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Monads and cohomology modules of rank 2 vector bundles

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Introduction

Monads are a useful tool to construct and study rank 2 vector bundles on the complex projective space $\mathbb{P}_n, n \geqslant 2$ (compare [O-S-S]). Horrocks' technique of eliminating cohomology [Ho 2] represents a given rank 2 vector bundle $\mathscr{E}$ as the cohomology of a monad

$$(M(\mathscr{E})) \quad \mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \mathscr{C}$$

as follows.

First eliminate the graded $S = \mathbb{C}[z_0, \dots, z_n]$-module $H^1\mathscr{E}(*) = \bigoplus_{m \in \mathbb{Z}} H^1(\mathbb{P}_n, \mathscr{E}(m))$ by the universal extension

$$0 \to \mathscr{E} \to \mathscr{Q} \to \tilde{L}_0 \to 0,$$

where

$$L_0 \to H^1\mathscr{E}(*) \to 0.$$

is given by a minimal system of generators ($\sim$ stands for sheafification).

If $n = 2$ take this extension as a monad with $\mathscr{A} = 0$.

If $n \geqslant 3$ eliminate dually $H^{n-1}\mathscr{E}(*)$ by the universal extension

$$0 \to \tilde{L}_0^\vee(c_1) \to \mathscr{K} \to \mathscr{E} \to 0$$

(where $c_1 = c_1(\mathscr{E})$ is the first Chern-class). Then notice, that the two extensions

* Partially supported by the DAAD.

can be completed to the display

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{L}_0^{\vee}(c_1) & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{E} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{L}_0^{\vee}(c_1) & \xrightarrow{\varphi} & \mathscr{B} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\
 & & & & \tilde{L}_0 & = & \tilde{L}_0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

of a monad

$$\tilde{L}_0^{\vee}(c_1) \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \tilde{L}_0$$

for $\mathscr{E}$.

To get a better understanding for $\mathscr{B}$, φ and ψ consider first the case $n = 2, 3$. Then $\mathscr{B}$ is a direct sum of line bundles by Horrocks' splitting criterion [Ho 1]. Taking cohomology we obtain a free presentation

$$B \xrightarrow{\psi} L_0 \longrightarrow H^1\mathscr{E}(*) \longrightarrow 0$$

with $B = H^0\mathscr{B}(*)$. The crucial point is that this is minimal [Ra]. Moreover, if $n = 3$, then B is self-dual [Ra]: $B^{\vee}(c_1) \simeq B$. We will see below that up to isomorphism φ is the dual map of ψ.

Let us summarize and slightly generalize. Consider an arbitrary graded S-module N of finite length with minimal free resolution (m.f.r. for short)

$$0 \longrightarrow L_{n+1} \xrightarrow{\alpha_n} L_n \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{\alpha_0} L_0 \longrightarrow N \longrightarrow 0.$$

If $n = 2$ then $N \simeq H^1\mathscr{E}(*)$ for some rank 2 vector bundle $\mathscr{E}$ on $\mathbb{P}_2$ iff $\operatorname{rk} L_1 = \operatorname{rk} L_0 + 2$ (compare [Ra]). In this case $\mathscr{E}$ is uniquely determined as $\widetilde{\ker \alpha_0}$:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \tilde{L}_0^{\vee}(c_1) & \xrightarrow{\alpha_0^{\vee}(c_1)} & \tilde{L}_1^{\vee}(c_1) & & \xrightarrow{\alpha_1} & & \tilde{L}_1 & \xrightarrow{\alpha_0} \tilde{L}_0 \longrightarrow 0. & \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & \mathscr{E} & & & & \\
 & & & & & \nearrow & & \searrow & & & \\
 & & & & 0 & & & & 0 & &
\end{array}
$$

(This sequence is self-dual by Serre-duality [Ho 1, 5.2], since $\mathscr{E}^\vee(c_1) \simeq \mathscr{E}$).

For $n = 3$ there is an analogous result. Answering Problem 10 of Hartshorne's list [Ha] we prove:

PROPOSITION 1. *N is the first cohomology module of some rank 2 vector bundle on $\mathbb{P}_3$ iff*

(1) $\operatorname{rk} L_1 = 2 \operatorname{rk} L_0 + 2$ *and*
(2) *there exists an isomorphism* $\Phi\colon L_1^\vee(c_1) \xrightarrow{\simeq} L_1$ *for some* $c_1 \in \mathbb{Z}$ *such that* $\alpha_0 \circ \Phi \circ \alpha_0^\vee(c_1) = 0$.

