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Extending Calabi's conjecture to complete noncompact Kähler manifolds which are asymptotically $\mathbb{C}^n$, $n > 2$

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Introduction

A fundamental problem of Riemannian geometry consists in looking for a Riemannian metric G with *prescribed* Ricci tensor $\mathbf{R}$ on a given manifold M. Introducing the Ricci curvature operator, which associates to each Riemannian metric on M its Ricci tensor $\mathbf{r}(g)$ (this is an, apparently determined, quasilinear second order differential system on M), one thus looks for a solution of the equation: $\mathbf{r}(g) = \mathbf{R}$. Although conceptually going back (at least) to Einstein's gravitation theory, this (inverse) problem has been systematically investigated only recently (see the survey [13], chapter III, and the references therein). Dennis DeTurck [10] has proved the local solvability of $\mathbf{r}(g) = \mathbf{R}$, say near the origin in $\mathbb{R}^n$, provided the given tensor $\mathbf{R}$ is *invertible as a linear map* at the origin; he has also found counterexamples when $\mathbf{R}(0)$ is *not* invertible. Actually, the *global* solvability of $\mathbf{r}(g) = \mathbf{R}$ may be compromised when $\mathbf{R}$ is positive-definite, because of various obstructions arising both for M compact and complete non-compact [19], [13], [11], [8].

On *Kähler* manifolds, though, the Ricci curvature operator $\mathbf{r}(g)$ has a much simpler expression and *global* results can be expected. Indeed, when M is a *compact* Kähler manifold the Calabi-Yau theorem [20] provides a necessary and sufficient condition for $\mathbf{R}$ to be the Ricci tensor of some *Kähler* metric G namely: $\mathbf{R}$ must be hermitian with respect to the complex structure J and the (real) (1,1)-form ρ defined by

$$\rho(U, V) := (2\pi)^{-1}\mathbf{R}(JU, V)$$

must belong to $\mathbf{c}_1(M)$, the first Chern class of M. This theorem actually yields a C^∞ map from $\mathbf{c}_1(M)$ to any pre-assigned Kähler class $\mathbf{C}$, let us denote it by $\mathrm{Cal}(\mathbf{C})$; the existence of $\mathrm{Cal}(\mathbf{C})$ was conjectured by Calabi in [2].

With the view of solving *globally* the Ricci curvature equation on some complete *non-compact* manifolds, we thus look for a tractable extension of Calabi's conjecture to such manifolds which are *Kähler*.

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In section 1, we introduce the notion of Kähler *asymptotically* $\mathbb{C}^n$ manifolds (here $n > 1$) and define on them *weighted* Fréchet spaces of exterior differential forms. For suitable weights, we recover the following basic result known on *compact* Kähler manifolds and obviously needed in the sequel: (1,1)-forms are exact if and only if they admit a Kähler potential (theorem 1(a)). Furthermore, the same result holds for *closed* (1,1)-forms (under an even weaker condition on the weight), provided the manifold has non-negative holomorphic bisectional curvature (theorem 1(b)); here, a much stronger result is that of [18] (p. 187).

Theorem 1 requires three ingredients: a decomposition of Hodge-De Rham type due to Kodaira [14] (see also [9] p. 165) combined with an L^2 density result of [18], Bochner's method as used in [1] and a linear elliptic theory in weighted Hölder spaces [4], [6].

In section 2, using theorem 1, we show how Calabi's conjecture extends meaningfully to those Kähler asymptotically $\mathbb{C}^n$ manifolds which are of (complex) dimension at least *three* and which flatten out at infinity with a suitable weight, and how it reduces again on such manifolds to inverting an elliptic complex Monge-Ampère operator, as it does on compact manifolds. Here, the Monge-Ampère operator acts smoothly between graded weighted Fréchet spaces: at the source lives the Kähler potential of the unknown Kähler deformation, at the target, that of the given Ricci form. Moreover, the Monge-Ampère operator is injective, and its Fréchet derivative is an *elliptic* isomorphism (theorem 2), in the sense of [7], a result already in favor of the conjecture.

