

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 76, n° 1-2 (1990), p. 163-196

http://www.numdam.org/item?id=CM_1990__76_1-2_163_0

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Complex analytic compactifications of $\mathbb{C}^3$

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Received 5 September 1988; accepted in revised form 28 August 1989

Introduction

The purpose of this paper is to give a proof of the announcement [7].

Let X be an n -dimensional compact connected complex manifold and Y an analytic subset of X . We call the pair (X, Y) a complex analytic compactification of $\mathbb{C}^n$ if $X - Y$ is biholomorphic to $\mathbb{C}^n$. By a theorem of Hartogs, Y is a divisor on X .

In this paper, we will consider only the case of $n = 3$. Let (X, Y) be a complex analytic compactification of $\mathbb{C}^3$. Assume that Y has at most isolated singularities. Then Y is normal. Thus, by Peternell-Schneider [18] (cf. Brenton [2]), X is projective. In particular, X is a Fano 3-fold of index r ($1 \leq r \leq 4$) with $b_2(X) = 1$. In the case of $r \geq 2$, such a (X, Y) is completely determined (cf. [3], [5], [6], [18]). In the case of $r = 1$, by a detailed analysis of the singularities of the boundary Y , we can prove that such a compactification (X, Y) does not exist. Thus, we have:

THEOREM. *Let (X, Y) be a complex analytic compactification of $\mathbb{C}^3$. Assume that Y has at most isolated singularities. Then X is a Fano 3-fold of index r ($2 \leq r \leq 4$) with $b_2(X) = 1$, and*

- (1) $r = 4 \Rightarrow (X, Y) \cong (\mathbb{P}^3, \mathbb{P}^2)$,
- (2) $r = 3 \Rightarrow (X, Y) \cong (\mathbb{Q}^3, \mathbb{Q}_0^2)$, where $\mathbb{Q}^3$ is a smooth quadric hypersurface in $\mathbb{P}^4$ and $\mathbb{Q}_0^2$ is a quadric cone in $\mathbb{P}^3$,
- (3) $r = 2 \Rightarrow (X, Y) \cong (V_5, H_5)$, where V_5 is a complete intersection of three hyperplanes in the Grassmannian $G(2, 5) \hookrightarrow \mathbb{P}^9$, and H_5 is a normal hyperplane section of V_5 with exactly one rational double point of A_4 -type.

This paper consists of five sections. In Section 1, we will prove that $(X, Y) \cong (V_{22}, H_{22})$ if such a (X, Y) exists in the case of $r = 1$, where $V_{22} \hookrightarrow \mathbb{P}H^0(V_{22}, \mathcal{O}(-K_{V_{22}})) \cong \mathbb{P}^{13}$ is a Fano 3-fold of degree 22 in $\mathbb{P}^{13}$ (index 1, genus 12) and H_{22} is a normal hyperplane section which is rational (Proposition 1.13). In Section 2, we will determine the singularities of $Y = H_{22}$ (Proposition 2.5). In Sections 3 and 4, we will prove that such a $(X, Y) = (V_{22}, H_{22})$ does

not exist. In Section 5, we will refer to a recent work of Peternell-Schneider [18] (c.f. Peternell [19]) on a projective compactification (X, Y) of $\mathbb{C}^3$ with $b_2(X) = 1$ (especially, the case where the boundary Y is non-normal), and prove that there is a compactification (X, Y) of $\mathbb{C}^3$ with a non-normal boundary Y in the case of the index $r = 1$.

Notations

- K_M : a canonical divisor on a projective manifold M .
- $b_i(M)$: the i th Betti number of M .
- $N_{C|M}$: the normal bundle of C in M .
- $c_1(\mathcal{F}_M)$: the first Chern class of a locally free sheaf $\mathcal{F}_M$ on M .
- $m(\mathcal{O}_{Y,x})$: the multiplicity of the local ring $\mathcal{O}_{Y,x}$ at x .

1. The structure in the case of $r = 1$

1. Let (X, Y) be an analytic compactification of $\mathbb{C}^3$ such that Y has at most isolated singularities. Assume that the index $r = 1$. Then X is a Fano 3-fold of index 1 with $\text{Pic } X \cong \mathbb{Z}c_1\mathcal{O}_X(Y)$ ([3], [9], [18]). Then, by Proposition 1, Proposition 2 and Proposition 3 in [3], we have:

- LEMMA 1.1. (1) $K_Y = 0$,
 (2) $H^1(Y, \mathcal{O}_Y) = 0, H^2(Y, \mathcal{O}_Y) \cong \mathbb{C}$,
 (3) $H^1(Y; \mathbb{Z}) = 0, H^2(Y; \mathbb{Z}) \cong \mathbb{Z}c_1(N_{Y|X})$.

Let $\text{Sing } Y$ be the singular locus of Y and put $S := \{y \in \text{Sing } Y; y \text{ is not a rational singularity}\}$. Let $\pi: \tilde{Y} \rightarrow Y$ be the minimal resolution of singularities of Y and Z be the fundamental cycle of S associated with the resolution $(\tilde{Y}, \pi)$. We put $E := \pi^{-1}(\text{Sing } Y)$, $C := \pi^{-1}(S) = \bigcup_{i=1}^s C_i$ (C_i 's are irreducible).

LEMMA 1.2. $S \neq \emptyset$

Proof. Let us consider the following exact sequence (see [2]):

$$\begin{aligned} \rightarrow H^1(Y; \mathbb{R}) \rightarrow H^1(\tilde{Y}; \mathbb{R}) \rightarrow H^1(E; \mathbb{R}) \rightarrow H^2(Y; \mathbb{R}) \rightarrow \\ \rightarrow H^2(\tilde{Y}; \mathbb{R}) \rightarrow H^2(E; \mathbb{R}) \rightarrow 0. \end{aligned} \quad (1.1)$$

Assume that $S = \emptyset$. Then we have $K_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}$ and $H^1(E; \mathbb{R}) = 0$. By Lemma 1.1(3), $H^1(\tilde{Y}; \mathbb{R}) = 0$. Thus $\tilde{Y}$ is a K -3 surface. Since $b_2(Y) = 1$ and Y is projective, we have $b^+(Y) = 1$. On the other hand, by Brenton [2], $b^+(Y) = b^+(\tilde{Y})$. Thus we have $b^+(\tilde{Y}) = 1$. This is a contradiction. Therefore $S \neq \emptyset$. Q.E.D.

Thus, by Umezū [22], we have:

LEMMA U.

(1) $K_{\tilde{Y}} = -\sum_{i=1}^s n_i C_i$ ($n_i > 0$, $n_i \in \mathbb{Z}$), and thus $\tilde{Y}$ is a ruled surface over a non-singular compact algebraic curve R of genus $q = \dim H^1(\tilde{Y}; \mathcal{O}_{\tilde{Y}})$ (namely, $\tilde{Y}$ is birationally equivalent to a $\mathbb{P}^1$ -bundle over R),

(2) if $q \neq 1$, then S consists of one point with $p_g := \dim(R^1 \pi_* \mathcal{O}_{\tilde{Y}})_s = q + 1$,

(3) if $q = 1$, then S consists of either one point with $p_g = 2$ or two points with $p_g = 1$. Moreover, in the second case of (3), both of the two points are simply elliptic.

LEMMA 1.3. S consists of one point with $p_g = q + 1$ and $b_2(\tilde{Y}) = b_2(E) + 1$.

Proof. Assume that S consists of two points. By Lemma U(3), these two points are simply elliptic, and $C = \pi^{-1}(S) = C_1 \cup C_2$, where C_1, C_2 are distinct sections of $\tilde{Y}$. Since $b_2(Y) = 1$, by (1.1), we have $b_1(\tilde{Y}) = b_1(E)$. Since

$$2 = b_1(\tilde{Y}) = b_1(E) \geq b_1(C) = b_1(C_1) + b_1(C_2) = 4,$$

we have a contradiction. Therefore S consists of one point with $p_g = q + 1$. Since $b_1(\tilde{Y}) = b_1(E)$ and $b_2(Y) = 1$, we have $b_2(\tilde{Y}) = b_2(Y) + b_2(E) = 1 + b_2(E)$.

Q.E.D.

2. Let U be a strongly pseudoconvex neighborhood of C in $\tilde{Y}$. A cycle D on U is an integral combination of the C_i , $D = \sum d_i C_i$ ($1 \leq i \leq s$), with $d_i \in \mathbb{Z}$. We denote the support of D by $|D| = \bigcup C_i, d_i \neq 0$. We put $\mathcal{O}_D = \mathcal{O}_U / \mathcal{O}_U(-D)$. Let K_U be a canonical divisor on U . We put $\chi(D) := \dim H^0(U, \mathcal{O}_D) - \dim H^1(U, \mathcal{O}_D)$. Then, by the Riemann-Roch theorem [21],

$$\chi(D) = -\frac{1}{2} \{(D \cdot D) + (D \cdot K_U)\}. \quad (1.2)$$

For two cycles A, B , we have, by (1.2),

$$\chi(A + B) = \chi(A) + \chi(B) - (A \cdot B). \quad (1.3)$$

LEMMA 1.4.

(1) $q = 0 \Rightarrow \tilde{Y}$ is a rational surface, and $K_{\tilde{Y}} = K_U = -Z$.

(2) $q \neq 0 \Rightarrow$ there is an irreducible component C_{i_1} of C such that C_{i_1} is a section of $\tilde{Y}$ and the rest $\overline{C - C_{i_1}} = \bigcup_{i \neq i_1} C_i (\neq \emptyset)$ is contained in the singular fibers of $\tilde{Y}$, and $-K_{\tilde{Y}} = Z + C_{i_1}$.

Proof. (1) Since $q = 0$, we have $p_g = 1$. Thus S consists of a minimally elliptic singularity. By Laufer [11], we have $K_{\tilde{Y}} = K_U = -Z$.

(2) Since $K_{\tilde{Y}} = -\sum_{i=1}^s n_i C_i$ ($n_i > 0, n_i \in \mathbb{Z}$), for a general fiber f of $\tilde{Y}$, we have

$$2 = (-K_{\tilde{Y}} \cdot f) = \sum_{i=1}^s n_i (C_i \cdot f).$$

Thus we have the following:

- (i) there is an irreducible component C_{i_1} of C such that $n_{i_1} = 2, (C_{i_1} \cdot f) = 2$ and $(C_i \cdot f) = 0$ ($i \neq i_1$),
- (ii) there are two irreducible components C_1, C_2 of C such that $n_1 = n_2 = 1, (C_i \cdot f) = 1$ ($i = 1, 2$), $(C_i \cdot f) = 0$ ($i \geq 3$), and
- (iii) there is an irreducible component C_1 of C such that $n_1 = 1, (C_1 \cdot f) = 2, (C_i \cdot f) = 0$ ($i \neq 1$).

Claim 1. The case (ii) can not occur.

Indeed, by the adjunction formula, the curve C_i ($i = 1, 2$) is a non-singular elliptic curve with $(C_1 \cdot C_2) = 0$ and there is no other irreducible component of C which intersects C_i ($i = 1, 2$). Thus $C = C_1 \cup C_2$ ($C_1 \cap C_2 = \emptyset$), namely, S consists of two points. This contradicts Corollary 1.4.

Claim 2. The case (iii) can not occur.

Indeed, by the adjunction formula, C_1 is a non-singular elliptic curve and there is no other irreducible component of C which intersects C_1 . By Corollary 1.4, we have $C = C_1$, hence, $K_{\tilde{Y}} = -C_1$. This contradicts Lemma 1 and Lemma 2 in Umezū [22].

Thus we have the case (i). In particular, C_{i_1} is a section of the ruled surface $\tilde{Y}$ and C_i 's ($i \neq i_1$) are all contained singular fibers of $\tilde{Y}$. We also have $C - C_{i_1} \neq \emptyset$ by the same reason as above. Since $n_{i_1} = 2$, we have $-K_{\tilde{Y}} = 2C_{i_1} + \sum_{i \neq i_1} n_i C_i$ ($n_i > 0$). We remark that the genus of C_{i_1} is equal to $q = h^1(\mathcal{O}_{\tilde{Y}}) \geq 1$.

Claim 3. $-K_{\tilde{Y}} = C_{i_1} + Z$.

Indeed, since $(-K_{\tilde{Y}} - C_{i_1}) \cdot C_i \leq 0$ ($1 \leq i \leq s$), by definition of the fundamental cycle, $-K_{\tilde{Y}} - C_{i_1} \geq Z$. Now, assume that $-K_{\tilde{Y}} = C_{i_1} + Z + D$, where $D > 0$. For a general fiber f of $\tilde{Y}$, we have $2 = (-K_{\tilde{Y}} \cdot f) = (C_{i_1} \cdot f) + (Z \cdot f) + (D \cdot f)$. Since $C_{i_1} \subset |Z|$, we have $(C_{i_1} \cdot f) = (Z \cdot f) = 1$, and $(D \cdot f) = 0$. This means that the support $|D|$ is contained in the singular fibers of $\tilde{Y}$. Since

$$H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-Z)) \cong H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-C_{i_1} - D)) = 0 \quad \text{and} \quad H^2(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = 0,$$

by the Riemann-Roch theorem, we have

$$0 \geq -\dim H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-Z)) = \frac{1}{2}(Z \cdot Z + Z \cdot K_{\tilde{Y}}) + 1 - q.$$

Since $K_M = K_U$, by (1.2), we have $\chi(Z) \geq 1 - q$. Since $H^0(U; \mathcal{O}_Z) \cong \mathbb{C}$ (cf. [11,

p. 1260]), $\chi(Z) = 1 - \dim H^1(U, \mathcal{O}_Z) \leq 1$. Since S does not consists of a rational singularity, $\chi(Z) \neq 1$ by Artin [1]. Thus we have

$$1 - q \leq \chi(Z) \leq 0. \quad (1.4)$$

Since $1 - q = \chi(C_{i_1}) = \chi(-K_U - C_{i_1}) = \chi(Z + D) = \chi(Z) + \chi(D) - (D \cdot Z)$, we have

$$\chi(Z) = -\chi(D) + 1 - q + (D \cdot Z). \quad (1.5)$$

Since $(D \cdot Z) \leq 0$, by (1.4), (1.5), we have $\chi(D) \leq 0$. On the other hand, the support $|D|$ is contained in the singular fibers of $\tilde{Y}$. Thus, the contraction of $|D|$ in $\tilde{Y}$ yields rational singularities. Hence $\chi(D) \geq 1$. This is a contradiction. Therefore $D = 0$, namely, $-K_{\tilde{Y}} = Z + C_{i_1}$. Q.E.D.

