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Introduction

This paper arose from an attempt to solve some questions which were posed at the seminar of A. N. Parchin when Deligne's program ([D]) was reviewed. These problems are related to hypothetical functorial and metrical versions of the Riemann-Roch-Hirzebruch theorem. One of the problems posed by Deligne is, for instance, the following construction:

Let a proper morphism of schemes $X \to S$ of relative dimension n and a polynomial $P(c_i(E_j))$ of absolute degree $n+1$ (where $\deg(c_i) = i$) in the Chern classes of vector bundles $E_1, \ldots, E_k$ be given. Construct a functor which to the vector bundles E_j on X associates a line bundle on S

$$I_{X/S} P(c_i(E_j)) \tag{1}$$

which is an 'incarnation' of $\int_{X/S} P(c_i(E_j)) \in CH^1(S)$. The functor (1) should be equipped with some natural transformations which correspond to well-known equalities between Chern classes (cf. [D, 2.1]). Further steps in Deligne's program are to equip the line bundles (1) with metrics, to prove a functorial version of the Riemann-Roch-Hirzebruch formula which provides an isomorphism between the determinant $\det(Rp_*(F))$ of the cohomology of a vector bundle F and a certain line bundle of type (1); and (finally) to compare the metric on the right side of the Riemann-Roch isomorphism and the Quillen metric on the determinant of the cohomology.

In [D], Deligne dealt with the case $n = 1$. He considered (1) as a closed expression. It is our strategy to give 'live' to each ingredient of (1). If one tries to do so, the ith Chern functor $\mathbf{c}_i(E)$ should take values in the ith Chow category $\mathbf{CH}^i(X)$. It is the aim of these notes to explain what we believe to be the best definition of the Chow category, and to define some of the basic functors between Chow categories.

Our proposal for (1) is the following expression:

$$\mathbf{p}_*(P(\mathbf{c}_i(E_j))), \tag{2}$$

where $p: X \to S$ is the morphism we have in consideration and $\mathbf{p}_*$ is a push-forward functor which will be introduced in §3. The most complicated ingredient of (2) is the Chern functor $\mathbf{c}_i(\cdot)$. Its construction has been outlined in [Fr1], and details are contained in the notes [Fr2] which I distributed in June 1988. We shall publish our results on Chern functors together with more considerations about the Riemann-Roch problem in a continuation of this paper.

One of the advantages of the approach to Deligne's program via Chow categories is that it allows us to state the functorial Riemann-Roch theorem in Grothendieck's form. Hence it should be possible to copy the standard proof for Riemann-Roch theorems. Our proposal for the Riemann-Roch-Grothendieck isomorphism is a canonical isomorphism

$$\mathbf{ch}(Rp_*E) \cup A \to \mathbf{p}_*(\mathbf{ch}(E) \cup Td(\Omega^{\cdot}_{X/S}{}^{\check{}}) \cup \mathbf{p}^!(A)) \tag{3}$$

for any local complete intersection p with relative cotangential complex $\Omega^{\cdot}_{X/S}$. The isomorphism (3) should be characterized by certain axiomatic properties.

To explain the ingredients of (3) further I mention that the Chern functor will not be a mere object of the Chow category but an intersection product

$$\mathbf{c}_i(E) \cup \cdot : \mathbf{CH}^k(X) \to \mathbf{CH}^{k+i}(X).$$

Therefore no regularity assumptions for X are necessary to define both sides of (3) as a functor with values in the quotient category $\mathbf{CH}^{\cdot}(S) \otimes Q$. The remaining ingredient of (3) is the Gysin functor $\mathbf{p}^!$. This is our first example of a non-trivial functor between Chow categories, and the most considerations of this paper are directly or indirectly devoted to its construction.

After recalling some basic properties of Quillen's spectral sequence in §1, we define the Chow categories and some of the basic functors in §2 and §3. §4 contains the construction of the Gysin functor. In §5 we use this Gysin functor to outline the construction of a functorial intersection product. As an example which lies outside Deligne's program, we apply the intersection product functor to construct a biextension between certain groups of algebraic cycles. This biextension generalizes the well-known autoduality of the Jacobian, and should be equivalent to a construction of Bloch.

I started my research on Chow categories while I was a postgraduate student in Moscow under the guidance of I. M. Gel'fand. I am much obliged to A. A. Beilinson, Ju. I. Manin, A. N. Parchin, V. V. Schechtman, and the participants of Parchin's seminar for many helpful discussions. In particular, Beilinson and Manin pointed out that the Chow category should provide an alternative construction of Bloch's biextension. Their proposal is carried out, at least partially, in §5.5.

Notations

Throughout this paper, schemes are assumed to be Noetherian, separated over Spec($\mathbb{Z}$) and universally catenary. Our notations of K-theory are as usual $K_i(X) = K_i(P(X))$ and $K'_i(X) = K_i(M(X))$, where $P(X)$ and $M(X)$ are the exact categories of vector bundles and of coherent $\mathcal{O}_X$-modules on X.

Products in K-theory are defined by Waldhausen's pairing $BQA \wedge BQB \to BQQC$ (cf. [W], [Gr]). The relation between the product and the boundary of the localization sequence is given by formula [Gr, Corollary (2.6)]. In particular, the boundary of the K-theoretic product of two invertible functions differs by a sign from the tame symbol.

1. The sheaves G_k

For a scheme X, denote by X_k the set of points of codimension k (i.e., of points x with dim $(\mathcal{O}_{X,x}) = k$) and by $X_{(k)}$ the set of points of X, equipped with the following topology. U is open in $X_{(k)}$ iff it is Zariski-open and for every $x \in X_l$ with $l < k$, we have either $x \in U$ or $x \notin \bar{U}$. In particular, $X_{(1)} = X_{\mathrm{Zar}}$. For a point $x \in X$, $k(x)$ is the residue field of x.

1.1. *Definition of* G_k

The descending filtration of $M(X)$ by $M_p(X) = \{$coherent sheaves on X with support in codimension $\geqslant p\}$ defines a spectral sequence (cf. [Q, (5.5)] or [G, p. 269]) with initial term

$$E_1^{p,q}(X) = K_{-p-q}(M_p/M_{p+1}) = \coprod_{x \in X_p} K_{-p-q}(k(x)) \tag{1}$$

converging to $K'_{-p-q}(X)$. In particular, $E_2^{p,q}(X)$ is the homology in the middle term of

$$\coprod_{x \in X_{p-1}} K_{1-p-q}(k(x)) \to \coprod_{x \in X_p} K_{-p-q}(k(x)) \to \coprod_{x \in X_{p+1}} K_{-p-q-1}(k(x)). \tag{2}$$

We are particularly interested in the groups $Z^k(X) = E_1^{k,-k}(X)$, $CH^k(X) = E_2^{k,-k}(X)$, and $G_k(X) = E_2^{k-1,-k}(X)$. By (2), they can be defined elementarily, using only K_0, K_1, and K_2 of fields.

The $E_k^{p,q}$ are presheaves on X_{Zar}. Furthermore, one checks easily that the restriction of G_k to $X_{(k)}$ is a sheaf. By (2), there is an exact sequence for U open in $X_{(k)}$

$$0 \to G_k(X - \bar{U})^0 \to G_k(X) \xrightarrow{\partial} G_k(U) \to Z^k(X, X - U) \to CH^k(X), \tag{3}$$

where $G_k(X - \bar{U})^0 = \text{image}(G_k(X) \to G_k(X - \bar{U}))$ and $Z^k(X, X - U) = \{x \in Z^k(X) \operatorname{supp}(x) \subset X - U\}$. The arrow $G_k(X) \to \ker(\partial)$ has a natural splitting.

1.2. *Mayer-Vietoris and localization sequences*

If X is the union of its open subsets U and V, we have

$$0 \to E_1^{p,q}(X) \to E_1^{p,q}(U) \oplus E_1^{p,q}(V) \to E_1^{p,q}(U \cap V) \to 0$$

and hence

$$\begin{aligned} &\to E_2^{p-1,q}(U \cap V) \to E_2^{p,q}(X) \to E_2^{p,q}(U) \oplus E_2^{p,q}(V) \\ &\quad \to E_2^{p,q}(U \cap V) \to E_2^{p+1,q}(X) \to. \end{aligned} \tag{4}$$

Let $Z \subset X$ be closed. We call Z of pure codimension d if $X_k \cap Z = Z_{k-d}$ for $k \in \mathbb{Z}$. Then the exact sequence

$$0 \to E_1^{p-d,q+d}(Z) \to E_1^{p,q}(X) \to E_1^{p,q}(X - Z) \to 0$$

gives rise to

$$\begin{aligned} &\to E_2^{p-1,q}(X - Z) \to E_2^{p-d,q+d}(Z) \to E_2^{p,q}(X) \to E_2^{p,q}(X - Z) \\ &\quad \to E_2^{p+1-d,q+d}(Z) \to \cdots \end{aligned} \tag{5}$$

1.3. *Flat pull-back*

If $f\colon Y \to X$ is a flat morphism, it defines an exact functor $f^*\colon M(X) \to M(Y)$ which maps $M_k(X)$ into $M_k(Y)$. Consequently, we have a homomorphism $f^*\colon E_k^{p,q}(X) \to E_k^{p,q}(Y)$ which commutes with the differentials d_k, and hence preserves (3), (4) and (5).

1.4. *Proper push-forward*

Let $f\colon X \to Y$ be a morphism of finite type. We call f of constant relative dimension $d \in \mathbb{Z}$ if for every $x \in X_l$ such that $\dim(\overline{f(x)}) = \dim(\bar{x})$ we have $f(x) \in Y_{l-d}$. The proof of the following lemma is straightforward:

LEMMA. *Let*

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a Cartesian diagram in which f is of constant relative dimension d. In each of the following cases, f' is also of constant relative dimension d:

(i) *If g is flat.*
(ii) *If g and g_X are l.c.i. (local complete intersections) and for every $x \in X'$, $d_x(g_X) = d_{f'(x)}(g)$, where $d_x(g)$ is the relative dimension of the lci-morphism g at x* (cf. [FL, p. 89] *or* [SGA6, VIII.1.9.]).

Proof. Since the question is local, we may assume in (i) that X is a closed subscheme of A^n_Y. Then f is of relative dimension d if and only if X is of codimension $n - d$ in A^n_Y, and this condition remains valid after flat base change. By (i), (ii) is reduced to the case of a regular closed immersion f in which it is trivial. □

Now we assume that f: $X \to Y$ is a proper morphism of constant relative dimension d. Then we have exact functors

$$f_*: M_p(X)/M_{p+1}(X) \to M_{p-d}(Y)/M_{p+1-d}(Y) \tag{6}$$

defining

$$f_*: E_1^{p,q}(X) \to E_1^{p-d,q+d}(Y). \tag{7}$$

The following theorem is similar to results of Gillet and Schechtman:

THEOREM. (i) *The homomorphism* (7) *commutes with the differential d_1 of the Quillen spectral sequence. Hence it defines f_*: $E_2^{p,q}(X) \to E_2^{p-d,q+d}(Y)$*

(ii) *The homomorphism f_* on the E_2-terms is compatible with the localization sequence* (3), *i.e., if U is open in $X_{(k)}$ and $V = Y - f(X - U)$ then we have a commutative diagram*

$$\begin{tikzcd} G_k(X) \arrow[r] \arrow[d, "f_*"] & G_k(U) \arrow[r] \arrow[d, "f_*"] & Z^k(X, X - U) \arrow[r] \arrow[d, "f_*"] & CH^k(X) \arrow[d, "f_*"] \\ G_{k-d}(Y) \arrow[r] & G_{k-d}(V) \arrow[r] & Z^{k-d}(Y, Y - V) \arrow[r] & CH^{k-d}(Y) \end{tikzcd}$$

Proof of (i). It is possible to copy the proof in [G, 7.22]. It should also be possible to apply the results of [GN].

Proof of (ii). This follows from (1) and the definition of (3).

1.5. *Specialization*

This is a modification of [F, Remark 2.3.], cf. also [G, 8.6.]. Let $D \subset X$ be a regular embedding of codimension 1, and assume that f is a section of $\mathcal{O}_X$ in some Zariski-neighbourhood U of D generating the sheaf of ideals defining $D \subset U$.

The existence of f is a serious restriction to the embedding $D \subset X$, for instance it implies the triviality of the conormal bundle of the immersion, which means that we are in the situation described in [F, Remark 2.3.].

We define homomorphisms

$$sp_f\colon E_k^{p,q}(X - D) \to E_k^{p,q}(D), \qquad k \geqslant 1 \tag{9}$$

as follows.

The tensor product $P(U - D) \times M_p(U - D) \to M_p(U - D)$ defines

$$[f] \cup \cdot\colon E_k^{p,q}(U - D) \to E_k^{p,q-1}(U - D), \tag{10}$$

where $[f]$ is the class of $f\cdot$ in $K_1(U - D)$. Let

$$\begin{aligned} M'_p = \{&\text{coherent } \mathcal{O}_U\text{-modules } F \text{ with } \operatorname{cod}_U(\operatorname{supp}(F)) \geqslant p \\ &\text{and } \operatorname{cod}_D(D \cap \operatorname{supp}(F)) \geqslant p\}. \end{aligned}$$

Then $M_p(U - D)/M_{p+k}(U - D) = (M'_p/M'_{p+k})/(M_p(D)/M_{p+k}(D))$, consequently we have

$$\partial\colon E_k^{p,q}(U - D) \to E_k^{p,q+1}(D). \tag{11}$$

If $k = 2$, ∂ is the boundary homomorphism in (5).

We define sp_g by the composition of

$$E_k^{p,q}(X - D) \xrightarrow{\text{restriction}} E_k^{p,q}(U - D) \xrightarrow{[f]\cup\cdot} E_k^{p,q-1}(U - D) \longrightarrow E_k^{p,q}(D). \tag{12}$$

On the line $p + q = 0$, sp_f is independent of f, and we obtain the homomorphism i^* described in [F, Remark 2.3.]. The composition

$$E_k^{p,q}(X) \longrightarrow E_k^{p,q}(X - D) \xrightarrow{\mathrm{sp}_f} E_k^{p,q}(D) \tag{13}$$

is also independent of f.

1.6. *Compatibilities*

Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a Cartesian square with g flat and f proper of constant relative dimension d. Then we have the base change identity

$$g^*f_* = f'_*g'^* \text{ in } \mathrm{Hom}(E_k^{p,q}(X), E_k^{p-d,q+d}(Y')), \quad k \in \{1, 2\}. \tag{14}$$

For the diagram of functors

$$\begin{tikzcd} M_p(X)/M_{p+1}(X) \arrow[r, "g'^*"] \arrow[d, "f_*"] & M_p(X')/M_{p+1}(X') \arrow[d, "f'_*"] \\ M_{p-d}(Y)/M_{p+1-d}(Y) \arrow[r, "g^*"] & M_{p-d}(Y')/M_{p+1-d}(Y') \end{tikzcd}$$

commutes up to a natural transformation.

Consider a fibre square

$$\begin{tikzcd} D' \arrow[r, phantom, "\subset"] \arrow[d, "p_D"] & X' \arrow[d, "p"] \\ D \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

in which $D \subset X$ and $D' \subset X'$ are regular embeddings of codimension 1. Let f be the same as in 1.5. If p is proper of constant relative dimension d, the lemma in 1.4 implies that p_D is of the same relative dimension d. We have

$$\mathrm{sp}_f p_* = p_{D*}\mathrm{sp}_{p^*(f)} \text{ in } \mathrm{Hom}(E_k^{p,q}(X'), E_k^{p-d,q+d}(D)), \qquad k \in \{1, 2\}. \tag{15}$$

If in the same fibre square p is flat, we have

$$p_D^* \mathrm{sp}_f = \mathrm{sp}_{p^*(f)} p^*. \tag{16}$$

(15) is a consequence of the commutative diagram

$$\begin{tikzcd}
E_1^{p,q}(X'-D') \arrow[r, "{[p^*(f)]\cup\cdot}"] \arrow[d, "p_*"] \arrow[dr, phantom, "\text{(A)}"] & E_1^{p,q-1}(X'-D') \arrow[r] \arrow[d, "p_*"] \arrow[dr, phantom, "\text{(B)}"] & E_1^{p,q}(D') \arrow[d, "p_*"] \\
E_1^{p-d,q+d}(X-D) \arrow[r, "{[f]\cup\cdot}"] & E_1^{p-d,q+1-d}(X-D) \arrow[r] & E_1^{p-d,q+d}(D)
\end{tikzcd}$$

The commmutativity of (A) follows from the fact that the diagram of bilinear functors

$$\begin{tikzcd}
P\times M \arrow[r] & p^*(P)\otimes M \\[-20pt]
P(X-D)\times(M_p/M_{p+1})(X'-D') \arrow[r] \arrow[d, "\mathrm{Id}\times p_*"] & (M_p/M_{p+1})(X'-D') \arrow[d, "p_*"] \\
P(X-D)\times(M_{p-d}/M_{p+1-d})(X-D) \arrow[r, "P\times M\to P\otimes M"] & (M_{p-d}/M_{p+1-d})(X-D)
\end{tikzcd}$$

commutes up to a natural transformation, and (B) commutes because p_* maps sheaves on X' whose support is of codimension p and meets D' in codimension p to sheaves on X with the similar property, and hence defines a morphism between the quasi-fibrations used to define (11).

The proof of (16) is similar.

If we have a commutative diagram

$$\begin{tikzcd}
D \arrow[rr, "i"] \arrow[dr, "g"'] & & X \arrow[dl, "h"] \\
& Z &
\end{tikzcd}$$

in which g and h are flat and D, X, and f satisfy the assumptions of 1.5, then

$$\mathrm{sp}_f h^* = g^*. \tag{17}$$

This can easily be reduced to the following general situation:

LEMMA. *Let the following objects be given:*

(i) *A sequence* $\mathscr{A} \xrightarrow{a} \mathscr{B} \xrightarrow{b} \mathscr{C}$ *of exact functors between exact categories such that* $ab \simeq 0$ *and* $BQ\mathscr{A} \to BQ\mathscr{B} \to BQ\mathscr{C}$ *is a fibration up to homotopy.*

(ii) *An exact functor $P \to P'$ between exact categories, an object Y of P and an endomorphism $f\colon Y \to Y$ in P which becomes an isomorphism in P'.*

(iii) *An exact category $\mathscr{D}$, biexact functors $\otimes\colon P \times \mathscr{D} \to \mathscr{B}$ and $P' \times \mathscr{D} \to \mathscr{C}$ such that*

$$\begin{tikzcd} P \times \mathscr{D} \arrow[r, "\otimes"] \arrow[d] & \mathscr{B} \arrow[d, "b"] \\ P' \times \mathscr{D} \arrow[r, "\otimes"] & \mathscr{C} \end{tikzcd}$$

commutes up to a natural transformation, and an exact functor $G\colon \mathscr{D} \to \mathscr{A}$ such that there is a functorial exact sequence in $\mathscr{B}$:

$$0 \to Y \otimes A \xrightarrow{f \otimes \mathrm{Id}_A} Y \otimes A \xrightarrow{\Pi} G(A) \to 0 \quad (A \in \mathrm{Ob}(\mathscr{D})).$$

Let $[f] \in K_1(P')$ be the class of f viewed as an automorphism in P'. Then

$$\partial([f] \cup \xi) = G_*(\xi), \qquad \xi \in K_i(\mathscr{D}), \tag{18}$$

where ∂: $K_{i+1}(\mathscr{C}) \to K_i(\mathscr{A})$ is the boundary defined by the fibration (i), *$\cup\colon K_1(P') \times K_i(\mathscr{D}) \to K_{i+1}(\mathscr{C})$ is the pairing defined by $\otimes$, and $G_*\colon K_i(\mathscr{D}) \to K_i(\mathscr{A})$ is defined by G.*

To derive (17) *from* (18), *we put $\mathscr{A} = M_p(D)/M_{p+1}(D)$, $\mathscr{B} = M'_p/M'_{p+1}$* (cf. 1.5.) *$\mathscr{C} = M_p(X)/M_{p+1}(X)$, $\mathscr{D} = M_p(Z)/M_{p+1}(Z)$, $P = P(X)$, $P' = P(X - D)$, $Y = \mathcal{O}_X$, and f = multiplication by f. Furthermore we put $G = g^*$ and define $\otimes\colon P \times \mathscr{D} \to \mathscr{B}$ by $(M, E) \to h^*(M) \otimes E$.*

Proof of Lemma. The class $[f]$ is given by the homotopy class of the map $S^2 \to |BPQ'|$ defined by the diagram

$$\begin{tikzcd} & 0 \arrow[dl, "o_Y^!"'] \arrow[dr, "o_Y^!"] & \\ Y \arrow[rr, "f"] & & Y. \\ & 0 \arrow[ul, "o_!^Y"] \arrow[ur, "o_!^Y"'] & \end{tikzcd} \tag{19}$$

Here we use the usual notations for morphisms in QP', and $o_Y = Y \to 0$, $o^Y = 0 \to Y$. To get S^2 from the diagram (19), identify its left and right boundary.

Consequently, the homotopy class $\Sigma^2 |BQ\mathscr{D}| \to |BQQ\mathscr{C}|$ obtained by applying Waldhausen's pairing to $[f]$ can be defined by the geometric realization of the map which associates to $A \in \mathscr{D}$ the following diagram of vertical morphisms in $QQ\mathscr{C}$

$$\begin{tikzcd}
& 0 \arrow[dl, "{}^v o^!_{Y\otimes A}"'] \arrow[dr, "{}^v o^!_{Y\otimes A}"] & \\
Y\otimes A \arrow[rr, "{}^v(f\otimes \mathrm{Id}_A)"] & & Y\otimes A \\
& 0 \arrow[ul, "{}^v o_!^{Y\otimes A}"] \arrow[ur, "{}^v o_!^{Y\otimes A}"'] &
\end{tikzcd} \tag{20}$$

(the left superscript v denotes vertical morphisms in QQ) and to a morphism in $Q\mathscr{D}$ the similar diagram of bimorphisms in $QQ\mathscr{C}$. The diagram (20) has an obvious lifting to $QQ\mathscr{B}$:

$$\begin{tikzcd}
& 0 \arrow[dl, "{}^v o^!_{Y\otimes A}"'] \arrow[dr, "{}^v[\mathrm{im}(f)]"] & \\
Y\otimes A \arrow[rr, "{}^v(f\otimes \mathrm{Id}_A)_!"] & & Y\otimes A \\
& 0 \arrow[ul, "{}^v o_!^{Y\otimes A}"] \arrow[ur, "{}^v o_!^{Y\otimes A}"'] &
\end{tikzcd} \tag{21}$$

(${}^v[\mathrm{im}(f)]$ = vertical morphism from 0 to $Y\otimes A$ defined by the subobject $\mathrm{im}(f\otimes \mathrm{Id}_A) \subset Y\otimes A$). The diagram of bimorphisms corresponding to (20) has a lifting to $QQ\mathscr{B}$ which is similar to (21). Our task is now to compute the difference between the two homotopy classes $\Sigma DBQ\mathscr{D}| \to |BQQ\mathscr{B}|$ defined by the arrows on the left and the right boundary of (21). Because ${}^v[\mathrm{im}(f)]$ is equal to the composition

$$0 \xrightarrow{{}^v o_!^{G(A)}} G(A) \xrightarrow{{}^v\pi^!} Y\otimes A,$$

the map $\Sigma|BQ\mathscr{D}| \to |BQQ\mathscr{B}|$ defined by the vertical morphisms on the right boundary of (21) and the related bimorphisms is homotopic to the map defined by the diagram

$$\begin{tikzcd}
0 \arrow[r, "{}^v o_!^{G(A)}"] \arrow[dr, "{}^v o_!^{Y\otimes A}"'] & G(A) \arrow[d, "{}^v\pi^!"] \\
& Y\otimes A
\end{tikzcd} \tag{22}$$

and the similar diagram of bimorphisms. By the commutative diagram

$$\begin{tikzcd}
G(A) \arrow[d, "{}^v\pi^!"'] & \\
Y\otimes A & 0, \arrow[ul, "{}^v o^!_{G(A)}"'] \arrow[l, "{}^v o^!_{Y\otimes A}"]
\end{tikzcd}$$

(22) is homotopic to

$$\begin{tikzcd}
 & G(A) & \\
0 \arrow[ur, "v_{O_!}^{G(A)}"] \arrow[dr, "v_{O_!}^{Y\otimes A}"'] & & 0. \arrow[ul, "v_{O^!_{G(A)}}"'] \arrow[dl, "v_{O^!_{Y\otimes A}}"] \\
 & Y\otimes A &
\end{tikzcd}$$

The bottom half of this diagram coincides with the left boundary of (21). By the very definition of the boundary operator ∂, we conclude that $\partial([f]\cup\cdot)$ is the homotopy class of the map $\Sigma\,|BQ\mathscr{D}| \to |BQQ\mathscr{A}|$ given by the diagram

$$\begin{tikzcd}
0 \arrow[r, bend left, "v_{O_!}^{G(A)}"] \arrow[r, bend right, "v_{O^!_{G(A)}}"'] & G(a)
\end{tikzcd}$$

and the similar diagram of bimorphisms. This is, however, the composition of $G_*\colon |BQ\mathscr{D}| \to |BQ\mathscr{A}|$ with the map $\Sigma\,|BQ\mathscr{A}| \to |BQQ\mathscr{A}|$ defined in [W, p. 197]. The proof of the lemma and of (17) is complete.