In section 3, we formulate *and establish* a particular version of the conjecture (theorem 4) on the simplest of our manifolds, namely $\mathbb{C}^n$; this is done by viewing $\mathbb{C}^n$ as a *tube-manifold* over $\mathbb{R}^n$ (much in the spirit of [3]) and by using there the asymptotically Euclidean analysis of the *real* Monge-Ampère operator performed in [5].

Extending this analysis to the complex Monge-Ampère operator yields the validity of the full conjecture on $\mathbb{C}^n$, $n > 2$ [12].

1. Kähler asymptotically $\mathbb{C}^n$ manifolds ($n > 1$) and invertibility of the Poincaré-Lelong operator

DEFINITION 1. *Let (M, J) be a non-compact complex manifold. A $\mathbb{C}^n$ structure of infinity on (M, J) is a biholomorphic map F sending the complement M_∞ of a compact subset of M to the complement E_R of a closed (Euclidean) ball B_R of radius $R > 0$ in $\mathbb{C}^n$. If such a structure F exists on M, (M, J, F) is called a complex manifold $\mathbb{C}^n$ at infinity.*

DEFINITION 2. *Let (M, J, F) be a complex manifold $\mathbb{C}^n$ at infinity. A smooth Riemannian metric g on M is called a Kähler asymptotically $\mathbb{C}^n$ metric on (M, J, F)*

if it is a Kähler metric on (M, J) *and if it satisfies the following conditions:*

(a) (strict ellipticity) *There exists a real* $\theta > 0$ *such that the metric* $[(F_* g) - \theta e]$ *is non-negative on* E_R *(e denoting the standard Kähler metric of* $\mathbb{C}^n$*);*
(b) (asymptotic flatness) *For any integer* k, $|z|^k |D^k (F_* g)(z)|$ *remains bounded on* E_R *(z is the generic point of* $\mathbb{C}^n$, *D is the Euclidean connection and* $|\cdot|$ *the Euclidean norm);*
(c) (existence of a limit, e) *The limit as* $r \uparrow \infty$ *of* $\sup|(F_* g) - e|$ *on* E_r, *exists and equals zero.*

If g is such a metric, (M, J, F, g) *is called a Kähler asymptotically* $\mathbb{C}^n$ *manifold.*

□

REMARKS. 1. Conditions (a) and (c) of definition 2 imply that a Kähler asymptotically $\mathbb{C}^n$ metric is *complete*.

2. Given a positive real number s, we may further define a Kähler asymptotically $\mathbb{C}^n$ manifold of order s, (M, J, F, g), by replacing conditions (b) and (c) of definition 2 by the sole following stronger condition:

(b.s) *for any integer* k, $|z|^{k+s} |D^k [(F_* g) - e](z)|$ *remains bounded on* E_R.

3. Let (M, J, F) be a complex manifold $\mathbb{C}^n$ at infinity. Let us denote by $K(M, J, F)$ the (convex) set of Kähler asymptotically $\mathbb{C}^n$ metrics on (M, J, F) and, given any positive real s, by $K_s(M, J, F)$ the subset of metrics in $K(M, J, F)$ which satisfy condition (b.s). If $K(M, J, F)$ is non-empty and $n > 1$, then any $\mathbb{C}^n$ structure of infinity F' on (M, J) such that $K(M, J, F) \cap K(M, J, F') \neq \varnothing$, differs from F only by a rigid motion of $\mathbb{C}^n$. Indeed, on one hand since $n > 1$, Hartog's continuation theorem implies that $T := (F' \cdot F^{-1})$ extends holomorphically to the whole of $\mathbb{C}^n$; on the other hand since, by condition (c), the Kähler metrics $h := (F_* g)$ and $h' := (F'_* g)$ are uniformly equivalent to e outside a large enough ball of $\mathbb{C}^n$, the identity $h = T^* h'$ implies that the (complex) derivative of T is a *bounded holomorphic* map on $\mathbb{C}^n$ hence, by Liouville's theorem, it is *constant*; furthermore, it belongs to $U(n)$ because both h and h' tend to e at infinity q.e.d. So, in particular, $|F|$ is intrinsic.

