^c_m \subset \theta_m$'.

3.9 LEMMA. *Let* $\underline{Y}$ be an object of **H(pro-Top)** *which admits a* θ^c_m*-equivalence* β: $\underline{Y} \to \underline{Y}'$ *onto a stable object* $\underline{Y}'$ *of* **H(pro-Top)**.

(a) *Any two* θ_m*-equal morphisms* φ_0, φ_1: $\underline{X} \to \underline{Y}$ *in* **H(pro-Top)** *are also* θ^c_m*-equal; i.e. for each* $\underline{X}$ the canonical ρ^c_m: **H(pro-Top)** $(\underline{X}, \underline{Y}) \to$ **H(pro-Top)**/$\theta^c_m(\underline{X}, \underline{Y})$ *induces a unique* ρ'_m: **H(pro-Top)**/$\theta_m(\underline{X}\ Y) \to$ **H(pro-Top)**/$\theta^c_m(\underline{X}, Y)$.

(b) ρ'_m *is a bijection provided* $\underline{X}$ *admits a* θ^*_m*-equivalence* α: $\underline{X} \to \underline{X}'$ *onto some* $\underline{X}' \in \mathrm{Ob}\,\theta^c_m$.

Proof. (a) Let $\underline{P} \in \mathrm{Ob}\,\theta^c_m$ and ψ: $\underline{P} \to \underline{X}$. We have to show $\varphi_0\psi = \varphi_1\psi$. Since β: $\underline{Y} \to \underline{Y}'$ is a θ^c_m-equivalence and $\underline{P} \in \mathrm{Ob}\,\theta^c_m$, it suffices to show $\beta\varphi_0\psi = \beta\varphi_1\psi$. Since $\underline{Y}'$ is stable in **H(pro-Top)**, the canonical map π: **H(pro-Top)** $(\underline{P}, \underline{Y}) \to$ **pro-HTop**$(\pi\underline{P}, \pi\underline{Y}')$ is a bijection. It therefore suffices to show $\pi(\beta)\pi(\varphi_0)\pi(\psi) = \pi(\beta)\pi(\varphi_1)\pi(\psi)$. We may assume that $\underline{P} = \{P_n, p_n\}$ with compact polyhedra P_n of dimension $\leqslant m$. We set $P^* = \bigcup_r P_r \times \{r\}$ and obtain (as in the proof of [13] 2.11) a 'canonical' morphism $\underline{f}$: $\varepsilon(P^*) \to \underline{P}$ in **pro-Top**. Let γ: $\varepsilon(P^*) \to \underline{P}$ be the image of $\underline{f}$ in **H(pro-Top)**. Since φ_0, φ_1 are θ_m-equal, $\varphi_0\psi\gamma = \varphi_1\psi\gamma$. Hence $\pi(\varphi_0)\pi(\psi)\pi(\gamma) = \pi(\varphi_1)\pi(\psi)\pi(\gamma)$, which implies $\pi(\varphi_0)\pi(\psi) = \pi(\varphi_1)\pi(\psi)$ (see again [13] 2.11). This completes the proof of (a).

(b) *Since* $\theta^c_m \subset \theta^*_m$, ρ^c_m: **H(pro-Top)** $\to$ **H(pro-Top)**/θ^c_m induces a full functor F_m: **H(pro-Top)**/$\theta^*_m \to$ **H(pro-Top)**/θ^c_m; since $\theta_m \subset \theta^*_m$, ρ_m: **H(pro-Top)** $\to$ **H(pro-Top)**/θ_m induces a full functor G_m: **H(pro-Top)**/$\theta^*_m \to$ **H(pro-Top)**/θ_m. Consider the following commutative diagram (cf. [13] 2.4, 2.7):

$$
\begin{array}{ccccc}
\mathbf{H(pro\text{-}Top)}/\theta^*_m(\underline{X}, \underline{Y}) & & \xleftarrow[\approx]{\alpha^\#} & & \mathbf{H(pro\text{-}Top)}/\theta^*_m(\underline{X}', \underline{Y}) \\
G_m \downarrow & & & & \approx\uparrow \\
\mathbf{H(pro\text{-}Top)}/\theta_m(\underline{X}, \underline{Y}) & \Big\downarrow F_m & & F_m \Big\downarrow & \mathbf{H(pro\text{-}Top)}(\underline{X}', \underline{Y}) \\
\rho'_m \searrow & & & & \approx\downarrow \\
\mathbf{H(pro\text{-}Top)}/\theta^c_m(\underline{X}, \underline{Y}) & & \xleftarrow[\approx]{\alpha^\#} & & \mathbf{H(pro\text{-}Top)}/\theta^c_m(\underline{X}', \underline{Y})
\end{array}
$$

We infer that the F_m are bijections; hence G_m is an injection. Since G_m is a full functor, (b) is proved.

3.10 THEOREM. *Let A be a σ-complete metrizable space and B be a σ-complete m-connected ANR.*

(a) $H_m U_{\mathrm{C}}$: $\mathbf{H}_m\mathbf{C}(A, B) \to \mathbf{H}_m\mathbf{C}_\infty(A, B)$ *is a bijection provided* $\varepsilon(A)$ *admits a* θ_m-*equivalence* φ: $\varepsilon(A) \to \underline{A}$ *onto some* $\underline{A} \in \mathrm{Ob}\,\theta_m$.
(b) $H_m U_{\mathrm{C}}$: $\mathbf{H}_m C(A, B) \to \mathbf{H}_m\mathbf{C}_\infty(A, B)$ *is a bijection provided* $\varepsilon(A)$ *admits a* θ_m^*-*equivalence* φ: $\varepsilon(A) \to \underline{A}$ *onto some* $\underline{A} \in \mathrm{Ob}\,\theta_m^c$ *and* $\varepsilon(B)$ *admits a* θ_m^c-*equivalence* ψ: $\varepsilon(B) \to \underline{B}$ *onto a stable object* $\underline{B}$ of **H(pro-top)**.

Proof. (1) Injectivity. A straightforward modification of the proof of [13] 2.14 yields injectivity in both cases.

(2) Surjectivity. We only prove (b), the other case is similar. By 3.7 and 3.9, ε induces a bijection ε_m': $\mathbf{H}_m\mathbf{C}_\infty(A, B) \to \mathbf{H(pro\text{-}Top)}/\theta_m^c(\varepsilon(A), \varepsilon(B))$. Since $\rho_m^c(\varphi)$ is an isomorphism in $\mathbf{H(pro\text{-}Top)}/\theta_m^c$, there exists φ': $\underline{A} \to \varepsilon(A)$ with $\rho_m^c(\varphi') = \rho_m^c(\varphi)^{-1}$.

Consider $\mathfrak{f} \in \mathbf{C}_\infty(A, B)$. By 3.2, there exists $\mathfrak{g} \in \mathbf{C}_\infty(A, B)$ such that $\varepsilon([\mathfrak{g}]) = \varepsilon([\mathfrak{f}])\varphi'\varphi$, where [ ] denotes equivalence class in $\mathbf{HC}_\infty$. We have $\varepsilon_m'([\mathfrak{g}]_m) = \rho_m^c\varepsilon([\mathfrak{g}]) = \rho_m^c\varepsilon([\mathfrak{f}]) = \varepsilon_m'([\mathfrak{f}]_m)$, where $[\ ]_m$ denotes equivalence class in $\mathbf{H}_m\mathbf{C}_\infty$. This implies $[\mathfrak{g}]_m = [\mathfrak{f}]_m$. But now the equation $\pi\varepsilon([\mathfrak{g}]) = \pi(\varepsilon([\mathfrak{f}])\varphi')\pi\varphi$ shows, as in the proof of [13] 2.14, that $\mathfrak{g}$ has the form $\mathfrak{g} = U_{\mathrm{C}}(G)$ with a complete G: $A \to B$.