COROLLARY 1.5. *Assume that $q \neq 0$. Then*

$$(1) (C_{i_1} \cdot Z) = 2 - 2q.$$

$$(2) (Z \cdot Z) \leq (C_{i_1} \cdot C_{i_1}).$$

Proof. Since $K_{\tilde{Y}} = -Z - C_{i_1}$, by the adjunction formula, we have (1) and (2). Q.E.D.

LEMMA 1.6.

$$(1) q \neq 0 \Rightarrow b_2(\tilde{Y}) \leq 9 - 4q + \sqrt{9 + 8q}.$$

$$(2) q = 0 \Rightarrow 11 \leq b_2(\tilde{Y}) \leq 13.$$

Proof. (1) By the Noether formula, we have

$$10 - 8q = (K_{\tilde{Y}} \cdot K_{\tilde{Y}}) + b_2(\tilde{Y}). \quad (1.6)$$

Since $K_{\tilde{Y}} = -Z - C_{i_1}$, we have

$$(K_{\tilde{Y}} \cdot K_{\tilde{Y}}) = (Z \cdot Z) + 2(Z \cdot C_{i_1}) + (C_{i_1} \cdot C_{i_1}). \quad (1.7)$$

By (1.6), (1.7) and Corollary 1.5, we have

$$b_2(\tilde{Y}) = 6 - 4q - (Z \cdot Z) - (C_{i_1} \cdot C_{i_1}). \quad (1.8)$$

$$\leq 6 - 4q - 2(Z \cdot Z). \quad (1.9)$$

Since $S = \{\text{one point}\}$ is a hypersurface singularity, we have

$$-(Z \cdot Z) \leq n := m(\mathcal{O}_{Y,S}) \quad (\text{Wagreich [23]}). \quad (1.10)$$

$$p_g \geq \frac{1}{2}(n-1)(n-2) \quad (\text{Yau [24]}). \quad (1.11)$$

Since $p_g = q + 1$, by (1.10), (1.11), we have

$$-(Z \cdot Z) \leq \frac{1}{2}(3 + \sqrt{9 + 8q}). \quad (1.12)$$

By (1.9), (1.12), we have the claim.

(2) By the Noether formula, we have

$$b_2(\tilde{Y}) = 10 - (K_{\tilde{Y}} \cdot K_{\tilde{Y}}). \quad (1.13)$$

Since $p_g = 1$ and S is a hypersurface singularity, by Laufer [11], we have $-3 \leq (Z \cdot Z) \leq -1$. By Lemma 1.4(1) and (1.13), we have the claim. Q.E.D.

COROLLARY 1.7. $0 \leq q \leq 3$.

Proof. Assume that $q \neq 0$. By Lemma 1.6(1), we have

$$2 \leq b_2(\tilde{Y}) \leq 9 - 4q + \sqrt{9 + 8q}. \text{ This implies } q \leq 3. \quad \text{Q.E.D.}$$

3. By the classification of Fano 3-Folds with the second Betti numbers one due to Iskovskih [9] (see also Mukai [15], [16]), we have:

(Table 1)

g	2	3	4	5	6	7	8	9	10	12
$\frac{1}{2}b_3(X)$	52	30	20	14	10	7	5	3	2	0

where $g := \frac{1}{2}(K_X^3) + 1 = \frac{1}{2}(Y^3) + 1$.

Since $2q = b_1(\tilde{Y}) = b_3(\tilde{Y}) = b_3(Y) = b_3(X)$ (cf. [2]), by Corollary 1.7, $0 \leq \frac{1}{2}b_3(X) \leq 3$. Thus, by the Table 1 above, we have $(g, q) = (9, 3), (10, 2)$ or $(12, 0)$.

LEMMA 1.8. $q \neq 3$.

Proof. Assume that $q = 3$. By Lemma 1.6(1), we have $2 \leq b_2(\tilde{Y}) \leq -3 + \sqrt{33} < 3$, namely, $b_2(\tilde{Y}) = 2$. Hence, $\tilde{Y}$ is a $\mathbb{P}^1$ -bundle over a smooth compact algebraic curve R of genus 3. Therefore, Y is a cone over R . This is a contradiction, by Table (I) in [3]. Q.E.D.

LEMMA 1.9. Assume that $q \neq 0$. Then there is exactly one exceptional curve of the first kind in every singular fiber of the ruled surface $\tilde{Y}$, and the other irreducible components of the singular fiber are all contained in $E := \pi^{-1}(\text{Sing } Y)$.

Proof. Since $q \neq 0$, the rest $\overline{E - C_{i_1}}$ must be contained in the singular fibers

of $\tilde{Y}$. Let $F_1, \dots, F_r$ be the singular fibers of $\tilde{Y}$, $1 + \alpha_i$ ($\alpha_i > 0$) the “number” of the irreducible components of F_i and δ_i the “number” of the irreducible components of F_i which are not contained in E . Then we have

$$\begin{cases} 1 + t = b_2(\tilde{Y}) = 2 + \sum_{i=1}^r \alpha_i. \\ \sum_{i=1}^r (1 + \alpha_i - \delta_i) + 1 = t. \end{cases}$$

Thus we have $\sum_{i=1}^r (1 - \delta_i) = 0$. Since each singular fiber F_i contains at least an exceptional curve of the first kind, we have $\delta_i \geq 1$ for $1 \leq i \leq r$, hence, $\delta_i = 1$ for $1 \leq i \leq r$. Q.E.D.

LEMMA 1.10. Assume that $q = 2$. Then the dual graphs of all the exceptional curves in $\tilde{Y}$ look like the Figure 1 below.

Proof. By Lemma 1.6(1), we have $2 \leq b_2(\tilde{Y}) \leq 6$. Since Y is not a cone (see Table 1 in [3], we have $b_2(\tilde{Y}) \neq 2$. If $b_2(\tilde{Y}) = 3$, then $\tilde{Y}$ contains two exceptional curves of the first kind in a singular fiber. This contradicts Lemma 1.9. Hence we have $4 \leq b_2(\tilde{Y}) \leq 6$. Thus, by (1.9), (1.12), we have $-4 \leq (Z \cdot Z) \leq -3$. We put $n := (C_{i_1} \cdot C_{i_1}) < 0$. Then, by Lemma 1.3 and (1.8), we have

- (i) $b_2(\tilde{Y}) = 6 \Rightarrow (n, t) = (-4, 5)$ and $(Z \cdot Z) = -4$.
- (ii) $b_2(\tilde{Y}) = 5 \Rightarrow (n, t) = (-3, 4)$ and $(Z \cdot Z) = -4$.
- (iii) $b_2(\tilde{Y}) = 4 \Rightarrow (n, t) = (-3, 3)$ and $(Z \cdot Z) = -3$,
 $\quad \quad \quad = (-2, 3)$ and $(Z \cdot Z) = -4$.

Thus, we have

- (a) $(Z \cdot Z) = -4 \Rightarrow (n, t) = (-2, 3), (-3, 4), (-4, 5)$.
- (b) $(Z \cdot Z) = -3 \Rightarrow (n, t) = (-3, 3)$.

Since $\text{Sing } Y - S$ consists of rational double points, by Lemma 1.9 and (a), (b), the configuration of the exceptional curves of $\tilde{Y}$ can be easily described. Thus we have the lemma. Q.E.D.

Notation. The vertex $\boxed{k}$ represents a non-singular compact algebraic curve of genus 2 with the self-intersection number $-k$, (which is corresponding to the section C_{i_1} of $\tilde{Y}$), $\textcircled{k}$ a non-singular rational curve with the self-intersection number $-k$. We denote $\textcircled{2}$ simply by $\bigcirc$. Adjacent to the graph, we write a basis $\{e_i\}$ ($0 \leq i \leq t$) of $H^2(\tilde{Y}; \mathbb{Z})$, where $t = \dim H^2(E; \mathbb{R})$.

LEMMA 1.11. Assume that $q = 2$. Then there is a canonical curve D of genus 10 and $\deg D = 18$ such that

- (i) $\text{Sing } Y \cap D = \emptyset$,
- (ii) $\mathcal{O}_Y(Y) = \mathcal{O}_Y(D)$.

Proof. Since $q = 2$, by Table 1, X is a Fano 3-fold of degree 18 in $\mathbb{P}^{11}$ and Y is a hyperplane section of X (see [9]). For a sufficiently general hyperplane section H , we put $D = H \cdot Y$, which is desired. Q.E.D.

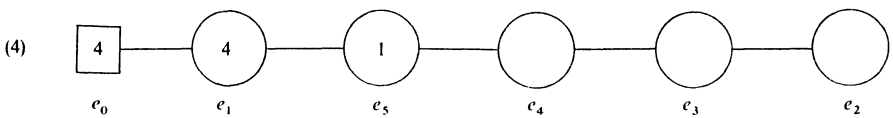
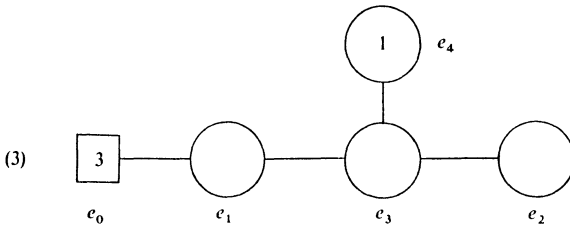
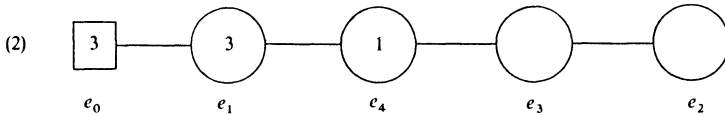
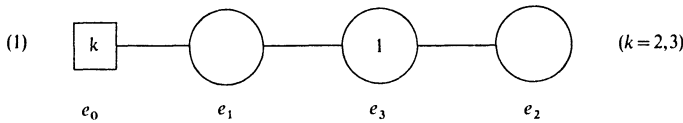
4. We put $\tilde{D} := \pi^{-1}(D) \subset \tilde{Y}$. Since $D \cap \text{Sing } Y = \emptyset$ by Lemma 1.11, we have $\tilde{D} \cong D$ (isomorphism), $(\tilde{D} \cdot \tilde{D}) = 18$ and $(\tilde{D} \cdot E_j) = 0$ for each irreducible component E_j of $E = \pi^{-1}(\text{Sing } Y)$.

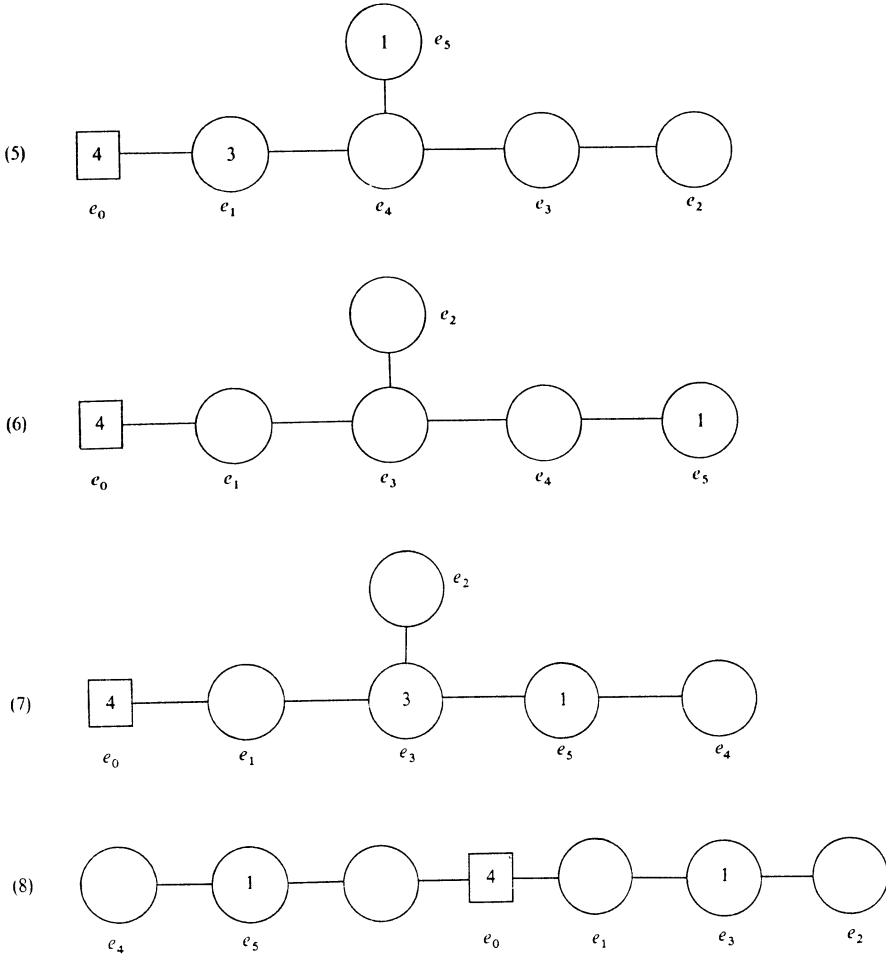
Let $\{e_i\}$ ($0 \leq i \leq t$) be a basis of $H^2(\tilde{Y}; \mathbb{Z}) \cong \mathbb{Z}^{t+1}$ (see Fig. 1). Then, we have

$$c_1(\mathcal{O}(\tilde{D})) = \sum_{i=0}^t \alpha_i e_i \quad (\alpha_i \in \mathbb{Z}), \quad (*)$$

where $c_1(\mathcal{O}(\tilde{D})) \in H^2(\tilde{Y}; \mathbb{Z})$ is the first Chern class of $\mathcal{O}(\tilde{D})$, and

- (i) the intersection number $e_i \cdot e_j$ is determined by the graph in Fig. 1,
- (ii) $c_1(\mathcal{O}(\tilde{D})) \cdot e_i = 0$ ($0 \leq i \leq t$),
- (iii) $c_1(\mathcal{O}(\tilde{D})) \cdot c_1(\mathcal{O}(\tilde{D})) = 18$,
- (iv) $d_{i_0} := c_1(\mathcal{O}(\tilde{D})) \cdot e_{i_0} \neq 0$, where e_{i_0} is a class corresponding to the exceptional curve of the first kind.