1.7. *A relation between two specializations*

Let $D_i \subset X$ be regular immersions of codimension one such that the sheaf of ideals defining D_i is in some neighbourhood of D_i trivialized by f_i, $i\in\{1; 2\}$. We suppose also that $D_1 \cap D_2 \subset X$ is a regular immersion of codimension two, i.e., that $f_i|_{D_j}$ is not a zero-divisor if $i \neq j$. Then we have the identity

$$\mathrm{sp}_{f_2|_{D_1}}\mathrm{sp}_{f_1} = \mathrm{sp}_{f_1|_{D_2}}\mathrm{sp}_{f_2} \text{ in } \mathrm{Hom}(E_k^{p,q}(X - D_1 - D_2), E_k^{p,q}(D_1 \cap D_2)). \tag{23}$$

This follows from the commutative diagram

$$\begin{tikzcd}[column sep=tiny]
 & & E_k^{p,q}(X - D_1 - D_2) \arrow[dl, "{[f_1]\cup\cdot}"'] \arrow[dr, "{[f_2]\cup}"] & & \\
 & E_k^{p,q-1}(X - D_1 - D_2) \arrow[dl, "\partial"'] \arrow[dr, "{[f_2]\cup\cdot}"] & & E_k^{p,q-1}(X - D_1 - D_2) \arrow[dl, "{-[f_1]\cup\cdot}"'] \arrow[dr, "\partial"] & \\
E_k^{p,q}(D_1 - D_2) \arrow[dr, "{[f_2]\cup}"'] & & E_k^{p,q-2}(X - D_1 - D_2) & & E_k^{p,q}(D_2 - D_1) \arrow[dl, "{-[f_1]\cup}"] \\
 & E_k^{p,q-1}(D_1 - D_2) \arrow[ur, "\partial"] \arrow[dr, "\partial"'] & & E_k^{p,q-1}(D_2 - D_1) \arrow[ul, "\partial"'] \arrow[dl, "-\partial"] & \\
 & & E_k^{p,q}(D_1 \cap D_2) & &
\end{tikzcd}$$

1.8. *A relation between specialization and restriction to closed subschemes*

Let X be a regular scheme satisfying Gersten's conjecture. Then we have an isomorphism

$$B\colon E_2^{p,q}(X) \to H^p(X, \mathcal{K}_{-q}) \tag{24}$$

defined by the well-known acyclic resolvent

$$\mathcal{K}_q \to \mathcal{E}_1^{\bullet,-q} \tag{25}$$

of the sheaf $\mathcal{K}_q$ associated to $U \to K_q(U)$. In (25), $\mathcal{E}_1^{p,q}$ is the Zariski-sheaf $U \to E_1^{p,q}(U)$.
If $i\colon Z \subset X$ is a closed regular subscheme of X, the composition

$$E_2^{p,q}(X) \xrightarrow{\ B\ } H^p(X, \mathcal{K}_{-q}) \xrightarrow{\text{restriction}} H^p(Z, \mathcal{K}_{-q}) \xrightarrow{\ B^{-1}\ } E_2^{p,q}(Z) \tag{26}$$

defines a homomorphism

$$i^*\colon E_2^{p,q}(X) \to E_2^{p,q}(Z). \tag{27}$$

PROPOSITION. *Let Z and D be closed regular subschemes of a regular scheme X satisfying Gersten's conjecture. We assume that $Z \cap D$ is regular and of codimension one in Z and that (X, D, f) satisfies the assumptions of* 1.5. *Then we have*

$$i_D^* \mathrm{sp}_f = \mathrm{sp}_f|_Z i^* \text{ in } \mathrm{Hom}(E_2^{p,q}(X - D), E_2^{p,q}(Z \cap D)), \tag{28}$$

where i and i_D are the inclusions $Z \subset X$ and $Z \cap D \subset D$.

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
E_2^{p,q}(X - D) \arrow[r, "{[f]\cup\cdot}"] \arrow[d, "B"] \arrow[dr, phantom, "(A)"] & E_2^{p,q-1}(X - D) \arrow[rr, "\partial"] \arrow[d, "B"] \arrow[drr, phantom, "(B)"] & & E_2^{p,q}(D) \arrow[d, "B"] \\
H^p(X - D, \mathcal{K}_{-q}) \arrow[r, "{(-1)^p[f]\cup\cdot}"] \arrow[d] & H^p(X - D, \mathcal{K}_{1-q}) \arrow[r, "\partial"] \arrow[d] \arrow[drr, phantom, "(C)"] & H_D^{p+1}(X, \mathcal{K}_{1-q}) \arrow[r, "(a)"] & H^p(D, \mathcal{K}_{-q}) \arrow[d] \\
H^p(Z - D, \mathcal{K}_{-q}) \arrow[r, "{(-1)^p[f]\cup\cdot}"] \arrow[d, "B"] \arrow[dr, phantom, "(A')"] & H^p(Z - D, \mathcal{K}_{1-q}) \arrow[r, "\partial"] \arrow[d, "B"] \arrow[drr, phantom, "(B')"] & H_{Z\cap D}^{p+1}(Z, \mathcal{K}_{1-q}) \arrow[r, "(a')"] & H^p(Z \cap D, \mathcal{K}_{-q}) \arrow[d, "B"] \\
E_2^{p,q}(Z - D) \arrow[r, "{[f]\cup\cdot}"] & E_2^{p,q-1}(Z - D) \arrow[rr, "\partial"] & & E_2^{p,q}(Z \cap D)
\end{tikzcd}$$

In this diagram, the arrow (a) and its symmetric counterpart (a′) are defined by the purity isomorphism

$$\mathscr{H}^p_D(X, \mathscr{K}_{q,X}) = \begin{cases} 0 \text{ if } p > 1 \\ \mathscr{K}_{q-1,0} \text{ if } p = 1, \end{cases} \tag{29}$$

where $\mathscr{H}^p_D$ is the derived functor of the sheaf of sections with support in D. The only non-zero isomorphism in (29) is normalized by the commutativity of

$$\begin{tikzcd}
K'_q(U - D) = K_q(U - D) \arrow[r] \arrow[d, "\partial"] & H^0(U - D, \mathscr{K}_q) \arrow[dr, "\partial"] & \\
& & H^1_{U \cap D}(U, \mathscr{K}_q). \\
K'_{q-1}(U \cap D) = K_{q-1}(U \cap D) \arrow[r] & H^0(U \cap D, \mathscr{K}_{q-1}) \arrow[ur, "(29)"'] &
\end{tikzcd} \tag{30}$$

The commutativity of the squares (B) and (B′) is therefore obvious. For the commutativity of (C), we denote by $\tilde{P}$ the category of sheaves F on U with the property

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_Z, F) = 0 \quad \text{if } i > 0.$$

There is an obvious diagram

$$\begin{tikzcd}
BQP(U \cap D) \arrow[r] \arrow[d] & BQ\tilde{P} \arrow[r] \arrow[d] & BQP(U - D) \arrow[d] \\
BQM(D \cap Z \cap U) \arrow[r] & BQM(Z \cap U) \arrow[r] & BQM((Z - D) \cap U)
\end{tikzcd} \tag{31}$$

in which the rows are fibrations up to homotopy. Since the boundary homomorphism of the top row coincides with the left vertical arrow in (30), (31) implies the commutativity of

$$\begin{tikzcd}
K_q(U - D) \arrow[r] \arrow[d, "\partial"] & K_q(Z \cap U - D) \arrow[d, "\partial"] \\
K_{q-1}(U \cap D) \arrow[r] & K_{q-1}(Z \cap U \cap D)
\end{tikzcd}$$

for every Zariski-open U in X. By (30), this proves the commutativity of (C). The commutativity of (A) and its counterpart (A′) follows from the diagram of resolvents

$$\begin{tikzcd}
\mathcal{K}_{q,X} \arrow[r] \arrow[d, "{[f]\cup\cdot}"] & \mathcal{E}_1^{0,-q} \arrow[r] \arrow[d, "{[f]\cup\cdot}"] & \mathcal{E}_1^{1,-q} \arrow[r] \arrow[d, "{[f]\cup\cdot}"] & \cdots \\
\mathcal{K}_{q+1,X} \arrow[r] & \mathcal{E}_1^{0,-q-1} \arrow[r] & \mathcal{E}_1^{1,-q-1} \arrow[r] & \cdots
\end{tikzcd}$$

in which all the squares except the first one are anti-commutative. The commutativity of the other squares in the diagram is obvious. The proof of (28) is complete.

1.9. *Homotopy invariance*

Let $p: E \to X$ be the projection of a vector bundle to its base. Then $p^*: E_2^{p,q}(X) \to E_2^{p,q}(E)$ is an isomorphism.

Proof. By the localization sequence and the five lemma, we may reduce the assertion to the case that all connected components of X are irreducible and hence equidimensional. In this case the assertion follows from [G, Theorem 8.3].

2. Definition of the Chow category by means of cycles

On a normal locally factorial scheme X, every Weil divisor $B \in E_1^{1,-1}(X)$ defines a line bundle $O(D)$, and isomorphisms between $O(D)$ and $O(D')$ correspond to rational functions f with $\mathrm{div}(f) = D' - D$. We try to generalize this to higher codimension.

Let $\mathbf{CH}_z^i(X)$ be the following category. Objects of $\mathbf{CH}_z^i$ are cycles $z \in E_1^{i,-i}(X)$. Homomorphisms between z and z' are elements of the factor set

$$\mathrm{Hom}_{\mathbf{CH}_z^i(X)}(z, z') = \{f \in E_1^{i-1,-i}(X) \mid d_1(f) = z' - z\}/d_1 E_1^{i-2,-i}(X). \tag{1}$$

The composition $\mathrm{Hom}(z, z') \times \mathrm{Hom}(z', z'') \to \mathrm{Hom}(z, z'')$ sends the equivalence classes of f and f' in (1) to the class of $f + f'$. It is easy to see that $\mathbf{CH}_z^i$ is a Picard category in the sense of [D, §4.1] if the sum is given by

$$z \oplus z' = z + z' \tag{2}$$

$$\text{class of } f \oplus \text{class of } f' = \text{class of } f + f'.$$

The commutativity and associativity law are simply identities between functors.

To admit also non-invertible arrows, we mention that $E_1^{i,-i}$ (being the free group generated by X_i) carries a natural ordering $\leqslant$, and define the extended Chow category $\mathbf{CH}_z^i(X)_e$ which has the same objects as $\mathbf{CH}_z^i(X)$ and

$$\mathrm{Hom}_{\mathbf{CH}_z^i(X)}(z, z') = \{f \in E_1^{i-1,-i}(X) \mid d_1(f) \geqslant z' - z\}/d_1 E_1^{i-2,-i}(X), \tag{3}$$

as morphisms between z and z'. The composition of arrows is defined by adding f and f'. (2) defines a sum in $\mathbf{CH}_z^i(X)_e$.

If X is normal and locally factorial, then $\mathbf{CH}_z^1(X)$ is (via $D \to O(D)$) equivalent to the category of line bundles and isomorphisms, while $\mathbf{CH}_z^1(X)_e$ is equivalent to the category of line bundles and inclusions of line bundles on X. The sum $\oplus$ corresponds to the tensor product of line bundles.

By the results of §1, there are flat pull-back, proper push-forward, and specialization functors between the categories $\mathbf{CH}_z^i$. If, for instance, $p\colon Y \to X$ is flat, the functor $\mathbf{p}^*$ sends the object z to $p^*(z)$ and the class of f in (1) to the class of $p^*(f)$. Using these functors, we could try to establish a functorial analogue of the usual intersection theory. We shall, however, prefer another definition of the Chow category which defines $\mathbf{CH}^i(X)$ as the category of principal homogeneous sheaves for G_i on $X_{(i)}$. We shall see in §3 that this definition is essentially equivalent to our previous definition. The advantages of the definition in §3 are that it is similar to the equivalence between line bundles and $\mathcal{O}_X^*$-principal homogeneous sheaves, that it is sometimes convenient to prove the commutativity of diagrams by computing images of so called 'rational sections', and that (in the case of manifolds over $\mathbb{C}$) it provides an easy definition of what a metric on an object of the Chow category should be.

3. The categories $\mathbf{CH}^k(X)$ and $\widetilde{\mathbf{CH}}^k(X)$

3.1. *Definition*

Let $k \geqslant 1$. Recall that $X_{(k)}$ is a topology on X consisting of sufficiently large Zariski open subsets. Let $\tilde{X}_{(k)}$ be the pretopology (cf. for instance [M]) on the category of open subsets in $X_{(k)}$ in which the U_i form a covering of U if and only if $U - U_i$ is of codimension $\geqslant k + 1$ in U. Then $G_k = E_2^{k-1,-k}$ is a sheaf on both $X_{(k)}$ and $\tilde{X}_{(k)}$. We recall from [M] that if G is a sheaf of groups over any site, then a G-principal homogeneous sheaf is a sheaf X of sets over this site which is equipped with a G-action such that the homomorphism

$$G \times X \to X \times X$$

$$(g, x) \to (x, gx)$$

is an isomorphism in category of sheaves of sets. A morphism in the category of G-principal homogeneous sheaves is a morphism in the category of sets which is compatible with the G-actions, such a morphism is automatically an isomorphism.

Let $\mathbf{CH}^k(X)$ (resp. $\widetilde{\mathbf{CH}}^k(X)$) be the category of G_k-principal homogeneous sheaves on $X_{(k)}$ (resp. $\tilde{X}_{(k)}$). If A is an object of one of these categories and if U is open in $X_{(k)}$, then the set of sections of A on U is denoted by $A(U)$. $\mathbf{CH}^k(X)$ is a full subcategory of $\widetilde{\mathbf{CH}}^k(X)$, and an object of $\widetilde{\mathbf{CH}}^k(x)$ belongs to $\mathbf{CH}^k(X)$ if and only if $X_{(k)}$ has a covering U_i such that $A(U_i)$ is not empty.

It is clear that the operation

$$A \oplus B = A \overset{G_k}{\times} B \tag{1}$$

defines the structure of a Picard category (in the sense of [D, §4]) on $\mathbf{CH}^k$ and $\widetilde{\mathbf{CH}}^k$. The commutativity law $A \oplus B \xrightarrow{\sim} B \oplus A$ sends $a \oplus b$ to $b \oplus a$, and the associativity law $(A \oplus B) \oplus C \xrightarrow{\sim} A \oplus (B \oplus C)$ sends $(a \oplus b) \oplus c$ to $a \oplus (b \oplus c)$ if a, b, and c are sections of A, B, and C on U. The zero object is G_k, and the isomorphism $G_k \oplus A \xrightarrow{\sim} A$ sends $g \oplus a$ to ga, where ga is the action of $g \in G_k(U)$ on $a \in A(U)$.

To admit also non-invertible arrows we define the following extended Chow category. Let $G_k^+(U)$ be the semi-group of self-homomorphisms of the zero object of $\mathbf{CH}_z^k(U)_e$. If $A \in \mathrm{Ob}(\widetilde{\mathbf{CH}}^k(X))$, put

$$A_e = G_k^+ \overset{G_k}{\times} A. \tag{2}$$

Homomorphisms from A to B in $\mathbf{CH}^k(X)_e$ (resp. $\widetilde{\mathbf{CH}}^k(X)_e$) are sheaf morphisms between A_e and B_e respecting the G_k^+-action.

Now we discuss the fundamental properties of these Chow categories.

3.2. *Relation to Line Bundles*

Since $X_{(1)} = X_{\mathrm{Zar}}$, the natural homomorphism $\mathcal{O}_X^* \to G_1$ defines a functor $\mathbf{c}_1$: (line bundles on X and isomorphisms) $\to \mathbf{CH}^1(X)$ and $\mathbf{c}_1$: (line bundles and $\mathcal{O}_X$-linear maps which are isomorphisms at the maximal points X_0) $\to \mathbf{CH}^1(X)_e$. This functor maps $\otimes$ between line bundles to $\oplus$ in $\mathbf{CH}^1$. It is faithful if X is reduced and an equivalence if X is normal.

3.3. *Rational sections and their cycles*

Let A be an object of $\mathbf{CH}^k(X)$. We define its sheaf of rational sections by

$$A_r(U) = \cup A(V), \tag{3}$$

where the union is over all V which are open in $U_{(k)}$ and meet every irreducible component of U. It is easy to check that $A_r(X)$ is not empty.

Every $a \in A_r(U)$ defines a cycle $\mathbf{c}(a)$ as follows. Choose a representative $a' \in A(V)$ for a. There exist a covering U_j of $\tilde{U}_{(k)}$ and sections $b_j \in A(U_j)$. Then $a'|_{V \cap U_j} - b_j|_{V \cap U_j} = c_j \in G_k(V \cap U_j)$. Let $z_j = \partial(c_j) \in Z^k(U_j)$, cf. 1(3). Then $z_i|_{U_i \cap U_j} = z_j|_{U_i \cap U_j}$, consequently (since Z^k is a sheaf on $\tilde{X}_{(k)}$) there exists $z \in Z^k(U)$ with $z|_{U_j} = Z_j$. We put $\mathbf{c}(a) = z$.

We have

$$\begin{aligned} A(U) &= \{a \in A_r(U) | \mathbf{c}(a) = 0\} \\ A_e(U) &= \{a \in A_r(U) | \mathbf{c}(a) \geqslant 0\} \end{aligned} \tag{4}$$

If X is irreducible, A_r is a constant sheaf.

We will often use (4) to construct objects of the Chow category by first constructing their sets of rational sections, then specifying the cycle map c on the set of rational cycles, and then defining the object itself by the first equation in (4).

An example is the group of rational sections of G_k. For an open $G_k(U) = \ker(E_1^{k-1,-k}(U) \to E_1^{k,-k}(U))/\mathrm{im}(E_1^{k-2,-k}(U) \to E_1^{k-1,-k}(U))$. Because points of X of codimension larger than k are elements of U, replacing X by U does not change $E_1^{k-1,-k}(U)$ or $E_1^{k-2,-k}(U)$. However, every element of $E_1^{k,-k}(X)$ vanishes on some open and dense subset U of $X_{(k)}$. Consequently, $(G_k)_r(X) = E_1^{k-1,-k}(X)/E_1^{k-2,-k}(X)$. It is easy to see that on this set c is given by the E_1-differential.

3.4. *Relations between the several definitions of CH^k*

Let $k > 0$. Then there is an equivalence of categories

$$\begin{aligned} &\mathbf{O}(\cdot)\colon \mathbf{CH}_z^k(X) \to \widetilde{\mathbf{CH}}^k(X) \\ &\mathbf{O}(z)(U) = \mathrm{Hom}_{\mathbf{CH}_z^k(U)}(0, z). \end{aligned} \tag{5}$$

$G_k(U)$, being the automorphism group of any object of $\mathbf{CH}_z^k(U)$, acts on the right side of (5). It is clear that a homomorphism from z to z' in $\mathbf{CH}_z^k(X)$ defines a homomorphism from $\mathbf{O}(z)$ to $\mathbf{O}(z')$ in $\widetilde{\mathbf{CH}}^k(X)$, that $\mathbf{O}(\cdot)$ is compatible with $\oplus$, and that $\mathbf{O}(\cdot)$ defines an equivalence of $\mathbf{CH}_z^k(X)_e$ and $\widetilde{\mathbf{CH}}^k(X)_e$.

An inverse to $\mathbf{O}(\cdot)$ may be constructed as follows: For every object A of $\widetilde{\mathbf{CH}}^k(X)$, fix a rational section a_A of A. The inverse functor associates the cycle $\mathbf{c}(a_A) \in E_1^{k,-k}(X)$ to A and the element $a_{A'} - \varphi(a_A) \in (G_k)_r(X) = E_1^{k-1,-k}(X)/E_1^{k-2,-k}(X)$ to a morphism $\varphi\colon A \to A'$.

To investigate the relation between **CH** and $\widetilde{\mathbf{CH}}$, consider the following assumption:

(LF_k) If $x \in X$, $CH^k(\mathcal{O}_{X,x}) = 0$.

It is clear that (LF_k) is true if the local rings of X satisfy Gersten's conjecture. If X is regular, it satisfies (LF_k) up to torsion by the result of [S], and (LF_1) by the Auslander-Buxbaum theorem. Let X satisfy (LF_k). We want to prove $\mathbf{CH}^k(X) = \widetilde{\mathbf{CH}}^k(X)$. For every object A of $\widetilde{\mathbf{CH}}^k(X)$, we have to find a covering U_i of X such that A has a section on U_i. By the above remark, it suffices to do this if $A = \mathbf{O}(z)$ for a codimension k cycle z on X. By (LF_k), for every $x \in X$ there exists $g_x \in E_1^{k-1,-k}(\operatorname{Spec} \mathcal{O}_{X,x})$ such that $(g_x) = z|_{\operatorname{Spec} \mathcal{O}_{X,x}}$. It is clearly possible to extend g_x to $g'_x \in E_1^{k-1,-k}(X)$. Let $U_x = X - \operatorname{supp}(z - \partial(g'_x))$. Then g'_x defines a section of $O(Z)$ on U_x. By our choice of g_x and g'_x, $X \in U_x$. Consequently, the U_x form a covering of $X_{(k)}$ on which $O(z)$ has sections.

3.5. *Convention*

For $k \leqslant 0$, we put $\mathbf{CH}^k(X) = \widetilde{\mathbf{CH}}^k(X) = \mathbf{CH}^k_z(X)$. If A is an object in $\mathbf{CH}^k(X)$, $k \leqslant 0$ and U open in $X_{(k)}$, $A_r(U)$ consists of a single element denoted by β. We put $\mathbf{c}(\beta) = A \in E_2^{0,0}(X) = E_1^{0,0}(X) = Z^0(X)$ if $k = 0$ and $\mathbf{c}(\beta) = 0$ if $k < 0$. $A(U)$ and $A_e(U)$ are defined by (4).

Note that $\mathbf{CH}^k(X) = \widetilde{\mathbf{CH}}^k(X)$ consists of only one zero object if $k < 0$.

3.6. *Definition of a fibred Picard Category*

Recall from [D, §4] that a commutative Picard category is a groupoid P together with a functor $\oplus: P \times P \to P$, an associativity law $a_{A,B,C}$. $(A \oplus B) \oplus C \xrightarrow{\sim} A \oplus (B \oplus C)$ and a commutativity law $c_{A,B}: A \oplus B \xrightarrow{\sim} B \oplus A$ satisfying the compatibilities [DM, (1.0.1) and (1.0.2)], such that the translation functor $X \oplus \cdot$ is an equivalence of categories for every $X \in \mathrm{Ob}(P)$. It follows that P has a zero object, which we assume to be fixed.

An additive functor is a functor $F: P \to P'$ between commutative Picard categories together with a functor-isomorphism $F(A \oplus B) \xrightarrow{\sim} F(A) \oplus F(B)$ satisfying the additive analogues of [DM, Definition 1.8]. An additive functormorphism is a natural transformation $F \to G$ satisfying the additive analogue of [DM, (1.12.1) and (1.12.2)].

Let K be a category. A fibred Picard category over K consists of:

(i) For every object $X \in K$, a commutative Picard category P_X.
(ii) For every homomorphism $f: X \to Y$ in K, an additive functor $f^*: P_Y \to P_X$.
(iii) For every pair of composable arrows f, g in K, an additive functormorphism

$\kappa_{f,g}\colon (fg)^* \xrightarrow{\sim} g^*f^*$ such that for every $X \xleftarrow{f} Y \xleftarrow{g} Z \xleftarrow{h} U$ in K, the diagram

$$\begin{tikzcd}
(fgh)^*(A) \arrow[r, "\kappa_{f,gh}"] \arrow[d, "\kappa_{fg,h}"'] & (gh)^*(f^*(A)) \arrow[d, "\kappa_{g,h}"] \\
h^*(gf)^*(A) \arrow[r, "h^*(\kappa_{f,g})"] & h^*(g^*(f^*(A)))
\end{tikzcd} \tag{6}$$

commutes.

Let $\tilde{P}$ be a second fibred Picard category over K with pull-back functors $f^!$ and natural transformations $\tilde{\kappa}_{f,g}$ for composable arrows in K. An admissible functor of fibred Picard categories is a pair (F, φ), consisting of:

(i) For every $X \in \mathrm{Ob}(K)$, an additive functor $F_X\colon P_X \to \tilde{P}_X$
(ii) For an arrow $f\colon X \to Y$ in K, an additive functor-morphism $\varphi_f\colon F_X \circ f^* \xrightarrow{\sim} f^! \circ F_Y$ such that the diagram

$$\begin{tikzcd}
F_X \circ (gf)^* \arrow[rr, "\varphi_{gf}"] \arrow[d, "F_X(\kappa_{g,f})"'] & & (gf)^! \circ F_Z \arrow[d, "\tilde{\kappa}_{g,f}"] \\
F_X \circ f^* \circ g^* \arrow[dr, "\varphi_f"'] & & f^! \circ g^! \circ F_Z \\
& f^! \circ F_Y \circ g^* \arrow[ur, "f^!(\varphi_g)"'] &
\end{tikzcd} \tag{7}$$

commutes for every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in K.

If P and P' are fibred Picard categories over K and F, F' are admissible functors from P to P', an admissible functor-isomorphism from F to F' is a family $(\psi_X\colon F_X \to F'_X)_{X \in \mathrm{Ob}(K)}$ of additive functor-isomorphisms such that for every arrow $f\colon X \to Y$ in K the diagram

$$\begin{tikzcd}
F_X f^* \arrow[r, "\varphi_f"] \arrow[d, "\psi_X"'] & f^! F_Y \arrow[d, "f^!((\psi_Y)"] \\
F'_X f^* \arrow[r, "\varphi'_f"] & f^! F'_Y
\end{tikzcd}$$

commutes.