In this case any Φ satisfying (2) *defines a monad*

$$(M_\Phi) \quad \tilde{L}_0^\vee(c_1) \xrightarrow{\Phi\circ\alpha_0^\vee(c_1)} \tilde{L}_1 \xrightarrow{\alpha_0} \tilde{L}_0$$

and $\mathscr{E}$ is a 2-bundle on $\mathbb{P}_3$ with $H^1\mathscr{E}(*) \simeq N$ (and $c_1 = c_1(\mathscr{E})$) iff $(M(\mathscr{E})) \simeq (M_\Phi)$ for some Φ.

To complete the picture let us mention a result of Hartshorne and Rao (not yet published). If $N \simeq H^1\mathscr{E}(*)$ as above then $L_0^\vee(c_1) \xrightarrow{\varphi} L_1$ is part of a minimal system of generators for $\ker \alpha_0$. In other words: There exists a splitting

$$0 \to L_4 \to L_3 \to L_2' \oplus L_0^\vee(c_1) \to L_1 \to L_0 \to H^1\mathscr{E}(*) \to 0$$

inducing the monad

$$(M(\mathscr{E})) \quad \tilde{L}_0^\vee(c_1) \to \tilde{L}_1 \to \tilde{L}_0$$

and the m.f.r.

$$0 \to \tilde{L}_4 \to \tilde{L}_3 \to \tilde{L}_2' \to \mathscr{E} \to 0$$

resp.

For $n \geqslant 4$ there is essentially only one indecomposable 2-bundle known on $\mathbb{P}_n$: The Horrocks-Mumford-bundle $\mathscr{F}$ on $\mathbb{P}_4$ with Chern-classes $c_1 = -1$, $c_2 = 4$. We prove:

PROPOSITION 2. *The* m.f.r. *of $H^2\mathscr{F}(*)$ decomposes as*

$$\begin{array}{c} 0 \to H_2 \xrightarrow{\beta_1} H_1 \xrightarrow{\begin{pmatrix}\beta_0' \\ \beta_0''\end{pmatrix}} L_0^\vee(c_1) \oplus L_1 \xrightarrow{\begin{pmatrix}0 & \alpha_0 \\ \alpha_0^\vee(c_1) & *\end{pmatrix}} L_0 \oplus L_1^\vee(c_1) \to \\ \searrow \quad \nearrow \\ B \\ \nearrow \quad \searrow \\ 0 \qquad\qquad 0 \\ \to H_1^\vee(c_1) \to H_2^\vee(c_1) \to H^2\mathscr{F}(*) \to 0 \end{array}$$

with $B = H^0\mathscr{B}(*)$, *inducing the monad*

$$(M(\mathscr{F}))\quad \tilde{L}_0^\vee(c_1) \to \mathscr{B} \to \tilde{L}_0$$

and the minimal free presentation

$$L_1 \xrightarrow{\alpha_0} L_0 \longrightarrow H^1\mathscr{F}(*) \longrightarrow 0.$$

The corresponding m.f.r. *decomposes as*

$$0 \to L_5 \to L_4 \to L_3' \oplus H_2 \xrightarrow{\begin{pmatrix} * & 0 \\ * & \beta_1 \end{pmatrix}} L_2' \oplus H_1 \xrightarrow{(* \;\; \beta_0'')} L_1 \xrightarrow{\alpha_0} L_0 \to$$
$$\to H^1\mathscr{F}(*) \to 0$$

inducing the m.f.r.

$$0 \to \tilde{L}_5 \to \tilde{L}_4 \to \tilde{L}_3' \to \tilde{L}_2' \to \mathscr{F} \to 0.$$

$(M(\mathscr{F}))$ is the monad given in [H-M]. Using its display we can compute the above m.f.r.'s explicitly. Especially we reobtain the equations of the abelian surfaces in $\mathbb{P}_4$ ([Ma 1], [Ma 2]).

Of course we may deduce from $\mathscr{F}$ some more bundles by pulling it back under finite morphisms $\pi\colon \mathbb{P}_4 \to \mathbb{P}_4$. The above result also holds for the bundles $\pi^*\mathscr{F}$ with $(M(\pi^*\mathscr{F})) = \pi^*(M(\mathscr{F}))$.