4. Provided $n > 1$, the only Kähler asymptotically $\mathbb{C}^n$ manifold which admits a holomorphic isometric immersion in $\mathbb{C}^N$ for some integer N, is the standard complex n-space. Indeed, let (M, J, F, g) be such a manifold, $j: M \to \mathbb{C}^N$ be a corresponding immersion. Then $f := (j \cdot F^{-1})$ is a holomorphic immersion of $E_R \subset \mathbb{C}^n$ into $\mathbb{C}^N$ such that the $\sup|f^* e - e|$ on E_r goes to zero as $r \uparrow \infty$. Using Hartog's and Liouville's theorems as in remark 3 shows at once that $f^* e \equiv e$, and also that the (complex) derivative of f is *constant*. Now, if (M, J, F, g) differed from the standard $\mathbb{C}^n$, the exterior domain in M embedded by j in an affine complex n-subspace of $\mathbb{C}^N$, would have a smooth non-empty compact boundary

Σ. By continuity of the sectional curvature of $j(M)$, the restriction of j to Σ should then be *minimal* into $\mathbb{C}^N$, which is impossible since Σ has no boundary. $\square$

Let (M, J, F, g) be a Kähler asymptotically $\mathbb{C}^n$ manifold, $n > 1$, with $F: M_\infty \to E_R$. Let ρ be a smooth real function on M satisfying: $\rho \equiv 1$ on $|F| \geqslant (R+1)$, $\rho > 0$ on $(R+1) \geqslant |F| > (R + \frac{1}{2})$, $\rho \equiv 0$ on $|F| \leqslant (R + \frac{1}{2})$ and on $M \backslash M_\infty$. Using ρ, to each couple $(k, p) \in \mathbb{N} \times \mathbb{R}$ we may associate the following norm, defined on smooth complex exterior forms on M:

$$|\omega|_{k;p} := |(1-\rho)\omega|_k + |\rho\omega|_{k;p}$$

where $|(1-\rho)\omega|_k$ stands for the $C^k(M, g)$ norm of the (compactly supported) form on M, $(1-\rho)\omega$, and where $|\rho\omega|_{k;p}$ stands for

$$\sum_{0 \leqslant j \leqslant k} \sup_{\mathbb{C}^n} \{|z|^{p+j} |D^j[F_*(\rho\omega)]|\}$$

which, by remark 3, does not depend on a particular choice of F. Fixing $p \in \mathbb{R}$, for any couple of integers (a, b) with $(a+b) \leqslant 2n$, let us denote by $\Omega^{a,b}_{p+a+b}$ the Fréchet space of smooth complex forms on M of type (a, b) whose topology is defined by the sequence of norms $(|\cdot|_{k;p+a+b})_{k\in\mathbb{N}}$. Different cut-off functions ρ readily yield *equivalent* norms, so the Fréchet spaces do not depend on a particular choice of ρ. We set C^∞_p for the *real* part of Ω^0_p. Note that ∂ (resp. $\bar\partial$) (the (1,0) and (0,1) parts of the exterior differentiation d) is a continuous linear map from $\Omega^{a,b}_{p+a+b}$ to $\Omega^{a+1,b}_{p+a+b+1}$ (resp. to $\Omega^{a,b+1}_{p+a+b+1}$). So, in particular, the vector space $Z^{1,1}_{p+2}$ of *real* forms in $\Omega^{1,1}_{p+2}$ which are *closed* on M, is a *closed* Fréchet subspace.

Hereafter we seek, for suitable p, a global characterization of the vector subspace $B^{1,1}_{p+2}$ of forms in $Z^{1,1}_{p+2}$ which are *exact* on M (of course, our results (on real forms) imply similar ones on *complex* forms).

THEOREM 1. *$B^{1,1}_{p+2}$ is a closed subspace of $Z^{1,1}_{p+2}$, (continuously) isomorphic to C^∞_p via the Poincaré-Lelong operator,*

$$f \in C^\infty_p \to i\partial\bar\partial f \in B^{1,1}_{p+2}$$

under each of the following assumptions,

(a) $p \in (n-2, 2n-2)$;

(b) $p \in (0, 2n-2)$ *and (M, J, g) has non-negative holomorphic bisectional curvature.*