4. The categorical complement theorems

The following result is basic for our purposes (see [13] for notation).

4.1 THEOREM. *Let M be an ANR and $X \subset M$ be an $(m+1)$-admissible compactum. Then* i: $U^*(X) \to U(X)$ *is a* θ_m^*-*equivalence in* **H(pro-Top)**.

The proof of 4.1 will be prepared by two technical Lemmas.

4.2. LEMMA. *Let $\tilde{P} = (P; P_n)$ be a filtered CW-complex(i.e. P is a CW-complex and each P_n is a subcomplex of P). For each subcomplex $Q \subset P$, the inclusion* i: $\tilde{P} \cap Q \to \tilde{P}$ *is a filtered cofibration, i.e. each filtered map* F: $\tilde{P} \times I \cap (P \times \{0\} \cup Q \times I) \to \tilde{X}$ *has a filtered extension* F': $\tilde{P} \times I \to \tilde{X}$.

The proof is a slight modification of the classical homotopy extension theorem for CW-complexes and is left to the reader.

4.3. LEMMA. *Let M be an ANR and $X \subset M$ be an m-admissible compactum.*

(a) *There is a cofinal subtower* $N(X) = \{N_r\}$ *of* $U(X)$ *such that* $\pi_k(N_{r+1}, N_{r+1} - X) \to \pi_k(N_r, N_r - X)$ *is trivial for all* $k = 0, \ldots, m+1$ *and all* r.

(b) *Let* $\tilde{P} = (P; P_n)$ *be a filtered CW-complex,* $Q \subset P$ *be a subcomplex with* $\dim(P - Q) \leqslant m + 1$, *and let* $f: \tilde{P} \to \operatorname{Tel} N(X)$ *be a filtered map such that* $f(P) \subset \operatorname{Tel}_{m+2} N(X)$ *and* $f(Q) \subset \operatorname{Tel} N^*(X)$, *where* $N^*(X) = \{N_r - X\}$. *There is a filtered map* $g: \tilde{P} \to \operatorname{Tel} N(X)$ *such that* $g(P) \subset \operatorname{Tel} N^*(X)$ *and* $f \simeq g$ *via a filtered homotopy rel.* Q.

Proof. (a) is trivial.

(b) We construct by induction filtered maps $f^{(i)}: \tilde{P} \to \operatorname{Tel} N(X)$ such that $f^{(i)}(P) \subset \operatorname{Tel}_{m+1-i} N(X)$, $f^{(i)}(P^{(i)} \cup Q) \subset \operatorname{Tel} N^*(X)$ and $f \simeq f^{(i)}$ via a filtered homotopy rel Q (then $g = f^{(m+1)}$ is the required map). The induction starts with $f^{(-1)} = f$. Given $f^{(i)}$, $i \leqslant m$, we have to construct $f^{(i+1)}$. Choose integers $0 = n_0 < n_1 < n_2 \cdots$. such that $f^{(i)}(P_{n_r}) \subset \operatorname{Tel}_{m+1-i+r} N(X)$. Consider an open $(i+1)$-cell σ of $P - Q$ with a characteristic map $\phi: D^{i+1} \to \operatorname{cl}(\sigma) \subset P$. Let $\partial\sigma = \phi(S^i) \subset P^{(i)}$ and $r(\sigma) = \max\{r \mid \sigma \subset P_{n_r}\}$. A homotopy h_σ: $\operatorname{cl}(\sigma) \times I \to \operatorname{Tel} N(X)$ is defined by $h_\sigma(x, t) = (f_1(x), (1 - t) f_2(x) + t(m - i + r(\sigma)))$, where $f^{(i)}(x) = (f_1(x), f_2(x)) \in \operatorname{Tel} N(X) \subset N_0 \times [0, \infty)$. Define $\lambda: D^{i+1} \times I \to D^{i+1} \times I$ by $\lambda(x, t) = (2x/(2 - t), t)$ for $\|x\| \leqslant (2 - t)/2$ and $\lambda(x, t) = (x/\|x\|, 2 - 2\|x\|)$ for $\|x\| \geqslant (2-t)/2$. Let $H = h_\sigma(\phi \times 1)\lambda: D^{i+1} \times I \to \operatorname{Tel}\ N(X)$. Since $H(\phi \times 1)^{-1}$ is single-valued, there is a homotopy H_σ: $\operatorname{cl}(\sigma) \times I \to \operatorname{Tel} N(X)$ with $H = H_\sigma(\phi \times 1)$. Let $D' = \{x \in D^{i+1} \mid \|x\| \leqslant \frac{1}{2}\}$ and $D = \phi(D') \subset \sigma$; D is an $(i+1)$-ball. By construction, H_σ is stationary on $\partial\sigma$ and $H_\sigma(x, 1) \in \operatorname{Tel}^*\ N(X)$ for $x \notin \mathring{D}$. Consider $g = H_{\sigma,1}|D$; this is a map into $N_{m+1-i+r(\sigma)} \times \{m - i + r(\sigma)\} \subset N_{m-i+r(\sigma)} \times \{m - i + r(\sigma)\} \subset \operatorname{Tel} N(X)$. Since $g(\partial D) \subset (N_{m+1-i+r(\sigma)} - X) \times \{m - i + r(\sigma)\}$, g is homotopic rel ∂D (in $N_{m-i+r(\sigma)} \times \{m - i + r(\sigma)\}$) to a map g': $D \to \operatorname{Tel} N(X)$ with $g'(D) \subset (N_{m-i+r(\sigma)} - X) \times \{m - i + r(\sigma)\}$. Combining this homotopy with H_σ, we obtain a homotopy G_σ: $\operatorname{cl}(\sigma) \times I \to \operatorname{Tel} N(X)$ which is stationary on $\partial\sigma$ and satisfies $G_{\sigma,0} = f^{(i)}|\operatorname{cl}(\sigma)$, $G_{\sigma,1}(\operatorname{cl}(\sigma)) \subset \operatorname{Tel} N^*(X)$ and $G_\sigma(\operatorname{cl}(\sigma) \times I) \subset \operatorname{Tel}_{m-i+r(\sigma)} N(X)$. Consider the homotopy G: $(\tilde{P} \cap (P^{(i+1)} \cup Q)) \times I \subset \operatorname{Tel}_{m-i} N(X)$ defined by $G_t|Q = f^{(i)}|Q$ and $G|\operatorname{cl}(\sigma) \times I = G_\sigma$ for σ in $P^{(i+1)} - Q$. Clearly, G is filtered. By 4.2, G extends to a filtered homotopy $H^{(i+1)}$: $\tilde{P} \times I \to \operatorname{Tel} N(X)$ such that $H_0^{(i+1)} = f^{(i)}$ and $H^{(i+1)}(\tilde{P} \times I) \subset \operatorname{Tel}_{m-i} N(X)$. By construction, $f^{(i+1)} = H_1^{(i+1)}$ has the desired properties.

We are now ready to prove 4.1:

Choose a cofinal subtower $N(X) \subset U(X)$ as in 4.3(a). It suffices to show that i: $N^*(X) \to N(X)$ is a θ_m^*-equivalence. Let $\underline{P} = \{P_n, p_n\}$ be a tower of polyhedra P_n, $\dim P_n \leqslant m$, and piecewise-linear bondings p_n. We have to show that i induces a bijection $\mathbf{H(pro\text{-}Top)}(\underline{P}, N^*(X)) \to \mathbf{H(pro\text{-}Top)}(\underline{P}, N(X))$. Since Tel: $\mathbf{H(tow\text{-}Top)} \to \mathbf{HF}_\infty$ is a full embedding, this is equivalent to showing that $\mathbf{HF}_\infty(\operatorname{Tel} \underline{P}, \operatorname{Tel} N^*(X)) \to \mathbf{HF}_\infty(\operatorname{Tel} \underline{P}, \operatorname{Tel} N(X))$ is a bijection. Since $\operatorname{Tel} \underline{P}$ has the structure of a filtered CW-complex of dimension $\leqslant m + 1$, this actually follows from 4.3(b) (note that we can always choose representatives for morphisms resp. homotopies in $\mathbf{F}_\infty$ which satisfy all assumptions in 4.3(b)).

