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# On Castelnuovo's regularity and Hilbert functions

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**Abstract.** New bounds for Castelnuovo's regularity are established. As a consequence, we show that a property of Hilbert functions stated by J. Harris and D. Eisenbud in [7], p. 82 is only true for curves and false for higher-dimensional subschemes. The letter of W. Vogel [25] gives rise to study this property again.

## 1. Introduction

Castelnuovo's regularity was first defined by D. Mumford [10], who attributes the idea to G. Castelnuovo, for coherent sheaves on projective spaces. In a more algebraic setting it was defined by D. Eisenbud and S. Goto [3] and A. Ooishi [13] (see Section 2). It comes out that Castelnuovo's regularity gives an upper bound for the maximal degrees of the syzygies in a minimal free resolution [3]. D. Bayer and M. Stillman [2] showed that an estimate of the regularity of an ideal gives a bound on the complexity of algorithms for computing syzygies. In [3], p. 93 D. Eisenbud and S. Goto stated the following well-known conjecture on such an estimation.

Let $X \subseteq \mathbb{P}^n_K$ ($K$ an algebraically closed field) be a nondegenerate, that is, $X$ is not contained in a hyperplane of $\mathbb{P}^n$, irreducible, reduced subvariety then holds

$$\operatorname{reg}(X) \leqslant \operatorname{degree}(X) - \operatorname{codim}(X) + 1.$$

So far, this conjecture has been proved for curves [6] and, if $\operatorname{char}(K) = 0$, for smooth surfaces [9] (see also [5]), for a large class of smooth threefolds in $\mathbb{P}^5_K$ [18], Theorem 3.3, and if $X$ is arithmetically Buchsbaum or $\operatorname{degree}(X) \leqslant \operatorname{codim}(X) + 2$ [21]. In the other cases only weaker results are known by using certain correction terms (see, e.g., [1], [22], [23], [5], [11], [18]). We also note that [11] and [12] describe applications of Castelnuovo's regularity. The aim of this paper is to describe a new approach for providing Castelnuovo bounds as presented in [23]. This provides new bounds for Castelnuovo's regularity which improve bounds of [23] in some cases. Moreover, we prove special cases of the conjectures of D. Eisenbud and S. Goto and of D. Bayer and M. Stillman [2] (see Theorem 2). In Section 4 we will apply these results in order to show that an assertion of J. Harris and D. Eisenbud [7], p. 82 on the equality of the abstract

Hilbert function and the Hilbert polynomial is not true in general (see Theorem 3). Providing our Theorem 3 we construct our counterexamples.

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## 2. Notations and preliminary results

We work over an algebraically closed field $K$. Let $S = K[x_0, \dots, x_n]$ be a polynomial ring and $\mathfrak{a} \subseteq S$ be a homogeneous ideal. We set $A = S/\mathfrak{a}$., that is, $A$ is a graded $K$-algebra. We denote by $P_A := \oplus_{n>0} A_n$ the irrelevant ideal of $A$. When there is no possibility of confusion we will denote $P_A$ simply by $P$. Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a graded $A$-module. The $i$-th local cohomology module of $M$ with support in $P$, denoted by $H^i_P(M)$, is also a graded $A$-module. Let $[M]_i$ denote the $i$-th graded part of $M$ for $i \in \mathbb{Z}$, i.e. $[M]_i = M_i$. Let $j$ be an integer then let $M(j)$ denote the graded $A$-module whose underlying module is the same as that of $M$ and whose grading is given by $[M(j)]_i = [M]_{i+j}$ for all $i \in \mathbb{Z}$. We set for an arbitrary $A$-module $M$: $e(M) := \sup\{t \in \mathbb{Z} : [M]_t \neq 0\}$

$$\operatorname{sgn}(M_n) := \begin{cases} 1, & \text{if } M_n \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$r_k(M) := \sup\{i + e(H^i_P(M)) : i \geqslant k\}.$$

For a finitely generated graded $A$-module $M$ we define Castelnuovo's regularity, denoted by $\operatorname{reg}(M)$, by $\operatorname{reg}(M) := r_0(M) = r_{\operatorname{depth}(M)}(M)$.

Let $\{x_1, \dots, x_m\}$ be a part of a system of parameters for $M$. It is said to be a filter-regular sequence if

$$e((x_1, \dots, x_{i-1})M :_M x_i/(x_1, \dots, x_{i-1})M) < \infty \quad \text{for } i = 1, \dots, m$$

(see, e.g., [20], appendix for further informations). We set $M^i = M/(x_1, \dots, x_i)M$ $(i = 0, \dots, m)$ for a filter-regular sequence $\{x_1, \dots, x_m\}$ for $M$. We have the following result.