Moreover, under assumption (b), $Z^{1,1}_{p+2} = B^{1,1}_{p+2}$. $\square$

REMARK 5. The solvability of the Poincaré-Lelong equation: $i\partial\bar{\partial}f = \omega$, on a complete non-compact Kähler manifold of non-negative holomorphic bisectional curvature, has been established in [18] (p. 187) assuming that the given (1,1)-form ω is closed, that $|\omega|$ vanishes at infinity at least like $[d(z_0, z)]^{-2}$, $d(z_0, \cdot)$ standing for the geodesic distance from some fixed point z_0 (this would correspond to $p = 0$ in condition (b) above), and that the volume of geodesic balls grows at least like that of the standard complex space of same dimension (this is of use to estimate the Green kernel; it is satisfied in particular by Kähler asymptotically $\mathbb{C}^n$ manifolds). This result is much stronger than theorem 1(b), of course, it requires also a lot more work.

Proof. Assume that $p \in (0, 2n - 2)$ unless otherwise indicated. Denote by L the Poincaré-Lelong operator $f \in C_p^\infty \to i\partial\bar{\partial}f \in Z_{p+2}^{1,1}$, by τ the map $\omega \in \Omega_{p+2}^{1,1} \to \operatorname{trace}(i\omega) \in C_{p+2}^\infty$, where the trace is taken with respect to the metric g, and by Δ the (complex) scalar Laplace operator of g, $\Delta = \tau \cdot L$. L, τ and Δ are continuous.

$\operatorname{Im}(L)$, the image of L, is included in $B_{p+2}^{1,1}$.

L is *injective*. Indeed, if $f \in C_p^\infty$ satisfies $L(f) = 0$, composing with τ yields: $\Delta f = 0$, which in turn implies $f = 0$ by the Maximum Principle (see e.g. [16], p. 135).

Let us show that $\operatorname{Im}(L)$ is a *closed* subspace of $Z_{p+2}^{1,1}$. Let $(f_\alpha)_{\alpha\in N}$ be a sequence in C_p^∞ such that the sequence $[\mathrm{L}(f_\alpha)]_{\alpha\in N}$ converges to ω in $Z_{p+2}^{1,1}$. By continuity of τ, the sequence $(\Delta f_\alpha)_{\alpha\in N}$ then converges in C_{p+2}^∞; since Δ is a continuous isomorphism from C_p^∞ to C_{p+2}^∞ [6], $(f_\alpha)_{\alpha\in N}$ actually converges to some f in C_p^∞, and by continuity of L, $\omega = L(f)$. So ω lies in $\operatorname{Im}(L)$. □

For any $\omega \in Z_{p+2}^{1,1}$ let us define $h(\omega) := [\omega - L(f)] \in Z_{p+2}^{1,1}$, where $f \in C_p^\infty$ stands for the solution of the equation $\Delta f = \tau(\omega)$ on M (given in [6]). g being Kähler, one easily checks that $h(\omega)$ is *co-closed* on (M, g).

Assume condition (a) and let $\omega \in B_{p+2}^{1,1}$. Since $p > n - 2$, $h(\omega)$ belongs to $\mathrm{L}^2(\Lambda^2 T^*M, g)$ as easily verified. Being exact, $h(\omega)$ is thus in the L^2 closure of the image by d of the vector space of smooth 1-forms with compact support, according to [17] (theorem 7.4). However, as noted, $h(\omega)$ is (closed and) co-closed on (M, g); the decomposition of Hodge-De Rham type due to Kodaira [14] (see also [9], p. 165) implies then $h(\omega) = 0$. Therefore $\omega = L(f)$ and $B_{p+2}^{1,1} \subset \operatorname{Im}(L)$; since the inclusion also holds the other way around, $B_{p+2}^{1,1} = \operatorname{Im}(L)$.