A cofibred Picard category over K is a fibred Picard category over I^{op}. The definition of an admissible functor between cofibred Picard categories is similar to the fibred case.

3.7. *Flat pull-back*

For a continuous mapping $f: X \to Y$ we denote the pull-back functor from sheaves on Y to sheaves on X by f^+. If $f: X \to Y$ is flat, the map $f: X_{(k)} \to Y_{(k)}$ is continuous. In §1 we defined a flat pull-back morphism $f^+: f^*G_{k,Y} \to G_{k,X}$.

If $A \in \mathrm{Ob}(\mathbf{CH}^k(Y))$ (resp. $\widetilde{\mathbf{CH}}^k(Y)$), $f^+(A)$ is a $f^+(G_{k,Y})$-torser on $X_{(k)}$ (resp. on $\tilde{X}_{(k)}$). We define $\mathbf{f}^*(A) \in \mathrm{Ob}(\mathbf{CH}^k(X))$ (resp. $\widetilde{\mathbf{CH}}^k(X)$) to be the image of $f^+(A)$ under $f^*: f^+(G_{k,Y}) \to G_{k,X}$. This defines a functor from $\mathbf{CH}^k(Y)$ to $\mathbf{CH}^k(X)$ and from $\mathbf{CH}^k(Y)_e$ to $\mathbf{CH}^k(X)_e$, and similar for $\widetilde{\mathbf{CH}}$. Every $a \in A(U)$ (resp. $A_e(U)$, resp. $A_r(U)$) defines $f^*(a) \in (\mathbf{f}^*A)(f^{-1}(U))$ (resp. $f^*(a) \in (\mathbf{f}^*A)_e(f^{-1}(U))$, resp. $f^*(a) \in (\mathbf{f}^*A)_r(f^{-1}(U))$ with $\mathbf{c}(f^*(a) = f^*(\mathbf{c}(a)))$.

If $k \leqslant 0$, $\mathbf{f}^*$ is defined to be the functor introduced in §2, and we put $f^*(\beta) = \beta$ (cf. Convention 3.5).

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be flat morphisms. There are a natural isomorphism $\mathbf{f}^*(A \oplus B) \xrightarrow{\sim} \mathbf{f}^*(A) \oplus \mathbf{f}^*(B)$ sending $f^*(a \oplus b)$ to $f^*(a) \oplus f^*(b)$, and a natural isomorphism $\mathbf{f}^*(\mathbf{g}^*(A)) \xrightarrow{\sim} (\mathbf{gf})^*(A)$ sending $f^*(g^*(a))$ to $(gf)^*(a)$. These data define on $\mathbf{CH}^k$ and $\widetilde{\mathbf{CH}}^k$ the structure of a fibred Picard category over (schemes, flat morphisms), and the functors

$$\mathbf{CH}^k_z \xrightarrow{\mathbf{O}(\cdot)} \widetilde{\mathbf{CH}}^k \xleftarrow{\text{inclusion}} \mathbf{CH}^k$$

can be extended to admissible functors of fibred Picard categories.

If $f: X \to Y$ is flat and $A \in \mathrm{Ob}(\mathbf{CH}^k(Y))$, we put $A(X) = (\mathbf{f}^*A)(X)$, $A_e(X) = (\mathbf{f}^*A)_e(X)$, and $A_r(X) = (\mathbf{f}^*A)_r(X)$. By the previous remarks, these are presheaves on Y_{fpqc}.

3.8. If $j: U \to X$ is an open immersion, we often write $|_U$ instead of j^*. With this notation, we have

PROPOSITION. *Let $k > 0$ and $A \in \mathrm{Ob}(\widetilde{\mathbf{CH}}^k(X))$. If X_{Zar} has a covering by open subsets U_i such that $A|_{U_i} \in \mathrm{Ob}(\mathbf{CH}^k(U_i))$, then $A \in \mathrm{Ob}(\mathbf{CH}^k(X))$.*

Proof. Let $r \in A_r(X)$ be a rational section. By replacing U_i by a covering of $(U_i)_{(k)}$, we may assume $A(U_i) \neq \varnothing$, with section $a_i \in A(U_i)$. Let $a_i = b_i + r|_{U_i}$ with $b_i \in (G_k)_r(U_j)$. Then b_i can be represented as the image of $c_i = (c_{i,x})_{x \in (U_i)_{k-1}} \in E_1^{k-1,-k}(U_i)$ in $(G_k)_r(U_i)$.

We define $c'_i = (c'_{i,x})_{x \in X_{k-1}} \in E_1^{k-1,-k}(X)$ by

$$c'_{i,x} = \begin{cases} 0 & \text{if } x \notin U_i \\ c_{i,x} & \text{if } x \in U_i \end{cases}.$$

Let b_i' be the image of c_i' in $(G_k)_r(X)$. Then $\mathbf{c}(b_i')|_{U_i} = \mathbf{c}(b_i)$, hence $\mathbf{c}(b_i' + r)|_{U_i} = 0$. If $V_i = X - \operatorname{supp}(\mathbf{c}(b_i' + r))$, then V_i is open in $X_{(k)}$ and contains U_i, consequently the V_i form a covering of $X_{(k)}$ on which A is trivial.

3.9. PROPOSITION. *Let p: $E \to X$ be the projection of a vector bundle E to its base X. Then* $\mathbf{p}^*\widetilde{\mathbf{CH}}^k(X) \to \widetilde{\mathbf{CH}}^k(E)$ and $\mathbf{p}^*$: $\mathbf{CH}^k(X) \to \mathbf{CH}^k(E)$ are equivalences of categories.

Proof. In §1, we proved that p^* is an isomorphism between the E_2-terms of the Quillen spectral sequences for X and E. Since the group of isomorphism classes of objects of $\widetilde{\mathbf{CH}}^k$ is $E_2^{k,-k}$, and since the group of automorphisms of any object of $\widetilde{\mathbf{CH}}^k$ is $E_2^{k-1,-k}$, the result follows for $\widetilde{\mathbf{CH}}^k$.

To prove the proposition for $\mathbf{CH}^k$, it suffices to prove that $A \in \operatorname{Ob}(\widetilde{\mathbf{CH}}^k(X))$ and $\mathbf{p}^*A \in \operatorname{Ob}(\mathbf{CH}^k(E))$ implies $A \in \operatorname{Ob}(\mathbf{CH}^k(X))$. By Proposition 3.8. We may assume E is trivial, i.e., $E = A_X^d$. By induction on d, we may also assume $d = 1$. Every $A \in \operatorname{Ob}(\widetilde{\mathbf{CH}}^k(X))$ is isomorphic to $\mathbf{O}(z)$ for some $z \in Z^k(X)$. Let $x \in X$. If $\mathbf{p}^*(\mathbf{O}(z)) \in \operatorname{Ob}(\mathbf{CH}^k(E))$, there exists a cycle $z' \in Z^k(E)$ with $(x, 0) \notin \operatorname{supp}(z')$ and $[p^*z] = \ [z']$ in $CH^k(E)$. If t is the coordinate on A_X^1 and sp_t the homomorphism defined in §1, this implies $[\operatorname{sp}_t(z')] = [\operatorname{sp}_t(p^*(z))] = [z]$ in $CH^k(X)$. But $(x, 0) \notin \operatorname{supp}(z')$, hence $x \notin \operatorname{supp}(\operatorname{sp}_t(z'))$, consequently $\mathbf{O}(z)$ is locally trivial.

3.10. *Proper Push-Forward*

Let f: $X \to Y$ be proper of constant relative dimension $d \in \mathbb{Z}$. We define push-forward functors $\mathbf{f}_*$: $\widetilde{\mathbf{CH}}^k(X) \to \widetilde{\mathbf{CH}}^{k-d}(Y)$ in the following manner. If $k - d < 0$ and $A \in \operatorname{Ob}(\widetilde{\mathbf{CH}}^k(X))$, $\mathbf{f}_*(A)$ is the only object of $\widetilde{\mathbf{CH}}^{k-d}(Y)$, every arrow in $\widetilde{\mathbf{CH}}^k(X)$ is mapped to the identity arrow, and $f_*(a) = \beta$ for every $a \in A_r(X)$.

Let $k = d$. Then every object in $\widetilde{\mathbf{CH}}^k(\mathbf{X})$ defines its class $[A] \in E_2^{k,-k}(X)$. We define $\mathbf{f}_*(A)$ to be the object of $\widetilde{\mathbf{CH}}^0(Y)$ defined by $f_*([A]) \in E_2^{0,0}(Y) = E_1^{0,0}(Y)$. Every arrow in $\widetilde{\mathbf{CH}}^k(X)$ is mapped to the identity arrow, and $f_*(a) = \beta$ for $a \in A_r(X)$.

Let $k < 0$ and $k - d > 0$. For $a \in \operatorname{Ob}(\widetilde{\mathbf{CH}}^k(X))$ (there is only one object, and only its identical arrow), we put $\mathbf{f}_*(A) = G_{k,Y}$ and $\mathbf{f}_*(\beta) = 0$. Finally we consider the case $k \geqslant 0$ and $k - d > 0$. For $g \in G_{k,X}(U)$, we have $f_*(g) \in G_{k-d,Y}(Y - f(X - U))$, hence f_*: $(G_{k,X})_r(X) \to (G_{k-d,Y})_r(Y)$. Also, we have f_*: $Z^k(X) \to Z^{k-d}(Y)$, and f_* is compatible with 1(3), hence with $\mathbf{c}$. Let $A \in \operatorname{Ob}(\widetilde{\mathbf{CH}}^k(X))$. We define $(\mathbf{f}_*A)_r(Y)$ to be the set of equivalence classes of pairs (g, a) with $g \in (G_{k-d,Y})_r(Y)$ and $a \in A_r(Y)$. Two pairs are equivalent if they are of the form $(g + f_*(h), a)$ and $(g, a + h)$ for some $h \in (G_{k,X})_r(X)$. The group $(G_{k-d,Y})_r(Y)$ acts on $(\mathbf{f}_*A)_r(Y)$ by the rule h: $(g, a) \to (g + h, a)$. We define the cycle of an element of $(\mathbf{f}_*A)_r(Y)$ by $\mathbf{c}((g, a)) = \mathbf{c}(g) + f_*\mathbf{c}(a)$. Now the sheaves $(\mathbf{f}_*A)_e$ and $\mathbf{f}_*A$ can be defined by (4). For $a \in A_r(X)$, the class of $(0, a)$ in $(\mathbf{f}_*A)_r(Y)$ is denoted by $f_*(a)$. If $a \in A(U)$, then $f_*(a) \in (\mathbf{f}_*A)(Y - f(X - U))$. If φ: $A \to B$ is a morphism in $\widetilde{\mathbf{CH}}^k(X)$, then $\mathbf{f}_*(\varphi)$ sends $f_*(a)$ to $f_*(\varphi(a))$.

In the remaining part of this paper we use the abbreviations c.r.d. for the condition 'constant relative dimension' and $\mathbf{CH}^{\bullet}(X)$, $\widetilde{\mathbf{CH}}^{\bullet}(X)$ for the direct sums of categories

$$\mathbf{CH}^{\bullet}(X) = \bigoplus_{k=-\infty}^{\infty} \mathbf{CH}^k(X), \qquad \widetilde{\mathbf{CH}}^{\bullet}(X) = \bigoplus_{k=-\infty}^{\infty} \widetilde{\mathbf{CH}}^k(X). \tag{8}$$

It is easy to see that the functors $\mathbf{f}_*$ constitute the structure of a cofibred Picard category over (schemes, proper morphisms of c.r.d.) on $\widetilde{\mathbf{CH}}^{\bullet}$. The transformation $\mathbf{f}_*(A) \oplus \mathbf{f}_*(B) \to \mathbf{f}_*(A \oplus B)$ maps $f_*(a) \oplus f_*(b)$ to $f_*(a \oplus b)$, and the isomorphism $\mathbf{g}_*(\mathbf{f}_*(A)) \to (\mathbf{gf})_*(A)$ maps $g_*(f_*(a))$ to $(gf)_*(a)$. It is easy to verify that the axioms of a cofibred Picard category are satisfied.

In the following cases, $\mathbf{f}_*(A)$ belongs to $\mathbf{CH}^{k-d}(Y)$:

(i) If Y satisfies $(LF)_{k-d}$.
(ii) *If* $k > 0$, f is a closed immersion and A belongs to $\mathbf{CH}^k(X)$.

3.11. *Definition of a bifibred Picard category*

For a bicategory K, we denote by Hor(K) (resp. Ver(K)) the category of objects of K with horizontal (resp. vertical) morphisms of K as morphisms. A bifibred Picard category over K consists of the following data:

(i) For every object $X \in \mathrm{Ob}(K)$, a commutative Picard category P_X.

(ii) The structure of a cofibred Picard category over Hor(K) for the P_X. We denote the push-forward functor of a horizontal morphism g by g_*.

(iii) The structure of a fibred Picard category, with pull-back functors f^*, over Ver(K) for the P_X.

(iv) For every bimorphism A in K with boundary

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "(A)"] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd},$$

an additive functor-isomorphism φ_A: $f^*g_* \to g'_*f'^*$. The following compatibility assumption must be satisfied: If A, B, C are bimorphisms fitting into a diagram

$$\begin{tikzcd} & \tilde{X} \arrow[r, "\tilde{g}"'] \arrow[d, "e'"'] \arrow[dr, phantom, "(B)"] & \tilde{Y} \arrow[d, "e"] \\ Z' \arrow[r, "h'"] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "(C)"] & X' \arrow[r, "g'"'] \arrow[d, "f'"'] \arrow[dr, phantom, "(A)"] & Y' \arrow[d, "f"] \\ Z \arrow[r, "h"'] & X \arrow[r, "g"'] & Y \end{tikzcd},$$

then the following diagrams commute:

$$\begin{tikzcd} f^*(gh)_* \arrow[r, "\text{can.}"] \arrow[d, "\varphi_{A_h \circ C}"] & f^*g_*h_* \arrow[r, "\varphi_A"] & g'_*f'^*h_* \arrow[d, "\varphi_C"] \\ (g'h')_*\tilde{f}^* \arrow[rr, "\text{can.}"] & & g'_*h'_*\tilde{f}^* \end{tikzcd} \tag{9}$$

and a similar diagram for the vertical composition $A_v \circ B$. In (9), can. denotes isomorphisms canonically defined by the datum (ii). If we are given a second bifibred Ricard category P over K with pull-back and push-forward functors $f^!$ and $g_!$ and base-change isomorphisms $\tilde{\varphi}_A$, an admissible functor of bifibred Picard categories consists of:

— For every object X, an additive functor F_X: $P_X \to P_X$.
— For every horizontal morphism $g: X \to Y$, an additive functor-isomorphism α_g: $F_Y g_* \to g_! F_X$ such that F becomes an admissible functor between cofibred Picard categories over Hor(K).
— For a vertical morphism $f: X \to Y$, the isomorphism β_f: $F_X f^* \to f^! F_Y$ making F an admissible functor between fibred Picard categories over Ver(K).

The following compatibility between α and β must be satisfied: For every bimorphism

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\mathrm{A}"] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd},$$

the following diagram commutes:

$$\begin{tikzcd} F_{Y'} f^* g_* \arrow[r, "F_{Y'}(\varphi_A)"] \arrow[d, "\beta_f"'] & F_{Y'} g'_* f'^* \arrow[d, "\alpha_{g'}"] \\ f^! F_Y g_* \arrow[d, "\alpha_g"'] & g'_! F_{X'} f'^* \arrow[d, "\beta_{f'}"] \\ f^! g_! F_X \arrow[r, "\tilde{\varphi}"] & g'_! f'^! F_X \end{tikzcd}.$$

If $G: P \to \tilde{P}$ is another admissible functor between bifibred Picard categories, then the collection ψ of additive functor-isomorphisms ψ_X: $F_X \to G_X$ is called a biadmissible functor-isomorphism if it is admissible for fibred Picard categories over $\mathrm{Hor}(K)$ and cofibred Picard categories over $\mathrm{Ver}(K)$.

Let $F\colon P \to \tilde{P}$ be an admissible functor between bifibred Picard categories over K. If for every object X the functor F_X is an equivalence of categories, it has an inverse F_X^{-1} together with a natural transformation $F_X F_X^{-1} \to \mathrm{Id}$, and this transformation determines F_X^{-1} up to unique functor-isomorphism. There is a unique way of giving F^{-1} the structure of an admissible functor between bifibred Picard categories such that the transformation $FF^{-1} \to \mathrm{Id}$ is biadmissible. In a similar fashion, admissible functors between fibred or cofibred Picard categories which are equivalences may be inverted.

In this paper we restrict our attention to bicategories K of the following type. Let C be a category and S_*, S^* be two distinguished families of morphisms in C which contain the identify morphisms and are closed under composition. We also assume that for $g \in S_*$ and $f \in S^*$ the fibre product

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd} \qquad f' \in S^*,\ g' \in S_*$$

exists. We denote by (C, S_*, S^*) the bicategory which has the same objects as C, elements of S_* (resp. S^*) as horizontal (resp. vertical) morphisms and precisely one bimorphism for every fibre square as above (with boundary given by g, $g' \in S_*$ and $f, f' \in S^*$). The composition of morphisms and bimorphisms is defined in the obvious manner.

3.12. *Base change*

Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

with f flat and g proper of constant relative dimension. For every $A \in \mathrm{Ob}(\widetilde{\mathbf{CH}}^k(Y'))$, there is a base-change isomorphism $\mathbf{f}^*\mathbf{g}_*A \xrightarrow{\sim} \mathbf{g}'_*\mathbf{f}'^*A$ sending $f^*(g_*(a))$ to $g'_*(f'^*(a))$ for $a \in A_r(X)$. That this definition is correct follows from the base change identity we proved for $E_2^{p,q}$ (and hence for G_k) in §1.

It is easy to check that these base-change isomorphisms constitute the structure of a bifibred Picard category (in the sense of 3.11.) on $\widetilde{\mathbf{CH}}^{\cdot}$ over (schemes, proper morphisms of c.r.d., flat morphisms).

3.13. *Specialization*

Let $D \subset X$ be a closed regular immersion of codimension 1, and that the ideal defining D is generated by f in some neighbourhood of D. We define a functor $\mathbf{sp}_f\colon \widetilde{\mathbf{CH}}^k(X - D) \to \widetilde{\mathbf{CH}}^k(D)$ in the following manner. Let $k \leqslant 0$. Then on the set of objects $\mathbf{sp}_f$ is defined using Convention 3.5 and the homomorphisms sp_f defined in §1. On the set of morphisms $\mathbf{sp}_f$ is trivial since there are only identity morphisms. Furthermore, put $\mathrm{sp}_f(\beta) = \beta$.

Let $k > 0$ and $A \in \mathrm{Ob}(\widetilde{\mathbf{CH}}^k(X - D))$. We define $(\mathbf{sp}_f(A))_r(D)$ as the set of equivalence classes of pairs (g, a) with $g \in (G_{k,D})_r(D)$ and $a \in A_r(X - D)$. Two pairs are equivalent iff they are of the form $(g + \mathrm{sp}_f(h), a)$ and $(g,\ h + a)$ with $g \in (G_{k,D})_r(D)$, $h \in (G_{k,X})_r(X - D)$, $a \in A_r(X - D)$. We put $\mathbf{c}((g, a)) = \mathbf{c}(g) + \mathrm{sp}_f(\mathbf{c}(a))$. The sheaves $(\mathbf{sp}_f A)_e$ and $\mathbf{sp}_f A$ are defined by (4). For $a \in A_r(X - D)$, we denote the equivalence class of $(0, a)$ by $\mathrm{sp}_f(a) \in (\mathbf{sp}_f(A))_r(D)$. We have $\mathbf{c}(\mathrm{sp}_f(a)) = \mathrm{sp}_f(\mathbf{c}(a))$. If $a \in A(U)$, then $\mathrm{sp}_f(a) \in (\mathbf{sp}_f(A))(D - (D \cap \overline{(X - U)}))$.

Let K_{sp} be the following category. Objects of K_{sp} are triples (D, X, f) satisfying the assumptions of 1.5. A morphism between (D, X, f) and (D', X', f') is a Cartesian diagram

$$\begin{array}{ccc} D' & \longrightarrow & X' \\ \Big\downarrow{\scriptstyle p_D} & & \Big\downarrow{\scriptstyle p} \\ D & \longrightarrow & X \end{array} \tag{10}$$

such that $f' = f \circ p = p^*(f)$. A morphism is called flat (proper) if p is flat (proper). If in the latter case p is of constant relative dimension, the lemma in 1.4. implies that p_D is of the same relative dimension.

As usual, we define an isomorphism $\mathbf{sp}_f(A \oplus B) \to \mathbf{sp}_f(A) \oplus \mathbf{sp}_f(B)$ which sends $\mathrm{sp}_f(a \oplus b)$ to $\mathrm{sp}_f(a) \oplus \mathrm{sp}_f(b)$. If in (10) p is proper and of constant relative dimension, we define an isomorphism $\mathbf{sp}_f(\mathbf{p}_*(A)) \to \mathbf{p}_{D*}(\mathbf{sp}_{f'}(A))$ which sends $\mathrm{sp}_f(p_*(a))$ to $p_{D*}(\mathrm{sp}_{f'}(a))$. If p is flat, the isomorphism $\mathbf{sp}_{f'}(\mathbf{p}^*(A)) \to \mathbf{p}_D^*(\mathbf{sp}_f(A))$ sends $\mathrm{sp}_{f'}(p^*(a))$ to $p_D^*(\mathrm{sp}_f(a))$. By (1.15) and (1.16), these definitions are correct.

By the results of 3.7, 3.10 and 3.12, the categories $\widetilde{\mathbf{CH}}^{\cdot}(X - D)$ and $\widetilde{\mathbf{CH}}^{\cdot}(D)$ are bifibred Picard categories over (K_{sp}, proper morphisms of c.r.d., flat morphisms). It is easy to check that $\mathbf{sp}_f$ is an admissible functor (in the sense of 3.11) between bifibred Picard categories over K_{sp}.

3.14. LEMMA. *Let $f\colon X \to Y$ be flat. For a Zariski-open $U \subset Y$, let $f_U\colon f^{-1}(U) \to U$ be the restriction of f. If for every Zariski-open subset $U \subset Y$ we are given an additive functor-automorphism φ_U of $\mathbf{f}_U^*\colon \widetilde{\mathbf{CH}}^k(U) \to \widetilde{\mathbf{CH}}^k(f^{-1}(U))$ such that the restriction of φ_U to $f^{-1}(V)$ is φ_V for every $V \subset U$, then $\varphi_U = \mathrm{Id}$ for every U.*

Proof. It suffices to prove $\varphi_Y = \mathrm{Id}$. Let $A \in \mathrm{Ob}(\widetilde{\mathbf{CH}}^k(Y))$, then there is a covering U_j of $\tilde{Y}_{(k)}$ such that the restriction of A to U_j is trivial. Since φ is additive, $\varphi_{U_j, A|_{U_j}} = \mathrm{Id}$. By our assumption, this implies that the action of $\varphi_{Y,A}$ on $\mathbf{f}^*(A)|_{f^{-1}(U_j)}$ is trivial. Since the $f^{-1}(U_j)$ form a covering of $\tilde{X}_{(k)}$, this proves the result.

3.15. *Some natural transformations involving* **sp**

If $(D, X, f) \in \mathrm{Ob}(K_{\mathrm{sp}})$ and if in the commutative diagram

$$\begin{tikzcd} D \arrow[rr] \arrow[dr, "g"'] & & X \arrow[dl, "h"] \\ & Z & \end{tikzcd} \tag{11}$$

g and h are flat, then there is an additive functor-isomorphism

$$\tau_{h,X,D,f}\colon \mathbf{sp}_f \mathbf{h}^* \to \mathbf{g}^* \tag{12}$$

sending $\mathrm{sp}_f(h^*(a))$ to $g^*(a)$. By (1.17), this definition is correct. It is easy to check that τ satisfies the following compatibility with flat and proper base-changes $Z' \to Z$: It is clear that (since g is flat) the assumptions to (11) remain true after arbitrary base change $Z' \to Z$. In particular, $\widetilde{\mathbf{CH}}^{\cdot}(Z')$, $\widetilde{\mathbf{CH}}^{\cdot}(X \times_Z Z')$ and $\widetilde{\mathbf{CH}}^{\cdot}(D \times_Z Z')$ are bifibred Picard categories over (Z-schemes, proper morphisms of c.r.d., flat morphisms).

Both sides of (12) are admissible functors between bifibred Picard categories. The compatibility with base-changes $Z' \to Z$ is the fact that τ is a biadmissible transformation. By Lemma 3.14, τ is characterized uniquely by this compatibility. This implies a compatibility of τ with flat maps $Z \to Z'$ and flat base-changes $X' \to X$.