LEMMA 1. *Let $M$ be a Noetherian graded $A$-module of dimension $d > 0$. Then there is a filter-regular sequence $\{l_1, \dots, l_d\}$ of forms $\in A_1$.*

*Proof.* It is sufficient to show that there is a filter-regular element $l \in A_1$ for $M$.

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of prime ideals $\mathfrak{p}$ of $\mathrm{Ass}_A(M)$ with Krull-dim$(A/\mathfrak{p}) \geqslant d$. Using, for example [16], Theorem 2.3 we can find an element $l \in A_1 \backslash (P \cdot A_1 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$ since $P \not\subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$. It follows from [20], Theorem 7 of the appendix that $l$ is a filter-regular element for $M$. Q.E.D.

Let $M$ be a Noetherian graded $A$-module of dimension $d \geqslant 1$. We denote by $h_0(M)$ the $(d-1)!$-fold of the leading coefficient of the Hilbert polynomial $p_M(t)$. We recall that $p_M(t) = \mathrm{rank}_K [M]_t$ for all $t \gg 0$. If $M = A = S/\mathfrak{a}$ we define

$$\mathrm{degree}(\mathfrak{a}) := \begin{cases} h_0(S/\mathfrak{a}), & \text{if Krull-dim}(S/\mathfrak{a}) > 0 \\ \mathrm{length}(S/\mathfrak{a}), & \text{if Krull-dim}(S/\mathfrak{a}) = 0. \end{cases}$$

Further, we set $\mathfrak{a}\colon \langle P \rangle = \{x \in S\colon \text{there is an integer } m \geqslant 0 \text{ with } P^m \cdot x \subseteq \mathfrak{a}\}$.

Let $X$ be a subscheme of $P^n_K$. Then we denote by $I(X)$ the defining ideal of $X$ in $S = K[x_0, \ldots, x_n]$. If $\mathfrak{a} \subseteq S$ is a homogeneous ideal let $V(\mathfrak{a})$ be the corresponding subscheme of $\mathbb{P}^n_K$. The ideal $\mathfrak{a}$ is said to be regular if $V(\mathfrak{a})$ is smooth. Note that degree $(X) = \mathrm{degree}(I(X))$. We set $r_k(X) := r_{k+1}(I(X)) = r_k(S/I(X)) + 1$ $(k \geqslant 0)$.

For a set $B$ we write card$(B)$ for its cardinality. Finally, we set for integers $a$, $b \geqslant 0$: $\{a/b\} := \inf\{t \in \mathbb{Z}\colon a \leqslant tb\}$. If $a > b$ we define a sum $\sum_{i=a}^{b} \ldots$ to be zero and a condition, say $B_i$, for $i = a, a+1, \ldots, b$ to be empty.

## 3. Castelnuovo bounds

Studying our integers $r_k$ we will prove a generalization of a theorem of D. Mumford [10], p. 99 and A. Ooishi [13], Theorem 2.

THEOREM 1. *Let $M$ be a finitely generated graded $A$-module of dimension $d$ and let $m$ and $k \geqslant 1$ be integers. Suppose that $[H^i_P(M)]_{m-i} = 0$ for all $i \geqslant k$. Then $r_k(M) \leqslant m-1$. Moreover,* reg$(M) \leqslant m-1$ *provides $A_i M_j = M_{i+j}$ for all $i \geqslant 0$ and $j \geqslant m-1$.*

*Proof.* We induct on $d$. In case of $d = 0$ the assertions are trivial since $H^0_P(M) = M$ and $H^i_P(M) = 0$ for all $i > 0$. Let $d > 0$. According to Lemma 1 we can choose a filter-regular element $l \in A_1$ for $M$. Then we get $H^i_P(M/0\colon l) \cong H^i_P(M)$ for $i > 0$ from the long exact cohomology sequence of $0 \to 0\colon_M l \to M \to M/0\colon_M l \to 0$. The exact sequence $0 \to M/0\colon l(-1) \overset{l}{\to} M \to M/lM \to 0$ gives rise to the cohomology sequence

