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Normal forms and moduli spaces of curve singularities with semigroup $\langle 2p, 2q, 2pq + d \rangle$

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The aim of this paper is to classify map germs $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ and germs of curve singularities in \mathbb{C}^2 given by an equation of the type $f = (x^p + y^q)^2 + \sum_{i q + j p > 2 p q} a_{i j} x^i y^j = 0$ with a fixed Milnor number $\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$. Here we always suppose $p < q$ and $\gcd(p, q) = 1$.

The moduli space $M_{p, q, \mu}$ of the map germs described above is an affine Zariski-open subset of $\mathbb{C}^{2(p-1)(q-1) - p - q + 2 + [q/p]}$ divided by a suitable action of μ_{2pq} (the group of $2pq$ -roots of unity) depending on $\mu(f)$.

The moduli space $T_{p, q, \mu}$ of all plane curve singularities described above (which is the moduli space of all plane curve singularities with the semigroup $\langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$ if μ is even) is $\mathbb{C}^{(p-2)(q-2) + [q/p] - 1}$ divided by a suitable action of μ_d , $d = \mu - (2p-1)(2q-1)$.

In both cases we also get an algebraic universal family. It turns out that the Tjurina-number $\tau(f) = \dim_{\mathbb{C}} \mathbb{C}[[x, y]] / (f, \partial f / \partial x, \partial f / \partial y) = \mu(f) - (p-1)(q-1)$ depends only on $\mu(f)$ and p and q .

Constructing the moduli spaces we use the graduation of $\mathbb{C}[[x, y]]$ defined by p, q : $\deg x^i y^j = i q + j p$.

We use the following idea to construct the moduli spaces: Let $\mu = (2p-1)(2q-1) + d$. We prove that for all f of the above type we can choose the same monomial base of $\mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$ (Lemma 2). We choose α, β and that $\alpha < p, \alpha q + \beta p = 2pq + d$. Hence $\mu(f_0) = \mu$ with $f_0 = (x^p + y^q)^2 + x^\alpha y^\beta$. Then we consider a universal μ -constant unfolding of f_0 as a "global" family (Lemma 3). The parameter space U of that unfolding is an affine open subset of $\mathbb{C}^{2(p-1)(q-1) - p - q + 2 + [q/p]}$. The group μ_{2pq} acts on U and $M_{p, q, \mu} = U / \mu_{2pq}$.

To construct $T_{p,q,\mu}$ we consider the Kodaira-Spencer map of the universal μ -constant unfolding. The Kernel of the Kodaira-Spencer map is a Lie-algebra acting on U . The integral manifolds of that Lie-algebra are the analytically trivial subfamilies of the unfolding.

We choose a suitable section transversal to those integral manifolds, which turns out to be isomorphic to $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}$. The group μ_d acts on the corresponding family and we prove that $T_{p,q,\mu} = \mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$.

1. A normal form for map germs $(\mathbb{C}^2, 0) \rightarrow \mathbb{C}$ with initial term $(x^p + y^q)^2$

LEMMA 1. *Let*

$$f = \left(x^p + y^q + \sum_{iq+jp > pq} h_{ij}x^i y^j \right)^2 + \sum_{iq+jp \geq 2pq+d} w_{ij}x^i y^j$$

then $\mu(f) \geq (2p - 1)(2q - 1) + d$, and $\mu(f) = (2p - 1)(2q - 1) + d$ iff

$$f_d := \sum_{iq+jp=2pq+d} (-1)^{[i/p]} w_{ij} \neq 0.$$

Proof. Either f is irreducible or the components of f have the same tangent direction. This implies that

$$\mu(\tilde{f}) = \mu(f) - 2p(2p - 1),$$

where \tilde{f} is the blowing up

$$\begin{aligned} \tilde{f} = \frac{f(xy, y)}{y^{2p}} &= \left(x^p + y^{q-p} + \sum h_{ij}x^i y^{i+j-p} \right)^2 + \sum w_{ij}x^i y^{i+j-2p} \\ &= \left(x^p + y^{q-p} + \sum_{i(q-p)+jp > (q-p)p} h_{i,j-i+p}x^i y^j \right)^2 + \\ &\quad + \sum_{i(q-p)+jp \geq 2(q-p)p+d} w_{i,j-i+2p}x^i y^j. \end{aligned}$$

Using induction we may assume that

$$\mu(\tilde{f}) \geq (2p - 1)(2(q - p) - 1) + d$$

and

$$\begin{aligned} \mu(\tilde{f}) &= (2p - 1)(2(q - p) - 1) + d \text{ iff } 0 \neq \sum_{i_q + j_p = 2pq + d} (-1)^{[i/p]} w_{ij} \\ &= \sum_{i(q-p) + j_p = 2(p-q)p + d} (-1)^{[i/p]} w_{i, j-i+2p}. \end{aligned}$$

This yields the if part of the result. Now if f is as above and $\mu(f) > (2p - 1)(2q - 1) + d$, the condition $f_d = 0$ says that $(x^p + y^q)$ divides $\sum_{i_q + j_p = 2pq + d} w_{ij} x^i y^j$, and adding $-\frac{1}{2} \sum_{i_q + j_p = 2pq + d} w_{ij} x^i y^j$ to the first part of f one gets $f = (x^p + y^q + \dots)^2 +$ terms of degree greater than $2pq + d$. Continuing this way, we get the result.

LEMMA 2. Let $f = (x^p + y^q)^2 + \sum_{i_q + j_p > 2pq} h_{ij} x^i y^j$ and $\mu(f) = (2p - 1)(2q - 1) + d$. Let γ, δ such that $\gamma q + \delta p = 3pq - q - p + d, \gamma < p$. Let $B = \{(i, j) \in N^2 / i < 2p - 1, j < q - 1\} \cup \{(i, j) \in N^2 / i < p, j < q\} \cup \{(i, j), (i, j) \in N^2 / i < \gamma, j < \delta + q\}$. Then $\{x^i y^j\}_{(i, j) \in B}$ is a base of $\mathbb{C}[[x, y]] / (\partial f / \partial x, \partial f / \partial y)$.