Assume condition (b) and let $\omega \in Z_{p+2}^{1,1}$. Write (with Einstein's convention)

$$h(\omega) = ih_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$$

and apply Bochner's method to $h(\omega)$. A direct computation in holomorphic chart, as in [1], yields

$$\tfrac{1}{2}\Delta(|h(\omega)|_g^2) \leqslant g^{\alpha\bar{\beta}} g^{\lambda\bar{\mu}} g^{\rho\bar{\sigma}} h_{\rho\bar{\mu}} (R_{\bar{\beta}}{}^{\bar{\tau}}{}_{\bar{\sigma}\alpha} h_{\lambda\bar{\tau}} - R^{\tau}_{\lambda\bar{\sigma}\alpha} h_{\tau\bar{\beta}}) \tag{1}$$

where $|.|_g$ stands for the norm of tensors in the metric g and $R^{\tau}_{\lambda\bar{\sigma}\alpha}$, for the curvature tensor of g (see e.g. [15], chapter 3 section 6). Fix a generic point $z_0 \in M$ and take at z_0 a holomorphic chart whose natural holomorphic tangential frame at z_0 is both orthonormal for g and along the n principal directions of the hermitian endomorphism

$$g^{\alpha\bar{\mu}} h_{\beta\bar{\mu}} \, dz^{\beta} \otimes (\partial/\partial z^{\alpha}).$$

In this chart, the right-hand side of inequality (1) reads at z_0,

$$\sum_{1 \leqslant \alpha, \beta \leqslant n} R_{\alpha\bar{\alpha}\beta\bar{\beta}} h_{\alpha\bar{\alpha}} (h_{\alpha\bar{\alpha}} - h_{\beta\bar{\beta}})$$
$$= \frac{1}{2} \sum_{1 \leqslant \alpha, \beta \leqslant n} R_{\alpha\bar{\alpha}\beta\bar{\beta}} (h_{\alpha\bar{\alpha}} - h_{\beta\bar{\beta}})^2$$

which is clearly non-positive under assumption (b). $|h(\omega)|^2_g$ thus satisfies on M the differential inequality,

$$\Delta(|h(\omega)|^2_g) \leqslant 0;$$

since it vanishes at infinity, it must vanish identically according to the Maximum Principle (e.g. [16] p. 135) (recall the *minus* sign in Δ). Therefore, arguing as above, $\mathrm{Im}(L) = Z^{1,1}_{p+2}$.

Lastly, the open mapping theorem shows that L factors through the isomorphism announced in theorem 1. □

DEFINITION 3. *The inverse of the Poincaré-Lelong operator is called the Kähler potential. Let us denote it by:*

$$\omega \in B^{1,1}_{p+2} \quad \text{or} \quad Z^{1,1}_{p+2} \to \Gamma(\omega) \in C^{\infty}_{p}.$$

2. Extension of Calabi's conjecture ($n > 2$) and invertibility of a complex Monge-Ampère operator

Assumptions

Let $n > 2$ be a fixed integer and let(M, J, F, g) be a Kähler asymptotically $\mathbb{C}^n$ manifold of order $(p + 2)$ satisfying: either $p \in (n - 2, 2n - 4)$, or (M, J, g) has non-negative holomorphic bisectional curvature and $p \in (0, 2n - 4)$. □

Weighted cohomology classes

Let $\Omega(g)$ and $\rho(g)$ denote respectively the Kähler form and the Ricci form of g. Let us define the 'first Chern class of order $(p+4)$ of (M, J, F)' as the (real) Fréchet affine space

$$\mathbf{c}_{1,p+4} := \rho(g) + B^{1,1}_{p+4}.$$

Recalling well-known properties of Kähler geometry (see e.g. [15], chapter 3 sections 6 and 7), for any $g' \in K_{p+2}(M, J, F)$,

$$\rho(g') \in \mathbf{c}_{1,p+4}.$$

Similarly, let us define the 'Kähler class of order $(p+2)$ of (M, J, F, g)' as the (real) Fréchet affine space

$$C_{p+2} := \Omega(g) + B^{1,1}_{p+2}$$

and let C^+_{p+2} be the open subset of C_{p+2} consisting of *positive* elements i.e. of all $\omega \in C_{p+2}$ such that, for all real tangent vector $V \neq 0$, $\omega(V, JV) > 0$. If $\omega \in C^+_{p+2}$ then $G(\omega) := 2\pi\ \omega(.\,, J.)$ defines a Kähler metric on M, whose Kähler form is precisely ω.

The geometric conjecture

Let us consider the following smooth nonlinear differential operator

$$\mathbf{R}\colon \omega \in C^+_{p+2} \to \mathbf{R}(\omega) := \rho[G(\omega)] \in \mathbf{c}_{1,p+4}.$$

CONJECTURE 1. (Extension of Calabi's, $n > 2$). *Under the assumptions made above, the map* $\mathbf{R}$ *is onto and smoothly invertible.*

Provided the conjecture holds, we let $\mathrm{Cal}(C_{p+2})$ denote the inverse of $\mathbf{R}$.