$$0 \to 0\colon_M l(-1) \to H^0_P(M)(-1) \overset{l}{\to} H^0_P(M) \to H^0_P(M/lM) \to H^1_P(M)(-1) \to \cdots \quad (*)$$

because we have $0\colon_M l \subseteq H^0_P(M)$ by the choice of $l$. Considering the following sequences of (*)

$$[H^i_P(M)]_{m-i} \to [H^i_P(M/lM)]_{m-i} \to [H^{i+1}(M)]_{m-i-1} \quad (i \geqslant 0)$$

we get $[H^i_P(M/lM)]_{m-1} = 0$ for all $i \geqslant k$ by assumption. Therefore the induction hypothesis provides $[H^i_P(M/lM)]_j = 0$ for all $i$ and $j$ with $i \geqslant k$ and $i + j \geqslant m$. The following sequences of (*)

$$[H^i_P(M)]_{m-i} \to [H^i_P(M)]_{m-i+1} \to [H^i_P(M/lM)]_{m+1-i} \quad (i \geqslant 0)$$

gives us $[H^i_P(M)]_{m+1-i} = 0$ for all $i \geqslant k$. By induction we therefore have the first assertion. Proving the second assertion we first note that $\operatorname{reg}(M) \leqslant m - 1$ and (*) yield $\operatorname{reg}(M/lM) \leqslant m - 1$. If we set $M' = M/lM$ and $A' = A/lA$ we have $A'_i M'_j = M'_{i+j}$ for $i \geqslant 0$ and $j \geqslant m - 1$ by induction hypothesis. Hence $A_i M_j + lM_{i+j-1} = M_{i+j}$. It follows from this by induction on $i$ that $M_{i+j-1} = A_{i-1} M_j$. We obtain $M_{i+j} = A_i M_j + lM_{i+j-1} = A_i M_j$. Q.E.D.

Moreover we get from the proof of Theorem 1. (see also [24], Lemma 2.3):

LEMMA 2. *Let $M$ be a finitely generated graded $A$-module and let $l \in A_1$ be a filter-regular element for $M$ then*

$$e(H^{i+1}_P(M)) < e(H^i_P(M/lM)) \leqslant \max\{e(H^i_P(M)), 1 + e(H^{i+1}_P(M))\} \quad (i \geqslant 0).$$

*Proof.* The assertions follows from the exact sequence (*) of the proof of Theorem 1. Q.E.D.

LEMMA 3. *Let $k \geqslant 1$ and $c$ be integers. Then we have for all finitely generated graded $A$-modules $M$ of dimension $d$*

$$r_k(M) \leqslant c - 1 + \sum_{i=k}^{d} \operatorname{card}\{t \in \mathbb{Z} : t \geqslant c - i \text{ and } [H^i_P(M)]_t \neq 0\}.$$

*Proof.* The assertion is trivial for $k > d$. Let $1 \leqslant k \leqslant d$. We have $\sum_{i=k}^{d} \operatorname{sgn}[H^i_P(M)]_{m-i} \geqslant 1$ for all $m \leqslant r_k(M)$ by Theorem 1. Thus it follows from all $c \in \mathbb{Z}$:

$$\begin{aligned} r_k(M) &\leqslant c - 1 + \sum_{m=c}^{r_k(M)} \sum_{i=k}^{d} \operatorname{sgn}[H^i_P(M)]_{m-i} \\ &= c - 1 + \sum_{i=k}^{d} \sum_{t=c-i}^{r_k(M)-i} \operatorname{sgn}[H^i_P(M)]_t \\ &= c - 1 + \sum_{i=k}^{d} \operatorname{card}\{t \in \mathbb{Z} : t \geqslant c - i \text{ and } [H^i_P(M)]_t \neq 0\} \end{aligned}$$

where the last equality follows from the definition of $r_k(M)$. Q.E.D.

In case of $M = A$ we obtain something more.

