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## Concordance of compacta

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### 1. Introduction and main results

The purpose of this paper is to prove a new complement theorem for compact subsets of a manifold. Consider two compacta embedded in the interior of a piecewise linear manifold  $M$  of dimension greater than four. We show that the compacta have homeomorphic complements in  $M$  provided that they satisfy the inessential loops condition, have fundamental codimension three, and are shape concordant in  $M \times [0, 1]$ .

Let us begin with some definitions. Throughout this note  $M$  denotes a closed manifold of dimension  $n \geq 5$ , and  $I$  the unit interval  $[0, 1]$ . If  $X$  is a subset of  $M \times I$  and  $t \in I$ , then  $X_t$  denotes the  $t$ -level  $X \cap (M \times \{t\})$  and  $X_\partial$  denotes  $X_0 \cup X_1$ . We will often identify  $M$  with  $M \times \{0\}$  or  $M \times \{1\}$  and thus consider  $X_0$  and  $X_1$  to be subsets of  $M$ . A compact subset  $X$  of  $M \times I$  is said to be a *shape concordance* from  $X_0$  to  $X_1$  if the inclusion  $X_i \hookrightarrow X$  is a shape equivalence for each  $i = 0, 1$ . We say that a compact subset  $Y$  of a space  $Z$  satisfies the *inessential loops condition* (abbreviated ILC) if for every neighborhood  $U$  of  $Y$  in  $Z$  there exists a neighborhood  $V$  of  $Y$  in  $U$  such that each loop in  $V - Y$  which is null-homotopic in  $V$  is also null-homotopic in  $U - Y$ . A subset  $X$  of  $M \times I$  satisfies the *proper inessential loops condition* if  $X_0$ ,  $X_1$  and  $X$  satisfy the ILC in  $M_0$ ,  $M_1$  and  $M \times I$  respectively. We use ‘ $\simeq$ ’ to denote ‘is homotopic to’ and ‘ $\cong$ ’ to denote ‘is homeomorphic to’. The statement  $\text{Sh}(X) = \text{Sh}(Y)$  means that  $X$  and  $Y$  have the same shape. The *fundamental dimension* of  $X$  is defined by  $\text{Fd}(X) = \min\{\dim Y \mid \text{Sh}(X) = \text{Sh}(Y)\}$ . We refer to [10] for other definitions related to shape theory.

The first *complement theorem* was proved by Chapman [1]. Let  $X_0$  and  $X_1$  be two compacta with  $\text{Sh}(X_0) = \text{Sh}(X_1)$ . If  $X_0$  and  $X_1$  are nicely embedded (as  $Z$ -sets) in the Hilbert cube, then their complements are homeomorphic [1]. In [15], a similar complement theorem is shown to hold if  $X_0$  and  $X_1$  are ILC embedded in  $n$ -dimensional Euclidean space,  $n \geq 5$  and  $2 \text{Fd}(X_0) + 2 \leq n$ . There are also complement theorems for compacta in manifolds; in a manifold it is necessary to add the additional assumption that the embeddings are homotopic.

Beyond the trivial range, complements of shape equivalent ILC compacta in Euclidean space need not be homeomorphic. For example, if  $X_0$  is the disjoint union of two copies of the 3-sphere and  $X_1 \cong X_0$ , it is possible to embed  $X_0$  and  $X_1$  into the 7-sphere in such a way that their complements do not even have the same homotopy type. Therefore additional conditions on the embeddings of  $X_0$  and  $X_1$  are needed in order to guarantee the existence of a homeomorphism of their complements. The main result of this paper supplies such conditions. It can be viewed as a version of Stallings' Neighborhood Isotopy Theorem for shape theory.

**THEOREM.** *Let  $X_0$  and  $X_1$  be two compact ILC subsets of the interior of the PL  $n$ -manifold  $M$ ,  $n \geq 5$ . If  $X_0$  and  $X_1$  are shape concordant in  $M \times I$  and  $\text{Fd}(X_0) \leq n - 3$ , then  $M - X_0 \cong M - X_1$ .*

**REMARK 1.** Partial results for  $n = 4$  have been given in [17] and [9], but the general theorem is not known to hold in dimension 4.

**REMARK 2.** The theorem is a generalization of the complement theorems in [11] and [8]. In those theorems additional hypotheses are imposed on the compacta: in [11] it is assumed that the compacta have the shape of a finite complex and in [8] restrictions are placed on the homotopy pro-groups. We emphasize that neither 1-movability of  $X_0$  nor the Mittag-Leffler condition on  $\text{pro-}\pi_*(X_0)$  is required here. Our theorem supplies a positive answer to [12, Question 2(a)].