Reduction to an analytic conjecture

Under our assumptions, theorem 1 holds both with p and with $(p+2)$. Therefore we may consider the mappings (recall definition 3)

$$\phi\colon \omega \in C^+_{p+2} \to \phi(\omega) := 2\pi\Gamma[\omega - \Omega(g)] \in C^\infty_p,$$
$$f\colon \rho' \in \mathbf{c}_{1,p+4} \to f(\rho') := 2\pi\Gamma[\rho(g) - \rho'] \in C^\infty_{p+2}.$$

Both are smooth diffeomorphisms, f is onto, ϕ ranges in an open subset $\phi(C^+_{p+2}) \subset C^\infty_p$ which we denote by U_p. Moreover, by well-known formulas of Kähler geometry [15], R factors on C^+_{p+2} through the following elliptic complex Monge-Ampère operator

$$P[\phi(\omega)] := \operatorname{Log}\{\det[G(\omega)][\det(g)]^{-1}\}$$

(a nonlinear *differential* operator in the variable ϕ on M) which makes commute the diagram

$$\begin{array}{ccc} C^+_{p+2} & \xrightarrow{\mathbf{R}} & \mathbf{C}_{1,p+4} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle f} \\ U_p & \xrightarrow{P} & C^\infty_{p+2}. \end{array}$$

Calabi's conjecture (in the form stated above) thus reduces to the invertibility of an elliptic complex Monge-Ampère operator, as it does on *compact* Kähler manifolds. Precisely, it is implied by the following

CONJECTURE 2 ($n > 1$). *Let* (M, J, F, g) *be a Kähler asymptotically* $\mathbb{C}^n$ *manifold of order* $(p + 2)$ *with* $n > 1$ *and* $p \in (0, 2n - 2)$. *Then, the elliptic complex Monge-Ampère operator*

$$P\colon U_p \subset C^\infty_p \to C^\infty_{p+2},$$

is a smooth diffeomorphism onto.

Albert Jeune has recently established conjecture 2 with $n > 2$, $p \in (0, 2n - 4)$, on the simplest manifold, namely $\mathbb{C}^n$ with its standard Kähler structure [12].

Well-posedness of conjecture 2.

As a first test in favor of conjecture 2, let us prove the following.

THEOREM 2. *The Monge-Ampère operator of conjecture* 2 *is injective and its Fréchet derivative is an elliptic* (*in the sense of* [7]) *isomorphism.*

Proof. A classical computations (see e.g. [20]) shows that, at a generic $\phi \in U_p$, the Fréchet derivative of P

$$\mathrm{d}P(\phi)\colon C^\infty_p \to C^\infty_{p+2}$$

equals minus the Laplacian of the Kähler metric $G(\omega)$, where $\omega \in C^+_{p+2}$ is such that $\phi(\omega) = \phi$. By assumption, $G(\omega)$ is Kähler asymptotically $\mathbb{C}^n$ of order $(p + 2)$; its Laplacian is thus an elliptic isomorphism [6].

To show that P is injective, let ϕ_0 and ϕ_1 be in U_p such that $P(\phi_0) = P(\phi_1)$. Note that U_p is convex. So $\phi = (\phi_1 - \phi_0) \in C_p^\infty$ solves on M the following *linear elliptic* equation,

$$\left[\int_0^1 \mathrm{d}P(\phi_t)\,\mathrm{d}t\right](\phi) = 0$$

where $\phi_t = t\phi_1 + (1 - t)\phi_0$. The Maximum Principle (e.g. [16] p. 135) implies $\phi \equiv 0$. □

3. Extending *and proving* Calabi's conjecture on $\mathbb{C}^n$ as a tube over $\mathbb{R}^n$, $n > 4$

Fréchet spaces with partial weights

Let us consider $\mathbb{C}^n$, $n > 2$, equipped with its standard Kähler structure, and let us view it as a *tube-manifold* over $\mathbb{R}^n$. Let *Re* denote the canonical projection $Re\colon \mathbb{C}^n \to \mathbb{R}^n$ whose fibers are invariant by pure imaginary translations. Let us call "tube-like" any tensor field on $\mathbb{C}^n$ which is invariant under any pure imaginary translation of $\mathbb{C}^n$ (for instance, the pull-back by *Re* of any tensor field on $\mathbb{R}^n$).