MAIN LEMMA: *Let $k \geqslant 0$ and $c$ be integers. Let $A$ be a graded K-algebra of Krulldimension $d$. Then:*

$$r_k(A) \leqslant c - 1 + \sum_{i=k}^{d} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i \text{ and } [H_P^i(A)]_t \neq 0\}.$$

*Proof.* We consider only the case $k = 0$ according to Lemma 3. Since $r_0(A) = \max\{e(H_P^0(A)), r_1(A)\}$ the assertion follows from Lemma 3 with $k = 1$ in assuming $e(H_P^0(A)) \leqslant r_1(A)$. We therefore suppose that $e(H_P^0(A)) > r_1(A)$. We set $A = S/\mathfrak{a} \cap \mathfrak{q}$ where $S = K[x_0, \ldots, x_n]$ and $\mathfrak{a}, \mathfrak{q} \subseteq S$ are homogeneous ideals such that $\mathfrak{a}: \langle P \rangle = \mathfrak{a}$, $\mathfrak{a} \nsubseteq \mathfrak{q}$ and $\mathfrak{q}$ is a primary ideal belonging to $P$. If we set $A' = S/\mathfrak{a}$ we get $\operatorname{depth}(A') \geqslant 1$. Hence $\operatorname{reg}(A') = r_1(A') = r_1(A)$, consequently $\operatorname{reg}(\mathfrak{a}) = 1 + r_1(A)$. It follows from the second assertion of Theorem 1. that $\mathfrak{a}$ is generated by forms of degree $\leqslant 1 + r_1(A)$. Since $H_P^0(A) = \mathfrak{a}/\mathfrak{a} \cap \mathfrak{q}$ we can deduce $[H_P^0(A)]_t \neq 0$ for all $t$ with $r_1(A) < t \leqslant e(H_P^0(A))$. This gives us

$$r_0(A) = e(H_P^0(A)) = r_1(A) + \operatorname{card}\{t \in \mathbb{Z}: t > r_1(A) \text{ and } [H_P^0(A)]_t \neq 0\}.$$

Therefore the assertion follows again from Lemma 3 with $k = 1$. Q.E.D.

COROLLARY 1. *Let $A$ be a graded K-algebra of Krulldimension $d$. Let $j$ and $k$ be integers such that $j + k \leqslant d$ and $\{l_1, \ldots, l_j\}$ be a filter-regular sequence for $A$. We set $A^j := A/(l_1, \ldots, l_j)A$. Then:*

$$r_k(A) \leqslant c - 1 + \sum_{i=k}^{k+j-1} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i \textit{ and } [H_P^i(A)]_t \neq 0\}$$

*for all $c > r_k(A^j)$.*

*Proof.* Lemma 2 gives us for $i = k, \ldots, d - j$

$$i + j + e(H_P^{i+j}(A)) \leqslant i + e(H_P^i(A^j)) \leqslant r_k(A^j).$$

Therefore the assertion follows from the Main Lemma. Q.E.D.

REMARKS (i) If we suppose $k \geqslant 1$ in Corollary 1 the above result remains true even for finitely generated graded $A$-modules because we can apply Lemma 3.

(ii) If we set $k = 1$ and $j = d - 1$ in Corollary 1 we obtain the main lemma of [23]. Hence we could deduce the Castelnuovo bounds of [23]. Here we want to state some new bounds.

THEOREM 2. *Let* $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m \subset S = K[x_0, \ldots, x_n]$ *be an intersection of* $m$ *equidimensional (homogeneous) prime ideals. Let* $d$ *be the Krulldimension of* $A = S/\mathfrak{a}$. *Then we have:*

(i) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-2} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i$ *and* $[H_P^i(A)]_t \neq 0\}$ *for all* $c \geqslant \operatorname{degree}(\mathfrak{a})$.

(ii) *In case of* $m = 1$ *we get:*

(ii.1) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-1} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i$ *and* $[H_P^i(A)]_t \neq 0\}$ *for all* $c > \{\operatorname{degree}(\mathfrak{a}) - 1/\operatorname{rank}_K[A]_1 - d\}$ *if* $\operatorname{char}(K) = 0$

(ii.2) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-2} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i$ *and* $[H_P^i(A)]_t \neq 0\}$ *for all* $c > \operatorname{degree}(\mathfrak{a}) + d - \operatorname{rank}_K[A]_1$

(ii.3) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-3} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i$ *and* $[H_P^i(A)]_t \neq 0\}$ *for all* $c > \operatorname{degree}(\mathfrak{a}) + d - \operatorname{rank}_K[A]_1$ *if* $\mathfrak{a}$ *is regular and* $\operatorname{char}(K) = 0$.