(Fig. 1).

By (*) and (i)–(iv) above, for each graph in the Fig. 1, we have the equations of α_i ($0 \leq i \leq t$) and d_{i_0} over $\mathbb{Z}$ below:

Case (1)

$$\alpha_1 - k\alpha_0 = 0$$

$$\alpha_3 - 2\alpha_1 + \alpha_0 = 0$$

$$\alpha_3 - 2\alpha_2 = 0$$

$$\alpha_2 - \alpha_3 + \alpha_1 = d_3$$

$$\alpha_3 \cdot d_3 = 18$$

(C-1)

$\therefore \alpha_0^2 = 36/2k - 1$ ($k = 2, 3$). Hence $\alpha_0 \notin \mathbb{Z}$.

Case (2).

$$\begin{aligned}
 \alpha_1 - 3\alpha_0 &= 0 \\
 \alpha_4 - 3\alpha_1 + \alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_2 &= 0 \\
 \alpha_4 - 2\alpha_3 + \alpha_2 &= 0 \\
 \alpha_1 - \alpha_4 + \alpha_3 &= d_4 \\
 \alpha_4 \cdot d_5 &= 18
 \end{aligned} \tag{C-2}$$

$\therefore \alpha_0^2 = \frac{27}{4}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (3).

$$\begin{aligned}
 \alpha_1 - 3\alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_0 &= 0 \\
 \alpha_4 + \alpha_1 - 2\alpha_3 + \alpha_2 &= 0 \\
 \alpha_3 - \alpha_4 &= d_4 \\
 \alpha_5 \cdot d_5 &= 18
 \end{aligned} \tag{C-3}$$

$\therefore \alpha_0^2 = 8$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (4).

$$\begin{aligned}
 \alpha_1 - 4\alpha_0 &= 0 \\
 \alpha_5 - 4\alpha_1 + \alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_2 &= 0 \\
 \alpha_4 - 2\alpha_3 + \alpha_2 &= 0 \\
 \alpha_5 - 2\alpha_4 + \alpha_3 &= 0 \\
 \alpha_1 - \alpha_5 + \alpha_4 &= d_5 \\
 \alpha_5 \cdot d_5 &= 18
 \end{aligned} \tag{C-4}$$

$\therefore \alpha_0^2 = \frac{24}{5}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (5).

$$\begin{aligned}
 \alpha_1 - 4\alpha_0 &= 0 \\
 \alpha_4 - 3\alpha_1 + \alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_2 &= 0 \\
 \alpha_4 - 2\alpha_3 + \alpha_2 &= 0 \\
 \alpha_5 + \alpha_1 - 2\alpha_4 + \alpha_3 &= 0 \\
 \alpha_4 - \alpha_5 &= d_5 \\
 \alpha_5 \cdot d_5 &= 18
 \end{aligned} \tag{C-5}$$

$\therefore \alpha_0^2 = \frac{81}{16}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (6).

$$\begin{aligned}
 \alpha_1 - 4\alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_2 &= 0 \\
 \alpha_4 + \alpha_1 - 2\alpha_3 + \alpha_2 &= 0 \\
 \alpha_5 - 2\alpha_4 + \alpha_3 &= 0 \\
 \alpha_4 - \alpha_5 &= d_5 \\
 a_5 \cdot d_5 &= 18
 \end{aligned} \tag{C-6}$$

$\therefore \alpha_0^2 = 6$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (7).

$$\begin{aligned}
 \alpha_1 - 4\alpha_0 &= 0 \\
 \alpha_0 - 2\alpha_1 + \alpha_3 &= 0 \\
 \alpha_3 - 2\alpha_2 &= 0 \\
 \alpha_5 + \alpha_1 - 3\alpha_3 + \alpha_2 &= 0 \\
 \alpha_5 - 2\alpha_4 &= 0 \\
 \alpha_4 - \alpha_5 + \alpha_3 &= d_5 \\
 \alpha_5 \cdot d_5 &= 18
 \end{aligned} \tag{C-7}$$

$\therefore \alpha_0^2 = \frac{16}{3}$. Hence $\alpha_0 \notin \mathbb{Z}$.

Case (8).

$$\begin{aligned}
 \alpha_1 - 4\alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_1 + \alpha_0 &= 0 \\
 \alpha_3 - 2\alpha_2 &= 0 \\
 \alpha_1 - \alpha_3 + \alpha_2 &= d_3 \\
 \alpha_5 - 2\alpha_4 &= 0 \\
 \alpha_4 - \alpha_5 &= d_5 \\
 \alpha_0 + \alpha_5 &= 0 \\
 \alpha_5 d_5 + \alpha_3 d_3 &= 18
 \end{aligned} \tag{C-8}$$

$\therefore \alpha_0^2 = 6$. Hence $\alpha_0 \notin \mathbb{Z}$.

By the computations (C-1)–(C-8), we find that these equations have no integral solutions. Thus, we have:

LEMMA 1.12. $q \neq 2$.

By Lemma 1.8, Lemma 1.12, and Table 1, we have the following

PROPOSITION 1.13. (cf. [18], [19]). *Assume that the index $r = 1$. Then,*

$(X, Y) \cong (V_{22}, H_{22})$, where V_{22} is a Fano 3-fold of degree 22 in $\mathbb{P}^{13}$ (index 1, genus 12) and H_{22} is a hyperplane section of V_{22} which is rational.

REMARK 1.14. Among Fano 3-folds of degree 22 in $\mathbb{P}^{13}$ (index 1, genus 12), there is a special one, $V'_{22} \hookrightarrow \mathbb{P}^{12}$, which has been overlooked by Iskovskih [8] (see Mukai-Umemura [14]).

Recently, Mukai has succeeded in classifying Fano 3-folds of index 1 with $b_2(X) = 1$, applying the theory of vector bundles on K-3 surfaces (see [15], [16]).

2. Determination of the boundary

1. Let $(X, Y) = (V_{22}, H_{22})$ be as in Proposition 1.13. Since $q = 0$, by Lemma 1.3, S consists of one point x with $p_g = 1$, namely, x is a minimally elliptic singularity. We put $\text{Sing } Y - \{x\} = \{y_1, \dots, y_k\}$ ($k \geq 0$), and $B := \pi^{-1}(\{y_1, \dots, y_k\})$. Then y_j 's are all rational double points.

By Lemma 1.6(2), we have:

$$b_2(B) + b_2(C) = 10 \quad \text{if} \quad (Z \cdot Z) = -1, \quad (2.1)$$

$$b_2(B) + b_2(C) = 11 \quad \text{if} \quad (Z \cdot Z) = -2, \quad (2.2)$$

$$b_2(B) + b_2(C) = 12 \quad \text{if} \quad (Z \cdot Z) = -3. \quad (2.3)$$

2. Let T_0 (resp. T_i) be a contractible neighborhood of x (resp. y_i) in Y . We may assume that T_0, T_i ($1 \leq i \leq k$) are disjoint. We put $T := \bigcup_{i=0}^k T_i$ and $\partial T := \bigcup_{i=0}^k \partial T_i$, where ∂T_i is the boundary of T_i . We put $T^* := T - \text{Sing } Y$ and $Y^* := Y - \text{Sing } Y$. Since $T^* \approx \partial T$ (deformation retract), by the Mayer-Vietoris exact sequence, we have

$$\begin{aligned} \rightarrow H_i(\partial T; \mathbb{Z}) &\rightarrow H_i(Y^*; \mathbb{Z}) \oplus H_i(T; \mathbb{Z}) \rightarrow \\ &\rightarrow H_i(Y; \mathbb{Z}) \rightarrow H_{i-1}(\partial T; \mathbb{Z}) \rightarrow. \end{aligned} \quad (2.4)$$

Since $\text{Sing } Y$ is isolated in Y , we have $H_2(Y^*; \mathbb{Z}) \cong H^2(\tilde{Y}, E; \mathbb{Z}) \cong H^2(Y, \text{Sing } Y; \mathbb{Z}) \cong H^2(Y; \mathbb{Z}) \cong \mathbb{Z}$. On the other hand, since $X = V_{22}$ is a Fano 3-fold of index 1 and the genus $g = 12$, we have $H^3(X; \mathbb{Z}) = 0$ (cf. [8], [15], [16]). Thus we have $H_3(Y; \mathbb{Z}) \cong H_3(X; \mathbb{Z}) \cong H^3(X; \mathbb{Z}) \cong 0$, and $H_1(Y^*; \mathbb{Z}) \cong H^3(Y; \mathbb{Z}) \cong H^3(X; \mathbb{Z}) \cong 0$ (cf. [2], [3]). Therefore we have finally the Poincaré's exact sequence:

$$0 \rightarrow H^2(Y; \mathbb{Z}) \xrightarrow{P_2} H_2(Y; \mathbb{Z}) \rightarrow H_1(\partial T; \mathbb{Z}) \rightarrow 0 \quad (\text{cf. [18]}). \quad (2.5)$$

By Lemma 2.5 in [18], we have

LEMMA 2.1 (cf. Peternell-Schneider [18]). $H_1(\partial T; \mathbb{Z}) = H_1(\partial T_0; \mathbb{Z}) \oplus \mathcal{H} \cong \mathbb{Z}_{22}$, where $\mathcal{H} = \bigoplus_{i=1}^k H_1(\partial T_i; \mathbb{Z})$, namely, we have:

(Table 2)

$H_1(\partial T_0; \mathbb{Z})$	$\mathbb{Z}_{22}$	$\mathbb{Z}_{11}$	$\mathbb{Z}_2$	0
$\mathcal{H}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_{11}$	$\mathbb{Z}_{22}$

LEMMA 2.2. For the rational double point $y_j \in \text{Sing } Y - \{x\}$ ($1 \leq j \leq k$), we have:

(Table 3)

Type of y_i	A_n	D_{2n} (resp. D_{2n+1})	E_6	E_7	E_8
$H_1(\partial T_i; \mathbb{Z})$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (resp. $\mathbb{Z}_4$)	$\mathbb{Z}_3$	$\mathbb{Z}_2$	0

Proof. Apply Lemma M below.

LEMMA M (Mumford [17]).

Let S be a smooth complex surface and consider a divisor $C = \bigcup_{i=1}^n C_i$ (C_i : a smooth rational curve) with normal crossings. Let ∂T be the boundary of a tubular neighborhood T of C in S . Then, $H_1(\partial T; \mathbb{Z})$ is generated by $\gamma_1, \dots, \gamma_n$ with the fundamental relations:

$$\sum_{j=1}^n (C_i \cdot C_j) \cdot \gamma_j \quad (j = 1, 2, \dots, n), \quad (\#)$$

where γ_j is a loop in ∂T which goes around C_j with positive orientation.

REMARK. By Lemma M, one can easily compute the homology group $H_1(\partial T_0; \mathbb{Z})$ for each exceptional divisor C in Table L-1–Table L-9 below.

By Lemma 2.1, Lemma 2.2, and (2.1), (2.2), we have easily the following

LEMMA 2.3.

- (1) $(Z \cdot Z) = -1 \Rightarrow H_1(\partial T_0; \mathbb{Z}) \not\cong 0, \mathbb{Z}_2$
- (2) $(Z \cdot Z) = -2 \Rightarrow H_1(\partial T_0; \mathbb{Z}) \not\cong 0$.

LEMMA 2.4.

- (1) The case of $(Z \cdot Z) = -1$. We have:
 - (i) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{11} \Rightarrow b_2(C) = 1, 3, 9$ (Table L-1)
 - (ii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{22} \Rightarrow b_2(C) = 2, 10$ (Table L-2)
- (2) The case of $(Z \cdot Z) = -2$. We have:
 - (i) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_2 \Rightarrow b_2(C) = 1$ (Table L-3)
 - (ii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{11} \Rightarrow b_2(C) = 2, 4, 10$ (Table L-4)
 - (iii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{22} \Rightarrow b_2(C) = 3, 11$ (Table L-5)

(3) The case of $(Z \cdot Z) = -3$. We have:

- (i) $H_1(\partial T_0; \mathbb{Z}) \cong 0 \Rightarrow b_2(C) = 1$ (Table L-6)
- (ii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_2 \Rightarrow b_2(C) = 2$ (Table L-7)
- (iii) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{11} \Rightarrow b_2(C) = 3, 5, 11$ (Table L-8)
- (iv) $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{22} \Rightarrow b_2(C) = 4, 12$ (Table L-9)

Proof. We will prove for the case (3)(iii). The proof for the other cases are similar. Since $(Z \cdot Z) = -3$ and $H_1(\partial T_0; \mathbb{Z}) \cong \mathbb{Z}_{11}$, we have $b_2(B) + b_2(C) = 12$ by (2.3), and $\mathcal{H} := \bigoplus_{i=1}^r H_1(\partial T_i; \mathbb{Z}) \cong \mathbb{Z}_2$. By the Table 3 Sing $Y - \{x\} = \{A_1\text{-type}\}, \{E_7\text{-type}\}$ or $\{A_1\text{-type} + E_8\text{-type}\}$, hence, $b_2(B) = 1, 7, 9$, respectively. By (2.3), we have $b_2(C) = 3, 5, 11$. Pick out the possible types of the dual graphs with $b_2(C) = 3, 5, 11$, from the Table 3 in Laufer [11], we have finally the Table L-8. We remark that there is no dual graph with $b_2(C) = 4$. Q.E.D.