Proof. We use the algorithm of Mora (cf. [3]) to compute a Groebner base of the ideal $(\partial f / \partial x, \partial f / \partial y)$. We consider $\mathbb{C}[[x, y]]$ as a graded ring with $\deg x = q, \deg y = p$. Let $f_1 = 1/2p(\partial f / \partial x)$ and $f_2 = 1/2q(\partial f / \partial y)$.

Consider $s(f_1, f_2) = y^{q-1} f_1 - x^{p-1} f_2$ and let f_3 be the reduction of $s(f_1, f_2) = y^{q-1} f_1 - x^{p-1} f_2$ with respect to the initial terms x^{2p-1} resp. $x^p y^{q-1}$ of f_1 resp. f_2 , i.e.

$$\begin{aligned} s(f_1, f_2) &= f_3 + h_1 f_1 + h_2 f_2 \\ f_3 &= \sum_{\gamma_i < p} l_i x^{\gamma_i} y^{\delta_i} \\ q\gamma_i + p\delta_i &= 3pq - q - p + i \end{aligned}$$

and the initial terms of h_1 resp. h_2 have degree $> pq - p$ resp. $> pq - q$. $f_3 \neq 0$ because of $\mu(f) < \infty$. Let k be the minimal such that $l_k \neq 0$, i.e. $l_k x^{\gamma_k} y^{\delta_k}$ is the initial term of f_3 . Consider now

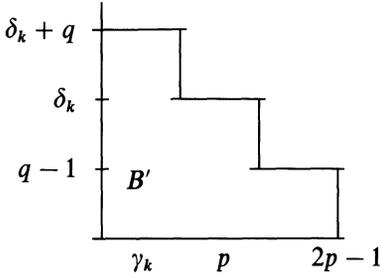
$$\begin{aligned} s(f_2, f_3) &= l_k y^{\delta_k - q + 1} f_2 - x^{p - \gamma_k} f_3 \\ &= l_k y^{\delta_k + q} + \text{terms of degree } > p\delta_k + pq \\ &=: f_4. \end{aligned}$$

It is not difficult to see that the reductions of $s(f_1, f_3)$ and $s(f_i, f_4) i = 1, 2, 3$ with respect to the initial terms of f_1, f_2, f_3, f_4 are zero, i.e. f_1, f_2, f_3, f_4 is a

Groebner base of $(\partial f/\partial x, \partial f/\partial y)$. This implies that

$$\{x^i y^j\}_{(i,j) \in B'}, B' = \{(i, j), i < 2p - 1, j < q - 1\} \cup \{(i, j), i < p, j < \delta_k\} \cup \{(i, j), i < \gamma_k, j < \delta_k + q\},$$

is a base of $\mathbb{C}[[x, y]]/(\partial f/\partial x, \partial f/\partial y)$.



This implies $\mu(f) = (p - 1)(q - 1) + q\gamma_k + p\delta_k$ and therefore $\gamma = \gamma_k$ and $\delta = \delta_k$ and $B = B'$. □

LEMMA 3. Let $f = (x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij} x^i y^j$ and $\mu(f) = (2p - 1)(2q - 1) + d$.

Let γ, δ be defined by

$$\gamma < p \text{ and } \gamma q + \delta p = 3pq - q - p + d$$

Let $B_0 = \{(i, j), iq + jp > pq, i \leq p - 2, j \leq q - 2\}$;

$$B_1 = \{(i, j), iq + jp \geq 2pq + d, i < p, j < \delta\} \cup \{(i, j), iq + jp \geq 2pq + d, i < \gamma, j < \delta + q\}.$$

There is an automorphism $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ such that

$$f(\varphi) = \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij} x^i y^j\right)^2 + \sum_{(i,j) \in B_1} w_{ij} x^i y^j$$

for suitable $h_{ij}, w_{ij} \in \mathbb{C}$.

Proof. Using Lemma 1 we may assume that

$$f = \left(x^p + y^q + \sum_{iq+jp > pq} b_{ij} x^i y^j\right)^2 + \sum_{iq+jp \geq 2pq+d} c_{ij} x^i y^j.$$

Assume that there is an automorphism $\varphi^{(k)}$ such that

$$f(\varphi^{(k)}) = \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq pq + k} b_{ij}^{(k)} x^i y^j \right)^2 + \\ + \sum_{(i,j) \in B_1} w_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq 2pq + k} c_{ij}^{(k)} x^i y^j,$$

$\varphi^{(1)} = \text{identity}$.

Now

$$\sum_{iq + jp = 2pq + k} c_{ij}^{(k)} x^i y^j = (x^p + y^q)H + \sum_{iq + jp = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} x^{i_0} y^{j_0}$$

for a suitable homogeneous H of degree $pq + k$ and $i_0 < p, i_0q + j_0p = 2pq + k$.

If $\sum_{iq + jp > 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} \neq 0$ and $(i_0, j_0) \notin B_1$ then $k \geq pq - q - p + d$ (Lemma 1) and $j_0 \geq \delta, i_0 \geq \gamma$ or $j_0 \geq \delta + q$.

Let α, β be defined by $q\alpha + p\beta = 2pq + d, \alpha < p$, then $w_{\alpha\beta} \neq 0$. Notice that $\alpha - 1 \equiv \gamma \pmod p$ and $\beta - 1 \equiv \delta \pmod q$.