*Proof*. First we show (i) and (ii.2). If $d = 0$ or $d = 1$ then $A$ is Cohen-Macaulay and we get $\operatorname{reg}(\mathfrak{a}) \leqslant \operatorname{degree}(\mathfrak{a}) + d - \operatorname{rank}_K[A]_1 + 1$ by [13], Proposition 13. This proves (i) and even (ii.2) since $\operatorname{rank}_K[A]_1 \geqslant d$, where equality holds if and only if $A$ is isomorphic to a polynomial ring over $K$. But in this case we have $\operatorname{degree}(\mathfrak{a}) = \operatorname{reg}(\mathfrak{a}) = 1$. Let $d \geqslant 2$. According to H. Flenner [4] there are generic linear forms $l_1, \ldots, l_{d-2} \in S$ such that $\mathfrak{a} + (l_1, \ldots, l_i)S$ is an intersection of $m$ prime ideals of dimension $d - i$ up to a primary component belonging to $P$ for $i = 0, \ldots, d-2$. Therefore $\{l_1, \ldots, l_{d-2}\}$ is a filter-regular sequence for $A$. If we set $A^{d-2} = A/(l_1, \ldots, l_{d-2})A$ and $\mathfrak{a}' = \mathfrak{a} + (l_1, \ldots, l_{d-2})S: \langle P \rangle$ we obtain $r_1(A^{d-2}) = r_1(S/\mathfrak{a}') = \operatorname{reg}(S/\mathfrak{a}')$. Thus we get $r_1(A^{d-2} \leqslant \operatorname{degree}(\mathfrak{a}') + 2 - \operatorname{rank}_K[S/\mathfrak{a}']_1$ by Theorem 1.1. of [6] for $m = 1$ and $r_1(A^{d-2}) \leqslant \operatorname{degree}(\mathfrak{a}') - 1$ according to the remark after the proof of Theorem 1.1. in [6] for $m \geqslant 1$. Therefore Corollary 1 with $k = 1$ and $j = d - 2$ proves (i) since $\operatorname{degree}(\mathfrak{a}) = \operatorname{degree}(\mathfrak{a}')$ by Bezout's theorem.

Proving (ii.2) we will show that $2 - \operatorname{rank}_K[S/\mathfrak{a}']_1 \leqslant d - \operatorname{rank}_K[A]_1$. Then we can apply Corollary 1. It follows from [22], Lemma 3 that

$$\begin{aligned}
\operatorname{rank}_K[\mathfrak{a}']_1 &= \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-3})S]_1 + 1\\
&\leqslant \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-3})S: \langle P \rangle]_1 + 1\\
&= \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-4})S]_1 + 2\\
&\leqslant \cdots \leqslant \operatorname{rank}_K[\mathfrak{a}]_1 + d - 2.
\end{aligned}$$

The proof of (ii.3) is analogous to the proof of (ii.2). For this we note that we can choose the linear forms $l_i$ according to [4] such that $\mathfrak{a}' := \mathfrak{a} + (l_1, \ldots, l_{d-3})S: \langle P \rangle$ is regular if $\mathfrak{a}$ is regular. Therefore the assertion follows from Corollary 1 with

$k = 1$ and $j = d - 3$ by using

$$\begin{aligned} r_1(A/(l_1, \dots, l_{d-3})A) &= \mathrm{reg}(S/\mathfrak{a}') \\ &\leqslant \mathrm{degree}(\mathfrak{a}') + 3 - \mathrm{rank}_K[S/\mathfrak{a}']_1 \quad \text{by [9] (or [5])} \\ &\leqslant \mathrm{degree}(\mathfrak{a}') + d - \mathrm{rank}_K[A]_1 \quad \text{by [22],} \end{aligned}$$

Lemma 3 and Bezout's theorem.

Now we show (ii.1). The assertion is trivial for $d = 0$ and $d = 1$. Let $d \geqslant 2$. Take the linear forms $l_1, \dots, l_{d-2}$ constructed in our proof of (ii.2). Consider a general linear form $l_{d-1}$ and set $\mathfrak{a}' = \mathfrak{a} + (l_1, \dots, l_{d-1})S : \langle P \rangle$. Then we get

$$r_1(A/(l_1, \dots, l_{d-1})A) = \mathrm{reg}(S/\mathfrak{a}') \leqslant \left\{ \frac{\mathrm{degree}(\mathfrak{a}') - 1}{\mathrm{rank}_K[S/\mathfrak{a}']_1 - 1} \right\}$$