**REMARK 3.** The theorem is also (indirectly) a generalization of the complement theorems of [6] and [7]. That is so because [5] can be used to supply the shape concordance needed in the hypothesis of our theorem. In order to see this, note that if  $\text{pro-}\pi_i(X_0)$  is stable for  $i < r$  and Mittag-Leffler for  $i = r$ , then  $X_0$  is pointed  $r$ -movable (for example, refer to [2, Theorem 4] and [10, Theorem 6, p. 203]). Thus two compacta  $X_0$  and  $X_1$  which satisfy the hypotheses of [6, Theorem 3] are actually shape concordant by [5, Theorem 1].

One way in which the proof given here differs from that in [8] is that we do not appeal to the theorem in [6], but give a new, independent proof of the complement theorem which uses only the Concordance Implies Isotopy Theorem of Hudson. The main Theorem is a direct consequence of the following Propositions. They spell out two technical facts about shape concordances which may be of independent interest: If ILC compacta are concordant at all then they are concordant via a proper ILC concordance and if the concordance satisfies the proper ILC, then the complement of the concordance is a product.

**PROPOSITION 1.** *Suppose  $X_0, X_1 \subset M^n$ ,  $n \geq 5$ , are ILC compacta which are shape concordant via a compactum  $X$  in  $M \times I$ . If  $\text{Fd}(X_0) \leq n - 3$ , then there is a concordance  $Y$  from  $X_0$  to  $X_1$  such that  $Y$  satisfies the proper inessential loops condition.*

**PROPOSITION 2.** *Suppose  $X_0, X_1 \subset M^n, n \geq 5$ , are compacta which are shape concordant via a compactum  $X$  in  $M \times I$ . If  $\text{Fd}(X_0) \leq n - 3$  and  $X$  satisfies the proper inessential loops condition, then  $(M \times I) - X \cong (M - X_0) \times I$ .*

**REMARK 4.** Proposition 2 is reminiscent of Hudson’s Concordance Implies Isotopy Theorem [3].

**2. Construction of proper ILC concordances**

In this section we will show how to replace a concordance whose ends are ILC embedded in  $M$  with a proper ILC one. Observe that without loss of generality we may assume in all the aforementioned statements that

- (a)  $X \cap (M \times [0, \frac{1}{3}]) = X_0 \times [0, \frac{1}{3}]$ , and
- (b)  $X \cap (M \times [\frac{2}{3}, 1]) = X_1 \times [\frac{2}{3}, 1]$ .

In the following, if  $A$  is a subset of  $B$ ,  $\iota_A$  will denote the inclusion map  $\iota: A \rightarrow B$ .

**LEMMA 1.** *Let  $X \subset M \times I$  be a shape concordance between  $X_0$  and  $X_1$  and let  $U$  be a given neighborhood of  $X$ . Then there is a smaller neighborhood  $V$  of  $X$  and a homotopy  $h_t: V \rightarrow U$  such that*

- (1)  $h_0 = \iota_V$ ,
- (2)  $h_1(V) \subset U_0$ , and
- (3)  $h_t|V_0 = \iota_{V_0}$  for every  $t \in I$ .

*Proof.* Since  $\iota_{X_0}: X_0 \rightarrow X$  is a shape equivalence, there is a neighborhood  $W$  of  $X$  in  $U$ , a neighborhood  $W'_0$  of  $X_0$  in  $W_0$  and a homotopy  $f_t: W \rightarrow U$  such that

- (1.1)  $f_0 = \iota_W$ ,
- (1.2)  $f_1(W) \subset U_0$ , and
- (1.3)  $f_1|W'_0 \simeq \iota_{W'_0}$  in  $U_0$ .

We must now improve the homotopy to achieve condition (3). Choose  $V$  to be a PL manifold neighborhood of  $X$  such that  $V \subset (W - W_0) \cup W'_0$ . By use of (1.3) and the homotopy extension property we can assume that

- (1.3)'  $f_1|V_0 = \iota_{V_0}$ .

We then use the technique in the proof of [14, Theorem 1.4.11] to find the homotopy  $h_t$  that we need.