There is some evidence (but so far no complete proof), that a splitting as in Proposition 2 occurs for every indecomposable 2-bundle on $\mathbb{P}_4$. This suggests a new construction principle for such bundles by constructing their H^2-module first.

Proof of Proposition 1

Let $n = 3$ and N be a graded S-module of finite length with m.f.r.

$$0 \to L_4 \to L_3 \to L_2 \to L_1 \xrightarrow{\alpha_0} L_0 \to N \to 0.$$

Suppose first that $N \simeq H^1\mathscr{E}(*)$ for some 2-bundle $\mathscr{E}$ on $\mathbb{P}_3$ (with first Chern-class c_1). As seen in the introduction, Horrocks' construction leads to a monad

$$(M(\mathscr{E}))\quad \tilde{L}_0^\vee(c_1) \xrightarrow{\varphi} \tilde{L}_1 \xrightarrow{\alpha_0} \tilde{L}_0$$

for $\mathscr{E}$. The dual sequence

$$\tilde{L}_0^\vee(c_1) \xrightarrow{\alpha_0^\vee(c_1)} \tilde{L}_1^\vee(c_1) \xrightarrow{\varphi^\vee(c_1)} \tilde{L}_0$$

is a monad for $\mathscr{E}^\vee(c_1) \simeq \mathscr{E}$. The induced presentation of N has to be isomorphic to that one given by the m.f.r.:

$$\begin{array}{ccccccc} L_1^\vee(c_1) & \xrightarrow{\varphi^\vee(c_1)} & L_0 & \to & N & \to & 0 \\ \Phi^\vee(c_1)\downarrow \wr\| & & \wr\| & & \wr\| & & \\ L_1 & \xrightarrow{\alpha_0} & L_0 & \to & N & \to & 0. \end{array}$$

Dualizing gives (2) since $\alpha_0 \circ \varphi = 0$ and thus also a monad (M_Φ) for $\mathscr{E}$, isomorphic to $(M(\mathscr{E}))$ (replace φ by $\Phi \circ \alpha_0^\vee(c_1)$).

Conversely if N satisfies (2), we obtain a monad (M_Φ) by sheafification. (Since $\tilde{N} = 0$, α_0 is a bundle epimorphism. Dually $\alpha_0^\vee(c_1)$ is a bundle monomorphism.) Let $\mathscr{E}$ be the cohomology bundle of (M_Φ). Then $H^1\mathscr{E}(*) \simeq N$. $\mathscr{E}$ has rank 2, if N satisfies (1). □

REMARK 1. (i) Let $N \simeq H^1\mathscr{E}(*)$ as above with induced splitting

$$0 \to L_4 \to L_3 \to L_2' \oplus L_0^\vee(c_1) \to L_1 \to L_0 \to H^1\mathscr{E}(*) \to 0$$

as in the introduction. Recall that $\mathscr{E}$ is stable iff $H^0(\mathbb{P}_3, \mathscr{E}(m)) = 0$ for $m \leqslant -c_1/2$. Thus $\mathscr{E}$ is stable iff L_2' has no direct summand $S(m)$ with $m \geqslant c_1/2$. Notice that this condition only depends on N.

(ii) If N satisfies (1) and has only one generator, then (2) is obviously equivalent to the symmetry condition $L_1^\vee(c_1) \simeq L_1$. Thus [Ra, 3.1] is a special case of Proposition 1.

EXAMPLES. (i) The well-known Null correlation bundles are by definition the bundles corresponding to the S-module $\mathbb{C}$. Consider the Koszul-presentation

$$4S \xrightarrow{\alpha_0} S(1) \to \mathbb{C} \to 0, \quad \alpha_0 = (z_0, z_1, z_2, z_3).$$

The isomorphisms $4S \xrightarrow{\Phi} 4S$ with $\alpha_0 \circ \Phi \circ \alpha_0^\vee(c_1) = 0$ are precisely the 4×4 skew symmetric matrices with nonzero determinant. Two such matrices give isomorphic bundles iff they differ by a scalar (use [O-S-S, II, Corollary 1 to 4.1.3]). The moduli space of Null correlation bundles is thus isomorphic to $\mathbb{P}_5 \backslash \mathbb{G}$, where $\mathbb{G}$ is the Plucker embedded Grassmanian of lines in $\mathbb{P}_3$.