Putting together the pieces collected so far, we obtain

4.4 THEOREM. ($\mathfrak{UNR}$, (Ad_{m+2}), $\mathbf{H}_{m+1}\mathbf{C}_\infty$) *are data of a categorical complement theorem for* $\mathbf{SSh}(\mathfrak{CM}_m)$.

Proof. Let $V(X)$ be the Vietoris system associated to a compactum X (cf. [5] 8.2.7). If $X \subset M$, *M an ANR*, there exists an isomorphism $U(X) \to V(X)$ in $\mathbf{H(pro\text{-}Top)}$ (note that $U(X)$ and $V(X)$ admit cofinal subtowers and apply 3.1). Since $\varepsilon(M - X)$ can be identified with a cofinal subsystem of $U^*(X)$, we infer from 4.1 the following.

(4.5) If $X \subset M$ is $(m + 2)$-admissible, then there exists a θ^*_{m+1}-equivalence $\vartheta_{(M,X)}$: $\varepsilon(M - X) \to V(X)$ in $\mathbf{H(pro\text{-}Top)}$.

Moreover, we clearly have

(4.6) If $FdX \leqslant m$, then $V(X) \in \mathrm{Ob}\, \theta^c_m \subset \mathrm{Ob}\, \theta_{m+1}$.

We can write $\mathbf{SSh}(X, Y) = \mathbf{H(pro\text{-}Top)}(V(X), V(Y))$, i.e. for $X \in \mathfrak{CM}_m$ (cf. [13] 2.4):

(4.7) $\mathbf{SSh}(X, Y) = \mathbf{H(pro\text{-}Top)}/\theta_{m+1}(V(X), V(Y))$.

The proof of 4.4 is now very similar to that of [13] 4.2: Define T: $\mathbf{H}_{m+1}\mathbf{C}_\infty(\mathfrak{CM}_m, \mathfrak{ANR}, (\mathrm{Ad}_{m+2})) \to \mathbf{SSh}(\mathfrak{CM}_m)$ by $T(\alpha) = [\vartheta_{(N,Y)}]\varepsilon_{m+1}(\alpha)[\vartheta_{(M,X)}]^{-1} \in \mathbf{SSh}(X, Y)$ for $\alpha \in \mathbf{H}_{m+1}\mathbf{C}_\infty(M - X, N - Y)$. Here [ ] denotes equivalence class in $\mathbf{H(pro\text{-}Top)}/\theta_{m+1}$. Clearly, T is an equivalence of categories. The following diagram may be useful to illustrate the definition of T.

$$
\begin{array}{ccc}
\mathbf{H}_{m+1}\mathbf{C}_\infty(M - X, N - Y) & \xrightarrow{\quad T \quad} & \mathbf{SSh}(X, Y) \\
\approx \downarrow \varepsilon_{m+1} & & \| \\
\mathbf{H(pro\text{-}Top)}/\theta_{m+1}(\varepsilon(M - X), \varepsilon(N - Y)) & & \mathbf{H(pro\text{-}Top)}/\theta_{m+1}(V(X), V(Y)) \\
\approx \searrow \vartheta_{(N,Y)\#} & & \vartheta^{\#}_{(M,X)} \swarrow \approx \\
& \mathbf{H(pro\text{-}Top)}/\theta_{m+1}(\varepsilon(M - X), V(Y)) &
\end{array}
$$

THEOREM A from the Introduction follows now immediately from 4.4 and 3.10(a).

4.8 REMARK. There is a modification of Theorem A with a slightly weaker embedding condition. Let $\mathfrak{T}(m)$ be the class of telescopes $\mathrm{Tel}(\{P_n, p_n\})$, where $\{P_n, p_n\}$ is a tower of compact polyhedra P_n with $\dim P_n \leqslant m - 1$ and piecewise-linear bondings p_n. We now obtain the *telescope m-homotopy categories* $\mathbf{H}_{\mathfrak{T}(m)}\mathbf{C}_\infty$ and $\mathbf{H}_{\mathfrak{T}(m)}\mathbf{C}$ (cf. §2). It is fairly obvious that ε: $\mathbf{HC}_\infty \to \mathbf{H(pro\text{-}Top)}$ induces a full embedding $\varepsilon_{\mathfrak{T}(m)}$: $\mathbf{H}_{\mathfrak{T}(m)}\mathbf{C}_\infty \to \mathbf{H(pro\text{-}Top)}/\theta^c_m$. Adapting the proof of 4.4, we see that ($\mathfrak{ANR}$, (Ad_{m+1}), $\mathbf{H}_{\mathfrak{T}(m+1)}\mathbf{C}_\infty$) are data of a categorical complement theorem for $\mathbf{SSh}(\mathfrak{CM}_m)$. Using a suitable version of 3.10, we infer moreover that $(\mathfrak{UNR}_{m+1}, (\mathrm{Ad}_{m+1}), \mathbf{H}_{\mathfrak{T}(m+1)}\mathbf{C})$ are data of a categorical complement theorem for $\mathbf{SSh}(\mathfrak{CM}_m)$.

Theorem A yields various corollaries (cf. §4 of [13]), for example

4.9 COROLLARY. *Let M be an $(m+1)$-connected ANR with a complete uniform structure. Then the strong shape category of $(m+2)$-admissible compacta $X \subset M$ with $FdX \leqslant m$ is isomorphic to the complete $(m+1)$-homotopy category of their complements $M - X$.*

4.10 COROLLARY. *Let M be an r-connected piecewise-linear manifold with a complete uniform structure such that $r \geqslant 0$ and $\dim M \geqslant 2$. Let $k \in \{0, \ldots, \min(r, d(M))\}$, where $d(M) = \max\{s \in \mathbb{N} \mid 2s + 2 \leqslant \dim M\}$, and let $m \in \{k, \ldots, \min(r, \dim M - 2 - k)\}$. Then the strong shape category of ILC compacta X in the interior of M with $FdX \leqslant k - 1$ is isomorphic to the complete m-homotopy category of their complements $M - X$. If $r \geqslant d(M)$, one can always choose $k = m = d(M)$.*

4.11 REMARK. If M is *compact*, one can replace the complete m-homotopy in the above two results by the *proper m-homotopy category.*

The rest of this section is devoted to compacta of *stable shape*, i.e. of the shape of a (not necessarily compact) polyhedron.

4.12 THEOREM. *Let $\mathfrak{CM}_m^{st}$ be the class of compacta $X \in \mathfrak{CM}_m$ which have stable shape. The following are data of a categorical complement theorem for $\mathbf{SSh}(\mathfrak{CM}_m^{st})$:*

(a) $(\mathfrak{ANR}, (\mathrm{Ad}_{m+1}), \mathbf{H}_m\mathbf{C}_\infty)$
(b) $(\mathfrak{ANR}_m, (\mathrm{Ad}_{m+1}), \mathbf{H}_m\mathbf{C})$

4.13 REMARK. Recall that the canonical functor $\mathbf{SSh}(\mathfrak{CM}_m^{st}) \to \mathbf{Sh}(\mathfrak{CM}_m^{st})$ is a category isomorphism.