**LEMMA 2.** *Let  $X \subset M \times I$  be a shape concordance from  $X_0$  to  $X_1$  and let  $U$  be a neighborhood of  $X$ . If  $X_\partial$  satisfies ILC in  $(M \times I)_\partial$ , then there exists a polyhedron  $Q$  in  $U_1$ , a PL map  $g: Q \times I \rightarrow U$  and a homotopy  $\beta_t: X \rightarrow U$  such that*

- (1)  $Q$  is a spine of a neighborhood of  $X_1$  in  $U_1 \subset M_1$  with  $\dim Q \leq \text{Fd}(X_1)$ ,
- (2)  $g(x, 1) = x$  for all  $x \in Q$ ,

- (3)  $g(x, 0) \in U_0$  for all  $x \in Q$ ,
- (4)  $\beta_0 = \iota_X: X \rightarrow U$ ,
- (5)  $\beta_1(X) \subset g(Q \times I) \cup P$ , where  $P$  is a spine of a neighborhood of  $X_0$  in  $U_0$ , and
- (6)  $\beta(X_\partial \times I) \subset U_\partial$ .

*Proof.* Let  $k$  denote  $\text{Fd}(X_0) = \text{Fd}(X_1) \leq n - 3$ . Let  $N_0$  be a neighborhood of  $X_0$  in  $U_0$  such that  $N_0 \searrow P$  where  $P$  is a polyhedron of dimension  $k$  from [16, Theorem 4.1]. Let  $V$  be a neighborhood of  $X$  in  $U' = (U - U_0) \cup (\text{Int } N_0)$  given by Lemma 1 such that  $\iota_V: V \rightarrow U'$  is homotopic (rel  $V_0$ ) to a map  $f: V \rightarrow U'_0 = \text{Int } N_0$ . Following this homotopy by the collapse  $N_0 \searrow P$  gives us a homotopy  $\alpha_t: V \rightarrow U' \subset U$  with  $\alpha_0 = \iota_V$  and  $\alpha_1(V) \subset P$ .

From [16, Theorem 4.1], there is a neighborhood  $N_1$  of  $X_1$  such that  $N_1 \times [\frac{2}{3}, 1] \subset V$  and  $N_1 \searrow Q$ , where  $Q$  is a subpolyhedron of dimension  $k$ . Let  $r_t$  denote a strong deformation retraction of  $N_1$  to  $Q$ . Let  $\varphi: [\frac{2}{3}, 1] \rightarrow [0, 1]$  be a PL map with  $\varphi([\frac{5}{6}, 1]) = 1$  and  $\varphi(\frac{2}{3}) = 0$ . Define a homotopy  $\mu_t: X \rightarrow V$  by  $\mu_t = \iota$  on  $X \cap (M \times [0, \frac{2}{3}])$ , and  $\mu_t(x, s) = (r_{t \cdot \varphi(s)}(x), s)$  if  $x \in X_1$  and  $\frac{2}{3} \leq s \leq 1$ . Notice that  $\mu_1(X) \cap (M \times [\frac{5}{6}, 1]) \subset Q \times [\frac{5}{6}, 1]$ .

Set  $X^1 = \mu_1(X) \cap (M \times [0, \frac{5}{6}])$ . Define a homotopy  $\lambda_t: \mu_1(X) \rightarrow U$  by  $\lambda_t|_{X^1} = \alpha_t|_{X^1}$ , and  $\lambda_t|_{Q \times [\frac{5}{6}, 1]}$  stretches out the fibers so that  $\lambda_t$  maps  $Q \times [\frac{5}{6}, 1]$  onto  $(Q \times [\frac{5}{6}, 1]) \cup \alpha((Q \times \{\frac{5}{6}\}) \times [0, t])$ . Define  $g$  in such a way that  $g(Q \times [0, 1]) = Q \times [\frac{5}{6}, 1] \cup \alpha((Q \times \{\frac{5}{6}\}) \times [0, 1])$  and, finally, define  $\beta$  to be the homotopy  $\mu$  followed by  $\lambda$ .

**DEFINITION.** If  $A \subset K \times I$ , we define the *shadow* of  $A$  by

$$\Sigma(A) = \{(x, t) \mid (x, s) \in A \text{ for some } s \geq t\}.$$

**LEMMA 3.** Let  $X \subset M^n \times I$ ,  $n \geq 5$ , be a shape concordance from  $X_0$  to  $X_1$  and let  $U$  be a neighborhood of  $X$ . If  $X_\partial$  is ILC in  $(M \times I)_\partial$  and  $\text{Fd}(X_0) = k \leq n - 3$ , then there is a subpolyhedron  $L$  of  $U$  such that

- (1)  $\dim L_\partial = k$  and  $\dim L = k + 1$ ,
- (2)  $L_1$  is a spine of a neighborhood of  $X_1$  in  $U_1$ ,
- (3)  $L \searrow L_0$ , and
- (4) there is a homotopy  $\beta_t: X \rightarrow U$  such that  $\beta_0 = \iota_X$ ,  $\beta_1(X) \subset L$  and  $\beta_t(X_\partial) \subset U_\partial$  for all  $t \in I$ .