(see, e.g., [22], Lemma 1). We have again $\mathrm{degree}(\mathfrak{a}) = \mathrm{degree}(\mathfrak{a}')$ by Bezout's theorem and $\mathrm{rank}_K[\mathfrak{a}']_1 \leqslant \mathrm{rank}_K[\mathfrak{a}]_1 + d - 1$. On the other hand Lemma 3 of [22] gives us

$$\begin{aligned} \mathrm{rank}_K[\mathfrak{a}']_1 &= \mathrm{rank}_K[\mathfrak{a} + (l_1, \dots, l_{d-2})S]_1 + \\ &\quad + 1 \geqslant \mathrm{rank}_K[\mathfrak{a}]_1 + d - 1. \end{aligned}$$

Putting all together we obtain

$$r_1(A^{d-1}) \leqslant \left\{ \frac{\mathrm{degree}(\mathfrak{a}) - 1}{\mathrm{rank}_K[A]_1 - d} \right\}. \tag{+}$$

Consequently Corollary 1 with $k = 1$ and $j = d - 1$ proves the assertion (ii.1). This completes the proof of Theorem 2. Q.E.D.

COROLLARY 2. *Let $X$ be a nondegenerate, irreducible and reduced subscheme of $\mathbb{P}^n$ of dimension $d$. Then we have:*

$$d + 1 + e(H_P^{d+1}(S/I(X))) \leqslant \left\{ \frac{\mathrm{degree}(X) - 1}{\mathrm{codim}(X)} \right\}.$$

*Proof.* The assertion follows from (+) of the proof of Theorem 2(ii.1) and Lemma 2. Q.E.D.

REMARKS. Theorem 2(ii.1) is Theorem 2(ii) of [23]. Theorem 2(i), (ii.2) and (ii.3) improve Theorem 2(i) of [23] in some special cases.

The assumption $\mathrm{char}(K) = 0$ is necessary in Theorem 2(ii.1) because the general position lemma does not remain true if $\mathrm{char}(K) > 0$ (see [17], Example 1.2).

The conjecture of D. Bayer and M. Stillman [2] gives $\mathrm{reg}(\mathfrak{a}) \leqslant \mathrm{degree}(\mathfrak{a})$. Therefore Theorem 2(i) and (ii.2) (see also [22], corollary) prove the conjectures of Bayer and Stillman and of Eisenbud and Goto in case of $\mathrm{depth}(S/\mathfrak{a}) \geqslant d - 1$. This means, for example, that the latter conjecture is true for surfaces in $\mathbb{P}^4$ if the homogeneous coordinate ring has depth $\geqslant 2$. Note that (see the introduction) singular surfaces in $\mathbb{P}^4$ are the simplest varieties such that the conjecture of D. Eisenbud and S. Goto is open.

## 4. Counterexamples to an assertion of J. Harris and D. Eisenbud

In this section we will apply Theorem 2 in order to study the equality between Hilbert functions and Hilbert polynomials. Let $X$ be a subscheme of $P^n$ and $A = S/I(X)$ be its homogeneous coordinate ring. We recall that the Hilbert function of $X$ is defined by $h_X(t) := \mathrm{rank}_K[A]_t$ for $t \geqslant 0$. The so-called Hilbert polynomial, denoted by $p_X(t)$, is given by $h_X(t)$ for $t \gg 0$. It is well-known that $p_X(t) = \sum_{i\geqslant 0}(-1)^i h^i(X, \mathcal{O}_X(t))$ where $h^i(X, O_X(t))$ is the dimension of $H^i(X, \mathcal{O}_X(t))$. Following [7] the function $h'_X(t) := h^0(X, \mathcal{O}_X(t))$ is said to be the abstract Hilbert function of $X$. The index of regularity of $X$, denoted by $r(X)$, is defined as $r(X) := \min\{t \in \mathbb{N} : h_X(i) = p_X(i) \text{ for all } i \geqslant t\}$. Moreover, we set

$$r'(X) := \min\{t \in \mathbb{N} : h'_X(i) = p_X(i) \text{ for all } i \geqslant t\}, \quad \mathrm{reg}(X) = \mathrm{reg}(I(X)).$$

LEMMA 4. (i) ([14], *Corollary* 2.2) $r(X) \leqslant \mathrm{reg}(X) - \mathrm{depth}(S/I(X))$,
(ii) $r'(X) \leqslant r_2(X) - \max\{2, \mathrm{depth}(S/I(X))\}$.

*Proof.* We have $h_X(t) - p_X(t) = \sum_{i\geqslant 0}(-1)^i \mathrm{rank}_K[H^i_P(S/I(X))]_t$ according to [19]. This proves (i). We obtain (ii) from the characterization of $p_X(t)$ as an Euler-Poincaré characteristic and the isomorphisms $H^i(X, \mathcal{O}_X(t)) \cong [H^{i+1}_P(S/I(X))]_t$ for $i > 0$. Q.E.D.