Unlike the case $n = 2$ the bundle is not uniquely determined by the module.

(ii) The S-module

$$6S \xrightarrow{\alpha_0} S(1) \oplus S(2) \to N \to 0, \; \alpha_0 = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 & 0 & 0 \\ 0 & 0 & z_0^2 & z_1^2 & z_2^2 & z_3^2 \end{pmatrix}$$

satisfies (1) and the symmetry condition $L_1^\vee \simeq L_1$, i.e. the necessary conditions of [Ra]. But N does not satisfy (2).

Cohomology modules of the Horrocks-Mumford-bundle $\mathscr{F}$

We first recall the construction of $\mathscr{F}$ [H-M]. Let

$$V = \mathrm{Map}(\mathbb{Z}_5, \mathbb{C})$$

be the vector space of complex valued functions on $\mathbb{Z}_5$. Denote by

$$H \subset N \subset \mathrm{SL}(5, \mathbb{C})$$

the Heisenberg group and its normalizer in $\mathrm{SL}(5, \mathbb{C})$ resp.

Let $V_0 = V, V_1, V_2, V_3$ and

$$W = \mathrm{Hom}_H(V_1, \Lambda^2 V)$$

be defined as in [H-M]. The V_i are irreducible representations of H and N of degree 5. W is an irreducible representation of N/H of degree 2. It is unimodular, so it comes up with an invariant skew symmetric pairing.

Let $\mathbb{P}_4 = \mathbb{P}(V)$ be the projective space of lines in V. The Koszul-complex on $\mathbb{P}(V)$ is the exact sequence

$$(K) \quad \cdots \to \mathcal{O}(-1) \otimes \Lambda^2 V \to \mathcal{O} \otimes \Lambda^3 V \to \cdots,$$
$$\begin{array}{ccccc} & \searrow & & \nearrow & \\ & & (\Lambda^2 \mathscr{T})(-3) & & \\ & \nearrow & & \searrow & \\ 0 & & & & 0 \end{array}$$

obtained by exterior multiplication with the tautological subbundle

$$\mathcal{O}(-1) \to V \otimes \mathcal{O}.$$

The exterior product provides (K) with a self-duality (with values in $\mathcal{O}(-1) \otimes \Lambda^5 V \simeq \mathcal{O}(-1)$).

This induces the natural pairings

$$(\Lambda^i \mathcal{T})(-i-1) \otimes (\Lambda^{4-i}\mathcal{T})(i-5) \xrightarrow{\wedge} (\Lambda^4 \mathcal{T})(-6) \simeq \mathcal{O}(-1)$$

and is compatible with the action of SL(5, $\mathbb{C}$).

It can be extended to $(K) \otimes W$ by tensoring with the invariant form, then being compatible with the action of N.

As in the proof of [H-M, Lemma 2.4] it follows, that $(K) \otimes W$ decomposes as

$$\begin{array}{ccccccc}
 & & \mathcal{O}(-1)\otimes V_1 & \xrightarrow{\begin{pmatrix} 0 & \alpha_0 \\ \alpha_0^\vee(-1) & \star \end{pmatrix}} & \mathcal{O}\otimes V_3 & & \\
\cdots & \rightarrow & \oplus & & \oplus & \rightarrow & \cdots, \\
 & & \mathcal{O}(-1)\otimes V_1 \otimes U & & \mathcal{O}\otimes V_3\otimes U & & \\
 & & & \searrow \qquad \nearrow & & & \\
 & & & (\Lambda^2 \mathcal{T})(-3)\otimes W & & & \\
 & & \nearrow & & \searrow & & \\
 & 0 & & & & 0 &
\end{array}$$

given by the splitting into irreducible N-modules. Moreover the induced

$$\mathcal{O}(-1)\otimes V_1 \rightarrow (\Lambda^2\mathcal{T})(-3)\otimes W \rightarrow \mathcal{O}\otimes V_3$$

is the self-dual Horrocks-Mumford-monad, whose cohomology is $\mathcal{F}$ (normalized such that $c_1\mathcal{F} = -1$).