Proof of 4.12. (a) By 3.9, we can define an equivalence of categories T: $\mathbf{H}_m\mathbf{C}_\infty(\mathfrak{CM}_m^{st}, \mathfrak{ANR}, (\mathrm{Ad}_{m+1})) \to \mathbf{SSh}(\mathfrak{CM}_m^{st})$ by $T(\alpha) = [\vartheta_{(N,Y)}]\varepsilon'_m(\alpha)[\vartheta_{(M,X)}]^{-1}$ for $\alpha \in \mathbf{H}_m\mathbf{C}_\infty(M - X, N - Y)$. Here, [ ] denotes equivalence class in $\mathbf{H}(\mathbf{pro\text{-}Top})/\theta_m^c$.

(b) This follows from (a).

Let us now say that a compactum X in an ANR M *satisfies the embedding condition* (Ad_m^{st}), if $X \subset M$ is m-admissible and $\pi\varepsilon(M - X)$ is stable in **pro-HTop**.

4.14 EXAMPLE. Let X be an ILC compactum in the interior of a piecewise-linear manifold M. If X is a subpolyhedron of M, or if $\dim M \geqslant 5$ and X has the shape of a compact polyhedron P with $\dim P \leqslant \dim M - 3$, then $X \subset M$ satisfies $(\mathrm{Ad}_{\dim M - 2 - FdX}^{st})$; see [16] Theorem 5.6. Notice that $X \subset M$ (in general) does not satisfy $(\mathrm{Ad}_{\dim M - 1 - FdX})$.

4.15 LEMMA. *Let $X \subset M$ satisfy* (Ad_m^{st}). *If* $X \in \mathfrak{CM}_m$, *then* $X \in \mathfrak{CM}_m^{st}$.

Proof. $\pi\varepsilon(M - X) \to U(X)$ is an Ω_m^*-equivalence in **pro-HTop** (cf. [13]). Hence, $U(X)$ is dominated in **pro-HTop** by the stable object $\pi\varepsilon(M - X)$ and a fortiori by a single space Z. Choose morphisms $\underline{u}: U(X) \to Z$ and $\underline{d}: Z \to U(X)$ such that $\underline{d}\underline{u} = 1$. Obviously, $\underline{u}$ factors through an *ANR* (the open neighbourhoods of X form a cofinal subsystem of $U(X)$). Hence, X is shape dominated by an ANR, and therefore X has stable shape.

4.16 THEOREM. *The following are data of a categorical complement theorem for* $\mathbf{SSh}(\mathfrak{CM}_m^{st})$:

(a) $(\mathfrak{ANR}, (\mathrm{Ad}_m^{st}), \mathbf{H}_m\mathbf{C}_\infty)$
(b) $(\mathfrak{ANR}_m, (\mathrm{Ad}_m^{st}), \mathbf{H}_m\mathbf{C})$

Proof. We have shown in [13] that $(\mathfrak{ANR}, (\mathrm{Ad}_m), \mathbf{wH}_m\mathbf{C}_\infty)$ and $(\mathfrak{ANR}_m, (\mathrm{Ad}_m), \mathbf{wH}_m\mathbf{C})$ are data of a categorical complement theorem for $\mathbf{Sh}(\mathfrak{CM}_m)$. It therefore suffices to show: If $X \subset M$, $Y \subset N$ and $\pi\varepsilon(N - Y)$ is stable, then

(a) $\mathbf{H}_m\mathbf{C}_\infty(M - X, N - Y) \to \mathbf{wH}_m\mathbf{C}_\infty(M - X, N - Y)$ is a bijection.
(b) $\mathbf{H}_m\mathbf{C}(M - X, N - Y) \to \mathbf{wH}_m\mathbf{C}(M - X, N - Y)$ is a bijection for m-connected $N - Y$.

It is clear that both maps are surjective since homotopy implies weak homotopy. Let $\mathfrak{f}_0, \mathfrak{f}_1 \in \mathbf{C}_\infty(M - X, N - Y)$ represent the same morphism in $\mathbf{wH}_m\mathbf{C}_\infty$, and let $\mathfrak{g} \in \mathbf{C}_\infty(P, M - X)$, where P is a σ-compact polyhedron of dimension $\leqslant m$. Then $\mathfrak{f}_0\mathfrak{g}$ and $\mathfrak{f}_1\mathfrak{g}$ represent the same morphism in $\mathbf{wHC}_\infty$. Hence $\pi\varepsilon([\mathfrak{f}_0\mathfrak{g}]) = \pi\varepsilon([\mathfrak{f}_0\mathfrak{g}])$, where [ ] denotes equivalence class in $\mathbf{HC}_\infty$ (cf. [13] 2.2). Since $\pi\varepsilon(N - Y)$ is stable, $\varepsilon([\mathfrak{f}_0\mathfrak{g}]) = \varepsilon([\mathfrak{f}_1\mathfrak{g}])$. This means $[\mathfrak{f}_0\mathfrak{g}] = [\mathfrak{f}_1\mathfrak{g}]$. Therefore, $\mathfrak{f}_0$, $\mathfrak{f}_1$ represent the same morphism in $\mathbf{H}_m\mathbf{C}_\infty$, which proves (a). Now let $f_0, f_1 \in \mathbf{C}(M - X, N - Y)$ represent the same morphism in $\mathbf{wH}_m\mathbf{C}$, and let $g \in \mathbf{C}(P, M - X)$, where P is as above. By (a), $[U_{\mathbf{C}}(f_0 g)] = [U_{\mathbf{C}}(f_0 g)]$. Then the argument of [13] 2.14 shows that $f_0 g, f_1 g$ are homotopic in $\mathbf{C}$. Thus, f_0, f_1 represent the same morphism in $\mathbf{H}_m\mathbf{C}$, which proves (b).

5. The Duality Theorem

In this section we prove Theorem B of the Introduction. In addition to our categorical complement theorems we shall need the following two ingredients.

5.1 THEOREM (Strong Shape S-duality Theorem of Q. Haxhibeqiri and S. Nowak [6]). *Let* $\mathbf{Stab\text{-}SSh}_n$ *be the full subcategory of the stable strong shape category* **Stab-SSh** *having as objects all compacta* $X \subset S^n$, *and let* $\mathbf{Stab\text{-}HTop}_n$ *be the full subcategory of the stable homotopy category* **Stab-HTop** having as objects *all complements* $S^n - X$ *of compacta* $X \subset S^n$. *Then there exist a contravariant category isomorphism* $D_n: \mathbf{Stab\text{-}SSh}_n \to \mathbf{Stab\text{-}HTop}_n$ *with* $D_n(X) = S^n - X$ *for all objects* X.

5.2 THEOREM (Strong Shape Suspension Theorem). *Let X be a compactum of fundamental dimension $FdX = m$, and let Y be an r-shape connected compactum. Then the suspension Σ: $\mathbf{SSh}(X, Y) \to \mathbf{SSh}(\Sigma X, \Sigma Y)$ is a surjection provided $m \leqslant 2r$ and a bijection provided $m \leqslant 2r - 1$.*

5.3 REMARK. Let **CM** be the category of compacta (=compact metrizable spaces) and continuous maps, and let S be the class of *strong shape equivalences* in **CM**. Then the strong shape category **SSh** is the quotient category $\mathbf{CM}\backslash S$ (see [2], [3]). We let $q\colon \mathbf{CM} \to \mathbf{SSh}$ denote the quotient functor. The suspension functor Σ on **CM** has the property $\Sigma(S) \subset S$; hence there is a unique functor $\Sigma = \Sigma_{\mathbf{SSh}}\colon \mathbf{SSh} \to \mathbf{SSh}$ with $\Sigma_{\mathbf{SSh}} q = q\Sigma$, called the *strong shape suspension.* Theorem 5.2 is the strong shape analogue of the classical suspension theorem in homotopy theory (see e.g. [18], [19]). We remark that the corresponding result in the Borsuk-Mardešić shape category **Sh** says that $\Sigma\colon \mathbf{Sh}(X, Y) \to \mathbf{Sh}(\Sigma X, \Sigma Y)$ is a surjection provided $m \leqslant 2r + 1$ and a bijection provided $m \leqslant 2r$ (this was essentially established by S. Nowak in [14]). A proof of Theorem 5.2 can be based on Yu. T. Lisica's description of **SSh** via the coherent homotopy category of towers (cf. [8]), applying the classical suspension theorem to maps *and* homotopies. Details are given in the Appendix.