From Lemma 2.4 and the Table L-1–Table L-9, we have directly the following

(Table L-1)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
1	Cu	-1	0
2	Tr	$-2, -2, -3$	$\mathbb{Z}_3$
3	$A_{5,****}$	$-2, -2, -2, -3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
4	$E_{8,*}$	-3	0

(Table L-2)

	Type of x	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
5	Ta	$-2, -3$	$\mathbb{Z}_2$
6	$A_{6,****}$	$-2, -2, -2, -3$	$\mathbb{Z}_4$

(Table L-3)

	Type of x	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
7	Cu	-2	" $\mathbb{Z}_2$ "

(Table L-4)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
8	Ta	$-2, -4$	$\mathbb{Z}_2^{\oplus 2}$
9	Ta	$-3, -3$	$\mathbb{Z}_5$

10	$A_{6,****}$	$-2, -2, -2, -4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$
11	$A_{6,****}$	$-2, -2, -3, -3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
12	$A_{6,****}$	$-2, -3, -3, -2$	$\mathbb{Z}_{17}$
13	$A_{*,o} + A_{*,o} + A_{*,o}$ $+ A_{4,***o}$	$-2, -2, -2$ $-2, -2$	$\mathbb{Z}_2^{\oplus 4}$
14	$A_{*,o} + A_{*,o} + E_{7,o}$	$-2, -2$	$\mathbb{Z}_2^{\oplus 2}$
15	$A_{*,o} + A_{n,***o}$ $+ A_{m,***o}$ $(m+n=4)$	$-2, -2, -2,$ $-2, -2$	$\mathbb{Z}_4^{\oplus 2}$ if $(m, n) = (1, 3)$ $\mathbb{Z}_2^{\oplus 4}$ if $(m, n) = (2, 2)$
16	$A_{*,o} + D_{7,*,o}$	$-2, -2$	$\mathbb{Z}_2^{\oplus 4}$
17	$A'_{7,***o}$	$-2, -2$	$\mathbb{Z}_5$
18	$A_{2,***o} + A'_{3,***o}$	$-2, -2, -2, -2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6$
19	$A_{1,***o} + D_{5,*,o}$	$-2, -2, -2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$
20	$A_{0,***o} + E_{7,o}$	$-2, -2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

(Table L-5)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
21	Tr	$-2, -2, -4$	$\mathbb{Z}_6$
22	Tr	$-2, -3, -3$	$\mathbb{Z}_8$
23	$A_{7,****}$	$-2, -2, -2, -4$	$\mathbb{Z}_2^{\oplus 3}$
24	$A_{7,****}$	$-2, -2, -3, -3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_6$
25	$A_{7,****}$	$-2, -3, -3, -2$	$\mathbb{Z}_6 \oplus \mathbb{Z}_9$
26	$A_{*,o} + A_{*,o} +$ $+ A_{*,o} + A_{5,***o}$	$-2, -2$ $-2, -2, -2$	$\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_4$
27	$A_{*,o} + A_{n,***o} +$ $+ A_{m,***o}$ $(m+n=5)$	$-2, -2, -2$ $-2, -2$	$\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_4$
28	$A_{3,***o} + A'_{3,***o}$	$-2, -2, -2, -2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_6$
29	$A_{2,***o} + D_{5,*,o}$	$-2, -2, -2$	$\mathbb{Z}_2^{\oplus 3}$
30	$A_{1,***o} + E_{7,o}$	$-2, -2$	$\mathbb{Z}_4$
31	$D_{9,*,o}$	-2	$\mathbb{Z}_2$

(Table L-6)

	Dual graph	$A_{\star} \cdot A_{\star}$	$H_1(\partial T_0; \mathbb{Z})$
32	Cu	-3	$\mathbb{Z}_3$

(Table L-7)

	Dual graph	$A_{\star} \cdot A_{\star}$	$H_1(\partial T_0; \mathbb{Z})$
33	Ta	$-2, -5$	$\mathbb{Z}_6$
34	Ta	$-3, -4$	$\mathbb{Z}_8$

(Table L-8)

	Dual graph	$A_{\star} \cdot A_{\star}$	$H_1(\partial T_0; \mathbb{Z})$
35	Tr	$-2, -2, -5$	$\mathbb{Z}_3^{\oplus 2}$
36	Tr	$-2, -3, -4$	$\mathbb{Z}_{13}$
37	Tr	$-3, -3, -3$	$\mathbb{Z}_4^{\oplus 2}$
38	$A_{1,****}$	$-2, -2, -2, -5$	$\mathbb{Z}_6^{\oplus 2}$
39	$A_{1,****}$	$-2, -2, -3, -4$	$\mathbb{Z}_{10}^{\oplus 2}$
40	$A_{1,****}$	$-2, -3, -3, -3$	$\mathbb{Z}_9^{\oplus 2}$
41	$A_{7,****}$	$-2, -2, -2, -5$	$\mathbb{Z}_6^{\oplus 2}$
42	$A_{7,****}$	$-2, -2, -3, -4$	$\mathbb{Z}_{10}^{\oplus 2}$
43	$A_{7,****}$	$-2, -3, -4, -2$	$\mathbb{Z}_{16}^{\oplus 2}$
44	$A_{7,****}$	$-2, -3, -3, -3$	$\mathbb{Z}_{15}^{\oplus 2}$
45	$A_{\star,o} + A_{\star,o} + A_{\star,o} +$ $+ A_{5,***,o}$	$-3, -2, -2,$ $-2, -2$	$\mathbb{Z}_8^{\oplus 2}$
46	$A_{\star,o} + A_{\star,o} + A_{\star,o} +$ $+ A_{5,***,o}$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{26}^{\oplus 2}$
47	$A_{\star,o} + A_{n,***,o} +$ $+ A_{m,***,o} (m + n = 5)$	$-3, -2, -2$ $-2, -2,$	$\mathbb{Z}_8^{\oplus 2}$
48	$A_{\star,o} + A_{n,***,o} +$ $+ A_{m,***,o} (m + n = 5)$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{18}^{\oplus 2}$ if $(m, n) = (1, 4)$ $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$ if $(m, n) = (2, 3)$ $\mathbb{Z}_{22}^{\oplus 2}$ if $(m, n) = (3, 2)$ $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$ if $(m, n) = (4, 1)$
49	$D_{\partial, \star, o}$	-3	$\mathbb{Z}_5$
50	$A_{3,***,o} + A'_{3,***,o}$	$-3, -2, -2, -2$	$\mathbb{Z}_{33}$

51	$A_{3,**,o} + A'_{3,**,o}$	$-2, -2, -2, -3$	$\mathbb{Z}_{14}^{\oplus 2}$
52	$A_{2,**,o} + D_{5,**,o}$	$-3, -2, -2$	$\mathbb{Z}_{10}^{\oplus 2}$
53	$A_{2,**,o} + D_{5,**,o}$	$-2, -2, -3$	$\mathbb{Z}_{30}^{\oplus 2}$
54	$A_{1,**,o} + E_{7,o}$	$-3, -2$	$\mathbb{Z}_9$
55	$A_{1,**,o} + A_{1,**,o} + E_{6,o}$	$-2, -2$	$\mathbb{Z}_3^{\oplus 2}$
56	$A_{1,**,o} + A_{7,**,o}$	$-2, -2$	$\mathbb{Z}_3 \oplus \mathbb{Z}_9$
57	$A_{4,**,o} + A_{4,**,o}$	$-2, -2$	$\mathbb{Z}_6^{\oplus 2}$

(Table L-9)

	Dual graph	$A_* \cdot A_*$	$H_1(\partial T_0; \mathbb{Z})$
58	$A_{8,*****}$	$-2, -2, -2, -5$	$\mathbb{Z}_{12}$
59	$A_{8,*****}$	$-2, -2, -3, -4$	$\mathbb{Z}_{20}$
60	$A_{8,*****}$	$-2, -3, -4, -2$	$\mathbb{Z}_{34}$
61	$A_{8,*****}$	$-2, -3, -3, -3$	$\mathbb{Z}_{48}$
62	$A_{*,o} + A_{*,o} + A_{*,o} + A_{6,**,o}$	$-3, -2, -2$ $-2, -2$	$\mathbb{Z}_8^{\oplus 3}$
63	$A_{*,o} + A_{*,o} + A_{*,o} + A_{6,**,o}$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{14}^{\oplus 3}$
64	$A_{*,o} + A_{n,**,o} + A_{m,**,o} (m+n=6)$	$-3, -2, -2$ $-2, -2$	$\mathbb{Z}_8^{\oplus 3}$
65	$A_{*,o} + A_{n,**,o} + A_{m,**,o} (m+n=6)$	$-2, -2, -2$ $-2, -3$	$\mathbb{Z}_{36}$ if $(m, n) = (1, 5)$ $\mathbb{Z}_{10}^{\oplus 3}$ if $(m, n) = (2, 4)$ $\mathbb{Z}_{44}$ if $(m, n) = (3, 3)$ $\mathbb{Z}_4 \oplus \mathbb{Z}^{\oplus 2}$ if $(m, n) = (4, 2)$ $\mathbb{Z}_{13} \oplus \mathbb{Z}_{52}$ if $(m, n) = (5, 1)$
66	$A_{4,**,o} + A'_{3,**,o}$	$-3, -2, -2, -2$	$\mathbb{Z}_6 \oplus \mathbb{Z}_{12}$
67	$A_{4,**,o} + A'_{3,**,o}$	$-2, -2, -2, -3$	$\mathbb{Z}_{28}$
68	$A_{3,**,o} + D_{5,**,o}$	$-3, -2, -2$	" $\mathbb{Z}_{22}$ "
69	$A_{3,**,o} + D_{5,**,o}$	$-2, -2, -3$	$\mathbb{Z}_{20}$
70	$A_{2,**,o} + E_{7,o}$	$-3, -2$	$\mathbb{Z}_{10}$
71	$A_{10,***,o}$	-2	$\mathbb{Z}_4$
72	$A_{4,**,o} + E_{6,o}$	-2	$\mathbb{Z}_6$

PROPOSITION 2.5. *Let $(X, Y) = (V_{22}, H_{22})$ be a compactification of $\mathbb{C}^3$ as in Proposition 1.13. Then,*

- (a) *Sing $Y = \{x\}$, where x is a minimally elliptic singularity of $A_{3,**0} + D_{5,**,0}$ - Type (Table L-9, (68)), or*
- (b) *Sing $Y = \{x, y\}$, where x is a minimally elliptic singularity of Cu-type (Table L-3, (7)) and y is a rational double point of A_{10} -type.*

In the Table L-1–Table L-9, we use the same terminology as that of the Table 1–Table 3 in Laufer [11, p. 1287–1294].

3. Non-existence of the case (a)

Assume that there is a compactification $(X, Y) = (V_{22}, H_{22})$ of the case (a) in Proposition 2.5. Let $\pi: \tilde{Y} \rightarrow Y$ be the minimal resolution of the singularity $x := \text{Sing } Y$ and Z the fundamental cycle of x associated with the resolution $(\tilde{Y}, \pi)$. By assumption, we have $K_{\tilde{Y}} = -Z$ and $(Z \cdot Z) = -3$. The dual graph of $\pi^{-1}(x)$ looks like the Fig. 2, where we denote by $\bigcirc$ (resp. $\bigcirc_{-3}$) a smooth rational curve with the self-intersection number -2 (resp. -3). We can represent $\tilde{Y}$ as a ruled surface $v: \tilde{Y} \rightarrow \mathbb{P}^1$ over $\mathbb{P}^1$ (see Fig. 3), where

$$\left\{ \begin{array}{l} v^{-1}(0) =: \tilde{C} \cup f_1 \cup f_2 \cup f_3 \cup f_4 \cup l_1 \text{ and } v^{-1}(\infty) =: \bigcup_{i=5}^{10} f_i \cup \tilde{B} \text{ are} \\ \text{singular fibers.} \end{array} \right. \quad (3.1)$$

$$l_2 \text{ is a section} \quad (3.2)$$

$$\left\{ \begin{array}{l} (l_1 \cdot l_1)_{\tilde{Y}} = (l_2 \cdot l_2)_{\tilde{Y}} = -3, \\ (f_i \cdot f_i)_{\tilde{Y}} = -2 (1 \leq i \leq 10), \\ (\tilde{C} \cdot \tilde{C})_{\tilde{Y}} = (\tilde{B} \cdot \tilde{B})_{\tilde{Y}} = -1. \end{array} \right. \quad (3.3)$$

$$\left\{ \begin{array}{l} (\tilde{D} \cdot \tilde{C})_{\tilde{Y}} = 2, (\tilde{D} \cdot \tilde{B})_{\tilde{Y}} = 3, \text{ where } D = \pi(\tilde{D}) \text{ is a canonical} \\ \text{hyperplane section such that } \text{Pic } Y \cong \mathbb{Z} \cdot \mathcal{O}_Y(D), \text{ in particular,} \\ \deg D = (D \cdot D)_Y = 22. \end{array} \right. \quad (3.4)$$

$$\pi^{-1}(x) = \bigcup_{i=1}^{10} f_i \cup l_1 \cup l_2, \quad (3.5)$$

$$\begin{aligned} Z = & f_4 + 2f_3 + 2f_2 + 2f_1 + l_1 + 2l_2 + \\ & + 3f_5 + 4f_6 + 2f_7 + 3f_8 + 2f_9 + f_{10}. \end{aligned} \quad (3.6)$$

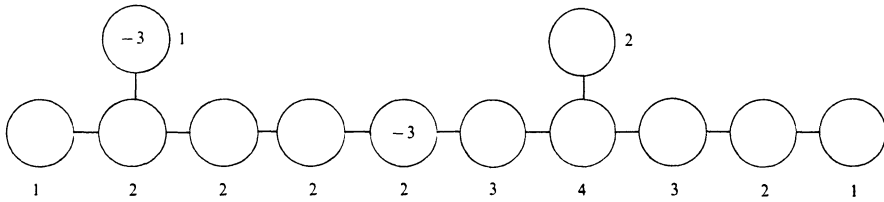


Fig. 2.