Given $q \in \mathbb{R}$ and any couple of integers (a, b) such that $(a + b) \leqslant 2n$, let us consider on $\mathbb{C}^n$ the Fréchet space $\Omega^{a,b}_{n,q+a+b}$ of smooth complex exterior differential forms of type (a, b), endowed with the topology defined by the sequence of norms $(|\,.\,|_{k;n,q+a+b})_{k \in N}$ where

$$|\omega|_{k;n,q+a+b} := \sum_{0 \leqslant j \leqslant k} \sup_{\mathbb{C}^n} (\{\sigma[\mathrm{Re}(z)]\}^{q+a+b+j}|D^j\omega|).$$

Here, for any $x \in \mathbb{R}^n$ we have set: $\sigma(x) := (1 + |x|^2)^{1/2}$ ($|\,.\,|$ standing for the usual euclidean norm). These norms have only *partial* weights, as in [6]. ∂ (resp. $\bar{\partial}$) is readily a continuous linear map from $\Omega^{a,b}_{n,q+a+b}$ to $\Omega^{a+1,b}_{n,q+a+b+1}$ (resp. $\Omega^{a,b+1}_{n,q+a+b+1}$); so the (real) vector subspace $Z^{1,1}_{n,q+2}$ of real forms in $\Omega^{1,1}_{n,q+2}$ which are closed is a *closed* Fréchet subspace of $\Omega^{1,1}_{n,q+2}$.

A tube-like analog of theorem 1

Let $C_q^\infty(\mathbb{R}^n)$ be the Fréchet space of smooth real functions on $\mathbb{R}^n$ whose topology is defined by the sequence of norms $(|\,.\,|_{k;q})_{k \in N}$ where

$$|f|_{k;q} := \sum_{0 \leqslant j \leqslant k} \sup_{\mathbb{R}^n} \{[\sigma(x)]^{q+j}|D^j f|\}$$

(D, the canonical euclidean connection).

THEOREM 3. *Let* $q \in (0, n-2)$. *Then the vector subspace* $Z^{t1,1}_{n,q+2}$ *of forms in* $Z^{1,1}_{n,q+2}$ *which are tube-like, is a closed Fréchet subspace, continuously isomorphic to* $C^{\infty}_{q}(\mathbb{R}^n)$ *via the map,*

$$\mu: f \in C^{\infty}_{q}(\mathbb{R}^n) \to \mu(f) := i\partial\bar{\partial}(\mathrm{Re}^* f) \in Z^{t}_{n,q+2}{}^{1,1} \qquad \square$$

The proof goes much like that of theorem 1 under assumption (b), so we just sketch it.

First, the map $\mu := L.\mathrm{Re}^*$ introduced in theorem 3 is obviously continuous; moreover, noting that the operator $\tau.\mu$ is just minus one half times the standard Laplacian Δ_R on $\mathbb{R}^n$, the injectivity and the closedness of the range of μ follow in the same way as above, but arguing on $\mathbb{R}^n$ and relying on [6].

Last, to show that μ is onto, given $\omega \in Z^{t1,1}_{n,q+2}$, note that $\tau(\omega)$ necessarily factors through a function $u \in C^{\infty}_{q+2}(\mathbb{R}^n)$: $\tau(\omega) = \mathrm{Re}^* u$. Let $f \in C^{\infty}_{q}(\mathbb{R}^n)$ be the solution of $\Delta_R f = -2u$ on $\mathbb{R}^n$ [6] and let $h(\omega) = [\omega - \mu(f)] \in Z^{t1,1}_{n,q+2}$. Then again, $|h(\omega)|^2$ factors through a smooth function w on $\mathbb{R}^n$ which vanishes at infinity. Moreover, the complex Laplacian of $|h(\omega)|^2$ on $\mathbb{C}^n$ equals one half the real Laplacian of w on $\mathbb{R}^n$, and it is identically *non-negative* as a direct computation shows (this is again Bochner's method). So $w \equiv 0$ by the Maximum Principle, hence $\omega = \mu(f)$. $\square$