If in the diagram

$$\begin{tikzcd} & D_1 \cap D_2 \arrow[dl, hook'] \arrow[dr, hook] & \\ D_1 \arrow[dr, hook'] & & D_2 \arrow[dl, hook] \\ & X & \end{tikzcd}$$

the triples (D_1, X, f_1), $(D_1 \cap D_2, D_2, f_1|_{D_2})$, (D_2, X, f_2), $(D_1 \cap D_2, D_1, f_2|_{D_1})$ belong to $\mathrm{Ob}(K_{\mathrm{sp}})$, we have a natural transformation

$$\omega_{f_1,f_2}\colon \mathbf{sp}_{f_1}|_{D_2}\mathbf{sp}_{f_2} \to \mathbf{sp}_{f_2}|_{D_1}\mathbf{sp}_{f_1} \tag{15}$$

sending $\mathrm{sp}_{f_1}(\mathrm{sp}_{f_2}(a))$ to $\mathrm{sp}_{f_2}(\mathrm{sp}_{f_1}(a))$. By (1.23), this is a correct definition. The following compatibilities are satisfied:

(*A*) *Compatibility with flat and proper base-changes.* Let K^b_{sp} be the category whose objects are 5-tuples (D_1, D_2, X, f_1, f_2) as in (14), and morphisms from $(D'_1, D'_2, X', f'_1, f'_2)$ to (D_1, D_2, X, f_1, f_2) are morphisms $X' \xrightarrow{h} X$ such that $f'_i = f_i \circ h$ and $D'_i = D_i \times_X X'$. In K^b_{sp} the fibre product of two morphisms exists if one of them is flat, hence the bicategory (K^b_{sp}, proper morphisms of c.r.d., flat morphisms) is well-defined. The Picard categories $\widetilde{\mathbf{CH}}^{\bullet}(X)$, $\widetilde{\mathbf{CH}}^{\bullet}(D_i)$, and $\widetilde{\mathbf{CH}}^{\bullet}(D_1 \cap D_2)$ are bifibred over (K^b_{sp}, proper morphisms of c.r.d., flat morphisms), and both sides of (15) are biadmissible functors. Then the transformation (15) is biadmissible.

(*B*). $\omega_{f_1,f_2}\omega_{f_2,f_1} = \mathrm{Id}$.

(*C*) *The Coxeter equality* $(\omega_{12}\omega_{23})^3 = \mathrm{Id}$. If $(D_i, X, f_i) \in \mathrm{Ob}(K_{\mathrm{sp}})$ for $i \in \{1; 2; 3\}$ and if for $i \neq j$ the embeddings $D_i \cap D_j \to D_j$ and $D_1 \cap D_2 \cap D_3 \to D_i \cap D_j$ are regular of codimension one, then the following diagram, in which each arrow is in an obvious manner constructed from the transformations (15), commutes:

$$\begin{tikzcd}
\mathbf{sp}_{f_3}|_{D_1\cap D_2}\mathbf{sp}_{f_2}|_{D_1}\mathbf{sp}_{f_1} \arrow[r] \arrow[d] & \mathbf{sp}_{f_3}|_{D_1\cap D_2}\mathbf{sp}_{f_1}|_{D_2}\mathbf{sp}_{f_2} \arrow[d] \\
\mathbf{sp}_{f_2}|_{D_1\cap D_3}\mathbf{sp}_{f_3}|_{D_1}\mathbf{sp}_{f_1} \arrow[d] & \mathbf{sp}_{f_1}|_{D_2\cap D_3}\mathbf{sp}_{f_3}|_{D_2}\mathbf{sp}_{f_2} \arrow[d] \\
\mathbf{sp}_{f_2}|_{D_1\cap D_3}\mathbf{sp}_{f_1}|_{D_3}\mathbf{sp}_{f_3} \arrow[r] & \mathbf{Sp}_{f_1}|_{D_2\cap D_3}\mathbf{sp}_{f_2}|_{D_3}\mathbf{sp}_{f_3}
\end{tikzcd} \tag{16}$$

(*D*) *Compatibility between* ω *and* τ. Let (Y, D, f) and (X, D_2, f_2) be objects of K_{sp}, and let $p\colon X \to Y$ be a flat projection whose restriction p_2 to D_2 remains flat. Then (D_1, D_2, X, f_1, f_2) belongs to K^b_{sp}, where $D_1 = p^{-1}(D)$ and $f_1 = f \circ p$. The following diagram commutes:

$$\begin{tikzcd}
\mathbf{sp}_{f_1}|_{D_2}\mathbf{sp}_{f_2}\mathbf{p}^* \arrow[r, "\tau_p"] \arrow[dd, "\omega_{f_1,f_2}"] & \mathbf{sp}_{f_1}|_{D_2}\mathbf{p}_2^* \arrow[dr] & \\
& & \mathbf{p}_{12}^*\mathbf{sp}_f, \\
\mathbf{sp}_{f_2}|_{D_1}\mathbf{sp}_{f_1}\mathbf{p}^* \arrow[r] & \mathbf{sp}_{f_2}|_{D_1}\mathbf{p}_1^*\mathbf{sp}_f \arrow[ur, "\tau_{p_1}"'] &
\end{tikzcd} \tag{17}$$

where p_1 and p_{12} are the restrictions of p to D_1 and $D_1 \cap D_2$. In the special case $(Y, D) = (D_2, D_1 \cap D_2)$, (such that $D_2 \to X$ is a section of p), p_2 is the identity, and (17) simplifies to

$$\begin{array}{ccc} \mathbf{sp}_{f_1}|_{D_2}\mathbf{sp}_{f_2}\mathbf{p}^* & \longrightarrow & \mathbf{sp}_{f_1}|_{D_2} \\ \downarrow & & \downarrow \\ \mathbf{sp}_{f_2}|_{D_1}\mathbf{sp}_{f_1}\mathbf{p}^* & \longrightarrow & \mathbf{sp}_{f_2}|_{D_1}\mathbf{p}_1^*\mathbf{sp}_{f_1}|_{D_2} \end{array} \tag{18}$$

4. Functoriality with respect to local complete intersections

For a better understanding of what follows, we suggest the reader to read the formulation of theorem 4.7. first.

4.1. LEMMA. *Let*

$$\begin{array}{ccc} D \cap Z & \subset & Z \\ \cap & & \cap \\ D & \subset & X \end{array} \tag{1}$$

be a Cartesian diagram in which the horizontal inclusions are regular embeddings of codimension one and the vertical inclusions are regular of codimension d. Let $X_{\widetilde{Z \cap D}}$ *be the blow-up of* X *along* $Z \cap D$. *For a regular embedding* $A \xrightarrow{j} B$, *let* N_A^B *be the normal bundle of* j. *For a vector bundle* E, *let* $\mathbb{P}(E)$ *be its projective fibration.*

(i) *We have* $(X_{\widetilde{Z \cap D}}) \times_X D = D_{\widetilde{Z \cap D}} \bigcup_{\mathbb{P}(N^D_{Z \cap D})} \mathbb{P}(N^X_{Z \cap D})$, *the components of the union are glued along the hyperplane* $\mathbb{P}(N^D_{Z \cap D})$ *of* $\mathbb{P}(N^X_{Z \cap D})$. *Let* $(X_{\widetilde{Z \cap D}})^\circ = (X_{\widetilde{Z \cap D}}) - D_{\widetilde{Z \cap D}}$. *Then the preimage of* D *in* $(X_{\widetilde{Z \cap D}})^\circ$ *can be identified with the vector bundle* $N^D_{Z \cap D} \otimes (N^Z_{Z \cap D})^{-1} = E$.

(ii) *There exists a unique lifting* $j: Z \to X_{\widetilde{Z \cap D}}$ *of the embedding* $Z \to X$. *The immersion* j *is regular of codimension* d, *and its restriction to* $Z \cap D$ *is (in the notations of (i)) the zero-section of the bundle* E.

(iii) *Let* $Z' \subset X$ *be a regular inclusion of codimension* $d - e$ *containing* Z, *and suppose that* $Z \subset Z'$ *is regular of codimension* e. *If* $\bar{Z}'$ *is the closure of* $Z' \cap (X - D)$ *in* $(X_{\widetilde{Z \cap D}})^\circ$, *then* $\bar{Z}' = (Z' \cap (X - D)) \cup N^{Z' \cap D}_{Z \cap D} \otimes (N^Z_{Z \cap D})^{-1}$, *and the inclusion* $Z \to \bar{Z}'$ *is regular of codimension* e.

(iv) *If* $\hat{X} \to X$ *is a morphism of schemes, we denote the base-change of objects on* X *to* $\hat{X}$ *by* $\hat{}$. *If the assumptions to the diagram* (1) *remain satisfied, with the same* d, *after base-change to* $\hat{X}$, *then* $(\hat{X}_{\widetilde{\hat{Z} \cap \hat{D}}})^\circ = (X_{Z \cap D})^\circ \times_X \hat{X} (= ((X_{Z \cap D})^\circ)\hat{})$. *If furthermore the assumption of (iii) remains satisfied, with the same* e, *after base-change to* $\hat{X}$, *then* $\hat{\bar{Z}}' = (\bar{Z}')\hat{}$.

(v) *If $Z \cap D$ and X are flat over a common base scheme S, then $X^{\sim}_{Z\cap D}$ is flat over S. If furthermore in (iii) Z' is flat over S*, then so is $\bar{Z}'$.

Proof. The first part of (ii) follows from the universal property of the blow-up ([H, II.7.14]), since $Z \cap D \subset Z$ is (by assumption) regular of codimension one. The proofs of (i), (iii), (iv), and of the second part of (ii) are straightforward computations involving Micali's theorem (cf. [FL, §IV.2]).

To prove the first part of (v), it suffices to prove that J^k is flat over S, where J is the sheaf of ideals defining $Z \cap D$ on X. Since a locally free $\mathcal{O}_{Z\cap D}$-module is flat over S, this follows by induction on k from the sequence $0 \to J^{k+1} \to J^k \to S^k(N^X_{Z\cap D}) \to 0$. The proof of the second part of (v) is similar.

4.2. *Deformation to the normal bundle*

(cf. [F, §5.1]). Throughout 4.2–4.4, the symbol $Y \overset{d}{\subset} X$ denotes a regular embedding of codimension d.

LEMMA. *Let $n \geqslant 0$. To each sequence $X_0 \overset{d_1}{\subset} X_1 \overset{d_2}{\subset} \cdots \overset{d_n}{\subset} X_n$ of regular immersions, one can construct a commutative diagram*

$$\begin{array}{ccccccc} M_0 & \overset{d_1}{\subset} & M_1 & \overset{d_2}{\subset} & \cdots & \overset{d_n}{\subset} & M_n \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & & & \downarrow{\scriptstyle \pi_n} \\ \mathbb{P}^1_{X_0} & \overset{d_1}{\subset} & \mathbb{P}^1_{X_1} & \overset{d_2}{\subset} & \cdots & \overset{d_n}{\subset} & \mathbb{P}^1_{X_n} \end{array} \tag{2}$$

with the following properties:

(i) *On M_i, there is an action of the affine group $Aff_1(X_n) = \{g \in \Gamma(X_n, \mathscr{PGL}_2) \mid g(\infty) = \infty\}$ which is compatible with the action of $Aff_1(X_n)$ on $\mathbb{P}^1_{X_n}$.*

(ii) *The restriction of π_j to $\pi_j^{-1}(A^1_{X_j})$ is an isomorphism $\pi_j^{-1}(A^1_{X_j}) \xrightarrow{\sim} A^1_{X_j}$, and π_0 is an isomorphism everywhere: π_0: $M_0 \xrightarrow{\sim} \mathbb{P}^1_{X_0}$.*

(iii) *For a section $t \in \mathbb{P}^1(X_n)$, put $M_i^{(t)} = \pi_i^{-1}(t)$. Then $M_i^{(\infty)} \overset{1}{\subset} M_i$, and $M^{(\infty)}$ is the sequence*

$$X_0 \overset{d_1}{\subset} N^{X_1}_{X_0} \overset{d_2}{\subset} N^{X_2}_{X_0} \subset \cdots \overset{d_n}{\subset} N^{X_n}_{X_0}.$$

(iv) *If p: $X_n \to Z$ is a flat morphism such that the restriction of p to X_i is flat for every i, then M_i is flat over Z for every i.*

(v) *For a morphism $\tilde{X}_n \xrightarrow{\varphi} X_n$, we use $\tilde{}$ to denote base change to $\tilde{X}_n$. If we have $\tilde{X}_0 \overset{d_1}{\subset} \tilde{X}_1 \cdots \overset{d_n}{\subset} \tilde{X}_n$ (with the same d_i as in* (2)!), *then we can define an isomorphism M_φ: $M_j \times_{X_n} X_n \xrightarrow{\sim} M_j(\tilde{X}.)$ (the last symbol means the construction of M_i, applied to the sequence $X.$) On $M_j^{(\infty)} \times_{X_n} \tilde{X}_n$, M_φ is an* *isomorphism of vector bundles. If we apply a second base change $\tilde{\tilde{X}}_n \xrightarrow{\psi} \tilde{X}_n \xrightarrow{\varphi} X_n$, then the diagram*

$$\begin{tikzcd}
M_j \times_{X_n} \tilde{\tilde{X}}_n \arrow[r] \arrow[d, "M_{\varphi\psi}"'] & (M_j \times_{X_n} \tilde{X}) \times_{\tilde{X}_n} \tilde{\tilde{X}}_n \arrow[d, "M_\varphi \times_{X_n} \tilde{\tilde{X}}_n"] \\
M_j(\tilde{\tilde{X}}.) & M_j(\tilde{X}.) \times_{\tilde{X}_n} \tilde{\tilde{X}}_n \arrow[l, "M_\psi"']
\end{tikzcd}$$

commutes.

Proof. Let M_n be the blow-up of $\mathbb{P}^1_{X_n}$ along $\infty(X_0)$, where ∞ is the infinity-section of $\mathbb{P}^1_{X_n} \cdot M_0$ is the image of the immersion j constructed in 4.1(ii), and M_j for $0 < j < n$ is constructed by applying 4.1(iii) to $Z' = \mathbb{P}^1_{X_j}$ (of course, with $D = \infty$ and $Z = \mathbb{P}^1_{X_0}$). The Lemma follows from the results of 4.1.

4.3. *An awkward proposition*

Let $K_{d_1,\ldots,d_n}$ be the following category. Its objects are sequences

$$X^{\bullet}\colon X_0 \overset{d_1}{\subset} X_1 \overset{d_2}{\subset} X_2 \cdots \overset{d_n}{\subset} X_n.$$

A homomorphism from $X^{\bullet}$ to $Y^{\bullet}$ is a morphism from X_n to Y_n such that $X_i = Y_i \times_{Y_n} X_n$, it is called flat (proper) if $X_n \to Y_n$ is flat (proper). Since in $X^{\bullet}$ the fibre product of a flat and an arbitrary morphism exists, the bicategory ($K_{d_1,\ldots,d_n}$, proper morphisms of c.r.d., flat morphisms) is well-defined.

Let $K_{d_1,\ldots,d_n,\mathrm{sp}}$ be the following category. Objects are triples $(D^{\bullet}, X^{\bullet}, f)$ where $D^{\bullet}$ and $X^{\bullet}$. form a Cartesian diagram

$$\begin{array}{ccccccc}
D_0 & \overset{d_1}{\subset} & D_1 & \overset{d_2}{\subset} & D_2 & \subset \cdots \overset{d_n}{\subset} & D_n \\
\cap_1 & & \cap_1 & & \cap_1 & & \cap_1 \\
X_0 & \overset{d_1}{\subset} & X_1 & \overset{d_2}{\subset} & X_2 & \subset \cdots \overset{d_n}{\subset} & X_n
\end{array}$$

and f is a section of $\mathcal{O}_{X_n}$ in some neighbourhood of D_n such that $(D_i, X_i, f|_{X_i})$ belongs to $\mathrm{Ob}(K_{\mathrm{sp}})$ for every i. A morphism from $(D!, X!, f')$ to $(D^{\bullet}, X^{\bullet}, f)$ is a morphism X'_n to X_n such that $X'_i = X_i \times_{X_n} X'_n$, $D'_i = D_i \times_{X_n} X'_n$, and $f' =$ (pull-back of f to X'_n), it is flat (proper) if $X'_n \to X_n$ is flat (proper). The bicategory ($K_{d_1,\ldots,d_n,\mathrm{sp}}$, proper morphisms of constant relative dimension, flat morphisms) is well-defined.

By the results of §3, $\widetilde{\mathbf{CH}}^{\bullet}(X_i)$ (cf. (3.13)) are bifibred Picard categories over ($K_{d_1,\ldots,d_n}$, proper mor. of c.r.d., flat mor.). Over ($K_{d_1,\ldots,d_n,\mathrm{sp}}$, proper mor. of c.r.d., flat mor.), $\widetilde{\mathbf{CH}}^{\bullet}(X_i)$ and $\widetilde{\mathbf{CH}}^{\bullet}(D_i)$ are bifibred, and $\mathbf{sp}_f|_{X_i}\colon \widetilde{\mathbf{CH}}^{\bullet}(X_i) \to \widetilde{\mathbf{CH}}^{\bullet}(D_i)$ is biadmissible.

We are now ready to formulate our awkward proposition. To stimulate the reader's patience, we mention beforehand that the functor $F_{X.}$ on K_{d_1} will play the role of the Gysin functor $\mathbf{i}^!$ for regular immersions of codimension d_1, that its uniqueness up to an additive functor-isomorphism on K_{d_1,d_2} will be used to construct the isomorphism $(\mathbf{ij})^! \to \mathbf{j}^!\mathbf{i}^!$, and that its uniqueness up to unique natural transformation on K_{d_1,d_2,d_3} will be used to verify (3.6).

PROPOSITION. *On $K_{d_1,\dots,d_n}$, there exists an admissible functor between bifibred Picard categories (cf.* 3.11.) $F_{X.}\colon \widetilde{\mathbf{CH}}^{\bullet}(X_n) \to \widetilde{\mathbf{CH}}^{\bullet}(X_0)$, *which preserves the graduation, together with the following data:*

(i) *If in the commutative diagram*

$$\begin{tikzcd} X_0 \arrow[r, phantom, "\overset{d_n}{\subset}"] \arrow[drr, "r_0"'] & X_1 \arrow[r, phantom, "\subset \cdots \overset{d_n}{\subset}"] \arrow[dr] & & X_n \arrow[dl, "r_n"] \\ & & Z & \end{tikzcd} \tag{3}$$

the projections r_i are flat, then there is an additive functor-isomorphism

$$\alpha_r\colon F_{X.} \circ \mathbf{r}_n^* \to \mathbf{r}_0^* \tag{4}$$

which satisfies the following compatibility with base-changes $Z' \to Z$:

If $X_i' = X_i \times_Z Z'$ and if $r_i'\colon X_i' \to Z'$ is the natural projection, then (since the r_i are flat) $X!$ is in $K_{d_1,\dots,d_n}$, $\widetilde{\mathbf{CH}}^{\bullet}(X_i)$ and $\widetilde{\mathbf{CH}}^{\bullet}(Z')$ are bifibred over (Z-schemes, proper mor. of const. rel. dim., flat *morphisms), and the functors $\mathbf{r}_i'^*$ and $F_{X'.}$ are biadmissible. The condition is that $\alpha_{r'}$ is a biadmissible transformation.*

(ii) *On $K_{d_1,\dots,d_n,\mathrm{sp}}$, there is a biadmissible transformation*

$$\mathbf{sp}_f|_{X_0} F_{X.} \to F_{D.}\mathbf{sp}_f \qquad (\text{denoted by } \beta_{f,X.,D.}) \tag{5}$$

with the following properties:

(ii.1) *Let $(D_{\bullet}^{(1)}, X_{\bullet}, f_1)$ and $(D_{\bullet}^{(2)}, X_{\bullet}, f_2)$ be objects of $K_{d_1,\dots,d_n,\mathrm{sp}}$ such that $(D_{\bullet}^{(12)}, D_{\bullet}^{(1)}, f_2|_{D_n^{(1)}})$ and $(D_{\bullet}^{(12)}, D_{\bullet}^{(2)}, f_1|_{D_n^{(2)}})$ are both objects of $K_{d_1,\dots,d_n}$, where $D_i^{(12)} = D_i^{(1)} \cap D_i^{(2)}$. Then the diagram*

$$\begin{tikzcd}[column sep=huge] \mathbf{sp}_{f_1}|_{D_0^{(2)}}\mathbf{sp}_{f_2}|_{X_0} F_{X.} \arrow[r, "\mathbf{sp}_{f_1|D_0^{(2)}}(\beta_{f_2,X.,D^{(2)}})"] \arrow[d, "\omega_{f_1|X_0,f_2|X_0}"] & \mathbf{sp}_{f_1}|_{D_0^{(2)}} F_{D_{\bullet}^{(2)}}\mathbf{sp}_{f_2} \arrow[d, "\beta_{f_1|D_n^{(2)},D^{(2)},D^{(12)}}"] \\ \mathbf{sp}_{f_2}|_{D_0^{(1)}}\mathbf{sp}_{f_1}|_{X_0} F_{X.} \arrow[d, "\mathbf{sp}_{f_2|D_0^{(1)}}(\beta_{f_1,X.,D^{(1)}})"] & F_{D_{\bullet}^{(12)}}\mathbf{sp}_{f_1}|_{D_n^{(1)}}\mathbf{sp}_{f_2} \arrow[d, "F_{D^{(12)}}(\omega_{f_1,f_2})"] \\ \mathbf{sp}_{f_2}|_{D_0^{(1)}} F_{D_{\bullet}^{(1)}}\mathbf{sp}_{f_1} \arrow[r, "\beta_{f_2|D_n^{(1)},D^{(1)},D^{(12)}}"'] & F_{D_{\bullet}^{(12)}}\mathbf{sp}_{f_2}|_{D_n^{(2)}}\mathbf{sp}_{f_1} \end{tikzcd}$$

commutes. The transformations ω have been defined in 3.15.

(ii.2) *If* $(D, Y, F) \in \mathrm{Ob}(K_{\mathrm{sp}})$, $X. \in \mathrm{Ob}(K_{d_1,\dots,d_n})$, *and* $r_n\colon X_n \to Y$ *is a flat projection whose restriction* r_i *to* X_i *remains flat, we put* $D_i = X_i \times_Y D$ *and denote the projection of* D_i *to* D *by* $r_{D,i}$. *Then the following diagram commutes:*

$$\begin{tikzcd}
F_{D.}\mathbf{sp}_{r_n^*(f)}\mathbf{r}_n^* \arrow[r] \arrow[dd, "\beta r_n^*(f),\, X.,\, D."] & F_{D.}\mathbf{r}_{D,n}^*\mathbf{sp}_f \arrow[dr] & \\
 & & \mathbf{r}_{D,0}^*\mathbf{sp}_f, \\
\mathbf{sp}_{r_0^*(f)}F_{X.}\mathbf{r}_n^* \arrow[r, "\alpha_r"] & \mathbf{sp}_{r_0^*(f)}\mathbf{r}_0^* \arrow[ur] &
\end{tikzcd} \tag{7}$$

UNIQUENESS. The data (i) and (ii) determine F_X up to a unique biadmissible transformation between biadmissible functors over $K_{d_1,\dots,d_n}$.

REMARK. In the special case $Y = X_0$, (7) simplifies to

$$\begin{tikzcd}
\mathbf{sp}_f F_{X.}\mathbf{r}_n^* \arrow[r] \arrow[d] & \mathbf{sp}_f \\
D_{D.}\mathbf{sp}_{r_n^*(f)}\mathbf{r}_n^* \arrow[r] & F_{D.}\mathbf{r}_{n,D}^*\mathbf{sp}_f \arrow[u]
\end{tikzcd} \tag{8}$$

Proof of Proposition

4.4. *Construction of* $F_{X.}$. For $X_0 \subset \cdots \subset X_n$, let $M_0 \subset \cdots \subset M_n$ be the result of construction 4.2. The projections π_n have been defined in (2). We put

$$\begin{aligned}
q_n &= \text{composition of } M_n \xrightarrow{\pi_n} \mathbb{P}^1_{X_n} \to X_n \\
q_{n,a} &= \text{restriction of } q_n \text{ to } \pi_n^{-1}(A^1_{X_n})
\end{aligned} \tag{9}$$

Furthermore, let

$$p_\infty\colon M_n^{(\infty)} \to M_0^{(\infty)} \tag{10}$$

be the bundle projection (cf. 4.2 (iii)). By 4.2 (v), the categories $\mathbf{CH}^{\boldsymbol{\cdot}}(M_i^{(\infty)})$ are bifibred over $K_{d_1,\dots,d_n}$, and $\mathbf{p}_\infty^*$ is a biadmissible functor. By 3.9, we know that it is also an equivalence of categories. Consequently it has a biadmissible inverse $(\mathbf{p}_\infty^*)^{-1}$ (cf. 3.11). Finally, let f_0 be a rational function on $\mathbb{P}^1_{\mathrm{Spec}(\mathbb{Z})}$ whose only zero is a simple zero along ∞ and whose pole does not intersect ∞. We denote the pull-back of f_0 to $\mathbb{P}^1_{X_n}$ by f and put

$$F_{X.} = (\mathbf{p}_\infty^*)^{-1}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}^*. \tag{11}$$

By 4.2(v) and our previous remarks, this is a biadmissible (over $K_{d_1,\dots,d_n}$) functor.