LEMMA 3.1.

- (1) *there is no line in X through the point $x = \text{Sing } Y \in X$.*
 (2) *$C_0 := \pi(C) \subset Y \subset X$ is a unique conic on X through the point x .*

Proof. Since the multiplicity $m(\mathcal{O}_{Y,x})$ is equal to 3 by Laufer [11] and $\text{Pic } X \cong \mathbb{Z} \cdot \mathcal{O}_X(Y)$, any line or any conic through the point x must be contained in Y . Now, since $(\tilde{C} \cdot \tilde{D})_Y = (C_0 \cdot D)_Y = 2$ and D is a hyperplane section, C_0 is a conic on X . Let F be a line or a conic on X through the point x . Then, we have $F \subset Y$.

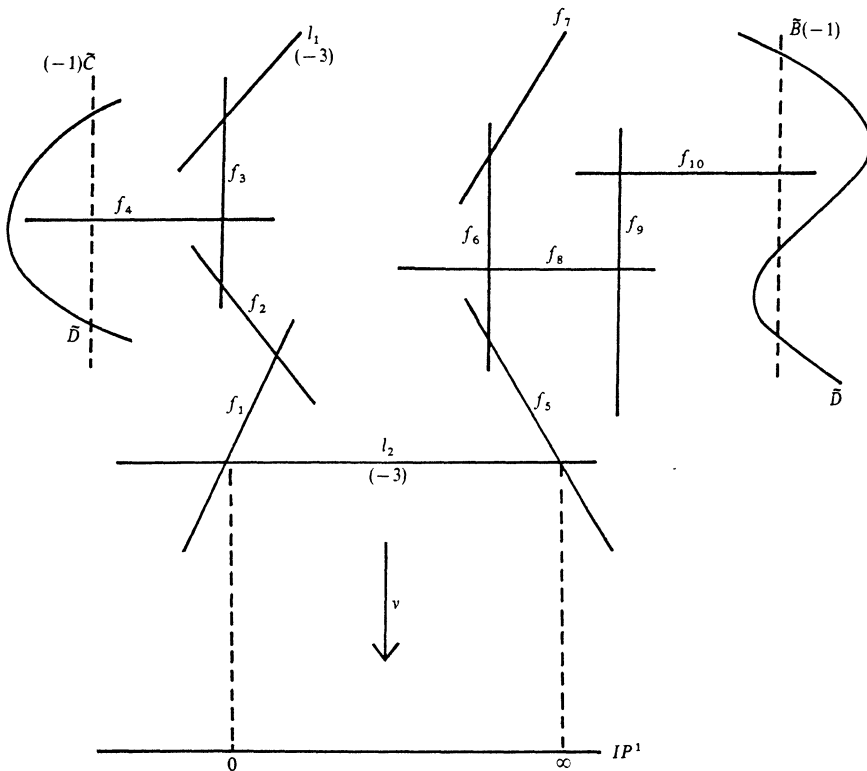


Fig. 3.

Let $\tilde{F}$ be the proper transform of F in $\tilde{Y}$. Since $\tilde{D}$ can be written as follows:

$$\begin{aligned}\tilde{D} = & 2\tilde{C} + 4f_4 + 6f_3 + 2l_1 + 6f_2 + 6f_1 + \\ & + 6l_2 + 12f_5 + 18f_6 + 9f_7 + 15f_8 + 12f_9 + 9f_{10} + 6\tilde{B},\end{aligned}\quad (3.7)$$

we have $(\tilde{D} \cdot \tilde{F}) \neq 1$, and also have $(\tilde{D} \cdot \tilde{F}) = 2$ if and only if $\tilde{F} = \tilde{C}$. This proves (1) and (2). Q.E.D.

2. Let us consider the triple projection of $X = V_{22}$ from the singularity $x = \text{Sing } Y \in X$. For this purpose, we will consider the linear system $|H - 3x| := |\mathcal{O}_X(H) \oplus \mathfrak{m}_{x,X}^3|$, where H is a hyperplane section of X and $\mathfrak{m}_{x,X}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Since the multiplicity $m(\mathcal{O}_{Y,x})$ is equal to 3, we have $Y \in |H - 3x|$ (c.f. [16a]).

Let $\sigma_1: X_1 \rightarrow X$ be the blowing up of X at the point x , and put $E_1 := \sigma^{-1}(x) \cong \mathbb{P}^2$. Let Y_1 be the proper transform of Y in X_1 . Since $-K_X = H$ and $Y \in |H - 3x|$, we have

$$-K_{X_1} = \sigma_1^* H - 2E_1 \quad (3.8)$$

$$Y_1 = \sigma_1^* H - 3E_1. \quad (3.9)$$

By the adjunction formula, we have

$$-K_{Y_1} = E_1|_{Y_1}. \quad (3.10)$$

LEMMA 3.2. $H^0(X_1, \mathcal{O}_{X_1}(\sigma_1^* H - 3E_1)) \cong \mathbb{C}^4$, and $H^i(X_1, \mathcal{O}_{X_1}(\sigma_1^* H - 3E_1)) = 0$ for $i > 0$.

Proof. Let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_{X_1} \rightarrow \mathcal{O}_{X_1}(Y_1) \rightarrow \mathcal{O}_{Y_1}(Y_1) \rightarrow 0. \quad (3.11)$$

Since $Y_1 = \sigma_1^* H - 3E_1$ and $H^i(X_1, \mathcal{O}_{X_1}) = 0$ for $i > 0$, we have only to prove $H^i(Y_1, \mathcal{O}_{Y_1}(Y_1)) = 0$ for $i > 0$ and $H^0(Y_1, \mathcal{O}_{Y_1}(Y_1)) \cong \mathbb{C}^3$.

By (3.10), we have

$$\begin{aligned}\mathcal{O}_{Y_1}(Y_1) &= \mathcal{O}_{Y_1}(\sigma_1^* H - 3E_1) \\ &= \mathcal{O}_{Y_1}(D_1 + 3K_{Y_1}),\end{aligned}\quad (3.12)$$

where $D_1 := \sigma_1^* H|_{Y_1}$ is linearly equivalent to the proper transform of D in Y_1 .

Claim. $\mathcal{O}_{Y_1}(D + 2K_{Y_1})$ is nef and big on Y_1 . Indeed, there exists a birational morphism $\mu_1: \tilde{Y} \rightarrow Y_1$ such that $\pi = (\sigma_1|_{Y_1}) \circ \mu_1$. Then, we have $\mu_1^*(D_1 + 2K_{Y_1}) = \tilde{D} - 2Z$. It is easy to see that $\tilde{D} - 2Z$ is nef and big on $\tilde{Y}$ (see (3.6), (3.7)). Thus $\mathcal{O}_{Y_1}(D_1 + 2K_{Y_1})$ is nef and big on Y_1 .

By the Kawamata–Vieweg vanishing theorem, we have $H^i(Y_1, \mathcal{O}_{Y_1}(D_1 + 3K_{Y_1})) = 0$ for $i > 0$, namely, $H^i(Y_1, \mathcal{O}_{Y_1}) = 0$ for $i > 0$. On the other hand, since $H^i(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D - 3Z)) = 0$ for $i > 0$, by the Riemann–Roch theorem, we have $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(D - 3Z)) \cong \mathbb{C}^3$. Q.E.D.

By Lemma 3.2, the linear system $|H - 3x|$ defines a rational map $\Phi := \Phi_{|H - 3x|}: X \dashrightarrow \mathbb{P}^3$, called a triple projection.

Let $\{g_1, g_2, g_3\}$ be a basis of $H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D} - 3Z))$ such that

$$\begin{cases} (g_1) = 11\tilde{C} + 10f_4 + 9f_3 + 2l_1 + 6f_2 + 3f_1 \\ (g_2) = 5\tilde{C} + 4f_4 + 3f_3 + 2f_2 + f_1 + 2f_5 + \\ \quad + 4f_6 + 2f_7 + 4f_8 + 4f_9 + 4f_{10} + 4\tilde{B} \\ (g_3) = 8\tilde{C} + 7f_4 + 6f_3 + l_1 + 4f_2 + 2f_1 + \\ \quad + f_5 + 2f_6 + f_7 + 2f_8 + 2f_9 + 2f_{10} + 2\tilde{B} \end{cases} \quad (3.13)$$

Since $2(g_3) = (g_1) + (g_2)$, $g := (g_1: g_2: g_3)$ defines a rational map $g: \tilde{Y} \dashrightarrow Q$ of $\tilde{Y}$ onto a conic $Q := \{w_2^2 = w_0 w_1\} \hookrightarrow \mathbb{P}^2(w_0: w_1: w_2)$. This implies that $\Phi(Y) = Q \cong \mathbb{P}^1$ and $W = \Phi(X)$ is a quadratic hypersurface in $\mathbb{P}^3$. Thus we have the following

LEMMA 3.3. *Let $\Phi := \Phi_{|H - 3x|}: X \dashrightarrow \mathbb{P}^3$ be the triple projection from the point x . Then the image $W := \Phi(X)$ is an irreducible quadric hypersurface in $\mathbb{P}^3$, and $Q = \Phi(Y)$ is a smooth hyperplane section, namely, a conic in $\mathbb{P}^2$.*

3. Next, we will study the resolution of the indeterminacy of the rational map $\Phi: X \dashrightarrow \mathbb{P}^3$.

Let $\Phi_{|\sigma_1^*H - 3E_1|}^{(1)}: X_1 \dashrightarrow \mathbb{P}^3$ be a rational map defined by the linear system $|\sigma_1^*H - 3E_1|$. Then we have the diagram:

$$\begin{array}{ccc} X_1 & & \\ \sigma_1 \downarrow & \searrow \Phi^{(1)} & \\ X & \xrightarrow{\Phi} & \mathbb{P}^3. \end{array}$$

Let $\Delta \subset X$ be a small neighborhood of x in X with a coordinate system

(Z_1, Z_2, Z_3) . By Laufer [11], we may assume that

$$\begin{cases} \Delta \cap Y = \{Z_2 \cdot Z_3^2 = Z_1^3 \cdot Z_2 + Z_1 Z_3^3 + Z_1 Z_2^4\}, \\ x = (0, 0, 0) \in \Delta. \end{cases} \quad (3.14)$$

By an easy computation, we find that Y_1 has two rational double points q_1 of A_4 -type and q_0 of D_6 -type (cf. [11, Theorem 3.15]). Let $\mu_1: \tilde{Y} \rightarrow Y_1$ be the birational map as above. Then we have $\mu_1^{-1}(q_1) = f_1 \cup f_2 \cup f_3 \cup f_4$, $\mu_1^{-1}(q_0) = \bigcup_{j=5}^{10} f_j$.

We put $l_i^{(1)} := \mu_1(l_i)$, and $C_1 := \mu_1(\tilde{C}) \hookrightarrow Y_1 \subset X_1$. Then C_1 is the proper transform of C_0 in X_1 , in particular, C_1 is a smooth rational curve in $Y_1 \subset X_1$ with $q_1 \in C_1$, $q_0 \notin C_1$. Moreover, we have $Y_1 \cdot E_1 = l_1^{(1)} + 2l_2^{(1)}$, where $l_1^{(1)}, l_2^{(1)}$ are two distinct lines in $E_1 \cong \mathbb{P}^2$, and $l_1^{(1)} \cap l_2^{(1)} = q_1 \in X_1$.

By (3.13), the base locus $Bs|\mathcal{O}_{Y_1}(Y_1)| = C_1 \ni q_1$. Since $H^1(X_1, \mathcal{O}_{X_1}) = 0$ by (3.12), we have the base locus $Bs|\sigma_1^*H - 3E_1| \ni C_1 \ni q_1$.