*Proof.* The proof is essentially the same as that of [8, Lemma 3], but for the sake of completeness we include an outline here. Let  $r = 2k + 2 - n$  and let  $V^{(0)} = U$ . Define inductively a sequence of neighborhoods of  $X$ :  $V^{(0)} \supset V^{(1)} \supset \dots \supset V^{(r)}$  such that each inclusion satisfies the conclusions of Lemma 1, above.

Let  $g_1: K \times I \rightarrow V^{(r)}$  be a map given by Lemma 2, where  $K$  is a spine of a neighborhood of  $X_1$  in  $V^{(r)}$ . Assume that  $g_1$  is in general position with singular set  $S^{(1)}$  of dimension  $\leq r - 1$ . We propose to construct  $L$  by attaching ‘fins’ to

$g_1(K \times I)$  along the shadow of the singular set. Let  $\Sigma^{(1)} \subset K \times I$  be the shadow of  $S^{(1)}$ ; then  $S^{(1)} \subset \Sigma^{(1)}$ ,  $\dim \Sigma^{(1)} \leq r$  and  $g_1(K \times I) \searrow g_1(\Sigma^{(1)} \cup (K \times \{0\}))$ . Let  $L^{(1)} = g_1(K \times I)$  and  $L^{(1),*} = L^{(1)} \cup (g_1(\Sigma^{(1)}) \times I) / \sim$  where  $\sim$  is defined by  $x \sim (x, 0)$  for every  $x \in g_1(\Sigma^{(1)})$ . By the choice of  $V^{(r)}$ , we can extend  $i: L^{(1)} \rightarrow V^{(r)}$  to a map  $g_2: L^{(1),*} \rightarrow V^{(r-1)}$  such that  $g_2(L^{(1),*}) \cap M_1 = K \times \{1\}$  and  $g_2(L^{(1),*}) \cap M_0 = g_2[g_1(K \times \{0\}) \cup (g_1(\Sigma^{(1)}) \cap K \times \{0\}) \times I \cup (g_1(\Sigma^{(1)}) \times \{1\})]$ . Assume that  $g_2$  is in general position, with singular set  $S^{(2)}$  of dimension  $\leq r-2$  and shadow  $\Sigma^{(2)}$ . Define  $L^{(2)} = g_2(L^{(1),*}) \subset V^{(r-1)}$  and  $L^{(2),*} = L^{(2)} \cup (g_2(\Sigma^{(2)}) \times I) / \sim$ . Continue to define  $L^{(3)} = g_3(L^{(2),*}) \subset V^{(r-2)}$ , and so on. Finally,  $L = g_{r+1}(L^{(r),*}) \subset V^{(0)}$  and  $L \searrow L_0$ .

From now on, whenever  $i$  is an integer, we will define  $\bar{i}$  to be 0 if  $i$  is even and 1 otherwise. If  $P$  is a subpolyhedron of a PL manifold,  $N(P)$  denotes a regular neighborhood of  $P$  (refer to [4]).

*Proof of Proposition 1.* Since  $M \times I$  is an ANR, the homotopy of  $X$  given by Lemma 3 may be extended to a neighborhood of  $X$  in  $M \times I$ . Applying Lemma 3 repeatedly gives a sequence of neighborhoods  $U^{(j)}$  of  $X$  and polyhedra  $L^{(j)} \subset U^{(j)}$  with  $X = \bigcap_{j=0}^{\infty} U^{(j)}$  such that for each  $j$  there is a homotopy  $\beta^{(j+1)}: U^{(j+1)} \times I \rightarrow U^{(j)}$  such that  $\beta_0^{(j+1)} = i_{U^{(j+1)}}$ ,  $\beta_1^{(j+1)}: U^{(j+1)} \rightarrow L^{(j)}$ ,  $\beta^{(j+1)}(U_\delta^{(j+1)} \times I) \subset U_\delta^{(j)}$ , and  $L^{(j)} \searrow L_j^{(j)}$ . Let  $r = 2 \text{Fd}(X_0) + 2 - n$ . For each  $j > r$  and  $k > j$ , we claim that there is an isotopy  $h_t$  of  $M \times I$  rel  $((M \times I) - U^{(j-r)}) \cup (M \times I)_\partial$  such that  $h_0 = id_{M \times I}$ , and  $h_1(L^{(k)}) \subset N(L^{(j)} \cup U_j^{(j-r)})$ .