The proof of Theorem B is now straightforward: Let $X, Y \subset S^n$ be ILC compacta such that $FdX, FdY \leqslant d(n) - 1$ and X, Y are $c(n)$-shape-connected. Using Theorem 5.2, Theorem 5.1 and the classical Suspension Theorem (noticing that $S^n - Y$ is $(n - FdY - 2)$-connected; cf. [13] 3.9), we obtain a bijection

$$D\colon \mathbf{SSh}(Y, X) \approx \textbf{Stab-SSh}(Y, X) \approx \textbf{Stab-HTop}(S^n - X, S^n - Y)$$
$$\approx \mathbf{HTop}(S^n - X, S^n - Y).$$

Finally, the category isomorphism theorem 4.10 yields a bijection $R\colon \mathbf{SSh}(Y, X) \approx \mathbf{H}_{d(n)}\mathbf{P}(S^n - Y, S^n - X)$. Now set $\Delta = RD^{-1}$; this yields the desired functor $\Delta\colon \mathbf{T}_n \to \mathbf{H}_{d(n)}\mathbf{P}$.

In the polyhedral case, there is a modification of Theorem B with a slightly improved connectivity condition.

5.4 THEOREM. *Let $c'(n) = \max\{k \mid 4k + 2 \leqslant n\}$, and let $\mathbf{T}'_n$ be the full subcategory of* **HTop** *having as objects all complements $S^n - X$ of $c'(n)$-connected piecewise-linear embedded compact polyhedra $X \subset S^n$ with $\dim X \leqslant d(n) - 1$. Then there is a contravariant full embedding*

$$\Delta'\colon \mathbf{T}'_n \to \mathbf{H}_{d(n)}\mathbf{P}$$

such that $\Delta'(S^n - X) = S^n - X$ for each object $S^n - X$.

Proof. There are bijections $D'\colon \mathbf{HTop}(Y, X) \approx \mathbf{HTop}(S^n - X, S^n - Y)$ and $R'\colon \mathbf{HTop}(Y, X) \approx \mathbf{H}_{d(n)}\mathbf{P}(S^n - Y, S^n - X)$.

Let us say that a map $f\colon A \to B$ is *homotopically associated in degree m* to a proper map $g\colon A \to B, f \sim_m g$, if the following holds: For each closed $A' \subset A$ such that $f|A'$ is proper, $f|A' \simeq_m g|A'$. Here $\simeq_m$ denotes proper m-homotopy.

The functor Δ' has the following remarkable property.

5.5 THEOREM. *For each* $\alpha \in \mathbf{T}'_n(S^n - X, S^n - Y)$, *there exist representatives* f: $S^n - X \to S^n - Y$ *of* α *and* $f^*\colon S^n - Y \to S^n - X$ *of* $\Delta'(\alpha) \in \mathbf{H}_{d(n)}\mathbf{P}(S^n - Y, S^n - X)$ *such that*

(1) ff^* *is a proper map with* $ff^* \simeq_{d(n)} 1$;
(2) $f^*f \sim_{d(n)} 1$.

5.6 REMARK. Since f^* is proper, $f^*(S^n - Y)$ is closed in $S^n - X$. It follows then from (1) that $f|f^*(S^n - Y)$ must be proper, which sheds some more light on (2). We may regard f and f^* as '*inverse to each other*' in a very weak sense; however, we emphasize that this strongly depends on the 'correct' choice of representatives $f \in \alpha, f^* \in \Delta'(\alpha)$.

Proof of 5.5. Let us first observe that we can extend the functor Δ' to the full subcategory $\mathbf{T}''_n$ of **HTop** having as objects all complements $S^n - W$ of $c'(n)$-connected *PL* embedded compact polyhedra $W \subset S^n$ with $FdW \leqslant d(n) - 1$ and $\dim W \leqslant n - 3$. We now consider $\beta \in \mathbf{HTop}(Y, X)$ and construct adequate representatives f of $D'(\beta)$ and f^* of $R'(\beta)$; see the proof of 5.4. Let $\beta = [g]$ with a PL map $g\colon Y \to X$, and let Z be the PL mapping cylinder of g. Clearly, the problem is unchanged if we move X and Y to other places by PL homeomorphisms of S^n. In particular, by the dimension hypothesis $\dim X$, $\dim Y \leqslant d(n) - 1$, we may assume that $X, Y \subset Z \subset S^{n-1} \subset S^n$, and that there is a collapsing map $r\colon Z \to X$ with $g = ri$ (where $i\colon Y \to Z$ is the inclusion). Note that Z is $c'(n)$-connected with $FdZ \leqslant d(n) - 1$ and $\dim Z \leqslant n - 3$.

Step 1. Clearly $D'([i])$ is represented by the inclusion $i^{\#}\colon S^n - Z \to S^n - Y$. Define $\lambda\colon S^n \to S^n_+ = \{(x_1, \ldots, x_{n+1}) \in S^n | x_{n+1} \geqslant 0\}$ by $\lambda(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, |x_{n+1}|)$. Let $\pi\colon S^n_+ \to D^n = \{(x_1, \ldots, x_n, 0) \in \mathbb{R}^{n+1} | \Sigma x_i^2 \leqslant 1\}$ be the homeomorphism $\pi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n, 0)$. Choose $\omega\colon D^n \to I$ such that $\omega^{-1}(1) = Y$ and define $\chi_Y\colon D^n \to D^n$ by $\chi_Y(x) = \omega(x)x$. Finally, define $i^*\colon S^n - Y \to S^n - Z$ by $i^*(x) = \pi^{-1}\chi_Y\pi\lambda(x)$. This is a proper map which can be pieced together with $i\colon Y \to Z$ to a continuous $F\colon S^n \to S^n$. But now [13] 4.9 and the proof of 4.16 above imply that $R'([i])$ is represented by i^*. By construction, $i^{\#}i^*$ is a proper map which can be pieced together with 1_Y to the above map F. Hence $R'([1_Y])$ is represented by $i^{\#}i^*$, i.e. $i^{\#}i^* \simeq_{d(n)} 1_{S^n - Y}$. Finally, let $\lambda_X\colon S^n - X \to S^n - X$, $\lambda_X(x) = \lambda(x)$. This is a proper map with $\lambda_X \simeq_{d(n)} 1$ (repeat the arguments above). It is easy to see that $i^*i^{\#} \sim_{d(n)} \lambda_X$ which implies $i^*i^{\#} \sim_{d(n)} 1$.

Step 2. We have shown in [11] that there exists a map $H\colon D^{n+1} \to D^{n+1}$ such that H carries $D^{n+1} - Z$ homeomorphically onto $D^{n+1} - X$, H restricts to a retraction $r': Z \to X$, and $H = \mathrm{id}$ outside of a regular neighbourhood N' of Z in D^{n+1}.