Let

$$g := x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq pq + k} b_{ij}^{(k)} x^i y^j$$

and

$$\omega := e \cdot x^\xi y^\eta \left(\frac{\partial g}{\partial y} \frac{\partial}{\partial x} - \frac{\partial g}{\partial x} \frac{\partial}{\partial y} \right), \quad \xi q + \eta p = k - pq + p + q - d$$

$$e := \frac{1}{(\alpha q + \beta p) w_{\alpha\beta}} \cdot \sum_{ip + jq = 2pq + k} (-1)^{[i/p] - [\alpha - 1 + \xi/p] + 1} c_{ij}^{(k)}$$

$$\text{with } (\xi, \eta) = \begin{cases} (i_0 - \gamma, j_0 - \delta) & \text{if } j_0 \geq \delta, \quad i_0 \geq \gamma \\ (i_0 - \gamma + p, j_0 - \delta - q) & \text{if } j_0 \geq \delta + q. \end{cases}$$

Let $\psi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ the automorphisms corresponding to the vector field ω , then $g(\psi) = g$.

Hence,

$$f(\psi \circ \varphi^{(k)}) = g^2 + \sum_{(i,j) \in B_0} w_{ij}^{(k)} x^i y^j + \sum_{iq + jp \geq 2pq + k} \bar{c}_{ij}^{(k)} x^i y^j$$

and

$$\begin{aligned} & \sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} \bar{c}_{ij}^{(k)} \\ &= \sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} + (-1)^{[\alpha - 1 + \zeta/p]} (\alpha q + \beta p) w_{\alpha\beta} \cdot e. \end{aligned}$$

If $(i_0, j_0) \notin B_1$ we may assume now that $\sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} = 0$.

Let $g_1 := g + \frac{1}{2}H$ and

$$\sum_{i_q + j_p = pq + k} b_{ij}^{(k)} x^i y^j + \frac{1}{2}H + m_k \frac{\partial g_1}{\partial x} + n_k \frac{\partial g_1}{\partial y} = \sum_{(i,j) \in B_0} d_{ij}^{(k)} x^i y^j.$$

The degree of the initial part of m_k resp. n_k is $q + k$ resp. $p + k$.

We define $\varphi^{(k+1)}$ by

$$\varphi^{(k+1)}(x) = \varphi^{(k)}(x) + m_k$$

$$\varphi^{(k+1)}(y) = \varphi^{(k)}(y) + n_k$$

and

$$h_{ij}^{(k+1)} = h_{ij}^{(k)} + d_{ij}^{(k)}$$

$$w_{ij}^{(k+1)} = w_{ij}^{(k)} \quad \text{if } (i, j) \neq (i_0, j_0)$$

$$w_{i_0, j_0}^{(k+1)} = w_{i_0, j_0}^{(k)} + \sum_{i_q + j_p = 2pq + k} (-1)^{[i/p]} c_{ij}^{(k)} \quad \text{if } (i_0, j_0) \in B_1.$$

Then

$$\begin{aligned} f(\varphi^{(k+1)}) &= \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}^{(k+1)} x^i y^j + \sum_{i_q + j_p \geq pq + k + 1} b_{ij}^{(k+1)} x^i y^j \right)^2 + \\ &+ \sum_{(i,j) \in B_1} w_{ij}^{(k+1)} x^i y^j + \sum_{i_q + j_p \geq 2pq + k + 1} c_{ij}^{(k+1)} x^i y^j \end{aligned}$$

for suitable $b_{ij}^{(k+1)}, c_{ij}^{(k+1)}$. □

LEMMA 4. Let

$$f_t = (x^p + y^q)^2 + \sum_{i_q + j_p > 2pq} a_{ij}(t) x^i y^j, \quad a_{ij}(t) \in \mathbb{C}[t]$$

and $\mu(f_t) = (2p - 1)(2q - 1) + d$ for $t \in \mathbb{C}$.

Let γ, δ, B_0, B_1 be as in Lemma 3. There is a $\mathbb{C}[t]$ -automorphism $\varphi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$ such that

$$f_t(\varphi_t) = \left(x^p + y^q + \sum_{(i,j) \in B_0} h_{ij}(t)x^i y^j \right)^2 + \sum_{(i,j) \in B_1} w_{ij}(t)x^i y^j$$

for suitable $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$.

The proof is similar to that of Lemma 3. □

Let us consider the family

$$F(x, y, H, W) = \left(x^p + y^q + \sum_{(i,j) \in B_0} H_{ij}x^i y^j \right)^2 + \sum_{(i,j) \in B_1} W_{ij}x^i y^j$$

depending on the parameters $H = (H_{ij})_{(i,j) \in B_0}$, $W = (W_{ij})_{(i,j) \in B_1}$ and define $N = \# B_0 + \# B_1$, then $\mu(F) = (2p - 1)(2q - 1) + d$ on the open set U defined by $W_{\alpha\beta} \neq 0$, $\alpha q + \beta p = 2pq + d$, in $\mathbb{C}^N = \text{Spec } \mathbb{C}[H, W]$. Notice that $N = 2(p - 1)(q - 1) - p - q + 2 + [q/p]$ is not depending on d !

The group of $2pq$ -roots of unity acts on U :

$$\lambda \in \mu_{2pq}, \lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})). \quad \square$$

THEOREM 1. U/μ_{2pq} is the moduli space of all functions

$$f = (x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij}x^i y^j$$

with $\mu(f) = (2p - 1)(2q - 1) + d$ and F is the universal family.

Proof. Using Lemma 3 we have to prove the following

LEMMA 5. Let φ be an automorphism $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ such that

$$F(\varphi(x), \varphi(y), \bar{h}, \bar{w}) = F(x, y, h, w) \quad (*)$$

for $(\bar{h}, \bar{w}), (h, w) \in U \subseteq \mathbb{C}^N$ then $\lambda \cdot (\bar{h}, \bar{w}) = (h, w)$ for a suitable $\lambda \in \mu_{2pq}$.