*Proof of the claim.* Make  $\beta^{(j+1)}$  a PL map and put the track  $\beta^{(j+1)}(L^{(k)} \times I) \subset V^{(j)}$  of the homotopy  $\beta^{(j+1)}$  in general position with respect to  $L^{(k)} \cup L^{(j)}$ . By piping, we can find a polyhedron  $\Sigma \subset \beta^{(j+1)}(L^{(k)} \times I)$  such that  $\Sigma$  contains the image of the singular set of  $\beta^{(j+1)}$ ,  $\dim \Sigma \leq \text{Fd}(X_0)$ , and  $\beta^{(j+1)}(L^{(k)} \times I) \searrow \Sigma \cup \beta_1^{(j+1)}(L^{(k)})$ . The collapse  $L^{(j)} \searrow L_j^{(j)}$  gives a homotopy of  $\Sigma \cap L^{(j)}$  down to  $U_j^{(j)}$ . Use Lemma 1 to attach cells to  $L^{(j)} \cup \Sigma \subset L^{(j)} \cup \beta^{(j+1)}(L^{(k)} \times I)$  as in the proof of Lemma 3 to obtain a polyhedron  $P \subset U^{(j-r)}$  with the following properties:

- (1)  $P \searrow L^{(j)} \cup P_j$ , and
- (2) there is an isotopy  $h_t$  which is rel  $(M \times I) - U^{(j-r)}$  and pushes  $L^{(k)}$  into a regular neighborhood of  $P$ .

By using thin collars on  $(M \times I)_\partial$  in  $M \times I$ , we may assume that  $h_t | (M \times I)_\partial = id$ . On the other hand, observe that  $P \cup U_j^{(j-r)} \searrow L^{(j)} \cup U_j^{(j-r)}$ . Therefore  $N(P \cup U_j^{(j-r)}) \cong N(L^{(j)} \cup U_j^{(j-r)})$  up to PL isotopy. This completes the proof of the claim.

We now return to the proof of Proposition 1. By the claim (taking subsequences, if necessary, and replacing  $N(L^{(j)} \cup U_j^{(j-r)})$  with  $h^{-1}(N(L^{(j)} \cup U_j^{(j-r)}))$ ), we may assume further that  $N(L^{(j+1)} \cup U_{j+1}^{(j+1)}) \subset$

$N(L^{(j)} \cup U_j^{(j)})$  for each  $j$ . Let  $Y = \bigcap_{j=0}^{\infty} N(L^{(j)} \cup U_j^{(j)})$ . Then

$$\bigcap_{j=0} N(L^{(j)} \cup U_j^{(j)}) = Y = \bigcap_{j=1} N(L^{(j)} \cup U_j^{(j)}).$$

Note that  $Y$  satisfies the proper ILC since any 2-dimensional polyhedron can be pushed off  $Y$  and that  $X_0 = Y_0 \hookrightarrow Y$  and  $X_1 = Y_1 \hookrightarrow Y$  are shape equivalences since  $N(L^{(j)} \cup U_j^{(j)}) \searrow U_j^{(j)}$ .

### 3. Complements of proper ILC concordances

In this section we will construct a sequence of neighborhoods of a proper ILC concordance. We will apply Hudson’s Concordance Implies Isotopy Theorem to gradually straighten out the neighborhoods and eventually prove that the complement is a product.

DEFINITION. Let  $P \subset L$  be two polyhedra in  $M \times I$ . A collapse  $L \searrow P$  is *proper* if the collapse induces a strong deformation retraction which keeps  $L_0$  in  $M \times \{0\}$  and  $L_1$  in  $M \times \{1\}$ .

LEMMA 4. Suppose  $X \subset M^n \times I$ ,  $n \geq 5$ , is a shape concordance from  $X_0$  to  $X_1$  which satisfies the proper ILC and  $\text{Fd}(X_0) = k \leq n - 3$ . For every neighborhood  $U$  of  $X$  there exists a compact PL manifold neighborhood  $V$  of  $X$  in  $U$  and a proper collapse  $V \searrow P$ , where  $P$  is a compact polyhedron of dimension  $\leq n - 2$ . Furthermore,  $\dim P_i = k$  and  $P_i$  is a spine of  $V_i$  for  $i = 0, 1$ .