REMARK 6. Let $q \in (0, \infty)$ and let $u \in \Omega^0_{n,q}$ be real, such that $L(u) \in Z^{t1,1}_{n,q+2}$. Given any pure imaginary vector $\delta \in \mathbb{C}^n$, the translated function $w: z \to w(z) := u(z+\delta)$ readily satisfies on $\mathbb{C}^n$, $L(u) = L(w)$. The Maximum Principle of [6] implies $u \equiv w$. So u is necessarily *tube-like* and theorem 3 completely characterizes $Z^{t1,1}_{n,q+2}$.

Let us denote by $\omega \in Z^{t1,1}_{n,q+2} \to \Gamma^t(\omega) \in C^{\infty}_{q}(\mathbb{R}^n)$, the inverse of the map μ considered in theorem 3.

A tube-like analog of conjecture 1

On $\mathbb{C}^n$, $n > 4$, equipped with its standard Kähler metric e, given $p \in (0, n-4)$, let us define the *tube-like first Chern class of* (*partial*) *order* $(p+4)$ as

$$\mathbf{c}^t_{\mathbf{1},\mathbf{n},p+4} := Z^{t1,1}_{n,p+4}$$

and the *tube-like Kähler class of* (*partial*) *order* $(p+2)$ *as*

$$C^t_{n,p+2} := \Omega(e) + Z^{t1,1}_{n,p+2}.$$

We let $C^{t+}_{n,p+2}$ denote the open subset of $C^t_{n,p+2}$ made of *positive* forms; if $\omega \in C^{t+}_{n,p+2}$, $G(\omega)$ is thus a tube-like Kähler asymptotically $\mathbb{C}^n$ metric of partial order $(p+2)$ (in the notations of section 2).

CONJECTURE AND THEOREM 4. *Under the preceding assumptions, the map*

$$\mathbf{R}^t\colon \omega \in C^{t+}_{n,p+2} \to \mathbf{R}^t(\omega) := \rho[G(\omega)] \in \mathbf{c}^t_{1,\mathbf{n},p+4}$$

is onto and smoothly invertible.

Proof. Since $p \in (0, n-4)$ we may apply theorem 3 both with p and $(p+2)$ and thus may consider the following maps:

$$\omega \in C^{t+}_{n,p+2} \to \phi^t(\omega) := 2\pi\Gamma^t[\omega - \Omega(e)] \in C^\infty_p(\mathbb{R}^n)$$
$$\rho \in \mathbf{c}^t_{1,\mathbf{n},p+4} \to f^t(\rho) := 2\pi\Gamma^t(-\rho) \in C^\infty_{p+2}(\mathbb{R}^n).$$

By theorem 3, both are smooth diffeomorphisms, f^t is onto, ϕ^t ranges in an open subset $\phi^t(C^{t+}_{n,p+2}) \subset C^\infty_p(\mathbb{R}^n)$ denoted by $U_p(\mathbb{R}^n)$. By well-known formulas of Kähler geometry [15], H^t factors on $C^{t+}_{n,p+2}$ through the following *real* elliptic Monge-Ampère operator (in the variable ϕ^t)

$$\phi \in U_p(\mathbb{R}^n) \to \operatorname{Log}[M(\tfrac{1}{2}|x|^2 + \phi)] \in C^\infty_{p+2}(\mathbb{R}^n) \tag{2}$$

where $x = \operatorname{Re}(z) \in \mathbb{R}^n$ and M is the standard (real) Monge-Ampère operator on $\mathbb{R}^n$,

$$M(u) := \det(\partial^2 u/\partial x^\lambda \partial x^\mu).$$

By [5] (theorem 1), the mapping defined by (2) is a smooth diffeomorphism onto; this result is established by means of a topological method, using the C^∞ inverse function theorem of [7] combined with the linear elliptic theory of [6] (in weighted Hölder spaces) and carrying out suitable weighted nonlinear *a priori* estimates. With this global result at hand, the proof of theorem 4 is complete. □

REMARK 7. The approach followed in this section is much in the spirit of [3].

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