Since $\text{Pic } X \cong \mathbb{Z}\mathcal{O}_X(Y)$, the linear system $|H - 3x|$ has no fixed component. Thus, we have the following

LEMMA 3.4. *The linear system $|\sigma_1^*H - 3E_1|$ on X_1 has no fixed component, but has the base locus $Bs|\sigma_1^*H - 3E_1| = C_1 \ni q_1$.*

We need the following

LEMMA Mo (Morrison [13]). *Let S be a surface with only one singularity x of A_n -type in a smooth projective 3-fold X . Let $E \subset S \subset X$ be a smooth rational curve in X . Let $\mu: \tilde{S} \rightarrow S$ be the minimal resolution of the singularity of S and put*

$$\mu^{-1}(x) = \bigcup_{j=1}^n C_j,$$

where C_j 's ($1 \leq j \leq n$) are smooth rational curves with

$$\begin{aligned} (C_j \cdot C_j)_S &= -2 & (1 \leq j \leq n), \\ (C_j \cdot C_{j+1})_S &= 1 & (1 \leq j \leq n-1), \\ (C_i \cdot C_j)_S &= 0 & \text{if } |i-j| \geq 2. \end{aligned}$$

Let $\tilde{E}$ be the proper transform of E in $\tilde{S}$. Assume that

- (i) $N_{E|S} \cong \mathcal{O}_E(-1)$,
- (ii) $\deg N_{E|X} = -2$,

where $N_{E|S}$ (resp. $N_{E|X}$) is the normal bundle of $\tilde{E}$ (resp. E) in $\tilde{S}$ (resp. X). Then we have

- (1) $N_{E|X} \cong \mathcal{O}_E \oplus \mathcal{O}_E(-2)$ if $x \in E$ and $(C_j \cdot \tilde{E}) = 1$ for $j = 1$ or n ,
- (2) $N_{E|X} \cong \mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$ if $x \notin E$.

Proof. In the proof of Theorem 3.2 in Morrison [13], we have only to replace the conormal bundle $N_{\tilde{E}|S}^* = \mathcal{O}_{\tilde{E}}(2)$ by $N_{E|S} = \mathcal{O}_E(1)$. Q.E.D.

The indeterminacy of the rational map $\Phi^{(1)}: X_1 \dashrightarrow \mathbb{P}^3$ can be resolved by the following way:

Let us consider the following sequence of blowing ups:

$$X_6 \xrightarrow{\sigma_6} X_5 \xrightarrow{\sigma_5} X_4 \xrightarrow{\sigma_4} X_3 \xrightarrow{\sigma_3} X_2 \xrightarrow{\sigma_2} X_1,$$

where

- (i) $\sigma_{j+1}: X_{j+1} \rightarrow X_j$ is the blowing up of X_j along $C_j \cong \mathbb{P}^1$ ($1 \leq j \leq 5$),
- (ii) C_{j+1} is the negative section of the $\mathbb{P}^1$ -bundle $C'_j = \mathbb{P}(N_{C_j|X_j}^*) \cong \mathbb{F}_2$ ($1 \leq j \leq 4$),
- (iii) C_6 is a section of $C'_5 \cong \mathbb{P}^1 \times \mathbb{P}^1$ with $(C_6 \cdot C_5) = 0$.

Then we have the morphism $\Phi: X_6 \rightarrow \mathbb{P}^3$ and a diagram:

$$\begin{array}{ccc} X_1 & \xleftarrow{\sigma} & X_6 \\ \sigma_1 \downarrow & \searrow \Phi^{(1)} & \downarrow \Phi \\ X & \dashrightarrow & \mathbb{P}^3, \\ & \Phi & \end{array} \quad (\text{D-1})$$

where $\sigma := \sigma_2 \circ \sigma_3 \circ \sigma_4 \circ \sigma_5 \circ \sigma_6$.

This is a desired resolution of the indeterminacy of the rational map

$$\Phi^{(1)}: X_1 \dashrightarrow \mathbb{P}^3 \text{ (or } \Phi: X \dashrightarrow \mathbb{P}^3 \text{)}.$$

4. We will prove the facts above.

Notations:

- Y_{j+1} : the proper transform of Y_j in X_{j+1} .
- E_{j+1} : the proper transform of E_j in X_{j+1} .
- C_{j+1} : a section of $C'_j = \mathbb{P}(N_{C_j|X_j}^*)$.
- q_j : the singularity of Y_j of A_{5-j} -type (A_0 -type means the smoothness).
- $\tilde{Y}_0$: the contraction of the exceptional set $\bigcup_{i=5}^{10} f_i$ in $\tilde{Y}$.
- $f^{(j+1)}$: a fiber of the $\mathbb{P}^1$ -bundle $C'_j \hookrightarrow X_{j+1}$.
- $l_i^{(j+1)}$: the proper transform of $l_i^{(1)}$ in X_{j+1} ($i = 1, 2$).

- $\mu_j: \tilde{Y}_0 \rightarrow Y_j$: a birational morphism with μ_j

$$\tilde{Y}_0 - \bigcup_{i=1}^4 f_i \xrightarrow{\sim} Y_j - \mu_j(\bigcup_{i=1}^4 f_i).$$

Step 1. Let $\sigma_2: X_2 \rightarrow X_1$ be the blowing up of X_1 along $C_1 \cong \mathbb{P}^1$. Since $(K_{X_1} \cdot C_1) = (\sigma_1^* H - 2E_1 \cdot C_1) = 0$, we have $\deg N_{C_1|X_1} = -2$. Since $q_1 \in C_1 \subset Y_1$ is the singularity of Y_1 of A_4 -type and $(\tilde{C} \cdot f_1)_{\tilde{Y}} = 1$, by Lemma Mo, we have

$$N_{C_1|X_1} \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_1}(-2). \quad (3.15)$$

Thus we have $C'_1 \cong \mathbb{F}_2$. It is easy to see that Y_2 has two rational double points q_2 of A_3 -type and q_0 of D_6 -type with $q_2 \in C_2 \subset Y_2, q_0 \notin C_2$. Since $(K_{X_2} \cdot C_2) = 0$, by Lemma Mo, we have

$$N_{C_2|X_1} \cong \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_2}(-2). \quad (3.16)$$

In particular, we have

$$\begin{aligned} \mu_2^{-1}(q_2) &= f_2 \cup f_3 \cup f_4, \\ \tilde{Y}_0 - (f_2 \cup f_3 \cup f_4) &\cong Y_2 - \{q_2\}, \\ \mu_2(\tilde{C}) &= C_2, \\ \mu_2(f_1) &= f^{(2)}, \\ \mu_2(l_i) &= l_i^{(2)} \quad (i = 1, 2). \end{aligned} \quad (3.17)$$

(*Step k, 2 ≤ k ≤ 5*). Let $\sigma_k: X_k \rightarrow X_{k-1}$ be the blowing up of X_{k-1} along $C_{k-1} \cong \mathbb{P}^1$. Then Y_k has two rational double points q_k of A_{5-k} -type and q_0 of D_6 -type with $q_k \in C_k \subset Y_k, q_0 \notin C_k$ ($k \leq 5$). Since $(K_{X_k} \cdot C_k) = 0$, we have $\deg N_{C_k|X_k} = -2$. By Lemma Mo, we have

$$\begin{aligned} N_{C_k|X_k} &\cong \mathcal{O}_{C_k} \oplus \mathcal{O}_{C_k}(-2) \quad (2 \leq k \leq 4) \\ N_{C_5|X_5} &\cong \mathcal{O}_{C_5}(-1) \oplus \mathcal{O}_{C_5}(-1) \quad (k = 5). \end{aligned} \quad (3.18)$$

In particular,

$$\begin{aligned} \mu_k^{-1}(q_k) &= f_k \cup \cdots \cup f_4 \\ \tilde{Y}_0 - (f_k \cup \cdots \cup f_4) &\cong Y_k - \{q_k\} \\ \mu_k(\tilde{C}) &= C_k, \\ \mu_k(f_{k-1}) &= f^{(k)}. \end{aligned} \quad (3.19)$$

Step 6. Let $\sigma_6: X_6 \rightarrow X_5$ be the blowing up of X_5 along $C_5 \cong \mathbb{P}^1$. By (3.18), we have $C'_5 = \sigma_5^{-1}(C_5) \cong \mathbb{P}^1 \times P^1$. Then we have an isomorphism $\mu_6: \tilde{Y}_0 \cong Y_6$. We identify $\tilde{Y}_0$ with Y_6 (see Fig. 6). Thus we put

$$\begin{aligned} \mu_6(\tilde{C}) &=: \tilde{C}, & \mu_6(\tilde{D}) &=: \tilde{D} \\ \mu_6(f_i) &=: f_i, & \mu_6(l_i) &=: l_i. \end{aligned} \quad (3.20)$$

Then $\tilde{C} = Y_6 \cdot C'_5$ gives another ruling on C'_5 . Let $\tilde{C}'_j$ ($1 \leq j \leq 4$) be the proper transform of C'_j in X_6 . Then we have Figure 4 (see also Pagoda (5.8) in Reid [20]).

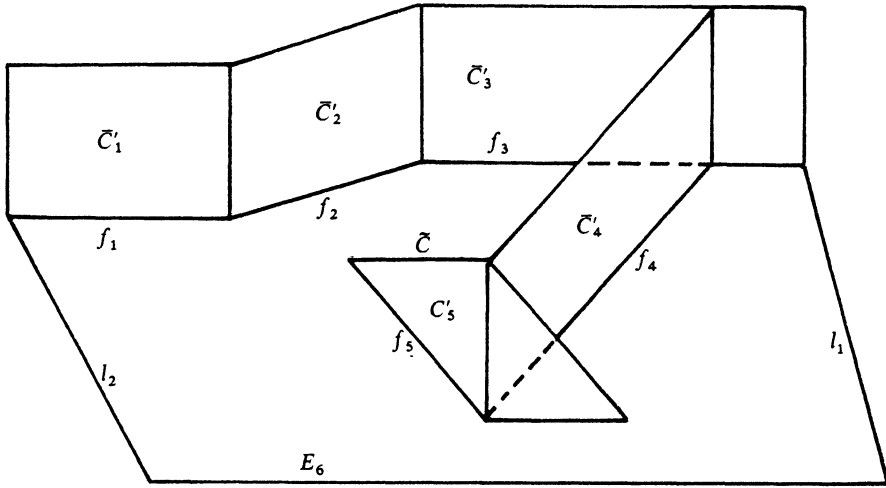


Figure 4

Now, since

$$\begin{aligned} Y_6 &= \sigma_6^* \sigma_5^* \sigma_4^* \sigma_3^* \sigma_2^* \sigma_1^* H - 3\sigma_6^* \sigma_5^* \sigma_4^* \sigma_3^* \sigma_2^* E_1 \\ &\quad - 5C'_5 - 4\tilde{C}'_4 - 3\tilde{C}'_3 - 2\tilde{C}'_2 - \tilde{C}'_1, \end{aligned} \quad (3.21)$$

we have

$$\begin{aligned} \mathcal{O}_{Y_6}(Y_6) &= \mathcal{O}_{Y_6}(\tilde{D} + 3K_{Y_6} - 5\tilde{C} - 4f_4 - 3f_3 - 2f_2 - f_1) \\ &\cong \mathcal{O}_{Y_6}(\tilde{D} - 3Z - 5\tilde{C} - 4f_4 - 3f_3 - 2f_2 - f_1) \\ &\cong \mathcal{O}_{Y_6}(2f), \end{aligned} \quad (3.22)$$

where f is a general fiber of $v: \tilde{Y} \rightarrow \mathbb{P}^1$ (see Fig. 4). This shows that the linear

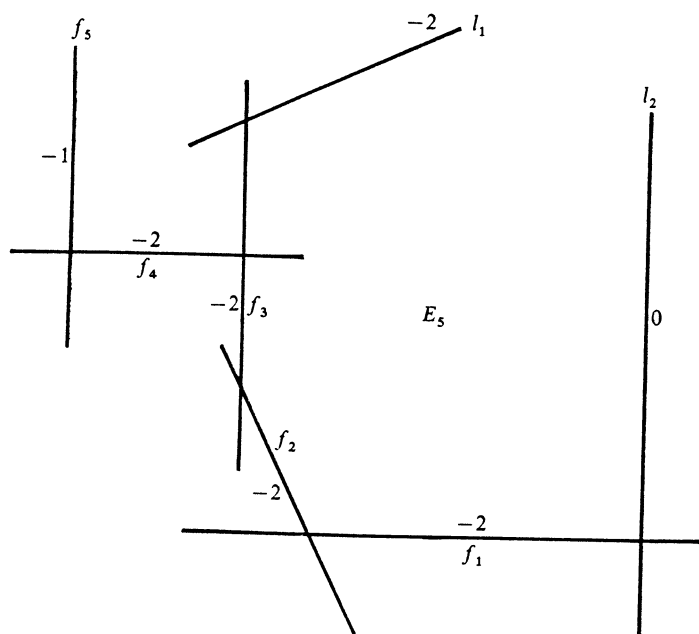


Figure 5

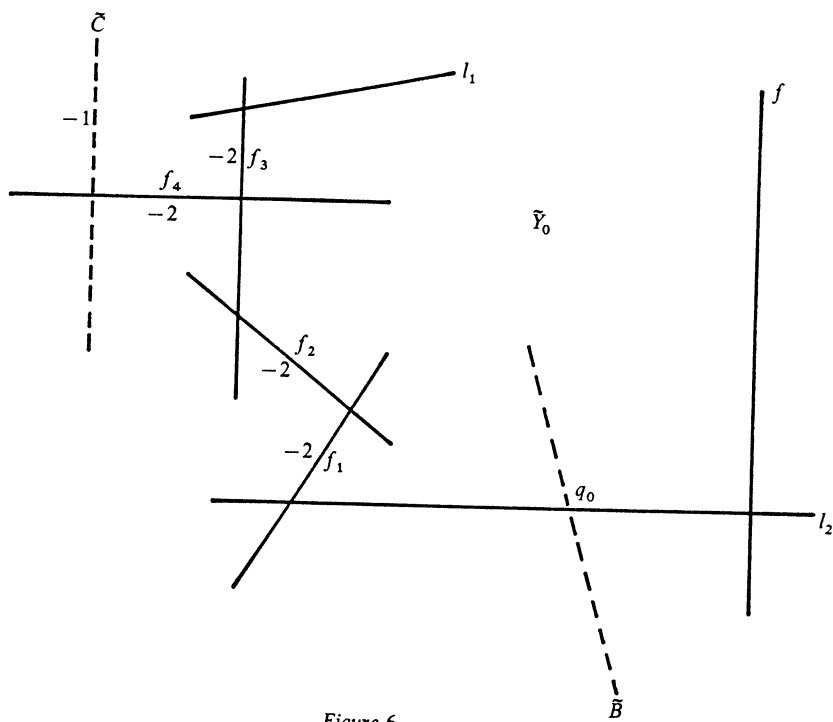


Figure 6

system $|\mathcal{O}_{Y_6}(Y_6)|$ has no fixed component and no base point. Therefore it defines a morphism $v_0: Y_6 = \tilde{Y}_0 \rightarrow Q$ of Y_6 onto a smooth conic $Q \cong \mathbb{P}^1$ in $\mathbb{P}^2$. Since $H^1(X_6, \mathcal{O}_{X_6}) = 0$ and $\text{Pic } X \cong \mathbb{Z}$, the linear system $|Y_6| = |\mathcal{O}_{X_6}(Y_6)|$ has no base locus. Therefore we have a morphism $\bar{\Phi} := \bar{\Phi}_{|Y_6|}: X_6 \rightarrow W \hookrightarrow \mathbb{P}^3$ defined by the linear system $|Y_6|$, and have the diagram (D-1), which is desired. It is easy to see that

$$\Phi(Y_6) = \Phi(Y) = v_0(Y_6) = Q \cong l_2 \quad (3.23)$$

$$\bar{\Phi}\left(\bigcup_{j=1}^4 \bar{C}'_j \cup C'_5\right) = \bar{\Phi}(f_5) \quad (\text{a line in } \mathbb{P}^3).$$