In [7], p. 82 J. Harris and D. Eisenbud assert for reduced and irreducible subschemes $X$ of $\mathbb{P}^n_K$ ($\mathrm{char}(K) = 0$):

$$r'(X) \leqslant \left\{\frac{\mathrm{degree}(X)}{\mathrm{codim}(X)}\right\}. \qquad (**)$$

The letter [25] gives rise to study this claim again. In this connection, we will prove the following theorem.

THEOREM 3. (i) *We have for nondegenerate, irreducible and reduced curves* $X$:

$$r'(X) \leqslant \left\{\frac{\text{degree}(X) - 1}{\text{codim}(X)}\right\} - 1,$$

*that is, the assertion* (**) *is true for such curves.*

(ii) *There are nondegenerate, irreducible and reduced subschemes* $X$ *of* $\mathbb{P}^n$, $n \geqslant 4$, *of dimension* $d$ *such that the assertion* (**) *is not true for all* $d \geqslant 2$.

*Proof.* (i) Corollary 2 gives us for $d = 1$

$$r_2(X) - 1 = 2 + e(H^2_{\mathfrak{P}}(S/I(X))) \leqslant \left\{\frac{\text{degree}(X) - 1}{\text{codim}(X)}\right\}.$$

Hence (i) follows from Lemma 4(ii).

(ii) We consider the following class of examples: Let $m \geqslant 3$ be an integer. Let $X_m \subseteq \mathbb{P}^4$ be the surface given parametrically by $\{u^m, u^{m-1}v, u^{m-2}vw, uw^{m-1}, w^m\}$. It follows from [8], Proposition 3 that degree$(X_m) = m + 1$. Moreover, Corollary 3.4(ii) of [24] shows $H^1_{\mathfrak{P}}(S/\mathfrak{p}_m) = 0$, that is depth$(S/\mathfrak{p}_m) \geqslant 2$ where $\mathfrak{p}_m \subset S = K[x_0, \ldots, x_4]$ denotes the defining prime ideal of $X_m$. That is why we can apply Theorem 2(ii.2) and obtain reg$(X_m) \leqslant$ degree$(X_m) -$ codim$(X_m) + 1 = m$. Hence $\mathfrak{p}_m$ is generated by forms of degree $\leqslant m$ according to Theorem 1. Thus we can compute a minimal basis of $\mathfrak{p}_m$ from its parametrization and obtain

$$\mathfrak{p}_m = (x_1x_4 - x_2x_3, x_0x_2^ix_4^{m-1-i} - x_1^ix_3^{m-i}, \quad i = 0, \ldots, m - 1).$$

Since $\mathfrak{p}_m$ needs a generator of degree $m$ we get reg$(X_m) = m$. Since $p_{X_m}(t) - h'_{X_m}(t) = \sum_{i>0}(-1)^i\text{rank}_K[H^{i+1}_{\mathfrak{P}}(S/\mathfrak{p}_m)]_t$ we obtain from Corollary 2 that

$$p_{X_m}(t) - h'_{X_m}(t) = -\text{rank}_K[H^2_{\mathfrak{P}}(S/\mathfrak{p}_m)]_t$$
$$\text{for } t \geqslant \left\{\frac{\text{degree}(X_m) - 1}{\text{codim}(X_m)}\right\} - 2 = \left\{\frac{m}{2}\right\} - 2.$$

Applying Theorem 1 we get from reg$(X_m) = m$ and Corollary 2 that $[H^2_{\mathfrak{P}}(S/\mathfrak{p}_m)]_t \neq 0$ for

$$\left\{\frac{m}{2}\right\} - 2 \leqslant t \leqslant m - 3. \text{ Hence } r'(X_m) = m - 2 > \left\{\frac{m+1}{2}\right\} = \left\{\frac{\text{degree}(X_m)}{\text{codim}(X_m)}\right\}$$

for $m \geqslant 7$. This shows (ii) in case $d = 2$.