Proof. Let $\bar{x} := \varphi(x)$, $\bar{y} := \varphi(y)$ then grouping the squared part of (*) one gets:

$$\begin{aligned} & \left(x^p + y^q + \bar{x}^p + \bar{y}^q + \sum \bar{h}_{ij} \bar{x}^i \bar{y}^j + \sum h_{ij} x^i y^j \right) \times \\ & \times \left(x^p + y^q - \bar{x}^p - \bar{y}^q - \sum \bar{h}_{ij} \bar{x}^i \bar{y}^j + \sum h_{ij} x^i y^j \right) \\ & = \sum \bar{w}_{ij} \bar{x}^i \bar{y}^j - \sum w_{ij} x^i y^j. \end{aligned}$$

This equation implies obviously that the degree of the initial term of $\varphi(x)$ is $\geq q$ and

$$\bar{x} = \lambda^q \left(x + \sum_{iq+jp>q} a_{ij}^{(1)} x^i y^j \right) \quad \bar{y} = \lambda^p \left(y + \sum_{iq+jp>p} a_{ij}^{(2)} x^i y^j \right), \lambda \in \mu_{2pq}.$$

We may assume that $\lambda = 1$ and prove $(\bar{h}, \bar{w}) = (h, w)$.

Let the degree of the leading parts of both sides of the above equation be $2pq + m$ and let r be the degree of φ , i.e. $a_{ij}^{(1)} = 0$ if $iq + jp < q + r$, $a_{ij}^{(2)} = 0$ if $iq + jp < p + r$ and $a_{ij}^{(1)} \neq 0$ or $a_{ij}^{(2)} \neq 0$ for suitable i, j with $iq + jp = q + r$ resp. $iq + jp = p + r$.

1. Step. We prove that

(a) $r \geq pq - p - q$

$$\sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} \cdot k, \quad \sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} \cdot k$$

(b) $h = \bar{h}$ and $w_{ij} = \bar{w}_{ij}$ if $iq + jp < 3pq - p - q + d$.

First of all $m \geq d + r$ because the leading part of the left side of the equation is divisible by $x^p + y^q$ and $m < d + r$ would imply that the leading part of the right side is a monomial. This implies $w_{ij} = \bar{w}_{ij}$ if $iq + jp < 2pq + d + r$. Now $h_{ij} = \bar{h}_{ij}$ if $iq + jp < pq + r$. Otherwise the leading part of the left side of the equation would be $2(x^p + y^q)(h_{ij} - \bar{h}_{ij})x^i y^j$ for some i, j with $iq + jp < pq + r$ and therefore of degree $2pq + r < 2pq + m$.

Now suppose $r < pq - p - q$. Then there is at most one monomial of degree $p + r$ resp. $q + r$.

If $iq + jp = pq + r$ for some $(i, j) \in B_0$ and

$$qi_0 + pj_0 = q + r$$

$$qi_1 + pj_1 = p + r$$

then

$$(h_{ij} - \bar{h}_{ij})x^i y^j - pa_{ioj_0}^{(1)} x^{i_0+p-1} y^{j_0} - qa_{i_1j_1}^{(2)} x^{i_1} y^{j_1+q-1} = 0$$

otherwise the leading part of the left side of the equation would have degree $2pq + r < 2pq + m$.

But $(i, j) \in B_0$, i.e. $i < p - 1$ and $j < q - 1$. This implies $h_{ij} = \bar{h}_{ij}$, $a_{ioj_0}^{(1)} = a_{i_1j_1}^{(2)} = 0$ (because of $r < pq - p - q$ we have $i_1 < p - 1$). This is a contradiction since $a_{ioj_0}^{(1)} \neq 0$ or $a_{i_1j_1}^{(2)} \neq 0$ by the definition of r .

Similarly one gets a contradiction if there is no $(i, j) \in B_0$ with $qi + pj = p + r$, resp. no i_0, j_0 with $qi_0 + pj_0 = q + r$ resp. no i_1, j_1 with $qi_1 + pj_1 = p + r$.

This proves that $r \geq pq - p - q$. With the same method we obtain

$$\sum_{iq+jp=p+r} a_{ij}^{(2)} x^i y^j = -\frac{1}{q} x^{p-1} k \quad \text{and} \quad \sum_{iq+jp=q+r} a_{ij}^{(1)} x^i y^j = \frac{1}{p} y^{q-1} k.$$

(b) is clear now by the choice of B_0 and the fact that $r \geq pq - p - q$.

2. Step. We prove that $r \geq 2pq - p - q$.

Assume that $r < 2pq - p - q$. Then $\deg k < pq$, i.e., k is a monomial.

The leading part of the left side of the above equation is divisible by $x^p + y^q$.

The leading part L of the right side is

$$(\bar{w}_{ij} - w_{ij})x^i y^j + \bar{w}_{\alpha\beta} \cdot k \left(\frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if $iq + jp = 2pq + m$ for some $(i, j) \in B_1$ or

$$\bar{w}_{\alpha\beta} \cdot k \left(\frac{\alpha}{p} x^{\alpha-1} y^{\beta+q-1} - \frac{\beta}{q} x^{\alpha+p-1} y^{\beta-1} \right)$$

if $iq + jp \neq 2pq + m$ for $(i, j) \in B_1$.

Let $k = \kappa \cdot x^\xi y^\eta$. If $\alpha + \xi - 1 < p$, then $i = \alpha + \xi - 1$ and $j = \beta + \eta + q - 1$. If $\alpha + \xi - 1 \geq p$, then $i = \alpha + \xi - 1 - p$ and $j = \beta + \eta + 2q - 1$. But $(\alpha + \xi - 1, \beta + \eta + q - 1) \notin B_1$ and $(\alpha + \xi - 1 - p, \beta + \eta + 2q - 1) \notin B_1$. This implies $L = \bar{w}_{\alpha\beta} \cdot \kappa x^{\alpha-1} y^{\beta-1} ((\alpha/p)y^q - (\beta/p)x^p)$ which is not divisible by $x^p + y^q$. This is a contradiction and therefore $r \geq 2pq - p - q$.