Let $h: S^n - Z \to S^n - X$ be the homeomorphism induced by H. Then $R'([r])$ is represented by h (note $[r] = [r']$ and argue as in Step 1). By construction, h is the identity outside a regular neighbourhood N of Z in S^n. Hence, h is homotopic to the inclusion $j: S^n - Z \to S^n - X$ (recall that $N - Z \approx \partial N \times (0, 1] \approx N - X$). We conclude that $D'([r])$ is represented by $h^{-1}: S^n - X \to S^n - Z$: Let $i': X \to Z$ be the inclusion, and let $\{\ \}$ denote stable homotopy classes. Then $\{[r]\} = \{[i']\}^{-1}$, thus $\{D'([r])\} = \{D'([i'])\}^{-1} = \{[j]\}^{-1} = \{[h]\}^{-1} = \{[h^{-1}]\}$.

Step 3. Set $f = i^{\#} h^{-1}$ and $f^* = hi^*$. Then $ff^* = i^{\#} i^*$ is a proper map such that $ff^* \cong_{d(n)} 1$. Moreover, $f^* f = hi^* i^{\#} h^{-1}$ is clearly homotopically associated in degree $d(n)$ to $h1h^{-1} = 1$. By construction, $[f] = [i^{\#}][h^{-1}] = D'([i])D'([r]) = D'([g])$ and $[f^*]_{d(n)} = [h]_{d(n)}[i^*]_{d(n)} = R'([r])R'([i]) = R'([g])$.

Appendix. Proof of the Strong Shape Suspension Theorem

We begin with some notation. Let $\partial I = \{0, 1\}$ be the boundary of $I = [0, 1]$. For each homotopy $H: A \times I \to B$, we define $H_t: A \to B$ by $H_t(x) = H(x, t)$. If f_0, $f_1: A \to B$ are maps and $A' \subset A$, the notation $H: f_0 \simeq f_1 \operatorname{rel} A'$ indicates that $H: X \times I \to Y$ is a homotopy such that $H|_{A' \times I}$ is stationary and $H_k = f_k$, $k = 0, 1$ (if $A' = \emptyset$, we write $H: f_0 \simeq f_1$). For homotopies $H^0, H^1: A \times I \to B$ we prefer to write $\theta: H^0 \cong H^1$ instead of $\theta: H^0 \simeq H^1 \operatorname{rel} A \dot{\times} \partial I$ (θ is then a *homotopy of homotopies*).

For each pointed space (A, a), let $p = p_{(A,a)}: A \times I \to A \times I/_{\{a\} \times I}$ denote the projection map, written as $p(x, t) = [x, t]$, and let $(A, a) \otimes I$ denote the pointed space $(A \times I/_{\{a\} \times I}, *)$ with $* = p(\{a\} \times I)$. By $(A, a) \otimes \partial I$ we mean the pointed subspace $(p(A \times \partial I), *)$. For each $t \in I$, let i_t: $(A, a) \to (A, a) \otimes I$ be the pointed map $i_t(x) = [x, t]$. If f_0, f_1: $(A, a) \to (B, b)$ are pointed maps, we also write $H: f_0 \simeq_* f_1$ instead of $H: f_0 \simeq f_1 \operatorname{rel}\{a\}$; H is then a *pointed homotopy*. Moreover, for two such pointed homotopies H^0, H^1 we also write $\theta: H^0 \cong_* H^1$ instead of $\theta: H^0 \simeq H^1 \operatorname{rel} A \times \partial I \cup \{a\} \times I$. Given any pointed homotopy H, we let $\hat{H}: (A, a) \otimes I \to (B, b)$ denote the unique pointed map satisfying $\hat{H}p = H$. We let $q = q_{(A,a)}$: $A \times I \to A \times I/_{\{a\} \times I \cup A \times \partial I}$ denote the projection map, written as $q(x, t) = \langle x, t\rangle$, and let $S(A, a) = (A \times I/_{\{a\} \times I \cup A \times \partial I}, *)$ denote the *reduced suspension* of (A, a), where $* = q(\{a\} \times I \cup A \times \partial I)$. It is easy to see that $\mu = \mu_{(A,a)}$: $S(A, a) \otimes I \to S((A, a) \otimes I)$, $\mu([\langle x, s\rangle, t]) = \langle [x, t], s\rangle$, defines a natural homeomorphism. Then for each pointed map G: $S((A, a) \otimes I) \to (B, b)$, the map $G^{\#} = G\mu p$: $S(A, a) \times I \to B$ is a pointed homotopy from $GS(i_0)$ to $GS(i_1)$.

Recall that the reduced suspension S and the loop space functor Ω are adjoint: In fact, the exponential law yields a natural bijection β: $\mathrm{Top}_0(S(A, a), (B, b)) \to \mathrm{Top}_0((A, a), \Omega(B, b))$ between sets of pointed maps. For $(B, b) = S(A, a)$ we obtain the map $\rho = \rho_{(A,a)} = \beta(1)$: $(A, a) \to \Omega S(A, a)$ which is a $(2r + 1)$-equivalence provided (A, a) is an r-connected pointed CW-complex; see [18] or [19].

A.1 PROPOSITION. *Let (B, b) be an r-connected pointed CW-complex, (A, a) be a pointed CW-complex, and f_0, f_1: $(A, a) \to (B, b)$ be pointed maps.*

(1) *Let* $\dim A \leqslant 2r$ *and* H: $S(f_0) \simeq_* S(f_1)$. *Then there exists* G:$f_0 \simeq_* f_1$ *such that* $S(\hat{G})^{\#} \cong_* H$.
(2) *Let* $\dim A \leqslant 2r - 1$ *and* G^0, G^1: $f_0 \simeq_* f_1$. *If* $S(\hat{G}^0)^{\#} \cong_* S(\hat{G}^1)^{\#}$, *then* $G^0 \cong_* G^1$.

Proof. (1) Define f: $(A, a) \otimes \partial I \to (B, b)$ by $f([x, k]) = f_k(x)$. One verifies $\rho_{(B,b)}f = \beta(\hat{H}\mu^{-1}{}_{(A,a)})|_{(A,a)\otimes\partial I}$. Since $\rho_{(B,b)}$ is a $(2r + 1)$-equivalence, there is a map g: $(A, a) \otimes I \to B$ such that $g|_{(A,a)\otimes\partial I} = f$ and $\rho_{(B,b)}g \simeq \beta(\hat{H}\mu^{-1}{}_{(A,a)})$ rel $(A, a) \otimes \partial I$ (cf. [18] p. 404). Let θ: $((A, a) \otimes I) \times I \to \Omega S(B, b)$ be such a homotopy; in particular, θ is a pointed homotopy. One readily verifies $(\beta^{-1}(\theta))^{\#}$: $\beta^{-1}(\rho_{(B,b)}g) \simeq \beta^{-1}\beta(\hat{H}\mu^{-1}_{(A,a)})$ rel $S((A, a) \otimes \partial I)$. Since $\beta^{-1}(\rho_{(B,b)}g) = S(g)$, we have found Γ: $S(g) \simeq \hat{H}\mu^{-1}_{(A,a)}$ rel $S((A, a) \otimes \partial I)$. Set $G = gp$: $A \times I \to B$ and $\Gamma^+ = \Gamma(\mu_{(A,a)}p_{(A,a)} \times 1_I)$: $S(A, a) \times I \times I \to S(B, b)$. Then G:$f_0 \simeq_* f_1$ and Γ^+: $S(\hat{G})^{\#} = S(g)^{\#} \cong_* H$.