To proof Proposition 2 consider the display

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1)\otimes V_1 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1)\otimes V_1 & \rightarrow & (\Lambda^2\mathcal{T})(-3)\otimes W & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}\otimes V_3 & = & \mathcal{O}\otimes V_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It first follows that $H^2\mathcal{F}(*) = W$ is a vector space, sitting in degree -2 (compare [H-M]). Its m.f.r. is the Koszul-complex obtained from $(K) \otimes W$ by taking global sections. So it decomposes, inducing the presentation

$$S(-1)\otimes V_1 \otimes U \xrightarrow{\alpha_0} S\otimes V_3 \longrightarrow H^1\mathscr{F}(*) \longrightarrow 0$$

and the Horrocks-Mumford-monad. But this is just

$$(M(\mathscr{F}))\qquad \mathcal{O}(-1)\otimes V_1 \to \mathscr{B} \to \mathcal{O}\otimes V_3:$$

apply e.g. [O-S-S, II, Corollary 1 to 4.1.3] (notice that $H^0\mathscr{F}=0$ implies $H^0\mathscr{B} = H^0\mathscr{B}^*(-1) = 0$ by construction of $(M(\mathscr{F}))$).

It remains to show that α_0 is minimal and that the corresponding m.f.r. of $H^1\mathscr{F}(*)$ decomposes, inducing the m.f.r., say,

$$0 \longrightarrow F_3 \xrightarrow{\gamma_2} F_2 \xrightarrow{\gamma_1} F_1 \xrightarrow{\gamma_0} F_0 \longrightarrow F \longrightarrow 0$$

of $F = H^0\mathscr{F}(*)$.

From the second row of the display we obtain the m.f.r. of $Q = H^0\mathscr{Q}(*)$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & S(-3)\otimes W & = & S(-3)\otimes W & & \\
 & & & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & & \\
 & & & & S(-2)\otimes V\otimes W & = & S(-2)\otimes V\otimes W & & \\
 & & & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta_0''} & & \\
0 & \to & S(-1)\otimes V_1 & \to & (S(-1)\otimes V_1)\oplus(S(-1)\otimes V_1\otimes U) & \to & S(-1)\otimes V_1\otimes U & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & S(-1)\otimes V_1 & \longrightarrow & H^0(\Lambda^2\mathscr{T})(*-3)\otimes W & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The third column of the display gives rise to the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & F_3 & & 0 & & & & \\
 & & \gamma_2\downarrow & & \downarrow & & & & \\
 & & F_2 & \xrightarrow{\alpha'_3} & S(-3)\otimes W & & & & \\
 & & \gamma_1\downarrow & & \downarrow\beta_1 & & & & \\
 & & F_1 & \xrightarrow{-\alpha'_2} & S(-2)\otimes V\otimes W & & & & \\
 & & \gamma_0\downarrow & & \downarrow\beta''_0 & & & & \\
 & & F_0 & \xrightarrow{\alpha'_1} & S(-1)\otimes V_1\otimes U & \xrightarrow{\alpha_0} & S\otimes V_3 \longrightarrow & H^1\mathscr{F}(*) & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \| & \| & \\
0 & \longrightarrow & F & \longrightarrow & Q & \longrightarrow & S\otimes V_3 \longrightarrow & H^1\mathscr{F}(*) & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

with exact columns and bottom row.

The induced

$$
0\to F_3 \xrightarrow{\gamma_2} F_2 \xrightarrow{\begin{pmatrix}\gamma_1\\ \alpha'_3\end{pmatrix}} F_1\oplus(S(-3)\otimes W) \xrightarrow{\begin{pmatrix}\gamma_0 & 0\\ \alpha'_2 & \beta_1\end{pmatrix}} F_0\oplus(S(-2)\otimes V\otimes W) \xrightarrow{(\alpha'_1\ \beta''_0)}
$$

$$
\to S(-1)\otimes V_1\otimes U \xrightarrow{\alpha_0} S\otimes V_3 \to H^1\mathscr{F}(*)\to 0
$$

is exact and it is minimal, iff $\alpha'_1, \alpha'_2, \alpha'_3$ have no entries in $\mathbb{C}\setminus\{0\}$. But since $H^0\mathscr{F}(1) = 0$ [H-M], these maps have only entries in degrees $\geqslant 1$. □

REMARK 2 (i) Let us describe $(M(\mathscr{F}))$ more explicitly by choosing convenient bases of $V_1, V_3 = V_1^*, W$ and forgetting the N-module structure (compare the proof of [H-M, Lemma 2.5].)