5. Since $N_{C_5|X_5} \cong \mathcal{O}_{C_5}(-1) \oplus \mathcal{O}_{C_5}(-1)$, by Reid [20], C'_5 can be blown down along C , and then the blowing downs can be done step-by-step. Finally, we have a smooth projective 3-fold V with $b_2(V) = 2$, morphisms $\bar{\Phi}_1: X_6 \rightarrow V$, $\bar{\Phi}_2: V \rightarrow W$, and a birational map $\rho: X_1 \rightarrow V$, called a flip, such that

- (i) $\bar{\Phi} = \bar{\Phi}_2 \circ \bar{\Phi}_1$,
- (ii) $X_1 - C_1 \stackrel{p}{=} (V - \bar{f}_3)$, where $\bar{f}_3 := \Phi_1(f_3)$, (see (D-2)).

$$(D-2) \quad \begin{array}{ccccc} & & X_6 & & \\ \sigma \swarrow & & & \searrow \Phi_1 & \\ X_1 & \cdots \cdots \cdots \rho & & & V \\ \downarrow \sigma_1 & & \bar{\Phi} & & \downarrow \bar{\Phi}_2 \\ X & \cdots \cdots \cdots \phi & & & W \hookrightarrow \mathbb{P}^3 \end{array}$$

Since $-K_{X_1} = Y_1 + E_1$, by (ii) above, we have $-K_V = A + \Sigma$, where $A := \bar{\Phi}(Y_6)$ and $\Sigma := \bar{\Phi}_1(E_6)$. For a general fiber F of $\bar{\Phi}_2: V \rightarrow W$, since $\deg(K_F) = (K_V \cdot F) = -(\Sigma \cdot F) \leq -1$, we have $F \cong \mathbb{P}^1$ and $(\Sigma \cdot F) = 2$. Since $\bar{\Phi}_1(l_1)$ is a smooth rational curve contained in Σ , and since $\bar{\Phi}_2 \circ \bar{\Phi}_1(l_1) = \bar{\Phi}(l_1)$ is a point, Σ is a meromorphic double section of $\bar{\Phi}_2: V \rightarrow W$.

Let G be a scheme-theoretic fiber. Then we have $(G \cdot \Sigma) = 2$. Since $V - (\Sigma \cup A) \cong X_1 - (Y_1 \cup E_1) \cong \mathbb{C}^3$ by assumption, it contains no compact analytic curve. Thus $\bar{\Phi}_2: V \rightarrow W$ is a conic bundle over W , and $\bar{\Phi}_2$ is the contraction of an extremal ray on V . Thus, W is smooth by Mori [12]. Since $\deg W = 2$, $W \cong \mathbb{P}^1 \times \mathbb{P}^1$, hence, $b_2(V) = 3$. This is a contradiction, since $b_2(V) = b_2(X_1) = 2$. Therefore we have:

Conclusion

The case (a) of Proposition 2.5 can not occur.

4. Non-existence of the case (b)

1. Assume that there is a compactification $(X, Y) = (V_{22}, H_{22})$ of the case (b) in Proposition 2.5. Then we have $\text{Sing } Y = \{x, y\}$, where x is a minimally elliptic singularity of Cu -type (Table L-3, (7)), and y is a rational double point of A_{10} -type. Let $\pi: \tilde{Y} \rightarrow Y$ be the minimal resolution of the singularities of Y and put $\tilde{C} := \pi^{-1}(x), \pi^{-1}(y) = \bigcup_{j=1}^{10} B_j$. Then $\tilde{C}$ is an irreducible rational curve with a cusp, and $K_{\tilde{Y}} = -\tilde{C}, (\tilde{C} \cdot \tilde{C})_{\tilde{Y}} = -2$. We can easily see that $\tilde{Y}$ can be obtained from $\mathbb{P}^2$ by succession of 11 blowing ups at a smooth point p on a cubic curve $C_0 \hookrightarrow \mathbb{P}^2$ with a cusp (infinitely near points allowed). Let $\mu: \tilde{Y} \rightarrow \mathbb{P}^2$ be the projection. Then $\tilde{C}$ is the proper transform of C_0 in $\tilde{Y}$ and $\mu^{-1}(p) = \bigcup_{j=1}^{11} B_j$, where B_{11} is the exceptional curve of the first kind (see Fig. 7).

We take sufficiently general hyperplane section H such that $D := H \cdot Y$ does not pass through the points x and y . Then D is a canonical curve of the genus $g = 12$ with $\deg D = 22$ in Y . Let $\tilde{D}$ be the proper transform of D in $\tilde{Y}$. Then we have

$$(\tilde{D} \cdot \tilde{D})_{\tilde{Y}} = \deg D = 22 \quad (4.1)$$

$$\tilde{D} = 3\tilde{C} + 2\tilde{G}, \quad (4.2)$$

where $\tilde{G}$ is the proper transform of a line $G \hookrightarrow \mathbb{P}^2$ with $p \notin G$ in $\tilde{Y}$. In particular, $(\tilde{G} \cdot \tilde{G})_{\tilde{Y}} = 1$.

LEMMA 4.1. *There is no line in X through the point x .*

Proof. Since the multiplicity $m(\mathcal{O}_{Y,x})$ is equal to two, any line through the point x is contained in Y . Let g be such a line in X , and $\tilde{g}$ be the proper transform of Y in $g \hookrightarrow \tilde{Y}$. Since $(D \cdot \tilde{g})_{\tilde{Y}} = (D \cdot g)_Y = (H \cdot g)_X = 1$, by (4.2), we have

$$3(\tilde{C} \cdot \tilde{g}) + 2(\tilde{G} \cdot \tilde{g}) = 1. \quad (4.3)$$

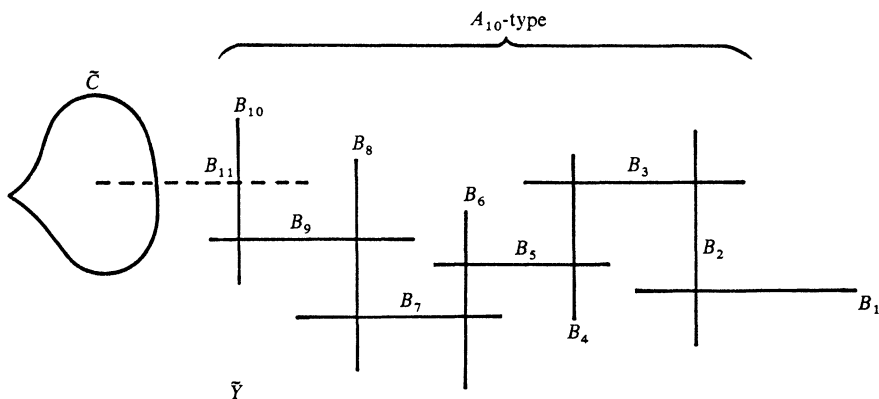


Figure 7

This is a contradiction.

Q.E.D.

2. Let $\sigma: X_1 \rightarrow X$ be the blowing up of X at the point x , and put $E := \sigma^{-1}(x) \cong \mathbb{P}^2$. Let $Y_1 = \sigma^*H - 2E$ be the proper transform of Y in X . Then we have:

- (i) $Y_1 \cdot E = 2l$, where l is a line in $E \cong \mathbb{P}^2$,
- (ii) $\text{Sing } Y_1 = l$,
- (iii) $N_{l|X} \cong \mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)$.

Let $\tau: X_2 \rightarrow X_1$ be the blowing up of X_1 along $l \cong \mathbb{P}^1$. By (iii) above, we have $L' := \tau^{-1}(l) \cong \mathbb{F}_2$. Let us denote the negative section (resp. a fiber) by s (resp. f). Let Y_2 be the proper transform of Y_1 in X_2 . Then we have $Y_2 = \tau^*Y_1 - 2L'$. Let $\tilde{Y}_0$ be the contraction of the exceptional curve $\bigcup_{j=1}^0 B_j$ in $\tilde{Y}$. Then $\tilde{Y}_0$ has a rational double point of A_{10} -type. By an easy computation, we have an isomorphism $\tilde{Y}_0 \xrightarrow{\sim} Y_2$. We identify Y_2 with $\tilde{Y}_0$ via v . For simplicity, we put $\tilde{C} := v(\tilde{C})$, $\tilde{D} := v(\tilde{D})$, $\tilde{G} := v(\tilde{G})$. Then we have

$$L' \cdot Y_2 = \tilde{C}, \quad (4.4)$$

$$(\tilde{C} \cdot f)_{L'} = 2, (\tilde{C} \cdot s)_{L'} = 0, \quad (4.5)$$

(see Fig. 8).

3. We will study the linear system $|\mathcal{L}| := |\tau^*Y_1 - L'|$ on X_2 .

Let us consider the exact sequence

$$0 \rightarrow \mathcal{O}_{X_2}(L') \rightarrow \mathcal{O}_{X_2}(\mathcal{L}) \rightarrow \mathcal{O}_{Y_2}(\mathcal{L}) \rightarrow 0. \quad (4.6)$$

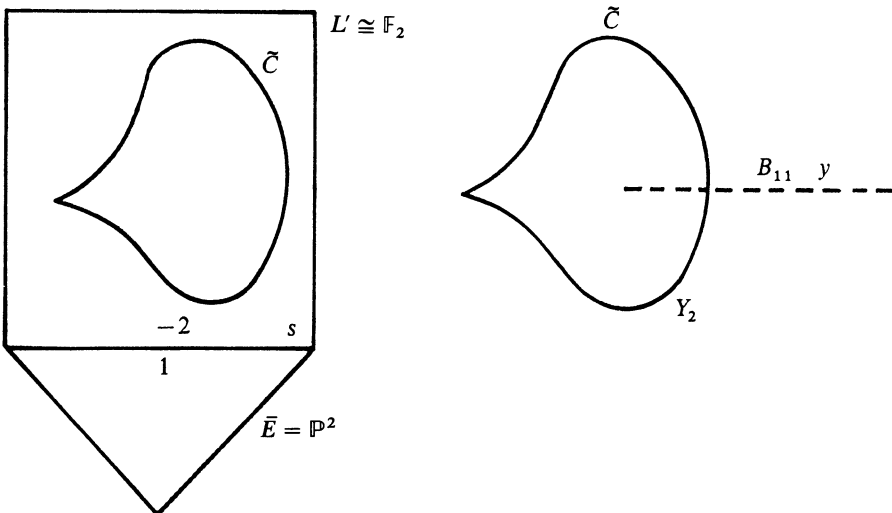


Figure 8

Since $\tau^*Y_1 - L' = \tau^*\sigma^*H - 2\bar{E} - 3L'$, we have

$$\mathcal{O}_{Y_2}(\mathcal{L}) = \mathcal{O}_{Y_2}(\tau^*Y_1 - L') = \mathcal{O}_{Y_2}(\tilde{D} - 3\tilde{C}) = \mathcal{O}_{Y_2}(2\tilde{G})$$

by (4.2), where $\bar{E} \cong \mathbb{P}^2$ is the proper transform of $E \cong \mathbb{P}^2$ in X_2 . Since $H^i(Y_2, \mathcal{O}_{Y_2}(2\tilde{G})) = 0$ for $i > 0$, by the Riemann–Roch theorem, we have $H^0(Y_2, \mathcal{O}_{Y_2}(\mathcal{L})) \cong \mathbb{C}^6$. Since $H^0(X_2, \mathcal{O}(X_2, \mathcal{O}_{Y_2}(L'))) \cong \mathbb{C}$ and $H^1(X_2, \mathcal{O}_{X_2}(L')) = 0$, we have finally the following exact sequence:

$$\begin{array}{ccccc} 0 \rightarrow H^0(X_2, \mathcal{O}_{X_2}(L')) & \rightarrow & H^0(X_2, \mathcal{O}_{X_2}(\mathcal{L})) & \rightarrow & H^0(Y_2, \mathcal{O}_{Y_2}(\mathcal{L})) \rightarrow 0 \\ & \int \parallel & & \int \parallel & \int \parallel \\ \mathbb{C} \cong H^0(X_2, \mathcal{O}_{X_2}) & & \mathbb{C}^7 & & \mathbb{C}^6 \end{array} \quad (4.6)$$

Since $\dim|\mathcal{L}| = 6$, we have a rational map $\Phi := \Phi|_{\mathcal{L}}: X_2 \rightarrow \mathbb{P}^6$ defined by the linear system $|\mathcal{L}|$.