Let u_n: $X_n \to Z$ be a flat morphism whose restriction u_i to X_i remains flat. 4.2(iv) proves that $M_i \to Z$ is flat, hence we can define α_u by

$$F_X\mathbf{u}_n^* = (\mathbf{p}_\infty^*)^{-1}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}^*\mathbf{u}_n^* \xrightarrow{\tau u_n q_{n,a}, M_n, M_n^{(\infty)}} (\mathbf{p}_\infty^*)^{-1}(\mathbf{u}_p\mathbf{p}_\infty)^*$$

$$\to (\mathbf{p}_\infty^*)^{-1}\mathbf{p}_\infty^*\mathbf{u}_0^* \to \mathbf{u}_0^*. \tag{12}$$

It is biadmissible over (Z-schemes) since (12) contains only biadmissible transformations.

Let $(D., X., g) \in \mathrm{Ob}(K_{d_1,\dots,d_n,\mathrm{sp}})$. We denote by $M.$ and $M.^{(D)}$ the construction 4.2, applied to $X.$ and $D.$ respectively. The analogues for $D.$ of the morphisms (9) and (10) are denoted by a superscript (D). By 4.2(v), we have $M_i^{(D)} \simeq M_i \times_{X_n} D_n \subset M_i$. There is a unique biadmissible (over $K_{d_1,\dots,d_n,\mathrm{sp}}$) transformation φ: $\mathbf{sp}_g(p_\infty^*)^{-1} \to (\mathbf{p}_\infty^{(D)*})^{-1}\mathbf{sp}_{p_\infty^*(g)}$ such that the diagram

$$\begin{tikzcd}
\mathbf{p}_\infty^{(D)*}\mathbf{sp}_g(\mathbf{o}_\infty^*)^{-1} \arrow[r, "\mathbf{p}_\infty^{(D)*}(\varphi)"] \arrow[d, "2"] & \mathbf{p}_\infty^{(D)*}(\mathbf{p}_\infty^{(D)*})^{-1}\mathbf{sp}_{p_\infty^*(g)} \arrow[d, "2"] \\
\mathbf{sp}_{p_\infty^*(g)}\mathbf{p}_\infty^*(\mathbf{p}_\infty^*)^{-1} \arrow[r] & \mathbf{sp}_{p_\infty^*(g)}
\end{tikzcd}$$

commutes. We also mention that $p^*(g) = q_n^*(g)|_{M_n^{(\infty)}}$ and that the sheaf of ideals defining $D_n \subset M_n$ is in some neighbourhood of D_n trivialized by $q_n^*(g)$. We define the datum (ii) by

$$\mathbf{sp}_g F_X = \mathbf{sp}_g(\mathbf{p}_\infty^*)^{-1}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}'^* \xrightarrow{\varphi} (\mathbf{p}_\infty^{(D)*})^{-1}\mathbf{sp}_{q_n^*(g)|M_n^{(\infty)}}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}^* \to$$

$$\xrightarrow{\omega_{f,g}} (\mathbf{p}_\infty^{(D)*})^{-1}\mathbf{sp}_{\pi_n^{(D)*}(f)}\mathbf{sp}_{q_n^*(g)}\mathbf{q}_{n,a}^* \to (\mathbf{p}_\infty^{(D)*})^{-1}\mathbf{sp}_{\pi_n^{(D)*}(f)}\mathbf{q}_{n,a}^{(D)*}\mathbf{sp}_g = F_{D.}\mathbf{sp}_g.$$

$$\tag{13}$$

Since (13) contains only biadmissible transformations, it defines a biadmissible (over $K_{d_1,\dots,d_n,\mathrm{sp}}$) functor-isomorphism. It is straightforward to check that (6) follows from (3.16) (applied to the three functions $\pi_n^*(f)$, $q_n^*(g^{(1)})$ and $q_n^*(g^{(2)})$ on M_n) and (7) follows from (3.17) (applied to a fibre diagram of the form

$$\left(\begin{tikzcd}
M_n^{(D)(\infty)} \arrow[r] \arrow[d] & M_n^{(\infty)} \arrow[d] \\
D \arrow[r] & Y
\end{tikzcd}\right).$$

4.5. Uniqueness of F

Let G be a functor with the properties (i) and (ii), and let F be the functor constructed in 4.4. We use the notations of 4.4 and put $M_i^{(a)} = \pi_i^{-1}(A^1_{X_i})$. Since q_0 is flat and $M_0^{(\infty)} = X_0$, (3.12) provides us with a biadmissible (over $K_{d_1,\dots,d_n}$) isomorphism

$$\psi\colon \mathbf{sp}_f|_{M_0}\mathbf{q}_{0,a}^* A \to A. \tag{14}$$

We define $\Phi_{X.}\colon G_{X.} \to F_{X.}$ by

$$\begin{aligned} &G_{X.} \to \mathbf{sp}_f|_{M_0}\mathbf{q}_{0,a}^* G_{X.} \xrightarrow{\text{(a)}} \mathbf{sp}_f|_{M_0} G_{M_\cdot^{(a)}}\mathbf{q}_{n,a}^* \xrightarrow{\beta_{\pi_n^*(f),M.,M_\cdot^{(\infty)}}} \\ &\to G_{M_\cdot^{(\infty)}}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}^* \xrightarrow{\text{(b)}} G_{M_\cdot^{(\infty)}}\mathbf{p}_\infty^*(\mathbf{p}_\infty^*)^{-1}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}^* \xrightarrow{\alpha_{p_\infty}} \\ &\to (\mathbf{p}_\infty^*)^{-1}\mathbf{sp}_{\pi_n^*(f)}\mathbf{q}_{n,a}^* = F_{X.}. \end{aligned} \tag{15}$$

In (15), (a) is base-change with respect to the flat morphism $q_{n,a}$, and (b) is canonical. Since (15) contains only biadmissible transformations, $\Phi_{X.}$ is biadmissible (over $K_{d_1,\dots,d_n}$). We have to check that it is compatible with the data (i) and (ii).

If we are given a commutative diagram (3), we define an additive functor-isomorphism $\chi_Z\colon \mathbf{r}_0^* \to \mathbf{r}_0^*$ by

$$\mathbf{r}_0^* \xrightarrow{(\alpha_r)^{-1}} G_{X.}\mathbf{r}_n^* \xrightarrow{\Phi_{X.}} F_{X.}\mathbf{r}_n^* \xrightarrow{\alpha_r} \mathbf{r}_0^*.$$

Since this definition contains only biadmissible (over Z-schemes transformations, we have $\chi_U = \chi_Z|_{r_0^{-1}(U)}$ for every Zariski-open $U \subset Z$. By 3.14, χ_Z is the identity, and Φ is compatible with the datum (i).

Diagram (16) implies that Φ is compatible with the datum (ii).

$$\begin{tikzcd}
\mathbf{sp}_g G_{X.} \arrow[r, "\mathbf{sp}_g(\psi)"] \arrow[ddd, "\beta"'] \arrow[ddr] \arrow[dr, phantom, "\text{(A)}"] & \mathbf{sp}_g\mathbf{sp}_f\mathbf{q}_{0,a}^* G_{X.} \arrow[r] \arrow[d, "\omega_{g,f}"'] \arrow[dr, phantom, "\text{NT}"] & \mathbf{sp}_g\mathbf{sp}_f G_{M_\cdot^{(a)}}\mathbf{q}_{n,a}^* \arrow[r, "\beta"] \arrow[d, "\omega_{g,f}"] & \mathbf{sp}_g G_{M_\cdot^{(\infty)}}\mathbf{sp}_f\mathbf{q}_{n,a}^* \arrow[d, "\beta"] \\
& \mathbf{sp}_f\mathbf{sp}_g\mathbf{q}_{0,a}^* G_{X.} \arrow[r] \arrow[d] & \mathbf{sp}_f\mathbf{sp}_g G_{m_\cdot^{(a)}}\mathbf{q}_{n,a}^* \arrow[d, "\beta"] \arrow[dr, phantom, "\text{(C)}"] & G_{M_\cdot^{(D,\infty)}}\mathbf{sp}_g\mathbf{sp}_f\mathbf{q}_{n,a}^* \arrow[d, "\omega_{g,f}", "g{,}f"'] \\
& \mathbf{sp}_f\mathbf{q}_{0,a}^{(D)*}\mathbf{sp}_g G_{X.} \arrow[d, "\beta"'] \arrow[r, phantom, "\text{(B)}"] & \mathbf{sp}_f G_{M_\cdot^{(D,a)}}\mathbf{sp}_g\mathbf{q}_{n,a}^* \arrow[r, "\beta"] \arrow[d] \arrow[dr, phantom, "\text{NT}"] & G_{M_\cdot^{(D,\infty)}}\mathbf{sp}_f\mathbf{sp}_g\mathbf{q}_{n,a}^* \arrow[d] \\
G_{D.}\mathbf{sp}_g \arrow[r, "\psi"] & \mathbf{sp}_f\mathbf{q}_{0,a}^{(D)*}G_{D.}\mathbf{sp}_g \arrow[r] & \mathbf{sp}_f G_{M_\cdot^{(D,a)}}\mathbf{q}_{n,a}^{(D)*}\mathbf{sp}_g \arrow[r] & G_{M_\cdot^{(D,\infty)}}\mathbf{sp}_f\mathbf{sp}_g\mathbf{q}_{n,a}^{(D)*}
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
\mathbf{sp}_g G_{M_!^{(\infty)}} \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[r] \arrow[d, "\beta"'] \arrow[dr, phantom, "\mathrm{NT}"] & \mathbf{sp}_g G_{M_!^{(\infty)}} \mathbf{p}_\infty^* (\mathbf{p}_\infty^*)^{-1} \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[r] \arrow[d, "\beta"] & \mathbf{sp}_g (\mathbf{p}^*)^{-1} \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[d, "\varphi"] \\
G_{M_!^{(D,\infty)}} \mathbf{sp}_g \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[r] \arrow[d, "\omega_{g,j}"'] \arrow[ddr] \arrow[dr, phantom, "(D)"] & G_{M_!^{(D,\infty)}} \mathbf{sp}_g \mathbf{p}_\infty^* (\mathbf{p}_\infty^*)^{-1} \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[d] \arrow[ur, phantom, "\mathrm{NT}"] & (\mathbf{p}_\infty^{(D)*})^{-1} \mathbf{sp}_g \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[d, "\omega_{g,f}"] \\
G_{M_!^{(D,\infty)}} \mathbf{sp}_f \mathbf{sp}_g \mathbf{q}_{n,a}^* \arrow[d] \arrow[ddr] \arrow[dr, phantom, "\mathrm{NT}"] & G_{M_!^{(D,\infty)}} \mathbf{p}_\infty^{(D)*} \mathbf{sp}_g (\mathbf{p}_\infty^*)^{-1} \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[d, "\varphi"] \arrow[uur, "\alpha"'] \arrow[ur, phantom, "\mathrm{NT}"] & (\mathbf{p}_\infty^{(D)*})^{-1} \mathbf{sp}_f \mathbf{sp}_g \mathbf{q}_{n,a}^* \arrow[d] \\
G_{M_!^{(D,\infty)}} \mathbf{sp}_f \mathbf{q}_{n,a}^{(D)*} \mathbf{sp}_g \arrow[ddr] \arrow[dr, phantom, "\mathrm{NT}"] & G_{M^{(D,\infty)}} \mathbf{p}_\infty^{(D)*} (\mathbf{p}_\infty^{(D)*})^{-1} \mathbf{sp}_g \mathbf{sp}_f \mathbf{q}_{n,a}^* \arrow[d, "\omega_{g,t}"] \arrow[uur, "\alpha"'] \arrow[ur, phantom, "\mathrm{NT}"] & (\mathbf{p}_\infty^{(D)*})^{-1} \mathbf{sp}_f \mathbf{q}_{n,a}^{(D)*} \mathbf{sp}_g \\
& G_{M_!^{(D,\infty)}} \mathbf{p}_\infty^{(D)*} (\mathbf{p}_\infty^{(D)*})^{-1} \mathbf{sp}_f \mathbf{sp}_g \mathbf{q}_{n,a}^* \arrow[d] \arrow[uur, "\alpha"'] & \\
& G_{M_!^{(D,\infty)}} \mathbf{p}_\infty^{(D)*} (\mathbf{p}_\infty^{(D)*})^{-1} \mathbf{sp}_f \mathbf{q}_{n,a}^{(D)*} \mathbf{sp}_g \arrow[uur] &
\end{tikzcd}
\tag{16}
$$

Glue the right boundary of the upper diagram with the left boundary of the lower diagram.

The notations in (16) are the same as in (13). To save space, the indices at the transformations α and β have been omitted, and the various pull-backs of f and g have been denoted by the same letter f or g. The vertical arrow at the left boundary of (16) is the isomorphism β for $G.$, and the outer right column of (16) is the sequence (13). The top and the bottom row of (16) are (15) for $X.$ and $D.$. The squares marked by 'NT' are commutative because they are of the form

$$
\begin{tikzcd}
F(A) \arrow[r, "\xi_A"] \arrow[d, "F(\eta)"'] & G(A) \arrow[d, "G(\eta)"] \\
F(B) \arrow[r, "\xi_B"'] & G(B)
\end{tikzcd}
$$

where $\xi\colon F \to G$ is a natural transformation between functors and $\eta\colon A \to B$ is a morphism.

The commutativity of (A) is (3.18). (B) commutes because β is admissible with respect to flat pull-backs in $K_{d_1,\ldots,d_n,\mathrm{sp}}$ (cf. the diagram at the end of 3.6). (C) is of type (6) and (E) of type (8). The commutativity of (D) is easily derived from the fact that $\mathbf{sp}$ is an admissible (with respect to flat pull-backs) functor.

4.6. *Uniqueness up to unique functor-isomorphism*

Suppose we are given a biadmissible automorphism Φ of F which is (in an obvious sense) compatible with the data (i) and (ii). From the compatibility with

(i) it follows tht $\Phi_{X.} = \mathrm{Id}$ if $X.$ is a sequence of vector bundles with base X_0. In particular, $\Phi_{M^{(\infty)}_{\cdot}} = \mathrm{Id}$, where $M^{(\infty)}_{\cdot}$ is the same as above. Since

$$\begin{tikzcd}
F_{X.} \arrow[r] \arrow[d, "\Phi_{X.}"] \arrow[dr, phantom, "\mathrm{NT}"] & \mathbf{sp}_f \mathbf{q}_{0,a}^* F_{X.} \arrow[r] \arrow[d, "\mathbf{sp}_f \mathbf{q}_{0,a}^*(\Phi_{X.})"] \arrow[dr, phantom, "(\mathrm{A})"] & \mathbf{sp}_f F_{M^{(a)}_{\cdot}} \mathbf{q}_{n,a}^* \arrow[r] \arrow[d, "\mathbf{sp}_f(\Phi_{M^{(a)}_{\cdot}})"] \arrow[dr, phantom, "(\mathrm{B})"] & F_{M^{(\infty)}_{\cdot}} \mathbf{sp}_f q_{n,a}^* \arrow[d, "\Phi_{M^{(\infty)}_{\cdot}} = \mathrm{Id}"] \\
F_{X.} \arrow[r] & \mathbf{sp}_f \mathbf{q}_{0,a}^* F_{X.} \arrow[r] & \mathbf{sp}_f F_{M^{(a)}_{\cdot}} \mathbf{q}_{n,a}^* \arrow[r] & F_{M^{(\infty)}_{\cdot}} \mathbf{sp}_f \mathbf{q}_{n,a}^*
\end{tikzcd}$$

commutes, we have $\Phi_{X.} = \mathrm{Id}$, and the proof of 4.3 is complete. NT has the same meaning as in (16), (A) commutes since Φ is biadmissible, and (B) commutes since Φ is compatible with the datum (ii).

4.7. *The Gysin-functor*

We are now ready to prove the main theorem of §4. Throughout this paper, 'lci' will be an abbreviation for local complete intersection. A morphism $X \to Y$ is called a smoothable lci-morphism (abbreviated: slci-morphism) if it has a factorization $X \to S \to Y$ where $S \to Y$ is smooth. Then it follows ([SGA 6, Exp. VIII] or [FL, IV.3.10.]) that $X \to S$ is a regular immersion. The relative dimension of a lci-morphism at $x \in X$ has been defined in [SGA 6] and [FL].

Let K_{lci} be the following category: Objects are triples (f, X, Y) with $f: X \to Y$ a slci-morphism. A morphism from (f', X', Y') to (f, X, Y) is a Cartesian diagram (which we denote by (p_X, p_Y))

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "p_X"'] & Y' \arrow[d, "p_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

such that for every $x \in X'$ $d_x(f') = d_{p_X(x)}(f)$, where $d_x(f)$ is the relative dimension of the lci-morphism f at the point x. A morphism in K_{lci} is called flat (resp. proper of c.r.d.) if so is p_Y (and hence p_X, cf. the lemma in 1.4). The bicategory (K_{lci}, proper morphisms of c.r.d., flat morphisms) is well-defined and will be denoted by $\mathbb{K}_{\mathrm{lci}}$.

Let $K_{\mathrm{lci,sp}}$ be the following category. Objects are 5-tuples (f, X, Y, D, λ) such that $f: X \to Y$ is a slci-morphism, $D \subset Y$ is a regular immersion of codimension one, λ is a section of $\mathcal{O}_Y$ in some neighbourhood of D which generates the sheaf of ideals defining D, and $D_X \to X$ is a regular immersion of codimension one, where $D_X = f^{-1}(D)$. A morphism in $K_{\mathrm{lci,sp}}$ from $(f', X', Y', D', \lambda')$ to (f, X, Y, D, λ) is a morphism (p_X, p_Y): $(f', X', Y') \to (f, X, Y)$ in K_{lci} such that $D' = p_Y^{-1}(D)$ and

$\lambda' = p_Y^*(\lambda)$. A morphism is said to be flat (proper), if so is p_Y. The bicategory ($K_{\mathrm{lci,sp}}$, proper morphisms of c.r.d., flat morphisms) is denoted by $\mathbb{K}_{\mathrm{lci,sp}}$.

Let $K_{\mathrm{lci,com}}$ be the category whose objects 5-tuples (f, g, X, Y, Z) where f: $X \to Y$ and g: $Y \to Z$ are lci-morphisms such that g and gf (and hence f too) are slci. A morphism from (f', g', X', Y', Z') to (f, g, X, Y, Z) is a triple (p_X, p_Y, p_Z), p_X: $X' \to X$, p_Y: $Y' \to Y$, p_Z: $Z' \to Z$ such that (p_Y, p_Z): $(g', Y', Z') \to (g, Y, Z)$ and (p_X, p_Y): $(f', X', Y') \to (f, X, Y)$ are morphisms in K_{lci}. It is flat (proper) if so is p_Z (and hence p_X and p_Y too). As usual, $\mathbb{K}_{\mathrm{lci,com}}$ refers to the bicategory ($K_{\mathrm{lci,com}}$, proper morphisms of c.r.d., flat morphisms).

The main result of §4 is the construction of an inverse image functor $f^!$ for local complete intersections f. Unlike the functors constructed in §3, it is no longer possible to define this functor directly. Instead, we describe it as a certain biadmissible functor, equipped with certain natural isomorphisms described in 4.7.1–4.7.3 which have to satisfy certain conditions explained in these paragraphs. The system of functors $f^!$ (for slci-morphisms f) is unique in the sense that, given another system of functors $f^?$ together with similar natural transformations satisfying the same conditions there exists a unique functor-isomorphism $f^! \to f^?$ respecting the natural transformations 4.7.1–4.7.3.

It should also be mentioned that the notions of a natural transformation and of a natural isomorphism are equivalent if applied to functors between groupoids, in particular to functors between Picard categories. Consequently, these two notions are used synonymously in the following text, and natural transformation is often abbreviated to transformations because confusions are impossible.

The main result of §4 is

THEOREM. *Let us denote objects of* $\mathbb{K}_{\mathrm{lci}}$ *by* (f, X, Y), *such that* $\widetilde{\mathbf{CH}}^\cdot(X)$ *and* $\widetilde{\mathbf{CH}}^\cdot(Y)$ *are bifibred Picard categories over* $\mathbf{K}_{\mathrm{lci}}$. *Then there is a biadmissible functor* $\mathbf{f}^!: \widetilde{\mathbf{CH}}^i(Y) \to \widetilde{\mathbf{CH}}^i(X)$ *between the bifibred Picard categories over* $\mathbb{K}_{\mathrm{lci}}\mathbf{CH}^\cdot(Y)$ *and* $\mathbf{CH}^\cdot(X)$, *together with the following data*:

4.7.1. For each flat morphism h: $Y \to Z$ such that hf is flat, we are given an isomorphism $\gamma_{f,h}$: $\mathbf{f}^!\mathbf{h}^* \to (\mathbf{hf})^*$ satisfying the following compatibility with flat and proper base changes $Z' \to Z$. For every $Z' \to Z$, f': $X' = Z' \times_Z X \to Z' \times_Z Y = Y'$ is slci (this is so because hf is flat), $\widetilde{\mathbf{CH}}^\cdot(X')$ $\widetilde{\mathbf{CH}}^\cdot(Y')$, $\widetilde{\mathbf{CH}}^\cdot(Z')$ are bifibred over (Z-schemes Z', proper morphisms of c.r.d., flat morphisms), and $\mathbf{f}'^!$, $\mathbf{h}'^*$ and $(\mathbf{h}'\mathbf{f}')^*$ are biadmissible functors between these categories. The condition is that $\gamma_{f',h'}$: $\mathbf{f}'^!\mathbf{h}'^* \to (\mathbf{h}'\mathbf{f}')^*$ is a biadmissible functor-isomorphism.

4.7.2. If we denote objects of $\mathbb{K}_{\mathrm{lci,sp}}$ by (f, X, Y, D, λ) and put $D_X = f^{-1}(D)$, then $\widetilde{\mathbf{CH}}^\cdot(X)$, $\widetilde{\mathbf{CH}}^\cdot(Y)$, $\widetilde{\mathbf{CH}}^\cdot(D)$, and $\widetilde{\mathbf{CH}}^\cdot(D_X)$ are bifibred Picard categories over $\mathbb{K}_{\mathrm{lci,sp}}$. The functors $\mathbf{sp}_\lambda$: $\widetilde{\mathbf{CH}}^\cdot(Y) \to \widetilde{\mathbf{CH}}^\cdot(D)$, $\mathbf{sp}_{f^*(\lambda)}$: $\widetilde{\mathbf{CH}}^\cdot(X) \to \widetilde{\mathbf{CH}}^\cdot(D_X)$, $\mathbf{f}^!$:

$\widetilde{\mathrm{CH}}^\cdot(Y) \to \widetilde{\mathrm{CH}}^\cdot(X)$, and $\mathbf{f}^!_D: \widetilde{\mathrm{CH}}^\cdot(D) \to \widetilde{\mathrm{CH}}^\cdot(D_X)$ are biadmissible (f_D is the restriction of f to $f^{-1}(D)$). The datum we require is a biadmissible functor-isomorphism

$$\delta_{\lambda,f}: \mathbf{f}^!_D \mathbf{sp}_\lambda \to \mathbf{sp}_{f^*(\lambda)} \mathbf{f}^!$$

which satisfies the following properties:

4.7.2.1. If $(f, X, Y, D_i, \lambda_i)$ $(i \in \{1,2\})$ are objects of $K_{\mathrm{lci,sp}}$ such that the immersions $D_{12} = D_1 \cap D_2 \to D_i$ and $D_{X,12} = D_{X,1} \cap D_{X,2} \to D_{X,i}$ $(i \in \{1,2\})$ are regular of codimension one, then the diagram similar to (6) commutes:

$$\begin{tikzcd}
\mathbf{sp}_{\lambda_2}\mathbf{sp}_{\lambda_1}\mathbf{f}^! \arrow[r] \arrow[d] & \mathbf{sp}_{\lambda_2}\mathbf{f}^!\mathbf{sp}_{\lambda_1} \arrow[r] & \mathbf{f}^!\mathbf{sp}_{\lambda_2}\mathbf{sp}_{\lambda_1} \arrow[d] \\
\mathbf{sp}_{\lambda_1}\mathbf{sp}_{\lambda_2}\mathbf{f}^! \arrow[r] & \mathbf{sp}_{\lambda_1}\mathbf{f}^!\mathbf{sp}_{\lambda_2} \arrow[r] & \mathbf{f}^!\mathbf{sp}_{\lambda_1}\mathbf{sp}_{\lambda_2}
\end{tikzcd} \tag{17}$$

(For the sake of simplicity, the various pull-backs of λ_i and restrictions of f have been denoted by the same letters.)