Choose the basis $e_0, \ldots, e_4$ of $V = \mathrm{Map}(\mathbb{Z}_5, \mathbb{C})$ given by $e_i(j) = \delta_{ij}$ and its dual basis $z_0, \ldots, z_4 \in V^*$.

Define

$$A = (a_{ij})_{\substack{0 \leqslant i \leqslant 4 \\ 0 \leqslant j \leqslant 1}} \quad \text{by}$$

$$\left.\begin{aligned} a_{i0} &= e_{i+2} \wedge e_{i+3} \\ a_{i1} &= e_{i+1} \wedge e_{i+4} \end{aligned}\right\} \quad i \bmod 5.$$

Then w_0, w_1, given by $w_j(e_i) = a_{ij}$ is a basis of W. Identifying $W \simeq \mathbb{C}^2$, the invariant form on W becomes the standard symplectic form $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\mathbb{C}^2$.

We thus may rewrite $(M(\mathscr{F}))$ as

$$(M(\mathscr{F})) \quad 5\mathcal{O}(-1) \xrightarrow{Q \cdot {}^tA} 2(\Lambda^2\mathscr{T})(-3) \xrightarrow{A} 5\mathcal{O},$$

the matrices operating by exterior multiplication.

(ii) From the explicit form of $(M(\mathscr{F}))$ we can compute α_0 explicitly. Choose a convenient basis of $\Lambda^2 V \otimes W = (V_1 \oplus U)$. Then

$$15S(-1) \xrightarrow{\alpha_0} 5S \longrightarrow H^1\mathscr{F}(*) \longrightarrow 0$$

is the matrix

$$\alpha_0 = \begin{pmatrix} 0 & z_3 & 0 & 0 & z_2 & 0 & 0 & z_1 & z_4 & 0 & z_0 & 0 & 0 & 0 & 0 \\ z_3 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 & z_2 & z_0 & 0 & z_1 & 0 & 0 & 0 \\ 0 & z_4 & 0 & z_0 & 0 & z_1 & 0 & 0 & 0 & z_3 & 0 & 0 & z_2 & 0 & 0 \\ 0 & 0 & z_0 & 0 & z_1 & z_4 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 \\ z_2 & 0 & 0 & z_1 & 0 & 0 & z_0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 & z_4 \end{pmatrix}.$$

Resolving it (use e.g. [B-S]), we obtain the m.f.r. of $H^1\mathscr{F}(*)$. Its shape is

$$0 \to 2S(-8) \to 20S(-6) \to 35S(-5) \oplus 2S(-3) \to (15S(-4) \oplus 4S(-3)) \oplus 10S(-2)$$
$$\to 15S(-1) \to 5S \to H^1\mathscr{F}(*) \to 0.$$

(iii) Consider the induced m.f.r. of F and its dual

$$\begin{array}{ccccccc} \cdots \to 35S(-5) \xrightarrow{\gamma_0} & 15S(-4) \oplus 4S(-3) & & \xrightarrow{\Gamma} & & 15S(3) \oplus 4S(2) & \xrightarrow{{}^t\gamma_0} 35S(4) \to \cdots \\ & & \searrow & & \nearrow & & \\ & & & F \simeq F^\vee(-1) & & & \\ & & \nearrow & & \searrow & & \\ & 0 & & & & 0\,. & \end{array}$$

Γ can be computed by resolving ${}^t\gamma_0$ (use again [B-S]). We thus obtain explicit bases for the spaces of sections $H^0\mathscr{F}(m)$. Especially we get the equations of the zero-schemes of sections of $\mathscr{F}(3)$, including the abelian surfaces in $\mathbb{P}_4$.

(iv) Let $\pi\colon \mathbb{P}_4 \to \mathbb{P}_4$ be a finite morphism and d^4 its degree. Then $\pi^*\mathscr{F}$ is a stable 2-bundle with Chern-classes $c_1 = -d$, $c_2 = 4d^2$. Proposition 2 and the above remarks also hold for $\pi^*\mathscr{F}$: Replace (K) by $\pi^*(K)$, $(M(\mathscr{F}))$ by $\pi^*(M(\mathscr{F})) = (M(\pi^*\mathscr{F}))$ and $z_0,\dots,z_4$ in α_0 by $f_0,\dots,f_4$, where $f_0,\dots,f_4$ are the forms of degree d defining π.

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