Since the linear system $|\mathcal{O}_{Y_2}(2\tilde{G})|$ has no base locus on Y_2 , neither does $|\mathcal{L}|$ by (4.6). Therefore $\Phi: X_2 \rightarrow \mathbb{P}^6$ is a morphism X_2 to $\mathbb{P}^6$ with

$$\Phi|_{Y_2} = \varphi := \varphi|_{\mathcal{O}_{Y_2}(2\tilde{G})}: Y_2 \rightarrow \mathbb{P}^5,$$

where $\varphi|_{\mathcal{O}_{Y_2}(2\tilde{G})}$ is a morphism defined by $|\mathcal{O}_{Y_2}(2\tilde{G})|$.

Thus we have the following:

LEMMA 4.2. $\Phi: X_2 \rightarrow \Phi(X_2) \hookrightarrow \mathbb{P}^6$ is a morphism of X_2 onto a 3-fold $\Phi(X_2)$ of degree 4 in $\mathbb{P}^6$. Moreover, the restriction $\Phi|_{Y_2}: Y_2 \rightarrow \Phi(Y_2) = \varphi(Y_2) \hookrightarrow \mathbb{P}^5$ gives an birational morphism of Y_2 onto a surface $\varphi(Y_2)$ of degree 4 in $\mathbb{P}^5$.

Proof. Since $(\tau^*Y_1 - L')^3 = 4$, we have $\deg \Phi(X_2) = 4$. Q.E.D.

$$\begin{array}{ccccc} 0 \rightarrow \mathcal{O}_{Y_2}(\tau^*Y_1 - 2L') & \rightarrow & \mathcal{O}_{X_2}(\mathcal{L}) & \rightarrow & \mathcal{O}_{L'}(\mathcal{L}) \rightarrow 0 \\ & \int \parallel & & \int \parallel & \\ \mathcal{O}_{X_2}(Y_2) & & \mathcal{O}_{L'}(s + 3f) & & \end{array} \quad (4.7)$$

Since $H^1(Y_2, \mathcal{O}_{Y_2}, \mathcal{O}_{Y_2}(Y_2)) \cong H^1(Y_2, \mathcal{O}_{Y_2}(2\tilde{G} - \tilde{C})) = 0$, we have a surjection

$$H^0(X_2, \mathcal{O}_{X_2}(\mathcal{L})) \twoheadrightarrow H^0(L', \mathcal{O}_{L'}(s + 3f)) \cong \mathbb{C}^6. \quad (4.8)$$

Since $\mathcal{O}_{L'}(s + 3f)$ is very ample on L' , the morphism $\rho := \rho|_{s+3f}: L' \rightarrow \rho(L') \hookrightarrow \mathbb{P}^5$ is an isomorphism of L' onto a smooth surface of degree 4 in $\mathbb{P}^5$. Thus we have the following

LEMMA 4.3. *The restriction $\Phi|_{L'}: L' \rightarrow \Phi(L') \hookrightarrow \mathbb{P}^5$ is an isomorphism of L' onto a smooth surface $\Phi(L')$ of degree 4 in $\mathbb{P}^5$.*

Finally, let us consider the exact sequence:

$$\begin{array}{ccc} 0 \rightarrow \mathcal{O}_{X_2}(\tau^*(Y_1 - E)) \rightarrow \mathcal{O}_{X_2}(\mathcal{L}) \rightarrow \mathcal{O}_E(\mathcal{L}) \rightarrow 0 \\ \int \parallel \qquad \qquad \qquad \int \parallel \\ \mathcal{O}_{X_2}(\tau^*(\sigma^*H - 3E)) \qquad \qquad \mathcal{O}_{\mathbb{P}^2}(1) \end{array} \quad (4.9)$$

Then we have $H^1(X_2, \mathcal{O}_{X_2}(\tau^*(\sigma^*H - 3E))) = 0$, namely, we have a surjection

$$H^0(X_2, \mathcal{O}_{X_2}(\mathcal{L})) \twoheadrightarrow H^0(\bar{E}, \mathcal{O}_E(\mathcal{L})) \cong \mathbb{C}^3. \quad (4.10)$$

Thus we have the following

LEMMA 4.4. *The restriction $\Phi|_{\bar{E}}: \bar{E} \rightarrow \mathbb{P}^2$ gives an isomorphism of $\bar{E}$ onto $\mathbb{P}^2$.*

5. Let γ be an irreducible curve in X_2 such that $(\tau^*Y_1 - L' \cdot \gamma) = 0$. Since $\tau^*Y_1 - L' = Y_2 + L'$, we have $(Y_2 \cdot \gamma) + (L' \cdot \gamma) = 0$. By Lemma 4.2, Lemma 4.3, Lemma 4.4, $\gamma \not\subset Y_2 \cup L'$. Thus $(Y_2 \cdot \gamma) = (L' \cdot \gamma) = 0$, namely, $(Y_1 \cdot \tau(\gamma)) = 0$. Hence, we have $Y_1 \cap \tau(\gamma) = \emptyset$ and $E \cap \tau(\gamma) \neq \emptyset$. This shows that there is no irreducible surface T in X_2 such that $\dim \Phi(T) \leq 1$. There are a finite numbers of conics in X through the point x (see [8]). Let γ be the proper transform of a conic in X through x . Then $\dim \Phi(\gamma) = 0$. In particular, there are a finite number of irreducible curves γ' in X_2 such that $\dim \Phi(\gamma') = 0$. Therefore we have the following

LEMMA 4.5. *$\Phi: X_2 \rightarrow W := \Phi(X_2) \hookrightarrow \mathbb{P}^6$ is a birational morphism of X_2 onto a 3-fold W of degree 4 in $\mathbb{P}^6$. In particular, $b_2(X_2) = b_2(W) = 3$.*

6. Since $\deg W = 4$ in $\mathbb{P}^6$, we have an equality

$$\deg W = \text{codim } W + 1. \quad (4.11)$$

Since there is a smooth rational curve γ in X_2 such that $\dim \Phi(\gamma) = 0$ and $b_2(X_2) = b_2(W)$, the 3-fold W has a finite number of isolated singularities. Thus, W is a cone over a rational scroll or a cone over the Veronese surface. Hence, $b_2(W) = 1$. This is a contradiction, since $b_2(W) = 3$ by Lemma 4.5. Thus, we have:

Conclusion

The case (b) of Proposition 2.5 can not occur.

We have proved in Section 3 that the case (a) of Proposition 2.5 can not occur.

Therefore, in the case of the index $r = 1$, such a compactification of $\mathbb{C}^3$ does not exist. Thus we have the Theorem (see the Introduction).

5. Remarks and an example

1. Let (X, Y) be an analytic compactification of $\mathbb{C}^3$. Then we have (cf. [2], [3]):

Y has at most isolated singularities.

$$\begin{array}{ccc} \Downarrow & & \\ Y \text{ is normal.} & \Rightarrow & Y \text{ is projective.} \\ & \Leftarrow & \end{array}$$

$$\begin{array}{ccc} \Downarrow & \nrightarrow & \\ Y \text{ is irreducible.} & \Leftrightarrow & b_2(X) = 1 \end{array}$$

In the case where Y is normal, we have determined the complete structure of such a (X, Y) (see Theorem in the Introduction).

On the other hand, we know that there is a non-normal hyperplane section E_5 of the Fano 3-fold V_5 such that $V_5 - E_5 \cong \mathbb{C}^3$ ([3]). This gives an example of a compactification (X, Y) of $\mathbb{C}^3$ with a non-normal irreducible boundary Y .

Recently, Peternell–Schneider [18] and Peternell [19] proved the following

THEOREM 5.1. *Let (X, Y) be a projective compactification of $\mathbb{C}^3$ with $b_2(X) = 1$. Assume that Y is non-normal. Then, X is a Fano 3-fold of the index r ($1 \leq r \leq 2$), and*

- (i) $r = 2 \Rightarrow (X, Y) \cong (V_5, E_5)$ (up to isomorphism).
- (ii) $r = 1 \Rightarrow X \cong V_{22} \hookrightarrow \mathbb{P}^{13}$ (or $V'_{22} \hookrightarrow \mathbb{P}^{12}$)
(Mukai–Umemura [14]).

2. Finally, we will prove that there is a non-normal hyperplane section H'_{22} of V'_{22} such that $V'_{22} - H'_{22} \cong \mathbb{C}^3$. Let $(a_0 : a_1 : \dots : a_{12})$ be a homogeneous coordinate of $\mathbb{P}^{12}$. Then $V'_{22} \hookrightarrow \mathbb{P}^{12}$ can be written as follow (see p. 506 in [14]):

$$\begin{aligned} a_0 a_4 - 4a_1 a_3 + 3a_2^2 &= 0 \\ a_0 a_5 - 3a_1 a_4 + 2a_2 a_3 &= 0 \\ 7a_0 a_6 - 12a_1 a_5 - 15a_2 a_4 + 20a_3^2 &= 0 \\ a_0 a_7 - 6a_2 a_5 + 5a_3 a_4 &= 0 \\ 5a_0 a_8 + 12a_1 a_7 - 42a_2 a_6 - 20a_3 a_5 + 45a_4^2 &= 0 \\ a_0 a_9 - 6a_1 a_8 - 6a_2 a_7 - 28a_3 a_6 + 28a_4 a_5 &= 0 \\ a_0 a_{10} + 12a_1 a_9 + 12a_2 a_8 - 76a_3 a_7 - 21a_4 a_6 + 72a_5^2 &= 0 \end{aligned}$$

$$\begin{aligned}
a_0 a_{11} + 24a_1 a_{10} + 90a_2 a_9 - 130a_3 a_8 - 405a_4 a_7 + 420a_5 a_6 &= 0 \\
a_0 a_{12} + 60a_1 a_{11} + 534a_2 a_{10} + 380a_3 a_9 - 3195a_4 a_8 - 720a_5 a_7 + 2940a_6^2 &= 0 \quad (*) \\
a_1 a_{12} + 24a_2 a_{11} + 90a_3 a_{10} - 130a_4 a_9 - 405a_5 a_8 + 420a_6 a_7 &= 0 \\
a_2 a_{12} + 12a_3 a_{11} + 12a_4 a_{10} - 76a_5 a_9 - 21a_6 a_8 + 72a_7^2 &= 0 \\
a_3 a_{12} - 6a_4 a_{11} - 6a_5 a_{10} - 28a_6 a_9 + 28a_7 a_8 &= 0 \\
5a_4 a_{12} + 12a_5 a_{11} - 42a_6 a_{10} - 20a_7 a_9 + 45a_8^2 &= 0 \\
a_5 a_{12} - 6a_7 a_{10} + 5a_8 a_9 &= 0 \\
7a_6 a_{12} - 12a_7 a_{11} - 15a_8 a_{10} + 20a_9^2 &= 0 \\
a_7 a_{12} - 3a_8 a_{11} + 2a_9 a_{10} &= 0 \\
a_8 a_{12} - 4a_9 a_{11} + 3a_{10}^2 &= 0
\end{aligned}$$

In the affine part $\{a_0 = 1\} \cong \mathbb{C}^{12}(a_1, \dots, a_{12})$, let us consider the following coordinate transformation:

$$\begin{aligned}
x_1 &= a_1 \\
x_2 &= a_2 \\
x_3 &= a_3 \\
x_4 &= a_4 - 4a_1 a_3 + 3a_2^2 \\
x_5 &= a_5 - 3a_1 a_4 + 2a_2 a_3 \\
x_6 &= 7a_6 - 12a_1 a_5 - 15a_2 a_4 + 20a_3^2 \\
x_7 &= a_7 - 6a_2 a_5 + 5a_3 a_4 \\
x_8 &= 5a_8 + 12a_1 a_7 - 42a_2 a_6 - 20a_3 a_5 + 45a_4^2 \\
x_9 &= a_9 - 6a_1 a_8 - 6a_2 a_7 - 28a_3 a_6 + 28a_4 a_5 \\
x_{10} &= a_{10} + 12a_1 a_9 + 12a_2 a_8 - 76a_3 a_7 - 21a_4 a_6 + 72a_5^2 \\
x_{11} &= a_{11} + 24a_1 a_{10} + 90a_2 a_9 - 130a_3 a_8 - 405a_4 a_7 + 420a_5 a_6 \\
x_{12} &= a_{12} + 60a_1 a_{11} + 534a_2 a_{10} + 380a_3 a_9 - 3150a_4 a_8 - 720a_5 a_7 + 2940a_6^2
\end{aligned} \quad (**)$$

Then the Jacobian $|\partial(x_1, \dots, x_{12})/\partial(a_1, \dots, a_{12})| = 35 \neq 0$, and further we have

$$V'_{22} \cap \{a_0 = 1\} \cong \{(x_1, \dots, x_{12}); x_j = 0 \ (4 \leq j \leq 12)\} = \mathbb{C}^3(x_1, x_2, x_3).$$

We put $H'_{22} := V_{22} \cap \{a_0 = 0\}$. Then H'_{22} is non-normal.

Therefore the pair (V'_{22}, H'_{22}) is a compactification of $\mathbb{C}^3$ with a non-normal boundary.

One can easily see that the singular locus of H'_{22} is a line in V'_{22} .

Question 1. Is there a non-normal hyperplane section E_{22} in $V_{22} (\neq V'_{22})$ such that $V_{22} - E_{22} \cong \mathbb{C}^3$?

Acknowledgements

The author would like to thank the Max-Planck-Institute für Mathematik in Bonn, especially Professor F. Hirzebruch for hospitality and encouragement, and would also like to thank Professor N. Nakayama for the stimulating conversations we had and for his valuable comments.

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