4.7.2.2. Let $(D, Z, \lambda) \in \mathrm{Ob}(K_{\mathrm{sp}})$, $p: X \to Y$ a slci-morphism and $q: Y \to Z$ be a flat morphism such that qp is flat. We denote by D_X and D_Y the pre-images of D in X and Y, by p_D the restriction of p to D_X, and by q_D the restriction of q to D_Y. The condition is that the following diagram (which is similar to (7)) commutes:

$$\begin{tikzcd}
\mathbf{p}^!_D\mathbf{q}^*_D\mathbf{sp}_\lambda \arrow[r] \arrow[d] & \mathbf{p}^!_D\mathbf{sp}_{q^*(\lambda)}\mathbf{q}^* \arrow[r] & \mathbf{sp}_{(qp)^*(\lambda)}\mathbf{p}^!\mathbf{q}^* \arrow[d] \\
(\mathbf{q}_D\mathbf{p}_D)^*\mathbf{sp}_\lambda \arrow[rr] & & \mathbf{sp}_{(qp)^*(\lambda)}(\mathbf{qp})^*
\end{tikzcd}\ . \tag{18}$$

4.7.3. Let us denote objects of $\mathbb{K}_{\mathrm{lci,com}}$ by (f, g, X, Y, Z), such that $\widetilde{\mathrm{CH}}^\cdot(X)$, $\widetilde{\mathrm{CH}}^\cdot(Y)$, $\widetilde{\mathrm{CH}}^\cdot(Z)$ are bifibred Picard categories over $\mathbb{K}_{\mathrm{lci,com}}$, and $\mathbf{f}^!$, $\mathbf{g}^!$, and $(\mathbf{gf})^!$ are biadmissible functors between them. The datum we need is a biadmissible functor-isomorphism

$$\varepsilon_{f,g}: \mathbf{f}^!\mathbf{g}^! \to (\mathbf{gf})^!$$

subject to the following conditions:

4.7.3.1. If (f, g, X, Y, Z) is an object of $\mathbb{K}_{\mathrm{lci,com}}$ and $(D, Z, \lambda) \in \mathrm{Ob}(K_{\mathrm{sp}})$ such that $(g^{-1}(D), Y, g^*(\lambda))$ and $((gf)^{-1}(D), X, (gf)^*(\lambda))$ are objects of K_{sp}. Then the following diagram commutes:

$$\begin{tikzcd}[column sep=small] \mathbf{f}^!\mathbf{g}^!\mathbf{sp}_\lambda \arrow[r] \arrow[d] & \mathbf{f}^!\mathbf{sp}_{g^*(\lambda)}\mathbf{g}_D^! \arrow[r] & \mathbf{sp}_{(gf)^*(\lambda)}\mathbf{f}_D^!\mathbf{g}_D^! \arrow[d] \\ (\mathbf{gf})^!\mathbf{sp}_\lambda \arrow[rr] & & \mathbf{sp}_{(gf)^*(\lambda)}(\mathbf{g}_D\mathbf{f}_D)^! \end{tikzcd} \quad . \tag{19}$$

The restrictions of f and g to $(gf)^{-1}(D)$ and $g^{-1}(D)$ have been denoted by f_D and g_D.

4.7.3.2. The analogue of 3.(6), applied to lci-morphisms f, g, h such that h, hg, and hgf exist and are slci, commutes (of course, $*$ is replaced by $^!$).

4.7.3.3. If we have a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

with f smooth and g slci, then the following diagram is commutative:

$$\begin{tikzcd} \mathbf{g}'^*\mathbf{f}^! \arrow[rr, "\text{base change}"] \arrow[d, "{\gamma_{g',\mathrm{Id}}}^{-1}"] & & \mathbf{f}'^!\mathbf{g}^* \arrow[d, "\mathbf{f}'^!(\gamma_{g,\mathrm{Id}})^{-1}"] \\ \mathbf{g}'^!\mathbf{f}^! \arrow[r, "\varepsilon_{g',f}"] & (\mathbf{fg}')^! = (\mathbf{gf}')^! & \mathbf{f}'^!\mathbf{g}^! \arrow[l, "\varepsilon_{f',g}"'] \end{tikzcd} \quad . \tag{20}$$

By 'base change' we mean the base change isomorphism defined by the coadmissible structure of $\mathbf{f}^!$over $\mathrm{Ver}(\mathbb{K}_{\mathrm{lci}})$.

4.7.4. UNIQUENESS. The data 4.7.1-3, determine a biadmissible functor $\mathbf{f}^!$ over $\mathbb{K}_{\mathrm{lci}}$ up to unique biadmissible functor-isomorphism.

Proof. The proof will be carried out in steps 4.8–4.17.

4.8. *$i^!$ for regular closed immersions.* Let $i: X_0 \to X_1$ be a regular closed immersion. Since the codimension of i is locally constant, it suffices to construct $\mathbf{i}^!$ if i is of constant codimension d. Then $X. = (X_0 \subset X_1)$ is an object of K_d, and we put $\mathbf{i}^! = F_{X.}$ (cf. 4.3). The data 4.7.1 and 4.7.2 are given by the isomorphisms α and β in Proposition 4.3.

To construct $\varepsilon_{i,j}$: $\mathbf{i}^!\mathbf{j}^! \to (\mathbf{ji})^!$, we may assume that i and j are of constant codimensions d_1 and d_2. Then $X. = (X_0 \xrightarrow{i} X_1 \xrightarrow{j} X_2) \in \mathrm{Ob}(K_{d_1,d_2})$, and both $\mathbf{i}^!\mathbf{j}^!$

and $(\mathbf{ji})^!$ are candidates for $F_{X.}$. Hence, Proposition 4.3 implies that there is a unique biadmissible (over K_{d_1,d_2}) isomorphism $\varepsilon_{i,j}\colon \mathbf{i}^!\mathbf{j}^! \to (\mathbf{ji})^!$ which is compatible with the datum 4.7.2, i.e., which satisfies 4.7.3.1 (that such an isomorphism is compatible with the datum 4.7.1 follows from 3.14).

If we denote objects of K_{d_1,d_2,d_3} by $X. = (X_0 \xrightarrow{i} X_1 \xrightarrow{j} X_2 \xrightarrow{k} X_3)$, then $(\mathbf{kji})^!$ and $\mathbf{i}^!\mathbf{j}^!\mathbf{k}^!$ satisfy the conditions for $F_{X.}$, hence a 'good' isomorphism between them is unique. This is 4.7.3.2. in the case of regular closed immersions.

4.9. *The isomorphisms* $\varphi_{j,p}$

Let

$$\begin{tikzcd} & S \arrow[d, "p"] \\ X \arrow[ur, "j"] \arrow[r, "i"'] & Y \end{tikzcd}$$

be a commutative diagram with regular closed immersions i and j and a smooth morphism p. We construct an isomorphism

$$\varphi_{j,p}\colon \mathbf{j}^!\mathbf{p}^* \to \mathbf{i}^! \tag{21}$$

as follows: Let $\tilde{S} = X \times_Y S$, $\tilde{p}\colon \tilde{S} \to X$ be the natural projection, and $\tilde{i}\colon \tilde{S} \to S$ be the base-change of i. The inclusion j determines a section $\tilde{j}$ of $\tilde{p}$:

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{p}"] & S \arrow[d, "p"] \\ X \arrow[u, bend left=60, "\tilde{j}"] \arrow[ur, "j"] \arrow[r, "i"'] & Y \end{tikzcd} \tag{22}$$

Then

$$\mathbf{j}^!\mathbf{p}^* \longrightarrow \tilde{\mathbf{j}}^!\tilde{\mathbf{i}}^!\mathbf{p}^* \longrightarrow \tilde{\mathbf{j}}^!\tilde{\mathbf{p}}^*\mathbf{i}^! \xrightarrow{\gamma_{\tilde{i},\tilde{p}}} \mathbf{i}^! \tag{23}$$

determines $\varphi_{j,p}$.

We need the following compatibilities:

(i) If $k\colon Z \to K$ is a regular immersion, we have

$$\varphi_{jk,p} = \mathbf{k}^!(\varphi_{j,p}) \tag{24}$$

(ii) If we have a commutative diagram

$$\begin{tikzcd} & T \arrow[d, "q"] \\ & S \arrow[d, "p"] \\ X \arrow[uur, "k"] \arrow[ur, "j"'] \arrow[r, "i"'] & Y \end{tikzcd} \tag{25}$$

with regular imbeddings i, j, k and smooth morphisms p, q, then

$$\varphi_{k,pq} = \varphi_{j,p}\varphi_{k,q}. \tag{26}$$

(iii) Compatibility with flat and proper base changes $Y' \to Y$.

(iv) If we have a commutative diagram

$$\begin{tikzcd} & \tilde{S} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{p}"] & S \arrow[d, "p"] \\ X \arrow[ur, "k"] \arrow[r, "j"'] & Y \arrow[r, "i"'] & Z \end{tikzcd} \tag{27}$$

in which the square is Cartesian, i and j are regular immersions, and p is smooth, then the following diagram commutes:

$$\begin{tikzcd} (\tilde{\mathbf{i}}k)^!\mathbf{p}^* \arrow[d, "\varphi_{ik,p}"'] & \mathbf{k}^!\tilde{\mathbf{i}}^!\mathbf{p}^* \arrow[l, "\varepsilon_{k,i}"'] \arrow[r] & \mathbf{k}^!\tilde{\mathbf{p}}^*\mathbf{i}^! \arrow[d, "\varphi_{k,p}"] \\ (\mathbf{ij})^! & & \mathbf{j}^!\mathbf{i}^! \arrow[ll] \end{tikzcd} \tag{28}$$

Since (23) contains only biadmissible transformations, (iii) is clear.

Proof of (i). We need the following

SUBLEMMA 1. *Let $k: Z \to X$ be a regular closed immersion, $\tilde{S} \xrightarrow{\tilde{p}} X$ a smooth morphism, $\bar{S} \xrightarrow{\bar{p}} Z$ the restriction of $\tilde{S}$* to Z, $\tilde{j}: X \to \tilde{S}$ a section of $\tilde{p}$, $\bar{j}$ its restriction to Z:

$$\begin{tikzcd} \bar{S} \arrow[r, "\tilde{k}"] \arrow[d, "\bar{p}"] & \tilde{S} \arrow[d, "\tilde{p}"] \\ Z \arrow[r, "k"'] \arrow[u, bend left=80, "\bar{\mathbf{j}}"] & X \arrow[u, bend right=80, "\tilde{j}"'] \end{tikzcd} \tag{29}$$

Then

$$\begin{tikzcd}
\mathbf{k}^! \arrow[r] \arrow[d] & \bar{\mathbf{j}}^!\bar{\mathbf{p}}^*\mathbf{k}^! \arrow[d] \\
\mathbf{k}^!\tilde{\mathbf{j}}^!\tilde{\mathbf{p}}^* \arrow[r] & \bar{\mathbf{j}}^!\tilde{\mathbf{k}}^!\tilde{\mathbf{p}}^*
\end{tikzcd} \tag{30}$$

commutes.

Proof of Sublemma 1. For $A \in \mathrm{Ob}(\mathbf{CH}^i(X))$, let $\delta_{X,A}$ be the unique automorphism of $\mathbf{k}^!A$ making (30) commutative. We have to show that $\delta_{X,A} = 0$.

First we assume $k\colon Z \to X$ is the zero-section of a vector bundle. Then $\mathbf{k}^!$ is an equivalence of categories, hence $\delta_{X,A} = \mathbf{k}^!(\delta'_{X,A})$ for a unique automorphism $\delta'_{X,A}$ of A. Since (30) contains only biadmissible transformations, $(\delta'_{X,A})|_U = \delta'_{U,A}|_U$ for every open $U \subset X$. By 3.14 (applied to $f = \mathrm{Id}$), this implies $\delta_{X,A} = 0$.

Now we return to the general situation. Let $\mathrm{M}_0 \to M_1$ be the deformation of $Z \to X$ to the normal bundle. We use the notations q_n and p_∞ as in (9) and (10), and also the isomorphism (14). By (14), we have $\delta_{X,A} = \mathbf{sp}_f\mathbf{q}_0^*(\delta_{X,A})$ as elements of $G_i(Z)$. Since the transformations in (30) are compatible with flat base change, $\mathbf{sp}_f\mathbf{q}_0^*(\delta_{X,A}) = \mathbf{sp}_f(\delta_{M_1^{(a)},\mathbf{q}_{1,a}^*A})$. Of course, $\delta_{M^{(a)},B}$ and $\delta_{M^{(\infty)},B}$ are defined by means of the pull-backs of $\tilde{S}$ to $M_1^{(a)}$ and $M_1^{(\infty)}$. Since the transformations in (30) are compatible with specialization, we get

$$\delta_{X,A} = \mathbf{sp}_f(\delta_{M_1^{(a)},\mathbf{q}_{1,a}^*\mathrm{A}}) = \delta_{M_1^{(\infty)},\mathbf{sp}_f\mathbf{q}_{1,a}^*A} = 0$$

by our previous considerations, and the proof of the sublemma is complete.

We return to the proof of (i). The composite diagram of (22) and (29) is

$$\begin{tikzcd}
\bar{S} \arrow[r, "\tilde{k}"] \arrow[d, "\bar{p}"] & \tilde{S} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{p}"'] & S \arrow[d, "p"] \\
Z \arrow[r, "k"'] \arrow[u, bend left=60, "\bar{j}"] & X \arrow[r, "i"'] \arrow[u, bend right=60, "\tilde{j}"'] \arrow[ur, "j"] & Y
\end{tikzcd} .$$

We have a diagram

$$\begin{tikzcd}
\mathbf{k}^!\mathbf{j}^!\mathbf{p}^* \arrow[r, "(a)"] \arrow[dr, "(a')"'] & \bar{\mathbf{j}}^!\tilde{\mathbf{k}}^!\tilde{\mathbf{i}}^!\mathbf{p}^* \arrow[r, "(b)"] \arrow[d] \arrow[dr, phantom, "\mathrm{NT}"] & \bar{\mathbf{j}}^!\tilde{\mathbf{k}}^!\tilde{\mathbf{p}}^*\mathbf{i}^! \arrow[r, "(c)"] \arrow[d] & \bar{\mathbf{j}}^!\bar{\mathbf{p}}^*\mathbf{k}^!\mathbf{i}^! \arrow[d, "(d)"] \\
& \mathbf{k}^!\tilde{\mathbf{j}}^!\tilde{\mathbf{i}}^!\mathbf{p}^* \arrow[r, "(b')"] & \mathbf{k}^!\tilde{\mathbf{j}}^!\tilde{\mathbf{p}}^*\mathbf{i}^! \arrow[r, "(c')"] & \mathbf{k}^!\mathbf{i}^!
\end{tikzcd} .$$

The commutativity of the triangle on the left is 4.7.3.2 for closed immersions, and the square on the right side is (30). It is clear that the composition of (a'), (b'), and (c') is $\mathbf{k}^!(\varphi_{j,p})$. Since $\bar{S} = Z \times_Y S$, the composition (d)(c)(b)(a) is $\varphi_{jk,p}$. The proof of (i) is complete.

Proof of (ii). We consider the diagram

$$\begin{tikzcd}
T \arrow[r, "\bar{i}"] \arrow[d, "\tilde{q}"'] & T \arrow[d, "q"] \\
S \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{p}"'] & S \arrow[d, "p"] \\
X \arrow[r, "i"'] \arrow[uu, bend left=60, "\tilde{k}"] \arrow[u, bend left, "\tilde{j}"] \arrow[ur, "j"] \arrow[uur] & Y
\end{tikzcd}$$

in which the squares are Cartesian. Our first aim is to show that the diagram

$$\begin{tikzcd}
\tilde{\mathbf{k}}^!(\tilde{\mathbf{p}}\tilde{\mathbf{q}})^* \arrow[r, "\varphi_{\tilde{k},\tilde{p}\tilde{q}}"] \arrow[d] & \mathrm{Id} \arrow[d, "\varphi_{\tilde{j},\tilde{p}}"] \\
\mathbf{k}^!\mathbf{q}^*\mathbf{p}^* \arrow[r, "\varphi_{\tilde{k},\tilde{q}}"] & \tilde{\mathbf{j}}^!\tilde{\mathbf{p}}^*
\end{tikzcd} \tag{31}$$

commutes. Since (31) contains only isomorphisms which are compatible with restriction to open subsets of X, this follows from 3.14.

Now we consider the diagram

$$\begin{tikzcd}
 & \tilde{\mathbf{k}}^!(\tilde{\mathbf{p}}\tilde{\mathbf{q}})^*\mathbf{i}^! \arrow[r, "(b)"] \arrow[dl] \arrow[dd, "\varphi_{\tilde{k},\tilde{q}}"] \arrow[ddr, phantom, "\mathrm{NT}"] & \tilde{\mathbf{k}}^!\tilde{\mathbf{q}}^*\tilde{\mathbf{i}}^!\mathbf{p}^* \arrow[r, "(c)"] \arrow[dd, "\varphi_{\tilde{k},\tilde{q}}"] \arrow[ddr, phantom, "\mathrm{(B)}"] & \tilde{\mathbf{k}}^!\bar{\mathbf{i}}^!\mathbf{q}^*\mathbf{p}^* \arrow[dr] & \\
\mathbf{i}^! \arrow[r, phantom, "\mathrm{(A)}"] & & & & \mathbf{k}^!\mathbf{q}^*\mathbf{p}^* \\
 & \tilde{\mathbf{j}}^!\tilde{\mathbf{p}}^*\mathbf{i}^! \arrow[r, "(b')"'] \arrow[ul] & \tilde{\mathbf{j}}^!\tilde{\mathbf{i}}^!\mathbf{p}^* \arrow[r, "(c')"'] & \mathbf{j}^!\mathbf{p}^* \arrow[ur] &
\end{tikzcd}$$

It is easy to check that $(a')(b')^{-1}(c')$ is $\varphi_{j,p}$ and that $(a)(b)^{-1}(c)^{-1}(d)^{-1}$ is $\varphi_{k,pq}$, such that (26) follows. It remains to prove that (32) commutes. For (A), this is an application of (31). For (B), form the fibre square

$$\begin{tikzcd}
\bar{T} \arrow[r, "\bar{j}"] \arrow[d, "\bar{q}"'] & \tilde{T} \arrow[d, "\tilde{q},"] \\
X \arrow[r, "\tilde{j}"'] \arrow[u, bend left=60, "\bar{k}"] \arrow[ur, "\tilde{k}"] & S
\end{tikzcd}$$

where $\bar{T} = X \times_S T = X \times_{\tilde{S}} \tilde{T}$ (with $X \to S$ and $X \to \tilde{S}$ given by j and $\tilde{j}$), and consider the isomorphisms

$$\mathbf{k}^!\mathbf{q}^* \longrightarrow \tilde{\mathbf{k}}^!\bar{\mathbf{i}}^!\mathbf{q}^* \longrightarrow \tilde{\mathbf{k}}^!\tilde{\mathbf{q}}^*\tilde{\mathbf{i}}^! \xrightarrow{(a)} \bar{\mathbf{k}}^!\bar{\mathbf{j}}^!\tilde{\mathbf{q}}^*\tilde{\mathbf{i}}^!$$

$$\xrightarrow{(b)} \bar{\mathbf{k}}^!\bar{\mathbf{q}}^*\tilde{\mathbf{j}}^!\tilde{\mathbf{i}}^! \xrightarrow{(c)} \tilde{\mathbf{j}}^!\tilde{\mathbf{i}}^! \longrightarrow \mathbf{j}^!$$

Their composition is $\varphi_{k,q}$, and the composition $(c)(b)(a)$ is $\varphi_{\tilde{k},\tilde{q}}$, which proves the commutativity of (B).

Proof of (iv). With the same notations as in (27), put $S' = S \times_Z X \xrightarrow{p'} X$ and consider the diagram

$$\begin{tikzcd}
S' \arrow[r, "\tilde{j}"] \arrow[d, "p'"] & \tilde{S} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{p}"] & S \arrow[d, "p"] \\
X \arrow[r, "j"'] \arrow[u, bend left=80, "\tilde{k}"] \arrow[ur] & Y \arrow[r, "i"'] & Z
\end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd}
(\mathbf{ik})^!\mathbf{p}^* \arrow[r] \arrow[d] \arrow[dr, phantom, "(A)"] & \mathbf{k}^!\mathbf{i}^!\mathbf{p}^* \arrow[r] \arrow[d] \arrow[dr, phantom, "NT"] & \mathbf{k}^!\mathbf{p}^*\mathbf{i}^! \arrow[d] \\
\mathbf{k}^!(\mathbf{ij})^!\mathbf{p}^* \arrow[r] \arrow[d] & \mathbf{k}^!\mathbf{j}^!\mathbf{i}^!\mathbf{p}^* \arrow[r] \arrow[d, phantom, "(B)"] & \mathbf{k}^!\mathbf{j}^!\mathbf{p}^*\mathbf{i}^! \arrow[d] \\
\mathbf{k}^!\mathbf{p}'^*(\mathbf{ij})^! \arrow[rr] \arrow[d] \arrow[drr, phantom, "NT"] & & \mathbf{k}^!\mathbf{p}'^*\mathbf{j}^!\mathbf{i}^! \arrow[d] \\
(\mathbf{ij})^! \arrow[rr] & & \mathbf{j}^!\mathbf{i}^!
\end{tikzcd}
.$$

(A) is 4.7.3.2., applied to the special case of regular immersions in which it has already been proved. (B) is the fact that $\varepsilon_{i,j}$ (cf. 4.7.3) is biadmissible at least in the special case of closed immersions. If we take the outer contour of this diagram and delete the vertices of the two middle rows, we get (28).

4.10. Now we are ready to construct $\mathbf{f}^!$. We choose a factorization

$$\delta\colon X \xrightarrow{i} S \xrightarrow{p} Y$$

of f, where p is smooth and i is a closed (and hence regular) immersion. We put

$$\mathbf{f}_\delta^! = \mathbf{i}^!\mathbf{p}^*. \tag{33}$$

Our task is to define the 'change of factorization'-isomorphism. First we assume we are given a new factorization δ' and a smooth morphism $r: S' \to S$ making

$$\begin{tikzcd} & S' \arrow[d, "r"] \arrow[dr, "p'"] & \\ X \arrow[ur, "i'"] \arrow[dr, "i"'] & S \arrow[r, "p"'] & Y \end{tikzcd}$$

commutative. We define $\varphi_{\delta',\delta,r}$ by

$$\mathbf{i}'^!\mathbf{p}'^* \longrightarrow \mathbf{i}'^!\mathbf{r}^*\mathbf{p}^* \xrightarrow{\varphi_{i',r}} \mathbf{i}^!\mathbf{p}^*.$$

The following facts are consequences of (i)–(iv) in 4.9:

(i) If $k: Z \to X$ is a regular immersion and if σ_k is the factorization of fk by $Z \xrightarrow{ik} S \xrightarrow{p} Y$, then

$$\varphi_{\sigma'_k,\sigma_k,r} = \mathbf{k}^!(\varphi_{\sigma',\sigma,r}). \tag{34}$$

(ii) If $\tilde{\sigma}$ is a factorization of f sitting in the top row of the diagram

$$\begin{tikzcd} & \tilde{S} \arrow[d, "q"] \arrow[dr] & \\ X \arrow[ur] \arrow[r, "i'"] \arrow[dr] & S' \arrow[r, "p'"] \arrow[d, "r"] & Y \\ & S \arrow[ur] & \end{tikzcd},$$

then we have

$$\varphi_{\tilde{\sigma},\sigma,rq} = \varphi_{\sigma',\sigma,r}\varphi_{\tilde{\sigma},\sigma',q}. \tag{35}$$

(iii) Compatibility with flat and proper base change $\tilde{Y} \to Y$.

If σ_1 and σ_2 are two factorizations of f, we denote by $\sigma_1 \times \sigma_2$ the factorization

$$X \to S_1 \times_Y S_2 \to Y,$$

by $r_{1,2}\colon S_1 \times_Y S_2 \to S_{1,2}$ the natural projections, and define $\varphi_{\sigma_1,\sigma_2}\colon \mathbf{f}^!_{\sigma_1} \to \mathbf{f}^!_{\sigma_2}$ by the composition of

$$\mathbf{f}^!_{\sigma_1} \xrightarrow{(\varphi_{\sigma_1\times\sigma_2,\sigma_1,r_1})^{-1}} \mathbf{f}^!_{\sigma_1\times\sigma_2} \xrightarrow{\varphi_{\sigma_1\times\sigma_2,\sigma_2,r_2}} \mathbf{f}^!_{\sigma_2}. \tag{36}$$

The identity $\varphi_{\sigma_1,\sigma_2}\varphi_{\sigma_2,\sigma_3} = \varphi_{\sigma_1,\sigma_3}$ follows from the diagram

$$\begin{tikzcd}
 & & \mathbf{f}^!_{\sigma} & & \\
 & \mathbf{f}^!_{\sigma_1\times\sigma_2} \arrow[ur] & & \mathbf{f}^!_{\sigma_1\times\sigma_3} \arrow[ul] & \\
 & & \mathbf{f}^!_{\sigma_1\times\sigma_2\times\sigma_3} \arrow[uu] \arrow[ul] \arrow[ur] \arrow[d] & & \\
\mathbf{f}^!_{\sigma_2} \arrow[uur] \arrow[urr] & & \mathbf{f}^!_{\sigma_2\times\sigma_3} \arrow[ll] \arrow[rr] & & \mathbf{f}^!_{\sigma_3} \arrow[uul] \arrow[ull]
\end{tikzcd}$$

in which each arrow is in an obvious manner of the form $\varphi_{\sigma',\sigma,r}$. The commutativity of the small triangles follows from (35).

The isomorphisms (36) enable us to identify the functors $\mathbf{f}^!_\sigma$ with a unique functor $\mathbf{f}^!$ By fact (iii) above, this functor is biadmissible over $\mathbb{K}_{\mathrm{lci}}$.

The construction of the data 4.7.2. and 4.7.1 is easy. If $q\colon Y \to Z$ is flat and if qf is also flat, then for every factorization σ as in (33) we have an isomorphism $\mathbf{f}^!_\sigma \mathbf{q}^* \to \mathbf{i}^!(\mathbf{qp})^* \to (\mathbf{qf})^*$. It is easy to see that the isomorphism $\mathbf{f}^!\mathbf{q}^* \to (\mathbf{qf})^*$ defined this way is biadmissible in the sense explained in 4.7.1. By 3.14, such an isomorphism is unique, hence it is independent of the choice of the factorization.