(2) $S(\hat{G}^k)^{\#}$, $k = 0, 1$, are pointed homotopies from $S(f_0)$ to $S(f_1)$. Let θ: $S(A, a) \times I \times I \to B$, θ: $S(\hat{G}^0)^{\#} \cong_* S(\hat{G}^1)^{\#}$. There is a unique map ϑ: $(S(A, a) \otimes I) \times I \to B$ satisfying $\vartheta(\mu_{(A,a)}p_{(A,a)} \times 1_I) = \theta$, and one easily verifies that ϑ: $S(\hat{G}^0) \simeq S(\hat{G}^1)$ rel $S((A, a) \otimes \partial I)$. In particular, ϑ is a pointed homotopy. Define $Z = (A, a) \otimes I \otimes \partial I \cup (A, a) \otimes \partial I \otimes I$ and define f: $Z \to B$ by $f([[x, t'], k]) = G^k(x, t')$, $f([[x, k], t]) = f_k(x)$. Straightforward computations show that $\rho_{(B,b)}f = \beta(\hat{\vartheta}\mu^{-1}_{(A,a)\otimes I})|_Z$. Since $\rho_{(B,b)}$ is a $(2r + 1)$-equivalence, there exists a map g: $(A, a) \otimes I \otimes I \to B$ such that $g|_Z = f$. Set $\psi = g(p_{(A,a)} \times 1_I)p_{(A,a)\otimes I}$: $A \times I \times I \to B$. Then one can check ψ: $G^0 \cong_* G^1$.

We shall now apply A.1 to the unpointed setting. For each space A, let ΣA denote its *unreduced suspension* and $q' = q'_A$: $A \times I \to \Sigma A$ the canonical quotient map, usually written as $q'(x\, s) = \langle x, s\rangle'$. We adopt the convention $\Sigma\varnothing = S^0$. Given a homotopy H: $A \times I \to B$, we define ΣH: $(\Sigma A) \times I \to \Sigma B$ by $\Sigma H(\langle x, s\rangle', t) = \langle H(x, t), s\rangle'$. Note that H:$f_0 \simeq f_1$ implies ΣH: $\Sigma f_0 \simeq \Sigma f_1$. Moreover, if we have pointed maps f_0, f_1: $(A, a) \to (B, b)$ and a pointed homotopy H: $f_0 \simeq_* f_1$ then $\pi_{(B,b)}\Sigma H = S(\hat{H})^{\#}(\pi_{(A,a)} \times 1_I)$, where $\pi = \pi_{(C,c)}$: $\Sigma C \to S(C, c)$ denotes the obvious quotient map, $C = A, B$.

A.2 THEOREM. *Let B be an r-connected CW-complex, A be a CW-complex, and f_0, f_1: $A \to B$.*

(1) *Let* $\dim A \leqslant 2r$ *and* H: $\Sigma f_0 \simeq \Sigma f_1$. *Then there exists* $G: f_0 \simeq f_1$ *such that* $\Sigma G \cong H$.

(2) *Let* $\dim A \leqslant 2r - 1$ *and* G^0, G^1: $f_0 \simeq f_1$. *If* $\Sigma G^0 \cong \Sigma G^1$, *then* $G^0 \cong G^1$.

Proof. The case $A = \varnothing$ is trivial. For $A \neq \varnothing$, let us fix basepoints $a \in A$ and $b \in B$.

(1) *Step 1.* Assume that f_0, f_1: $(A, a) \to (B, b)$ are pointed maps and $H(\langle a, s\rangle', t) = \langle b, s\rangle'$ for all s, t.

Then H induces a pointed homotopy H': $S(A, a) \times I \to S(B, b)$, $H': S(f_0) \simeq_* S(f_1)$, characterized by $H'(\pi_{(A,a)} \times 1_I) = \pi_{(B,b)}H$. By A.1, there is $G: f_0 \simeq_* f_1$ with $S(G)^{\#} \cong_* H'$. We shall show $\Sigma G \cong H$. Let $\theta: S(A, a) \times I \times I \to S(B, b)$, $\theta: S(\hat{G})^{\#} \cong_* H'$. Define $Z = \Sigma X \times I \times \partial I \cup \Sigma X \times \partial I \times I$ and $F: Z \to \Sigma B$ by $F(\xi, t', 0) = (\Sigma G)(\xi, t')$, $F(\xi, t', 1) = H(\xi, t')$ and $F(\xi, k, t) = (\Sigma f_k)(\xi)$. One checks $\pi_{(B,b)}F = \theta(\pi_{(A,a)} \times 1_{I \times I})|_Z$. Since $\pi_{(B,b)}$ is a homotopy equivalence, there is $\psi: \Sigma A \times I \times I \to \Sigma B$ with $\psi|_Z = F$. It is obvious that $\psi: \Sigma G \cong H$.

Step 2. Assume that f_0, f_1: $(A, a) \to (B, b)$ are pointed maps.

Then, since the inclusion $\Sigma A \times \partial I \cup \Sigma A \times I \to \Sigma A \times I$ is a cofibration, and ΣB is 1-connected, one can use a standard homotopy extension argument to find a homotopy H': $\Sigma f_0 \simeq \Sigma f_1$ such that $H' \cong H$ and $H'(\langle a, s\rangle', t) = \langle b, s\rangle'$. Now apply Step 1.

Step 3. General case.

Since $a \in A$ is nondegenerate and B is path-connected, f_0, f_1 are homotopic to pointed maps f'_0, f'_1: $(A, a) \to (B, b)$. This reduces the problem to Step 2: Let $h^k: f_k \simeq f'_k$ and set $H' = (\Sigma h^1) \circ H \circ (\Sigma h^0)^{-1}$, where '$\circ$' denotes juxtaposition of homotopies (starting from the right side) and '$^{-1}$' denotes the inverse homotopy. Then $H': \Sigma f'_0 \simeq \Sigma f'_1$, and we find $G': f'_0 \simeq f'_1$ with $\Sigma G' \cong H'$. The desired homotopy G is defined by $G = (h^1)^{-1} \circ G' \circ h^0$.

(2) This is proved by similar arguments and left to the reader (the case $r = 0$ is trivial; for $r \geqslant 1$ the first step is to assume that f_0, f_1: $(A, a) \to (B, b)$ are pointed maps, G^0, G^1: $f_0 \simeq_* f_1$ and $\theta: \Sigma G^0 \cong \Sigma G^1$ with $\theta(\langle a, s\rangle', t', t) = \langle b, s\rangle$ for all s, t', t).

We now come to the proof of the Strong Shape Suspension Theorem. We first need explicit descriptions of **SSh** and $\Sigma_{\mathbf{SSh}}$. Σ extends in an obvious way to a functor Σ: **pro-Top** $\to$ **pro-Top**; if f is a level homotopy equivalence in **pro-Top**, then so is Σf. Hence, Σ induces a functor Σ: **H(pro-Top)** $\to$ **H(pro-Top)**. The Vietoris functor V: **CM** $\to$ **pro-Top** (cf. [5] §8, [10] III §9) induces a full embedding V: **SSh** $\to$ **H(pro-Top)** with the property $V\Sigma = \Sigma V$ (we identify $V(\Sigma X)$ and $\Sigma V(X)$; cf. [5]). Now, a category **K** is defined as follows. The objects are triples $(X, \underline{X}, \underline{p})$, where X is a compactum, $\underline{X}$ is an inverse system of spaces and $\underline{p}$: $V(X) \to \underline{X}$ is an isomorphism in **H(pro-Top)**. The morphisms are defined by $\mathbf{K}(X, \underline{X}, \underline{p}), (X', \underline{X}'\underline{p}) = \mathbf{H(pro\text{-}Top)}(\underline{X}, \underline{X}')$. It is obvious that the suspension functor on **H(pro-Top)** induces a suspension functor $\Sigma: \mathbf{K} \to \mathbf{K}$ (for the objects $\Sigma(X, \underline{X}, \underline{p}) = (\Sigma X, \Sigma \underline{X}, \Sigma \underline{p})$). A functor $\phi: \mathbf{K} \to \mathbf{SSh}$ is defined as follows. For the objects $\phi(X, \underline{X}, \underline{p}) = X$; for the morphisms $\underline{f} \in \mathbf{K}((X, \underline{X}, \underline{p}), (X', \underline{X}', \underline{p}'))$, $\phi(\underline{f}) = V^{-1}((\underline{p}')^{-1}\underline{f}\underline{p})$. Obviously ϕ is an equivalence of categories which satisfies $\Sigma\phi = \phi\Sigma$.