Now $iq + jp \leq 4pq - 2p - 2q + d$ for $(i, j) \in B_1$ then $w_{ij} = \bar{w}_{ij}$ for all $(i, j) \in B_1$. □

2. The construction of the moduli space

We will construct the moduli space of all plane curve singularities given by an equation $(x^p + y^q)^2 + \sum_{iq+jp > 2pq} a_{ij} x^i y^j = 0$ with fixed Milnor number μ .

For μ being even we get especially the moduli space for all irreducible plane curve singularities with the semigroup $\Gamma = \langle 2p, 2q, \mu - 2(p-1)(q-1) + 1 \rangle$.

We use the family

$$V(F) \subseteq U \times \mathbb{C}^2 \rightarrow U$$

constructed in Theorem 1.

U admits a \mathbb{C}^* -action defined by

$$\lambda \circ ((h_{ij}), (w_{ij})) := ((\lambda^{iq+jp-pq} h_{ij}), (\lambda^{iq+jp-2pq} w_{ij})).$$

We get

$$F(\lambda^q x, \lambda^p y, h, w) = \lambda^{2pq} F(x, y, \lambda \circ (h, w)).$$

If $\mu = (2p - 1)(2q - 1) + d$ and $\alpha q + \beta p = 2pq + d, \alpha < p$, then $U \subseteq \mathbb{C}^N$ was defined by $W_{\alpha, \beta} \neq 0$.

For the construction of the moduli space it is enough to consider the restriction of our family to the transversal section to the orbits of the \mathbb{C}^* -action defined by $W_{\alpha, \beta} = 1$.

Let W' be defined by $W = (W_{\alpha, \beta}, W')$ and $G(x, y, H, W') = F(x, y, H, 1, W')$. The parameter space of G is $\mathbb{C}^{N-1} = \text{Spec } \mathbb{C}[H, W']$.

The group μ_d of d th roots of unity acts on the family

$$V(G) \subseteq \mathbb{C}^2 \times \mathbb{C}^{N-1} \rightarrow \mathbb{C}^{N-1}$$

induced by the above \mathbb{C}^* -action

$$G(\lambda^q x, \lambda^p y, h, w') = \lambda^{2pq} G(x, y, \lambda \circ (h, w')) \quad \lambda \in \mu_d.$$

LEMMA 6. *Let $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ be an automorphism and $u \in \mathbb{C}[[x, y]]$ a unit such that*

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

then there is a $\lambda \in \mu_d$ such that (h, w') and $\lambda \circ (\bar{h}, \bar{w}')$ are contained in an analytically trivial subfamily of $V(G) \rightarrow \mathbb{C}^{N-1}$.

Proof. Let

$$\varphi(x) = \sum a_{ij}^{(1)} x^i y^j \quad \text{and} \quad \varphi(y) = \sum a_{ij}^{(2)} x^i y^j, \quad u = \sum u_{ij} x^i y^j.$$

$$u \cdot G(\varphi(x), \varphi(y), h, w') = G(x, y, \bar{h}, \bar{w}')$$

implies

- (1) $a_{ij}^{(1)} = 0$ if $iq + jp < q$
- (2) $a_{1,0}^{(1)2p} = a_{0,1}^{(2)2q} = a_{1,0}^{(1)p} a_{0,1}^{(2)q} = u_{0,0}^{-1}$.

Let $a_{1,0}^{(1)} = \lambda^q$ and $a_{0,1}^{(2)} = \lambda^p$.

We will prove later that $\lambda^d = 1$.

Now we may assume that $\lambda = 1$ and prove that (h, w') and (\bar{h}, \bar{w}') are contained in an analytically trivial subfamily of $V(G) \rightarrow \mathbb{C}^{N-1}$.

We choose

- (1) $u(t) \in \mathbb{C}[t][[x, y]]$ with the following properties $u(0) = 1$, $u(1) = u$ and u is a unity for all $t \in \mathbb{C}$.
- (2) $\varphi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$ with the following properties $\varphi_0 = \text{identity}$, $\varphi_1 = \varphi$ and φ_t is an automorphism of positive degree for all $t \in \mathbb{C}$.

Let $H(t) := u(t)G(\varphi_t(x), \varphi_t(y), h, w')$ and apply Lemma 4. There is an $\mathbb{C}[t]$ -automorphism $\Phi_t: \mathbb{C}[t][[x, y]] \rightarrow \mathbb{C}[t][[x, y]]$ such that

$$H(\Phi_t) = F(x, y, h(t), w(t))$$

for suitable $h_{ij}(t), w_{ij}(t) \in \mathbb{C}[t]$ with the property

$$h(0) = h$$

$$w(0) = (1, w').$$

$H(\Phi_t)$ has a constant Milnor number, i.e. $w_{\alpha, \beta}(t)$ has to be constant.

This implies

$$H(\Phi_t) = G(x, y, h(t), w'(t)).$$

But,

$$G(x, y, h(1), w'(1)) = H(\Phi_1) = G(\Phi_1(x), \Phi_1(y), \bar{h}, \bar{w}').$$

Using Lemma 5 and the fact that Φ_1 has positive degree we get

$$\bar{h} = h(1)$$

$$\bar{w}' = w'(1),$$

i.e. (h, w') and (\bar{h}, \bar{w}') are in the trivial family

$$G(x, y, h(t), w'(t)) = u(\Phi_t)G(\Phi_t \varphi_t, h, w').$$

To finish the proof of Lemma 6 we have to prove

LEMMA 7. *Let*

$$f_k = \left(x^p + y^q + \sum_{i+jp > pq} a_{ij}^{(k)} x^i y^j \right)^2 + x^\alpha y^\beta + \sum_{i+jp > 2pq+d} b_{ij}^{(k)} x^i y^j, \quad k = 1, 2$$

$$\alpha < p, \alpha q + \beta p = 2pq + d.$$

Let $\varphi: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y]]$ be an automorphism with the property

$$\varphi(x) = \lambda^q x + \text{terms of degree } > q$$

$$\varphi(y) = \lambda^p y + \text{terms of degree } > p$$

and u a unit, such that

$$f_1(\varphi) = f_2 \cdot u$$

then $\lambda^d = 1$.