Let $j \geqslant 0$ be an integer. We denote by $Y_m$ the projective cone over $X_m$ in $\mathbb{P}^{j+4}$. Then we get $d := \dim(Y_m) = \dim(X_m) + j = 2 + j$, degree$(Y_m) =$ degree$(X_m)$ and

$\operatorname{depth}(S/I(Y_m)) = j + 2$ where $S = K[x_0, \ldots, x_{j+4}]$. Moreover, Lemma 2 gives us $\operatorname{reg}(Y_m) = \operatorname{reg}(X_m) = m$. Therefore we obtain from Corollary 2 as above:

$$r'(Y_m) = m - 2 - j > \left\{\frac{m+1}{2}\right\} = \left\{\frac{\operatorname{degree}(Y_m)}{\operatorname{codim}(Y_m)}\right\}$$

for $m \geqslant 2j + 7$ and for all $d = 2 + j$. Q.E.D.

REMARKS. (i) Using results of [6] the subschemes $Y_m$ show that the conjecture of D. Eisenbud and S. Goto is sharp in the sense that there are nondegenerate, irreducible and reduced varieties $X$ with $\operatorname{reg}(X) = \operatorname{degree}(X) - \operatorname{codim}(X) + 1$ in any dimension $\geqslant 1$ and of any degree $\geqslant 4$.

(ii) If the assertion (**) were true we could deduce $r_2(X) \leqslant \{\operatorname{degree}(X)/\operatorname{codim}(X)\} + 2$. But this is also not true in general as the varieties $Y_m$ show.

(iii) (**) is true in assuming, for example, that the subschemes $X$ are arithmetically Buchsbaum, i.e., that the homogeneous coordinate ring $S/I(X)$ is a graded Buchsbaum $K$-algebra. In this case we obtain from [21], Theorem 1. $\operatorname{reg}(X) \leqslant \{\operatorname{degree}(X) - 1/\operatorname{codim}(X)\} + 1$. Therefore Lemma 4(ii) yields

$$r'(X) < \left\{\frac{\operatorname{degree}(X) - 1}{\operatorname{codim}(X)}\right\}.$$

(iv) The varieties $Y_m$ are not arithmetically Buchsbaum for $m \geqslant 3$ due to [24], Lemma 4.11 and Corollary 4.7 and even not locally Buchsbaum for $m \geqslant 4$ because $Y_m \subset \mathbb{P}^{j+4}$ has a singularity in the point $\mathfrak{p} = (x_1, \ldots, x_{j+4})$ which is not Buchsbaum for $m \geqslant 4$. Otherwise $(S/I(Y_m))_{\mathfrak{p}}$ and consequently also $(S/I(Y_m) + x_3 S)_{\mathfrak{p}}$ would be Buchsbaum. Since $(I(Y_m) + x_3 S)_{\mathfrak{p}} = (x_3, x_1 x_4, x_2^{m-1}, x_2^{m-2} x_4, \ldots, x_2 x_4^{m-2}, x_4^{m-1})_{\mathfrak{p}} = (x_2^{m-1}, x_3, x_4)_{\mathfrak{p}} \cap (x_1, x_3, x_2^{m-1}, x_2^{m-2} x_4, \ldots, x_4^{m-1})_{\mathfrak{p}}$ we have $x_2(x_2^{m-1}, x_3, x_4)_{\mathfrak{p}} \nsubseteq (I(Y_m) + x_3 S)_{\mathfrak{p}}$ for $m \geqslant 4$. We immediately obtain a contradiction to a Buchsbaum ring property. (For the facts on Buchsbaum rings used here we refer to [20].)

(v) We can not apply the results of [9] or [5] in order to obtain the bound $\operatorname{reg}(X_m) \leqslant \operatorname{degree}(X_m) - \operatorname{codim}(X_m) + 1$ since the varieties $X_m$ are singular.

(vi) Since our counterexamples are singular varieties it is an open problem if (**) is true in the case of smooth varieties, see [26].

(vii) In [15], p. 370 P. Philippon considers the ideal $\mathfrak{a} := (x_1 x_4 - x_2 x_3, x_0 x_2^2 - x_1^2 x_3, x_0 x_4^2 - x_3^3)$. He asserts that $\mathfrak{a}$ is a prime ideal used in his computations. But it follows from $x_4(x_0 x_2 x_4 - x_1 x_3^2) \in \mathfrak{a}$ that $\mathfrak{a}$ is even not a primary ideal because no power of $x_4$ and $x_0 x_2 x_4 - x_1 x_3^2$ is contained in $\mathfrak{a}$. (Note that $\mathfrak{p}_3 = \mathfrak{a} + (x_0 x_2 x_4 - x_1 x_3^2)S$.)

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