Let $(f, X, Y, D, \lambda) \in \mathrm{Ob}(K_{\mathrm{lci,sp}})$. We choose a factorization $X \xrightarrow{i} S \xrightarrow{p} Y$ of f, put $D_X = f^{-1}(D)$, $D_S = p^{-1}(D)$, and denote by i_D and p_D the restrictions of i to D_X and p to D_S. By our assumptions to objects of $K_{\mathrm{lci,sp}}$, the restriction f_D of f to D_X is lci, and $f_D = p_D i_D$ is a factorization of f_D into a smooth morphism and a regular immersion. We define

$$\begin{aligned} \delta_{\lambda,f}\colon \mathbf{f}^!_D \mathbf{sp}_\lambda \to \mathbf{sp}_{f^*(\lambda)}\mathbf{f}^! \text{ by } & \mathbf{i}^!_D \mathbf{p}^*_D \mathbf{sp}_\lambda \\ & \to \mathbf{i}^!_D \mathbf{sp}_{p^*(\lambda)} \mathbf{p}^* \to \mathbf{sp}_{f^*(\lambda)} \mathbf{i}^! \mathbf{p}^*. \end{aligned}$$

Since our definition of the transformations $\varphi_{j,r}$, $\varphi_{\sigma,\sigma',r'}$, and $\varphi_{\sigma_1,\sigma_2}$ contains only isomorphisms compatible with specialization, $\delta_{\lambda,f}$ is independent of the factorization. The verification of 4.7.2.1 and 4.7.2.2 is easy.

4.11. Let $(f, g, X, Y, Z) \in \mathrm{Ob}(K_{\mathrm{lci,com}})$. For two Z-schemes A, B we denote $A \times_Z B \to B$ by $A_B \to B$. By our assumption, there exists a closed immersion k_0 of X into a smooth Z-scheme A. We have an induced immersion $k\colon X \to A_Y$ and a factorization $X \xrightarrow{k} A_Y \xrightarrow{\pi} Y$ of f, which we denote by α. We choose a factorization $\sigma\colon Y \xrightarrow{l} S \xrightarrow{p} Z$ of g and consider the diagram

$$
\begin{tikzcd}
 & & A_S \arrow[dr, "\pi'"] & & \\
 & A_Y \arrow[ur, "l"] \arrow[dr, "\pi"] & & S \arrow[dr, "p"] & \\
X \arrow[ur, "k"] \arrow[rr, "f"'] & & Y \arrow[ur, "l"] \arrow[rr, "g"'] & & Z
\end{tikzcd}
\tag{37}
$$

Let $\alpha * \sigma$ be the factorization of gf over A_S. We define

$$
\varepsilon_{f,g,\alpha,\sigma}\colon \mathbf{f}^!_\alpha \mathbf{g}^!_\sigma = \mathbf{k}^!\boldsymbol{\pi}^*\mathbf{l}^!\mathbf{p}^* \to \mathbf{k}^!\mathbf{l}'^!\boldsymbol{\pi}'^*\mathbf{p}^* \to (\mathbf{l}'\mathbf{k})^!(\mathbf{p}\boldsymbol{\pi}')^* = (\mathbf{gf})^!_{\alpha*\sigma} \tag{38}
$$

Let i_0 be another closed immersion of X into a smooth Z-scheme B, with induced factorization $\beta\colon X \xrightarrow{i} B_Y \xrightarrow{p} Y$ of f, and let $\tau\colon Y \xrightarrow{j} T \xrightarrow{q} Z$ be another factorization of g. Consider the morphisms

$$
\begin{tikzcd}
A_Y \arrow[rr, "m'"] \arrow[dr, "l'"] \arrow[ddd, "\pi"'] & & A_{S\times T} \arrow[dl, "\bar{q}"'] \arrow[ddd, "\pi''"] \\
 & A_S \arrow[d, "\pi'"] & \\
 & S & \\
Y \arrow[ur] \arrow[rr, "m"'] & & S\times T \arrow[ul, "\tilde{q}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A_{S\times T} \arrow[d, "\bar{q}"] \\
X \arrow[r, "k"] & A_Y \arrow[ur, "m'"] \arrow[r, "l'"'] & A_S
\end{tikzcd}
\;. \tag{39}
$$

We get a commutative diagram

$$
\begin{tikzcd}
\mathbf{k}^!\boldsymbol{\pi}^*\mathbf{m}^!\tilde{\mathbf{q}}^* \arrow[r] \arrow[d, "k^!\pi^*(\psi_{m,q})"'] \arrow[dr, phantom, "(A)"] & \mathbf{k}^!\mathbf{m}'^!\bar{\mathbf{q}}^*\boldsymbol{\pi}'^* \arrow[r] \arrow[d] \arrow[dr, phantom, "(B)"] & (\mathbf{m}'\mathbf{k})^!\bar{\mathbf{q}}^*\boldsymbol{\pi}'^* \arrow[d] \\
\mathbf{k}^!\boldsymbol{\pi}^*\mathbf{l}^! \arrow[r] & \mathbf{k}^!\mathbf{l}'^!\boldsymbol{\pi}'^* \arrow[r] & (\mathbf{l}'\mathbf{k})^!\boldsymbol{\pi}'^*
\end{tikzcd}
\;. \tag{40}
$$

(A) commutes by 4.9(iii) since in the left diagram (39) the upper triangle is the base change of the lower triangle by the flat map $A_{S\times T} \to S \times Y$, and (B) follows from 4.9(i), applied to the right side of (39).

If we apply (40) to $\mathbf{p}^*A$ for $A \in \mathrm{Ob}(\mathbf{CH}^\bullet(Z))$, we get a diagram whose outer contour is

$$
\begin{tikzcd}
\mathbf{f}^!_\alpha \mathbf{g}^!_{\sigma\times\tau} A \arrow[r] \arrow[d] & (\mathbf{fg})^!_{\alpha*(\sigma\times\tau)} \arrow[d] \\
\mathbf{f}^!_\alpha \mathbf{g}^!_\sigma A \arrow[r] & (\mathbf{fg})^!_{\alpha*\sigma} A
\end{tikzcd}
\;, \tag{41}
$$

where $(\mathbf{fg})^!_{\alpha^*(\sigma\times\tau)} \to (\mathbf{fg})^!_{\alpha^*\sigma}$ is defined by the commutative diagram of factorizations

$$\begin{tikzcd}
 & A_{S\times T} \arrow[d] \arrow[dr] & \\
X \arrow[ur] \arrow[r] & S\times T \arrow[r] & Z
\end{tikzcd} .$$

Let us consider the morphisms ($A\times B$ means $A\times_Z B$)

$$\begin{tikzcd}
 & (A\times B)_Y \arrow[r, "m''"] \arrow[d, "\rho'"] & (A\times B)_{S\times T} \arrow[d, "oA"] & & & \\
X \arrow[ur, "n"] \arrow[r, "k"'] & A_Y \arrow[r, "m'"'] & A_{S\times T} \arrow[r, "\pi''"'] & S\times T \arrow[r, "\tilde{q}"'] & S \arrow[r, "p"'] & Z
\end{tikzcd} .$$

By 4.9(iv), the diagram

$$\begin{tikzcd}
(\mathbf{m}''\mathbf{n})^! \arrow[r] \arrow[d] & \mathbf{n}^!\mathbf{m}''^!\boldsymbol{\rho}''^* \arrow[r] & \mathbf{n}'^!\boldsymbol{\rho}^*\mathbf{m}'^! \arrow[d] \\
(\mathbf{m}'\mathbf{k})^! \arrow[rr] & & \mathbf{k}^!\mathbf{m}^!
\end{tikzcd}$$

commutes. Applying this to objects of the form $\boldsymbol{\pi}''^*\mathbf{q}^*\mathbf{p}^*A$, $A\in \mathrm{Ob}(\mathbf{CH}^{\cdot}(Z))$, we get

$$\begin{tikzcd}
\mathbf{f}^!_{\alpha\times\beta}\mathbf{g}^!_{\sigma\times\tau}A \arrow[r] \arrow[d] & (\mathbf{gf})^!_{(\alpha\times\beta)*(\sigma\times\tau)}A \arrow[d] \\
\mathbf{f}^!_{\alpha}\mathbf{g}^!_{\sigma\times\tau}A \arrow[r] & (\mathbf{gf})^!_{\alpha*(\sigma\times\tau)}A
\end{tikzcd} . \tag{42}$$

Gluing (41) and (42) and using 4.10(ii), we get

$$\begin{tikzcd}
\mathbf{f}^!_{\alpha\times\beta}\mathbf{g}^!_{\sigma\times\tau} \arrow[r] \arrow[d] & (\mathbf{gf})^!_{(\alpha\times\beta)*(\sigma\times\tau)} \arrow[d] \\
\mathbf{f}^!_{\alpha}\mathbf{g}^!_{\sigma} \arrow[r] & (\mathbf{gf})^!_{\alpha*\sigma}
\end{tikzcd} . \tag{43}$$

Applying (43) another time, with the roles of α and σ and of β and τ interchanged, and using the definition (36), we arrive at the commutativity of

$$\begin{tikzcd} \mathbf{f}^!_\alpha \mathbf{g}^!_\sigma \arrow[r] \arrow[d, "\varphi_{\alpha,\beta}\mathbf{f}^!(\varphi_{\sigma,\tau})"] & (\mathbf{gf})^!_{\alpha * \sigma} \arrow[d] \\ \mathbf{f}^!_\beta \mathbf{g}^!_\tau \arrow[r] & (\mathbf{gf})^!_{\beta * \tau} \end{tikzcd} . \tag{44}$$

This proves that the transformations (38) fit together and define $\varepsilon_{f,g}\colon \mathbf{f}^!\mathbf{g}^! \to (\mathbf{gf})^!$.

4.12. We omit the proof of 4.7.3.1 since it is straightforward. To prove 4.7.3.2, we consider lci-morphisms $U \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ such that U, X, and Y admit closed immersions into smooth Z-schemes. We want to prove

$$\begin{tikzcd} \mathbf{f}^!\mathbf{g}^!\mathbf{h}^! \arrow[r] \arrow[d] & \mathbf{f}^!(\mathbf{hg})^! \arrow[d] \\ (\mathbf{gf})^!\mathbf{h}^! \arrow[r] & (\mathbf{hgf})^! \end{tikzcd} . \tag{45}$$

We choose closed Z-immersions of U, X, and Y into smooth Z-schemes A, B, and S. Then we have the following factorizations of f, g, and h:

$$\alpha\colon \quad U \xrightarrow{\ i\ } A_X \xrightarrow{\ p\ } X$$

$$\beta\colon \quad X \xrightarrow{\ j\ } B_Y \xrightarrow{\ q\ } Y$$

$$\sigma\colon \quad Y \xrightarrow{\ k\ } S \xrightarrow{\ r\ } Z.$$

It suffices to prove

$$\begin{tikzcd} \mathbf{f}^!_\alpha \mathbf{g}^!_\beta \mathbf{h}^!_\sigma \arrow[r] \arrow[d] & \mathbf{f}^!_\alpha(\mathbf{hg})^!_{\beta * \sigma} \arrow[d] \\ (\mathbf{gf})^!_{\alpha * \beta}\mathbf{h}^!_\sigma \arrow[r] & (\mathbf{hgf})^!_{\alpha * \beta * \sigma} \end{tikzcd} \tag{46}$$

We consider the morphisms ($A \times B$ means $A \times_Z B$):

$$\begin{tikzcd}[column sep=small] & & & (A \times B)_S & & & \\ & & (A \times B)_Y \arrow[ur, "k''"] \arrow[dr, "p'"] & & B_S \arrow[ul, "p''"'] & & \\ & A_X \arrow[ur, "j'"] \arrow[dr, "p"] & & B_Y \arrow[ur, "k'"] \arrow[dr, "q"] & & S \arrow[ul, "q'"'] \arrow[dr, "r"] & \\ U \arrow[ur, "i"] \arrow[rr, "f"'] & & X \arrow[ur, "j"] \arrow[rr, "g"'] & & Y \arrow[ur, "k"] \arrow[rr, "h"'] & & Z \end{tikzcd} .$$

We have a commutative diagram

$$
\begin{tikzcd}
\mathbf{p}^*\mathbf{j}^!\mathbf{q}^*\mathbf{k}^!C \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{NT}"] & \mathbf{p}^*\mathbf{j}^!\mathbf{k}'^*\mathbf{q}'^!C \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{A})"] & \mathbf{p}^*(\mathbf{k}'\mathbf{j})^!\mathbf{q}'^*C \arrow[dd] \\
\mathbf{j}'^!\mathbf{p}'^*\mathbf{q}^*\mathbf{k}^!C \arrow[r] \arrow[dd] & \mathbf{j}'^!\mathbf{p}'^*\mathbf{k}'^!\mathbf{q}'^*C \arrow[d] & \\
(\mathrm{B}) & \mathbf{j}'^!\mathbf{k}''^!\mathbf{p}''^*\mathbf{q}'^*C \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{NT}"] & (\mathbf{k}''\mathbf{j}')^!\mathbf{p}''^*\mathbf{q}'^*C \arrow[d] \\
\mathbf{j}'^!(\mathbf{q}\mathbf{p}')^*\mathbf{k}^!C \arrow[r] & \mathbf{j}'^!\mathbf{k}''^!(\mathbf{q}'\mathbf{p}'')^*C \arrow[r] & (\mathbf{k}''\mathbf{j}')^!(\mathbf{q}'\mathbf{p}'')^*C
\end{tikzcd}
\tag{47}
$$

The commutativity of (A) belongs to the conditions which were used to characterize the isomorphism $\mathbf{j}^!\mathbf{k}'^! \to (\mathbf{k}'\mathbf{j})^!$ defined in 4.8, and (B) is of type 3(7) (applied to the biadmissible functor $F = \mathbf{k}^!$). If we insert $C = \mathbf{r}^*(.)$ in (46) and apply $\mathbf{i}^!$, we get a diagram whose outer contour can be identified with (46).

Now we prove 4.7.3.3. Since (20) is clear for a smooth morphism g, it suffices to consider the case of a regular closed immersion g. In this case, the proof consists of two parts:

SUBLEMMA 1. *If in $\sigma: X \xrightarrow{i} Y \xrightarrow{p} Z$ i is a regular immersion and p is smooth, then*

$$
\begin{tikzcd}
\mathbf{i}^!\mathbf{p}^* \arrow[d, equal] & \mathbf{i}^!\mathbf{p}^! \arrow[l, "\mathbf{i}^!(\gamma_{p,\mathrm{Id}})"'] \arrow[d, "\varepsilon_{i,p}"] \\
(\mathbf{pi})^!_\sigma \arrow[r] & (\mathbf{pi})^!
\end{tikzcd}
\tag{48}
$$

commutes.

SUBLEMMA 2. *We suppose that in a Cartesian diagram*

$$
\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d] & Y' \arrow[d, "p"] \\
X \arrow[r, "i"] & Y
\end{tikzcd}
$$

p is smooth and i is a regular immersion. Then $\sigma\colon X' \xrightarrow{i'} Y' \xrightarrow{p} Y$ *is an admissible factorization of the lci-morphism* ip'. *With these notations, the diagram*

$$\begin{tikzcd}
\mathbf{p}'^*\mathbf{i}^! \arrow[rr, "\text{base change}"] & & \mathbf{i}'^!\mathbf{p}^* \arrow[d, equal] \\
\mathbf{p}'^!\mathbf{i}^! \arrow[u, "\gamma_{p',\mathrm{Id}}"] \arrow[r, "\varepsilon_{p',i}"'] & (\mathbf{ip}')^! & (\mathbf{ip}')^!_\sigma \arrow[l]
\end{tikzcd} \tag{49}$$

commutes.

It is clear that (20) *for a regular immersion g follows from* (48) *and* (49).

Proof of Sublemma 1. In 4.11, we choose for k_0 the immersion of X into the smooth Z-scheme Y, and put $S = Y$. Then (37) becomes

$$\begin{tikzcd}
 & & Y\times Y \arrow[dr, "p_1"] & & \\
 & Y\times Y \arrow[ur, equal] \arrow[dr, "p_1"] & & Y \arrow[dr] & \\
X \arrow[ur, "\tilde{i}"] \arrow[rr, "i"'] & & Y \arrow[ur, equal] \arrow[rr, "p"'] & & Z
\end{tikzcd}$$

($Y\times Y = Y\times_Z Y$, and $p_1 =$projection to the first factor).

Hence $\varepsilon_{i,p}$ is

$$\begin{tikzcd}
\mathbf{i}^!\mathbf{p}^* \arrow[r] \arrow[d, equal] & \mathbf{i}^!\mathbf{p}_1^*\mathbf{p}^* \arrow[r] & \mathbf{i}^!(\mathbf{pp}_1)^* = (\mathbf{pi})^!_{\sigma\times\sigma} \arrow[r] & (\mathbf{pi})^! \\
(\mathbf{pi})^!_\sigma & & &
\end{tikzcd} \tag{50}$$

By the definition made at the beginning of 4.10, the isomorphism $(\mathbf{pi})^!_\sigma \to (\mathbf{pi})^!_{\sigma\times\sigma}$ in (50) is $\varphi^{-1}_{\sigma\times\sigma,\sigma,p_1}$. So it remains to prove $\varphi_{\sigma\times\sigma,\sigma,p_1} = \varphi_{\sigma\times\sigma,\sigma}$. By (4.10(ii), the following diagram commutes:

$$\begin{tikzcd}
(\mathbf{pi})^!_\sigma \arrow[d, "\varphi_{\sigma\times\sigma,\sigma,p_1}"] & (\mathbf{pi})^!_{\sigma\times\sigma\times\sigma} \arrow[l, "\varphi_{\sigma\times\sigma\times\sigma,\sigma_1,p_1}"'] \arrow[d, "\varphi_{\sigma\times\sigma\times\sigma,\sigma\times\sigma\times\sigma,s_{12}}"] \arrow[dr, "\varphi_{\sigma\times\sigma\times\sigma,\sigma,p_3}"] & \\
(\mathbf{pi})^!_{\sigma\times\sigma} & (\mathbf{pi})^!_{\sigma\times\sigma\times\sigma} \arrow[l, "\varphi_{\sigma\times\sigma\times\sigma_1\sigma\times\sigma_1 p_{23}}"] \arrow[r, "\varphi_{\sigma\times\sigma\times\sigma,\sigma,p_3}"'] & (\mathbf{pi})^!_\sigma,
\end{tikzcd}$$

where $p_{23}\colon Y\times Y\times Y \to Y\times Y$ is projection to the last two factors and $s_{12}\colon Y\times Y\times Y \to Y\times Y\times Y$ interchanges the first two factors. Since the triangle on the right side commutes, the right vertical arrow is the identity. Using this and definition (36), we see that the commutativity of the square implies the desired equality $\varphi_{\sigma\times\sigma,\sigma,p_1} = \varphi_{\sigma\times\sigma,\sigma}$.

Proof of Sublemma 2. By a special choice of factorizations in the application of 4.11 to the composition $X' \xrightarrow{p'} X \xrightarrow{i} Y$ (namely, $A = Y'$ and $S = Y$), (37) becomes

$$\begin{tikzcd}
 & & Y' = Y' \times_Y Y \arrow[dr, "p"] & & \\
 & X \times_Y Y' = X' \arrow[ur, "i'"] \arrow[dr, "p'"] & & Y \arrow[dr] & \\
X' \arrow[ur, equal] \arrow[rr, "p'"] & & X \arrow[ur, "i"] \arrow[rr, "i"] & & Y,
\end{tikzcd}$$

and (49) follows from definition (38).

4.13. Our previous considerations in 4.8–4.12 prove the existence of functors $\mathbf{f}^!$ with the properties required in 4.7.1–4.7.3. It remains to prove the uniqueness assertion 4.7.4. Let $\mathbf{f}^?$ be another collection of functors satisfying the same conditions. We proceed in several steps:

4.13.1. Let i be a regular closed immersion of codimension of codimension d, defining an object $X_0 \xrightarrow{i} X_1$ of K_d; we denote this object by $X_.$. Then both $\mathbf{i}^!$ and $\mathbf{i}^?$ satisfy the conditions for $F_{X.}$ in 4.3. By the uniqueness result in Proposition 4.3, there is a unique biadmissible functor-isomorphism $\mathbf{i}^! \to \mathbf{i}^?$ which respects the datum 4.7.2. By 3.14, this isomorphism automatically respects 4.7.1. Since it was mentioned in 4.8 that the composition law $\mathbf{i}^!\mathbf{j}^! \to (\mathbf{ij})^!$ is the unique one which is biadmissible and respects the datum 4.7.2, it follows that our isomorphism $\mathbf{i}^! \to \mathbf{i}^?$ respects the composition law 4.7.3. Since the codimension of a regular immersion is locally constant, we get the isomorphism $\mathbf{i}^! \to \mathbf{i}^?$ for any regular closed immersion i. If in the next considerations an isomorphism $\mathbf{i}^! \to \mathbf{i}^?$ is used without any comment, it is supposed to be the isomorphism constructed here.

4.13.2. Let

$$\begin{tikzcd}
 & S \arrow[d, "p"] \\
X \arrow[ur] \arrow[r, "i"'] & Y
\end{tikzcd}$$

be a commutative diagram in which p is smooth and i is a regular closed immersion. We want to prove that the transformation

$$\mathbf{j}^!\mathbf{p}^* \to \mathbf{j}^?\mathbf{p}^* \xrightarrow{\mathbf{j}^?(\gamma_{p,\mathrm{Id}_X})^{-1}} \mathbf{j}^?\mathbf{p}^? \xrightarrow{\varepsilon_{j,p}} (\mathbf{pj})^? = \mathbf{i}^? \to \mathbf{i}^! \tag{51}$$

must be (21). Since confusions are impossible, the transformations 4.7.1–3 for $\mathbf{f}^!$ and $\mathbf{f}^?$ are denoted by the same letters γ, δ, ε. In the following diagram

$$\begin{tikzcd}
\mathbf{j}^?\mathbf{p}^* \arrow[d, "\gamma_{p,\mathrm{Id}}"'] \arrow[r, phantom, "\mathrm{NT}"] & \tilde{\mathbf{j}}^?\tilde{\mathbf{i}}^?\mathbf{p}^* \arrow[l, "\varepsilon_{\tilde{j},\tilde{i}}"'] \arrow[d] \arrow[r, phantom, "\text{(A)}"] & \tilde{\mathbf{j}}^?\tilde{\mathbf{p}}^*\mathbf{i}^? \arrow[l, "(b)"'] \arrow[d] \arrow[dd, bend left, "\gamma_{\tilde{j},\tilde{p}}"] \\
\mathbf{j}^?\mathbf{p}^? \arrow[drr, "\varepsilon_{j,p}"'] \arrow[r, phantom, "\text{(C)}"] & \tilde{\mathbf{j}}^?\tilde{\mathbf{i}}^?\mathbf{p}^? \arrow[l] \arrow[dr] & \tilde{\mathbf{j}}^?\tilde{\mathbf{p}}^?\mathbf{i}^? \arrow[d, "\varepsilon_{\tilde{j},\tilde{p}}"'] \arrow[d, phantom, shift left=4, "\text{(B)}"] \\
& & \mathbf{i}^?
\end{tikzcd}$$

All notations are the same as in (22)!

The commutativity of the pentagon (A) is (20), commutativity of (B) follows from 3.14, and the commutativity of (C) follows from 4.7.3.2. If we identify $\mathbf{k}^?$ with $\mathbf{k}^!$ by 4.13.1, $\varepsilon_{j,p}\gamma_{\tilde{j},\mathrm{Id}}^{-1}$ becomes (51) and $\gamma_{\tilde{j},p}(b)\varepsilon_{\tilde{j},\tilde{i}}$ is (21), which proves our claim.

4.13.3. Let $f\colon X \to Y$ be a slci-morphism with a factorization $\sigma\colon X \xrightarrow{i} Z \xrightarrow{p} Y$ into a smooth projection and a regular immersion. We define $\Gamma_\sigma\colon \mathbf{f}^? \to \mathbf{f}^!$ by

$$\mathbf{f}^? \xrightarrow{\varepsilon_{i,p}^{-1}} \mathbf{i}^?\mathbf{p}^? \xrightarrow{\mathbf{i}^?(\gamma_{p,\mathrm{Id}})} \mathbf{i}^?\mathbf{p}^* \to \mathbf{i}^!\mathbf{p}^* = \mathbf{f}^!. \tag{52}$$

By the result of 4.13.2 and the definition of $\varphi_{\tilde{\sigma}\Gamma\sigma,r}$ in 4.10, if $\tilde{\sigma}\colon X \xrightarrow{\tilde{i}} \tilde{Z} \xrightarrow{\tilde{p}} Y$ is another factorization of f and $r\colon \tilde{Z} \to Z$ is smooth such that $r\tilde{i} = i$ and $pr = \tilde{p}$, then

$$\begin{tikzcd}
& \mathbf{f}^!_{\tilde{\sigma}} \arrow[d, "\varphi_{\tilde{\sigma},\sigma,r}"] \\
\mathbf{f}^? \arrow[ur, "\Gamma_\sigma"] \arrow[r, "\Gamma_\sigma"'] & \mathbf{f}^!_\sigma
\end{tikzcd}$$

commutes. By definition (36), this implies $\varphi_{\sigma_1,\sigma_2}\Gamma_{\sigma_1} = \Gamma_{\sigma_2}$ for any two factorizations σ_1 and σ_2 of f. Consequently, (52) defines an unique isomorphism $\mathbf{f}^? \to \mathbf{f}^!$. This isomorphism is biadmissible over $\mathbb{K}_{\mathrm{lci}}$ since its definition contains only biadmissible transformations. In the case of a closed immersion we obtain the isomorphism constructed in 4.13.1. In paragraph 4.13.4, any isomorphism $\mathbf{f}^? \to \mathbf{f}^!$ will be the isomorphism constructed here.