Proof. $u = \lambda^{2pq} + \text{terms of higher degree}$.

$$u \cdot f_2 = \lambda^{2pq} \left(x^p + y^q + \sum_{iq+jp > pq} \bar{a}_{ij}^{(2)} x^i y^j \right)^2 + \lambda^{2pq} x^\alpha y^\beta + \sum_{iq+jp > 2pq+d} \bar{b}_{ij}^{(2)} x^i y^j$$

$$f_1(\varphi) = \lambda^{2pq} \left(x^p + y^q + \sum_{iq+jp > pq} \bar{a}_{ij}^{(1)} x^i y^j \right)^2 + \lambda^{2pq+d} x^\alpha y^\beta + \sum_{iq+jp \geq 2pq+d} \bar{b}_{ij}^{(1)} x^i y^j$$

for suitable $\bar{a}_{ij}^{(k)}, \bar{b}_{ij}^{(k)}$.

This implies

$$\begin{aligned} & \left(2x^p + 2y^q + \sum_{iq+jp > pq} (\bar{a}_{ij}^{(1)} + \bar{a}_{ij}^{(2)}) x^i y^j \right) \cdot \sum_{iq+jp > pq} (\bar{a}_{ij}^{(1)} - \bar{a}_{ij}^{(2)}) x^i y^j \\ &= (1 - \lambda^d) x^\alpha y^\beta + \sum_{iq+jp > 2pq+d} \lambda^{-2pq} (\bar{b}_{ij}^{(2)} - \bar{b}_{ij}^{(1)}) x^i y^j. \end{aligned}$$

Because the leading term of the left side of the equation is divisible by $x^p + y^q$, we get $\lambda^d = 1$. □

We consider now the Kodaira-Spencer map of the family

$$V(G) \rightarrow \mathbb{C}^{N-1}:$$

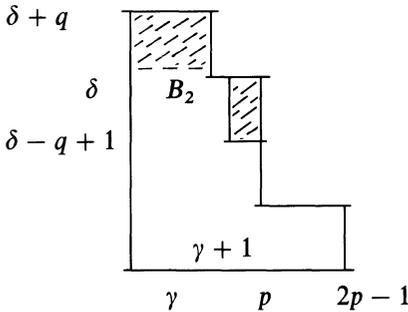
$$\rho: \text{Der}_{\mathbb{C}} \mathbb{C}[H, W'] \rightarrow \mathbb{C}[H, W'][[x, y]] / \left(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

defined by

$$\rho(\delta) = \text{class}(\delta G).$$

The kernel of the Kodaira-Spencer map is a Lie-algebra L and along the integral manifolds of L the family is analytically trivial. We will choose a transversal section to the integral manifolds of L and divide by the action of μ_d to get the moduli space. To describe this transversal section we choose a suitable subset of B_1 :

$$B_2 = \{(i, j) \in B_1, i \leq \gamma, j \leq \delta\} \cup \{(i, j) \in B_1, j \leq \delta - q\}.$$



Let $M := \#B_0 + \#B_2 = N - (p - 1)(q - 1) = (p - 2)(q - 2) + [q/p] - 1$. Let $W'' := (W_{ij})_{(i,j) \in B_2}$ and $\mathbb{C}^M = \text{Spec } \mathbb{C}[H, W'']$

$$G_u(x, y, H, W'') := \left(x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j \right)^2 + x^\alpha y^\beta + \sum_{(i,j) \in B_2} W_{ij} x^i y^j$$

As before μ_d acts on the family $V(G_u) \subseteq \mathbb{C}^2 \times \mathbb{C}^M \rightarrow \mathbb{C}^M$.

THEOREM 2. \mathbb{C}^M/μ_d is the moduli space of all plane curve singularities defined by an equation

$$(x^p + y^q)^2 + \sum_{iq + jp > 2pq} a_{ij} x^i y^j = 0$$

with Milnor numbers $\mu = (2p - 1)(2q - 1) + d$ and G_u is the corresponding universal family.

Especially the Tjurina number $\tau = \mu - (p - 1)(q - 1)$ only depends on μ for these singularities.

COROLLARY. Let $\Gamma = \langle 2p, 2q, 2pq + d \rangle$, d odd, a semigroup.

Then $\mathbb{C}^{(p-2)(q-2)+[q/p]-1}/\mu_d$ is the moduli space of all irreducible plane curve singularities with the semigroup Γ .

G_u is the corresponding universal family.

Proof. To prove the theorem we compute generators of the kernel of the

Kodaira-Spencer map.

Let

$$G^{(0)} = x^p + y^q + \sum_{(i,j) \in B_0} H_{ij} x^i y^j$$

$$G^{(1)} = x^a y^b + \sum_{\substack{(i,j) \in B_1 \\ iq + jp > 2pq + d}} W_{ij} x^i y^j, \text{ i.e.}$$

$$G = G^{(0)2} + G^{(1)}.$$

Let $\delta \in \text{Der}_{\mathbb{C}} \mathbb{C}[H, W']$ be a vector field which belongs to the kernel of the Kodaira-Spencer map, i.e.

$$\delta G \in \left(G, \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

Now

$$\delta G = 2G^{(0)} \sum_{(i,j) \in B_0} \delta H_{ij} x^i y^j + \sum_{\substack{(i,j) \in B_1 \\ iq + jp > 2pq + d}} \delta W_{ij} x^i y^j = S \cdot G \text{ mod } \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$$

for a suitable $S \in \mathbb{C}[H, W'][[x, y]]$.