For $-1 \leqslant r, m \leqslant \infty$ let $T(r, m)$ be the class of towers $\underline{X} = \{X_n\}$ of compact r-connected polyhedra X_n with $\dim X_n \leqslant m$.

Given a compactum X with $FdX = m$ and an r-shape-connected compactum Y, there exist $\underline{X} \in T(-1, m)$ and $\underline{Y} \in T(r, \infty)$ admitting isomorphisms $\underline{p}$: $V(X) \to \underline{X}$ and $\underline{q}$: $V(Y) \to \underline{Y}$ in **H(pro-Top)**. By the above considerations, the Strong Shape Suspension Theorem follows from

(A.3) Let $\underline{X} \in T(-1, m)$ and $\underline{Y} \in T(r, \infty)$. Then Σ: **H(tow-Top)**$(\underline{X}, \underline{Y}) \to$ **H(tow-Top)**$(\Sigma\underline{X}, \Sigma\underline{Y})$ is a surjection if $m \leqslant 2r$ and a bijection if $m \leqslant 2r - 1$.

To prove (A.3), we shall employ Lisica's description [8] of **H(tow-Top)** via the coherent homotopy category of towers, **Coh**, which is defined as follows. Objects are all towers $\underline{X} = \{X_n\}$ of spaces. A pre-morphism $\underline{f}$: $\underline{X} \to \underline{Y}$ consists of a strictly increasing index function φ: $\mathbb{N} \to \mathbb{N}$, maps f_n: $X_{\varphi(n)} \to Y_n$ and homotopies h_n: $X_{\varphi(n+1)} \times I \to Y_n$, h_n: $\mathrm{bond} f_{n+1} \simeq f_n \mathrm{bond}$. Pre-morphisms $\underline{f} = \{\varphi, f_n, h_n\}$ and $\underline{f}' = \{\varphi', f'_n, h'_n\}$ are homotopic if there exists a pre-morphism $\underline{H} = \{\chi, F_n, H_n\}$: $\underline{X} \times I \to \underline{Y}$ such that $\chi \geqslant \varphi$, φ', F_n: f_n bond $\simeq f'_n$ bond and $H_n(x, 0, t) = h_n(\mathrm{bond}(x), t)$, $H_n(x, 1, t) = h'_n(\mathrm{bond}(x), t)$. A morphism of **Coh** is then a homotopy class of pre-morphisms; composition comes from the obvious composition of pre-morphisms (see [8] for details). There is an obvious suspension functor Σ: **Coh** $\to$ **Coh**: $\Sigma\underline{X} = \{\Sigma X_n\}$, $\Sigma\underline{f} = \{\varphi, \Sigma f_n, \Sigma h_n\}$ for a pre-morphism $\underline{f}$. Moreover, the canonical functor λ: **tow-Top** $\to$ **Coh** given by $\lambda(\underline{X}) = \underline{X}$ and $\lambda(\{\varphi, f_n\}) = \{\varphi, f_n, \text{stationary homotopy}\}$, is readily seen to induce a category isomorphism λ: **H(tow-Top)** $\to$ **Coh** such that $\lambda\Sigma = \Sigma\lambda$. Note also that the *strong shape theories* based on **H(tow-Top)** (cf. [5]) and on **Coh** (cf. [8]) are already known to be equivalent by [9], Theorems 2 and 3. We have now seen that (A.3) is equivalent to

(A.4) Let $\underline{X} \in T(-1, m)$ and $\underline{Y} \in T(r, \infty)$. Then Σ: **Coh**$(\underline{X}, \underline{Y}) \to$ **Coh**$(\Sigma\underline{X}, \Sigma\underline{Y})$ is a surjection if $m \leqslant 2r$ and a bijection if $m \leqslant 2r - 1$.

In the proof of (A.4) we need the following easily established fact.

(A.5) Given a pre-morphism $\underline{f} = \{\varphi, f_n, h_n\}$: $\underline{X} \to \underline{Y}$ and homotopies Φ_n: $X_{\varphi(n)} \times I \to Y_n$, Φ_n: $f_n \simeq f'_n$. Then the pre-morphism $\underline{f}^* = \{\varphi, f'_n, \Phi_n(\mathrm{bond} \times 1_I) \circ h_n \circ \mathrm{bond}\, \Phi_{n+1}^{-1}\}$ is homotopic to $\underline{f}$.

Proof of (*A*.4).

(1) Let $m \leqslant 2r$. Consider any pre-morphism $\underline{f} = \{\varphi, f_n, h_n\}$: $\Sigma\underline{X} \to \Sigma\underline{Y}$. Each f_n: $\Sigma X_{\varphi(n)} \to \Sigma Y_n$ desuspends, i.e. there is g_n: $X_{\varphi(n)} \to Y_n$ with $f_n \simeq \Sigma g_n$. By (A.5) we may assume that already $f_n = \Sigma g_n$. But then h_n: $(\Sigma\,\text{bond})(\Sigma g_{n+1}) \simeq (\Sigma g_n)(\Sigma\text{bond})$. By A.2, there is h'_n: bond $g_{n+1} \simeq g_n$ bond with $\Sigma h'_n \cong h_n$. It is now obvious that $\underline{g} = \{\varphi, g_n, h'_n\}$: $\underline{X} \to \underline{Y}$ is a pre-morphism such that $\Sigma\underline{g}$ is homotopic to $\underline{f}$.

(2) Let $m \leqslant 2r - 1$. Consider pre-morphisms $\underline{f} = \{\varphi, f_n, h_n\}, \underline{f}' = \{\varphi', f'_n, h'_n\}$: $\underline{X} \to \underline{Y}$ such that $\Sigma\underline{f}$ and $\Sigma\underline{f}'$ are homotopic. This homotopy is realized by a pre-morphism $\underline{H} = \{\chi, F_n, H_n\}$: $\Sigma\underline{X} \times I \to \Sigma\underline{Y}$. We may assume $\chi = \varphi = \varphi'$. By A.2, the homotopies F_n: $\Sigma f_n \simeq \Sigma f'_n$ desuspend, i.e. there are ϕ_n: $f_n \simeq f'_n$ with $\Sigma\phi_n \cong F_n$. By (A.5) the pre-morphism $\underline{f}^* = \{\varphi, f'_n, h^*_n = \phi_n(\text{bond} \times 1_I) \circ h_n \circ \text{bond}\, \phi^{-1}_{n+1}\}$ is homotopic to $\underline{f}$; hence $\Sigma\underline{f}^*$ is homotopic to $\Sigma\underline{f}$ and therefore homotopic to $\Sigma\underline{f}'$. The homotopy between $\Sigma\underline{f}^*$ and $\Sigma\underline{f}'$ is now realized by a premorphism $\underline{H}^* = \{\varphi, F^*_n, H^*_n\}$ where each F^*_n: $f'_n \simeq f'_n$ is a stationary homotopy. But then the H^*_n may be regarded as homotopies of homotopies $\Sigma h^*_n \cong \Sigma h'_n$. By A.2, we infer $h^*_n \cong h'_n$. But this implies that $\underline{f}^*$ and $\underline{f}'$ are homotopic, i.e. that $\underline{f}$ and $\underline{f}'$ represent the same morphism of **Coh**.

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