4.13.4. It remains to prove that the isomorphism $\mathbf{f}^? \to \mathbf{f}^!$ is compatible with the isomorphisms 4.7.1–4.7.3. For 4.7.1, this follows from 3.14. To prove compatibility with the isomorphisms $\varepsilon_{p,q}$, i.e., with 4.7.3, it suffices by 4.7.3.2 to consider the following four cases:

(α) p and q are smooth. Then the assertion follows from 3.14.

(β) p and q are regular immersions. Then the compatibility follows from the considerations in 4.13.1.

(γ) $p = i$ is a regular closed immersion, and q is smooth. Then the assertion is a trivial consequence of definition (52).

(σ) $q = i$: $Y \to Z$ is a regular closed immersion, $A \xrightarrow{p'} Z$ is smooth, $X = A_Y = A \times_Z Y$, and $p: X \to Y$ is the natural projection:

$$
\begin{tikzcd}
A_Y = X \arrow[r, "i'"'] \arrow[d, "p"'] & A \arrow[d, "p'"] \\
Y \arrow[r, "i"'] & Z
\end{tikzcd}
$$

Case (δ) is a consequence of the following diagram:

$$
\begin{tikzcd}
\mathbf{p^*i^!} \arrow[rrrrr, "\text{flat base change}"] \arrow[dddd, "\text{NT}"] & & & & & \mathbf{i'^!p'^*} \arrow[dddd, "\text{NT}"'] \\
& \mathbf{p^!i^!} \arrow[ul, "\gamma_{p',\mathrm{Id}}"'] \arrow[r, "\varepsilon_{p,i}"] \arrow[d] & \mathbf{(ip)^! = (p'i')^!} \arrow[dd] \arrow[u, phantom, "\text{(A)}"] & \mathbf{i'^!p'^!} \arrow[l, "\varepsilon_{i',p'}"'] \arrow[urr, "\gamma_{p',\mathrm{Id}}"'] \arrow[dr] & & \\
& \mathbf{p^!i^?} \arrow[r, phantom, "(?)"] \arrow[d] \arrow[ddl, "\gamma_{p,\mathrm{Id}}"'] & \arrow[r, phantom, "\text{(C)}"] & {} & \mathbf{i'^?p'^!} \arrow[dl] \arrow[ddr, "\gamma_{p,\mathrm{Id}}"] & \\
\arrow[r, phantom, "\text{(B)}"] & \mathbf{p^?i^?} \arrow[r, "\varepsilon_{p,i}"] \arrow[dl, "\gamma_{p',\mathrm{Id}}"] & \mathbf{(ip)^? = (p'i')^?} \arrow[d, phantom, "\text{(D)}"] & \mathbf{i'^?p'^?} \arrow[l, "\varepsilon_{i',p'}"'] \arrow[drr, "\gamma_{p,\mathrm{Id}}"'] \arrow[r, phantom, "\text{(E)}"] & {} & \\
\mathbf{p^*i^?} \arrow[rrrrr, "\text{flat base change}"'] & & {} & & & \mathbf{i'^?p'^*}
\end{tikzcd}
$$

The outer contour commutes by the axiomatic description of $\mathbf{i}^! \to \mathbf{i}^?$ in 4.13.1 (biadmissibility). (A) and (D) are (20), (B) and (E) commute by 3.14, applied to p and p'. By case (γ), (C) commutes. It follows that (?) commutes, which is the desired result. The proof of case (δ) is complete.

It remains to prove the compatibility of $\mathbf{f}^! \to \mathbf{f}^?$ with the datum 4.7.2, i.e., with the isomorphisms $\delta_{\lambda,f}$. If f is smooth, this follows from 4.7.2.2. If f is a regular immersion, this is among the conditions characterizing the isomorphism defined in 4.13.1. By 4.7.3.1, the general case follows from these two cases since it has already been proved that our isomorphism commutes with $\varepsilon_{p,q}$.

4.14. *An alternative description of* $\mathbf{i}^! A$

Let $Z \subset X$ be a closed regular submanifold of a regular quasi-projective manifold X over an infinite field F. Since in this case Gersten's conjecture is known to be true, we have a homomorphism i^*: $E_2^{p,q}(X) \to E_2^{p,q}(Z)$ (cf. 1.8.). For $A \in \mathrm{Ob}(\mathbf{CH}^k(X)) = \mathrm{Ob}(\widetilde{\mathbf{CH}}^k(X))$ and $U \subset Z_{(k)}$ open we put

$$(\mathbf{i}^{!,0}\Lambda)(U) = \{(g, V, a)\}/\sim, \tag{53}$$

where:

— The triple (g, V, a) consists of $g \in G_k(U)$, an open subset $V \subset X_{\mathrm{Zar}}$ such that $V \cap Z = U$, and $a \in A(V)$.

— Two triples (g, V, a) and (g', V', a') are equivalent if and only if $i^*(a' - a) = g - g'$, where $a - a'$ is defined as an element of $G_k(V \cap V')$.

It is clear that (53) defines a separated presheaf on $Z_{(k)}$. By the moving lemma, every point $z \in Z_{(k)}$ has a neighbourhood U in $Z_{(k)}$ such that $(\mathbf{i}^{!,0}A)(U)$ is not empty. The group $G_k(U)$ acts on $(\mathbf{i}^{!,0}A)(U)$ by γ: $(g, V, a) \to (\gamma + g, V, a)$, and it acts simply transitive if $(\mathbf{i}^{!,0}A)(U)$ is not empty. For $a \in A(V)$, we denote the class of $(0, V, a)$ in (53) by $i^!(a)$.

Let $\mathbf{i}^! A$ be the sheaf associated to the presheaf $\mathbf{i}^{!,0}A$ on $Z_{(k)}$. From the previous remarks it is clear that $\mathbf{i}^! A$ is an object of $\mathbf{CH}^k(Z)$. We want to prove that this new functor $\mathbf{i}^!$ is canonically isomorphic to the old one constructed in Theorem 4.7. A careful examination of the proof of 4.3 shows that in 4.5 and 4.6 we do not leave the category of regular manifolds over a field if the closed immersions we consider belong to this category. Therefore it suffices to equip the functor $\mathbf{i}^!$ defined by (53) with the data 4.7.1–4.7.3:

4.14.1. Let

$$\begin{tikzcd} Z \arrow[rr, "i"] \arrow[dr, "p"'] & & X \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

be a commutative diagram of regular manifolds over F, with a closed immersion i and flat morphisms p and q. If W is open in $Y_{(k)}$, then $U = p^{-1}(W)$ and $V = q^{-1}(W)$ are open in $Z_{(k)}$ and $X_{(k)}$ and satisfy the assumptions of (53). There is a unique isomorphism γ: $\mathbf{p}^*A \to \mathbf{i}^!\mathbf{q}^*A$ with the property that for every $a \in A(W)$ we have $\gamma(p^*(a)) = i^!(q^*(a))$.

4.14.2. Let $i: Z \to X$ be a closed immersion of codimension d between regular manifolds, $D \subset X$ a regular submanifold of codimension one such that $Z \cap D$ is also regular and of codimension one in Z, λ a section of $\mathcal{O}_X$ in some neighbourhood of D which trivializes the sheaf of ideals defining D.

Let $x \in D$ and $A \in \mathrm{Ob}(\mathbf{CH}^k(X - D))$. Since the restriction $CH^k(X) \to CH^k(X - D)$ is surjective, it is possible to extend A to an object of $\mathbf{CH}^k(X)$. Therefore the moving lemma implies the existence of an open $V \subset X - D$ such that $A(V)$ is not empty and $K =$ (closure of $X - D - V$ in X) meets $Z \cap D$ in codimension $\geqslant k$ and $x \notin K$. Let $U = Z - D - (K \cap Z)$, $W = D - (K \cap D)$, $Y = Z \cap D - (K \cap Z \cap D)$. If $a \in A(V)$, then

$$\begin{array}{ll} \mathrm{sp}_\lambda(a) \in (\mathbf{sp}_\lambda A)(W) & i_D^!(\mathrm{sp}_\lambda(a)) \in (\mathbf{i}_D^!\mathbf{sp}_\lambda A)(Y) \\ \mathbf{i}^!(a) \in (\mathbf{i}^{!,0}A)(U) \subset (\mathbf{i}^!A)(U) & \mathrm{sp}_\lambda|_Z(i^!(a)) \in (\mathbf{sp}_\lambda|_Z\mathbf{i}^!A)(Y), \end{array}$$

where i_D is the inclusion $Z \cap D \to D$.

There exists a unique isomorphism δ: $\mathbf{i}_D^!\mathbf{sp}_\lambda A \to \mathbf{sp}_\lambda|_Z\mathbf{i}^!A$ with the property

$$(i_D^!(\mathrm{sp}_\lambda(a))) = \mathrm{sp}_\lambda|_Z(i^!(a)). \tag{54}$$

Indeed, it follows from Proposition 1.8 that the $G_k(Y)$-equivariant isomorphism $\delta: (\mathbf{i}_D^!\mathbf{sp}_\lambda A)(Y) \to (\mathbf{sp}_\lambda|_Z\mathbf{i}^!A)(Y)$ characterized by (54) is independent of $a \in A(V)$. Consequently, definition (54) is correct.

4.14.3. If $j: Y \to Z$ is another embedding satisfying our assumptions, then $\varepsilon_{i,j}: \mathbf{j}^!\mathbf{i}^!A \to (\mathbf{ij})^!A$ is characterized by $j^!i^!(a) \to (ij)^!(a)$.

It is easy to see that the isomorphisms 4.14.1–3 satisfy all assumptions of Theorem 4.7. If we replace the word 'slci-morphism' by 'locally closed embedding of regular manifolds over a field'. Consequently, (53) is an alternative description of the functors $\mathbf{i}^!$ defined in 4.7. We shall use this description in a forthcoming paper when metrics on objects of $\mathbf{CH}^k$ are investigated.

5. Intersection products. Bloch's biextension

Here we restrict our considerations to the case of regular manifolds over a field F. For the sake of simplicity we shall also assume that these manifolds are connected. By working with methods similar to [F, §20.2], we could also deal with the case of smooth schemes over a Dedekind domain. The smoothness assumption is absolutely essential only for 5.3.

5.1. *The cross product*

Let X and Y be manifolds and $p_{1,2}$ the projections of $X \times Y$ to X and Y. The biexact functor

$$\begin{aligned} M_p(X) \times M_q(Y) &\longrightarrow M_{p+q}(X \times Y) \\ (A; B) &\longrightarrow p_1^* A \otimes p_2^* B \end{aligned} \tag{1}$$

defines a product

$$\times \colon E_1^{p,q}(X) \times E_1^{s,t}(Y) \to E_1^{p+q,s+t}(X \times Y) \tag{2}$$

which satisfies

$$d_1(a \times b) = d_1(a) \times b + (-1)^{p+q} a \times d_1(b), \qquad a \in E_1^{p,q}. \tag{3}$$

Because of $E_1^{p+1,-p} = 0$, (3) implies that the following product is well-defined:

$$\begin{aligned} &\times \colon (E_1^{p-1,-p}(X)/d_1(E_1^{p-2,-p}(X)) \times E_1^{q,-q}(Y) \\ &\quad \to E_1^{p+q-1,-p-q}(X \times Y)/\mathrm{im}(d_1). \end{aligned}$$

In our terminology, this is a pairing

$$\times \colon (G_p)_r(X) \times Z^q(Y) \to (G_{p+q})_r(X \times Y). \tag{4}$$

(3) implies:

$$\mathbf{c}(g \times z) = \mathbf{c}(g) \times z. \tag{5}$$

In a similar manner, we get

$$\times \colon Z^p(X) \times (G_q)_r(Y) \to (G_{p+q})_r(X \times Y) \tag{6}$$

with the property

$$\mathbf{c}(z \times g) = z \times \mathbf{c}(g). \tag{7}$$

For $A \in \mathrm{Ob}(\mathbf{CH}^k(X))$, $B \in \mathrm{Ob}(\mathbf{CH}^1(Y))$, we define $A \boxtimes B \in \mathrm{Ob}(\mathbf{CH}^{k+1}(X \times Y))$ as follows:

$$(A \boxtimes B)_r(X \times Y) = \{(g, a, b)\}/\sim, \tag{8}$$

where:

— The entries of the triple are $g \in (G_{k+1})_r(X \times Y)$, $a \in A_r(X)$, $b \in B_r(Y)$.
— $(g, a, b) \sim (g', a', b')$ if and only if

$$\begin{aligned} g' - g &= \mathbf{c}(a) \times (b - b') + (a - a') \times \mathbf{c}(b') \\ &= \mathbf{c}(a') \times (b - b') + (a - a') \times \mathbf{c}(b). \end{aligned} \tag{9}$$

That the two expressions on the right hand side of (9) agree is a consequence of the identity

$$\mathbf{c}(g) \times h = g \times \mathbf{c}(h) \quad \text{in } (G_{k+1})_r(X \times Y) \tag{10}$$

if $g \in (G_k)_r(X)$, $h \in (G_1)_r(Y)$. To prove (10), we choose representatives $\gamma \in E_1^{k-1,-k}(X)$, $\eta \in E_1^{l-1,-l}(Y)$ for g and h. Then a representative for $\mathbf{c}(g) \times h - g \times \mathbf{c}(h)$ is

$$d_1(\gamma) \times \eta - \gamma \times d_1(\eta) = d_1(\gamma \times \eta)$$

by (3). This proves (10) and completes the explanation of (8).

We define $\mathbf{c}$: $(A \boxtimes B)_r(X \times Y) \to Z^{k+1}(X \times Y)$ by

$$\mathbf{c}((g, a, b) \bmod \sim) = \mathbf{c}(g) + \mathbf{c}(a) \times \mathbf{c}(b). \tag{11}$$

By (9), (5), and (7), this is independent of the choice of the representative. Finally we define $(A \boxtimes B)(U)$, $U \subset (X \times Y)_{(k+1)}$ open, by formula 3(4):

$$(A \boxtimes B)(U) = \{a \in (A \boxtimes B)_r(U) | \mathbf{c}(a)|_U = 0\}. \tag{12}$$

For $a \in A_r(X)$, $b \in B_r(Y)$ we denote the equivalence class of $(0, a, b)$ in (8) by $a \times b$. Then $\mathbf{c}(a \times b) = \mathbf{c}(a) \times \mathbf{c}(b)$. If $U \subset X_{(k)}$, $V \subset Y_{(1)}$ are open and $a \in A(U)$, $b \in B(V)$, then $a \times b \in (A \boxtimes B)(X \times Y - (X - U) \times (Y - V))$.

5.2. *Natural transformations involving* $\boxtimes$

We give a mere list of them without going into detail.

5.2.1. Biadditivity. For A, $B \in \mathrm{Ob}(\mathbf{CH}^k(X))$, C, $D \in \mathrm{Ob}(\mathbf{CH}^1(Y))$, we have isomorphisms

$$A \boxtimes (C \oplus D) \to A \boxtimes C \oplus A \boxtimes D \tag{13}$$

$$(A \oplus B) \boxtimes C \to A \boxtimes C \oplus B \boxtimes C \tag{14}$$

which satisfy the additivity condition [DM, 1.8] in each of the two variables and make the diagram

$$\begin{array}{ccc} (A \oplus B) \boxtimes (C \oplus D) & \longrightarrow & A \boxtimes (C \oplus D) \oplus B \boxtimes (C \oplus D) \\ \downarrow & & \downarrow \\ (A \oplus B) \boxtimes C \oplus (A \oplus B) \boxtimes D & \longrightarrow & A \boxtimes C \oplus A \boxtimes D \oplus B \boxtimes C \oplus B \boxtimes D \end{array} \tag{15}$$

commutative. If a, b, c, d are rational sections of A, B, C, D, then the isomorphisms (13) and (14) send $a \times (c \oplus d)$ to $a \times c \oplus a \times d$ and $(a \oplus b) \times c$ to $a \times c \oplus b \times c$.

5.2.2. Symmetry. Let s: $X \times Y \to Y \times X$ be the permutation of the factors. Since s interchanges the two expressions on the right hand side of (9), there is a canonical isomorphism $\mathbf{s}^*(A \boxtimes B) \to B \boxtimes A$ sending $s^*(a \times b)$ to $b \times a$.

5.2.3. Compatibility with pull-back. If f: $Z \to X$ is flat, $A \in \mathrm{Ob}(\mathbf{CH}^k(X))$, $B \in \mathrm{Ob}(\mathbf{CH}^1(Y))$, then there is an isomorphism $(\mathbf{f}^*A) \boxtimes B \to (\mathbf{f} \times \mathbf{Id}_Y)^*(A \boxtimes B)$ which maps $f^*(a) \times b$ to $(f \times \mathrm{Id}_Y)^*(a \times b)$.

Similarly, if i: $Z \to X$ is a closed immersion and if X and Y are quasiprojective, there is an isomorphism $(\mathbf{i}^! A) \boxtimes B \to (\mathbf{i} \times \mathbf{Id}_Y)^!(A \boxtimes B)$ which maps $i^!(a) \times b$ to $(i \times \mathrm{Id}_Y)^!(a \times b)$ if $a \in A_r(X)$ is a rational section whose cycle meets Z properly (cf. 4.14).

Finally, if X, Y, and Z are quasi-projective regular manifolds, we may factor g: $Z \to X$ into $Z \xrightarrow{i} S \xrightarrow{f} Y$, with a closed immersion i and f smooth and quasi-projective. We get an isomorphism

$$(\mathbf{g}^! A) \boxtimes B \to (\mathbf{g} \times \mathbf{Id}_Y)^!(A \boxtimes B)$$

by composing

$$\begin{aligned} \mathbf{g}^! A \boxtimes B &\to \mathbf{i}^! \mathbf{f}^* A \boxtimes B \to (\mathbf{i} \times \mathbf{Id}_Y)^!(\mathbf{f}^* A \boxtimes B) \\ &\to (\mathbf{i} \times \mathbf{Id}_Y)^!(\mathbf{f} \times \mathbf{Id}_Y)^*(A \boxtimes B) \to (\mathbf{g} \times \mathbf{Id}_Y)^!(a \boxtimes B). \end{aligned}$$

It is easy to see that this isomorphism does not depend on the factorization of g.

5.2.4. Compatibility with push-forward. If f: $Z \to X$ is proper, $A \in \mathrm{Ob}(\mathbf{CH}^k(Z))$, $B \in \mathrm{Ob}(\mathbf{CH}^1(Y)$, there is an isomorphism

$$(\mathbf{f}_* A) \boxtimes B \to (\mathbf{f} \times \mathbf{Id}_Y)_*(A \boxtimes B)$$

which maps $f_*(a) \times b$ to $(f \times \mathrm{Id}_Y)_*(a \times b)$.

There are some obvious compatibilities between the isomorphisms 5.2.1–4. We do not list them explicitly because they are not essential for our approach to Deligne's program.

5.3. The intersection product. Let X be a regular manifold, $A \in \mathrm{Ob}(\mathbf{CH}^k(X))$, $B \in \mathrm{Ob}(\mathbf{CH}^1(X))$. By $\Delta: X \to X \times X$ we denote the diagonal. This is a regular closed immersion. We put

$$A \cup B = \Delta^!(A \boxtimes B) \in \mathrm{Ob}(\mathbf{CH}^{k+1}(X)). \tag{16}$$

If a and b are rational sections of A and B, we put $a \cup b = \Delta^!(a \times b)$. This is a rational section of $A \cup B$ if the supports of $\mathbf{c}(a)$ and $\mathbf{c}(b)$ intersect properly. If $a \in A(U)$, $b \in B(V)$, then $a \cup b \in (A \cup B)(U \cup V)$.

The following isomorphisms are derived from 5.2:

5.3.1. Biadditivity. Isomorphisms $A \cup (C \oplus D) \to A \cup C \oplus A \cup D$ and $(A \oplus B) \cup C \to A \cup C \oplus B \cup C$. They are additive in each variable and satisfy the analogue of (15).

5.3.2. Symmetry. An isomorphism $s: A \cup B \oplus B \cup A$ defined by composing

$$A \cup B = \Delta^!(A \boxtimes B) = (\mathbf{s}\Delta)^!(A \boxtimes B) \to \Delta^! \mathbf{s}^*(A \boxtimes B) \to \Delta^!(B \boxtimes A) = B \cup A.$$

The symmetry $s: A \cup A \to A \cup A$ may be different from the identity!

5.3.3. An associativity law and compatibilities with pull-back and push-forward. We do not define them explicitly because we shall not need them later.

5.4. *Example*

Let $p: X \to S$ be a proper morphism of relative dimension one between regular manifolds over a field. To line bundles L, M on X, Deligne associates the line bundle $\langle L, M \rangle$ on S which is Zariski-locally on S generated by sections $\langle l, m \rangle$, where l and m are rational sections of L and M whose divisors do not intersect. The relations

$$\langle gl, m \rangle = g(\mathrm{div}(m))\langle l, m \rangle; \ \langle l, gm \rangle = g(\mathrm{div}(l))\langle l, m \rangle$$

are satisfied. Cf. [D, 6.1].

Since $\mathbf{CH}^1(X)$ is the category of line bundles on X, we can consider the intersection product $L \cup M \in \mathrm{Ob}(\mathbf{CH}^2(X))$. There is a canonical isomorphism

$$\langle L, M \rangle \to \mathbf{p}_*(L \cup M)$$

$$\langle l, m \rangle \to p_*(l \cup m).$$

5.5. *Bloch's biextension*

By the identification $CH^p(X) = H^p(X, \mathcal{K}_p)$ and $C_q(X) = H^{q-1}(X, \mathcal{K}_q)$, there is a product $CH^p(X) \times G_p(X) \to G_{p+q}(X)$. Following the methods of [Gr], it is easy to show that this product coincides up to sign with the product defined by the biadditive functor $\cup$. More precisely, if $g \in G_q(X)$ is viewed as an automorphism of $B \in \mathrm{Ob}(\mathbf{CH}^q(X))$, then for any $A \in \mathrm{Ob}(\mathbf{CH}^p(x))$ the automorphism $\mathrm{Id}_A \cup g$ of $A \cup B$ is given by $[A] \cdot g \in G_{p+q}(X)$, where $[A] \in CH^p(X)$ is the class of A.

If $f: X \to S$ is a proper morphism of relative dimension n between regular manifolds over a field, we put

$$\begin{aligned} CH^p(X/S)^v = \{a \in CH^p(X) |\ & \text{For every Zariski-open } U \subset S \text{ and every} \\ & g \in G_{n+1-p}(f^{-1}(U)),\ f_*(a \cdot g) = 0 \text{ in } G_1(U) \\ & (\text{i.e., } = 1 \text{ in } \mathcal{O}(U)^*)\}. \end{aligned} \tag{17}$$

Let $p + q = n + 1$, $a \in CH^p(X/S)^v$, $b \in CH^q(X/S)^v$. We choose representatives $A \in \mathrm{Ob}(CH^p(X))$, $B \in \mathrm{Ob}(CH^q(X))$ for a and b. We have a line bundle

$$L_{A,B} = f_*(A \cup B) \tag{18}$$

on S. If $g \in G_p(X)$ is any automorphism of A, then $\mathbf{f}_*(g \cup \mathrm{Id}_B) = f_*([\mathrm{B}] \cdot g) = 1$. Consequently, if A' is another representative for A, then all isomorphisms $A \to A'$ define the same isomorphism $L_{A,B} \to L_{A',B}$. Since the same is true for representatives of b, the $L_{A,B}$ can be identified with one line bundle $L_{a,b}$.

By the biadditivity of our intersection product, we get isomorphisms

$$\begin{aligned} & L_{a,c} \otimes L_{b,c} \to L_{a+b,c} \\ & L_{a,c} \otimes L_{a,d} \to L_{a,c+d} \end{aligned} \tag{19}$$

satisfying the axioms of [SGA 7.I, Exp. VII, 2.1]. For instance, it follows from the analogue of (15) for the functor $\cup$ that the isomorphisms (19) satisfy [SGA 7.I, Exp. VII, (2.1.1)].

If $\mathcal{CH}^p(X/S)^v$ is the sheaf on S_{Zar} associated to the presheaf $U \to CH^p(f^{-1}(U)/U)^v$, we get a biextension of $\mathcal{CH}^p(X/S)^v \times \mathcal{CH}^{n+1-p}(X/S)^v$ by $\mathcal{O}_S^*$. This biextension is unique up to unique isomorphism of biextensions.

Using the crucial lemma 1 in Bloch's paper [B], it is easy to see that $CH^p(X\,?\,S)^v$ is contained in the group of cycles which are homologically equivalent to zero on each geometric fibre (i.e. of cycles z on X such that for every geometric point s of S and every prime 1 prime to the characteristic of $k(s)$, the fundamental class of z in $H^{2p}(X_s, \mathbb{Z}_1(p))$ vanishes). If Bloch's biextension is also defined (i.e., if S and X are smooth and quasi-projective over a field), then there exists a canonical isomorphism between Bloch's biextension and the biextension constructed here.

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