We will associate to any monomial $x^a y^b$, $(a, b) \neq (0, 0)$, a vector field $\delta_{a,b} \in \text{Der}_{\mathbb{C}}[H, W']$ such that

$$\delta_{a,b} G = x^a y^b G \text{ mod } \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

Obviously $\{\delta_{a,b}\}$ generate the kernel of the Kodaira-Spencer map as $\mathbb{C}[H, W']$ -module.

Now consider

$$x^a y^b G = x^a y^b G^{(0)2} + x^a y^b G^{(1)}.$$

$$\text{Let } x^a y^b G^{(0)} = \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + L_1 \frac{\partial G^{(0)}}{\partial x} + L_2 \frac{\partial G^{(0)}}{\partial y}$$

for suitable $E_{ij}^{ab} \in \mathbb{C}[H, W']$, $L_1, L_2 \in \mathbb{C}[H, W'][[x, y]]$,

$$L_1 = \frac{1}{p} x^{a+1} y^b + \text{terms of higher degree}$$

$$L_2 = \frac{1}{q} x^a y^{b+1} + \text{terms of higher degree,}$$

then

$$x^a y^b G = G^{(0)} \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right).$$

The leading term of

$$x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y}$$

is $-(d/2pq)x^{\alpha+a}y^{\beta+b}$.

Using Lemma 2 we get

$$\begin{aligned} & x^a y^b G^{(1)} - \frac{1}{2} L_1 \frac{\partial G^{(1)}}{\partial x} - \frac{1}{2} L_2 \frac{\partial G^{(1)}}{\partial y} \\ &= \sum_{\substack{(i,j) \in B_1 \\ iq + jp \geq 2pq + d + aq + bp}} D_{ij}^{ab} x^i y^j \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right) \end{aligned}$$

for suitable $D_{ij}^{ab} \in \mathbb{C}[H, W']$.

This implies

$$x^a y^b G = G^{(0)} \sum_{(i,j) \in B_0} E_{ij}^{ab} x^i y^j + \sum_{\substack{(i,j) \in B_1 \\ iq + jp \geq 2pq + d + aq + bp}} D_{ij}^{ab} x^i y^j \bmod \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y} \right)$$

We define for $(a, b) \neq (0, 0)$

$$\delta_{a,b}(H_{ij}) := \frac{1}{2} E_{ij}^{ab}$$

$$\delta_{a,b}(W_{ij}) := D_{ij}^{ab}, \text{ i.e.,}$$

$$\delta_{a,b} = \frac{1}{2} \sum E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \sum D_{ij}^{ab} \frac{\partial}{\partial W_{ij}}$$

The vector fields $\delta_{a,b}$ have the following properties:

- (1) $\delta_{a,b}$ is zero if $aq + bp > 2pq - 2p - 2q$
- (2) $\delta_{a,b}(W_{ij}) = 0$ if $iq + jp < 2pq + d + aq + bp$
- (3) $\delta_{a,b}(W_{ij}) = -d/2pq$ if $(i, j) = (\alpha + a, \beta + b)$ or $(i, j) = (\alpha + a - p, \beta + b + q)$ (in this case $iq + jp = 2pq + d + aq + bp$).

- (4) $\delta_{a,b}(H_{ij}) = 0$ for all $(i, j) \in B_0$ if $aq + bp \geq pq - 2p - 2q$
- (5) For $iq + jp \geq 3pq + d - q$ and $(i, j) \in B_1$ there is (a', b') such that

$$(i, j) = (\alpha + a', \beta + b') \text{ or } (i, j) = (\alpha + a' - p, \beta + b' + q),$$

i.e. $\delta_{a',b'}(W_{ij}) = -d/2pq$.

- (6) For any (a, b) , $aq + bp < pq - q$, always $(\alpha + a, \beta + b)$ or $(\alpha + a - p, \beta + b + q) \in B_1$, i.e. $\delta_{a,b}(W_{ij}) = -d/2pq$ for the corresponding $(i, j) \in B_1$.

(1) and (4) hold because of the fact that

$$iq + jp \leq 2pq - 2p - 2q \quad \text{if } (i, j) \in B_0$$

$$iq + jp \leq 4pq + d - 2p - 2q \quad \text{if } (i, j) \in B_1.$$

(2) and (3) hold because of the fact that the leading term of $G^{(1)}$ has degree $2pq + d$ and because of Lemma 2. To prove (5) we consider two cases

1. Case $i \geq \alpha - 1$

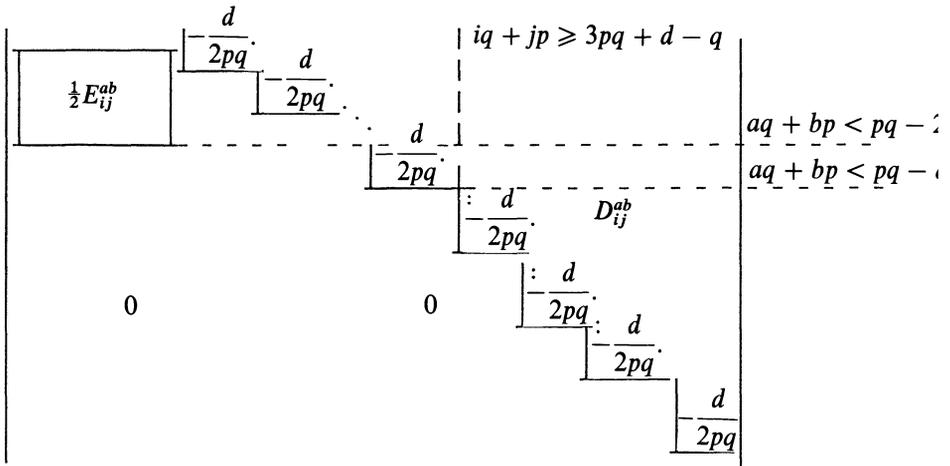
In this case $(i, j) \in B_1$ implies $i \leq p - 1$. But $iq + jp \geq 3pq + d - q$ implies $j \geq \beta$. Then $a' = i - \alpha, b' = j - \beta$ have the required properties. Notice that $i < \alpha - 1$ and $iq + jp \geq 3pq + d - q$ implies $(i, j) \notin B_1$

2 Case $i < \alpha - 1$

Now $iq + jp \geq 3pq + d - q$ implies $j \geq \beta + q$ then $a' = p + i - \alpha, b' = j - \beta - q$ have the required properties. (6) is similar to (5):

We may assume that $(\alpha + a, \beta + b) \notin B_1$. This implies $2p - 3 \geq \alpha + a \geq p$ and $\alpha \geq 2$ because $b \leq q - 2, a \leq p - 2$. Suppose $(\alpha + a - p, \beta + b + q) \notin B_1$ then $\beta + b + q \geq \beta + 2q - 1$, i.e. $b \geq q - 1$, or $\alpha + a - p \geq \alpha - 1$, i.e. $a \geq p - 1$, but this is not possible. □

For the coefficients to the vectorfields $\delta_{a,b}$ we get, because of (1)–(6), the following matrix:



This implies that the kernel of the Kodaira–Spencer map is generated (as $\mathbb{C}[H, W']$ -module) by the vector fields

$$\delta'_{ij} := \frac{\partial}{\partial W_{ij}} \quad (i, j) \in B_1, \quad iq + jp \geq 3pq + d - q$$

and

$$\begin{aligned} \delta'_{l,m} = & -\frac{pq}{d} \sum_{(i,j) \in B_0} E_{ij}^{ab} \frac{\partial}{\partial H_{ij}} + \frac{\partial}{\partial W_{l,m}} + \\ & + \sum_{\substack{(i,j) \in B_1 \\ 2pq + d + aq + bp < iq + jp < 3pq + d - q}} \left(-\frac{2pq}{d} \right) D_{ij}^{ab} \frac{\partial}{\partial W_{ij}} \\ & \times (l, m) \in B_1 \setminus (B_2 \cup \{(\alpha, \beta)\}), \quad lq + mp < 3pq + d - q, \end{aligned}$$

with $(l, m) = \begin{cases} (\alpha + a, \beta + b) & \text{if } l \geq \alpha \\ (\alpha + a - p, \beta + b + q) & \text{else.} \end{cases}$

The vectorfields $\delta'_{l,m}$ act nilpotently on $\mathbb{C}[H, W']$. Namely, if we consider $\mathbb{C}[H, W']$ as a graded algebra defined by $\deg H_{ij} = pq - iq - jp < 0$, $\deg W_{ij} = 2pq - iq - jp < 0$ then the E_{ij}^{ab} resp. D_{ij}^{ab} are polynomials in $\mathbb{C}[H, W']$ of degree $\geq aq + bp + pq - iq - jp$ resp. $\geq aq + bp + 2pq - iq - jp$. Notice that their degree is always ≤ 0 . Let $A \in \mathbb{C}[H, W']$ be any polynomial of degree $0 \geq \deg A = s$ ($\deg A =$ minimum of the degrees of the monomials in A). Then the degree of $\delta'_{lm}(A) > s$. Therefore there is some n with $\delta'^n_{lm}(A) = 0$.

LEMMA 8. *Let A be a ring of finite type over a field k . $L \subseteq \text{Der}_k(A)$ a Lie-Algebra.*

Let $\delta_1, \dots, \delta_r$ vector fields with the following properties:

- (1) $\delta_1, \dots, \delta_r \in L$ and $L \subseteq \sum \delta_i A$
- (2) $[\delta_i, \delta_j] \in \sum_{k > \max\{i,j\}} \delta_k A$
- (3) *There are $x_1, \dots, x_r \in A$ such that*

$$\delta_i(x_i) = 1 \text{ and } \delta_j(x_i) = 0 \quad j > i$$

- (4) $\delta_1, \dots, \delta_r$ act nilpotently on A .

Then $A^L[x_1, \dots, x_r] = A$.

The Lemma is not difficult to prove. A similar lemma was used in the construction of the moduli space for curve singularities with the semi-group $\langle p, q \rangle$ (cf. [1], [2]).

Obviously A^L is the ring of all elements of A being invariant under $\delta_1, \dots, \delta_r$.

Now $A^{\delta_r}[x_r] = A$ and the conditions (2)–(4) of the lemma are satisfied for $\delta_1, \dots, \delta_{r-1}$ acting on A^{δ_r} .

Now we may apply the Lemma 8 to the kernel of the Kodaira-Spencer map and its generators $\{\delta'_{lm}\}$.

Because of the lemma the geometric quotient of $\mathbb{C}^{N-1} = \text{Spec } \mathbb{C}[H, W']$ by the action of the kernel of the Kodaira-Spencer map exist and is isomorphic to the transversal section to the maximal integral manifolds (which intersect therefore each of these integral manifolds exactly in one point) defined by

$$W_{l,m} = 0, \quad (l, m) \in B_1 \setminus (B_2 \cup \{(\alpha, \beta)\}).$$

Now we use Lemma 6 and get Theorem 2. Notice that

$$\{G^{(0)}x^i y^j\}_{(i,j) \in B_0} \cup \{x^i y^j\}_{(i,j) \in B_2} \cup \{x^i y^j, iq + jp \leq 2pq, (i, j) \in B\}$$

is a base of the free $\mathbb{C}[H, W']$ -module $\mathbb{C}[H, W'][[x, y]]/(G, \partial G/\partial x, \partial G/\partial y)$. This implies $\mu - \tau = \#(B_1 \setminus B_2) = (p - 1)(q - 1)$. \square

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