*Proof.* Since  $X_\partial$  satisfies ILC in  $(M \times I)_\partial$ , we can use [16, Theorem 4.1] to find a PL manifold neighborhood  $V$  of  $X$  in  $U$  such that

- (a)  $V \cap (M \times [0, \frac{1}{6}]) = V_0 \times [0, \frac{1}{6}]$ ,
- (b)  $V \cap (M \times [\frac{5}{6}, 1]) = V_1 \times [\frac{5}{6}, 1]$ , and
- (c) for each  $i = 0, 1$ ,  $V_i \searrow P_i$ , where  $P_i$  is a polyhedron of dimension  $k$ .

It is then clear that there is a proper collapse  $V \searrow \tilde{P}$  for some  $n$ -dimensional polyhedron  $\tilde{P}$  with  $\tilde{P} \cap V_i = P_i$  ( $i = 0, 1$ ). A triangulation of  $\tilde{P}$  determines, in a natural way, a handle decomposition of  $V$  modulo  $V_0 \cup V_1$ . This handle decomposition has handles of index  $\leq n$ . We wish to eliminate the handles of index  $n$  and  $n - 1$ ; a spine of the remaining manifold is the polyhedron  $P$  we seek.

First, consider an  $n$ -handle. Its cocore is an arc whose endpoints miss  $X$ . Since  $\text{Fd}(X) \leq n - 3$ , we can push the arc off  $X$  with an isotopy and thus eliminate the  $n$ -handle. Next, consider an  $(n - 1)$ -handle. Its cocore is a 2-disk whose boundary misses  $X$ . We claim that it is possible to push this cocore off  $X$  with an ambient isotopy as well and thus eliminate the  $(n - 1)$ -handle. In order to

demonstrate this, we must produce a homotopy which pushes the disk off  $X$ , keeping its boundary fixed. Let  $X' = X \cap (M \times [\frac{1}{6}, \frac{5}{6}])$ . Since  $X'$  is in the interior of  $M \times I$ , it is possible to push the cocore off  $X'$  in  $M \times I$  [15, Lemma 1]. Projecting  $M \times I$  onto  $M \times [\frac{1}{6}, \frac{5}{6}]$  gives a homotopy which pushes the cocore into  $((U - X) \cap (M \times [\frac{1}{6}, \frac{5}{6}])) \cup U_{1/6} \cup U_{5/6}$ . Now apply [15, Lemma 1] again to push the cocore off  $X_t$  in  $U_t$  for  $t = \frac{1}{6}$  and  $\frac{5}{6}$ .

For convenience we make the following technical definition.

DEFINITION. A *collapsible defining sequence* for a concordance  $X$  in  $M \times I$  is a sequence of PL manifold neighborhoods  $N^{(i)}$  of  $X$  in  $M \times I$  such that

- (1) each  $N^{(i+1)} \subset \text{Int } N^{(i)}$ ,
- (2)  $X = \bigcap_{i=0}^{\infty} N^{(i)}$ , and
- (3)  $N^{(i)} \searrow N_{\bar{i}}^{(i)}$  for each  $i$ .

LEMMA 5. Let  $X \subset M \times I$  be a proper ILC concordance from  $X_0$  to  $X_1$ . If  $\dim M = n \geq 5$  and  $\text{Fd}(X_0) = k \leq n - 3$ , then there is a collapsible defining sequence for  $X$  in  $M \times I$ .

*Proof.* Choose a sequence of compact PL manifold neighborhoods  $\{U^i \mid i = 0, 1, \dots\}$  of  $X$  such that  $X = \bigcap_{i=0}^{\infty} U^i$ . We will inductively construct  $N^{(i)}$  so that  $N^{(0)} \subset U^0$  and  $N^{(i)} \subset U^i \cap \text{Int } N^{(i-1)}$  for  $i > 0$ . We will describe the construction for  $i$  even (say  $i = 0$ ). The case  $i$  odd is handled similarly by just interchanging the subscripts 0 and 1.

Let  $r = 2 \text{Fd}(X_0) + 2 - n$ . Use Lemma 3 to choose a subpolyhedron  $L$  of  $U^r$  such that  $L \searrow L_0 \subset U_0^r$ . By Lemma 4 there is a small neighborhood  $V$  of  $X$  which has a codimension-3 spine  $P$ . Lemma 3 gives a homotopy  $\beta_t: P \rightarrow U^r$  satisfying the following properties:

- (1)  $\beta_0 = \iota_P$ ,
- (2)  $\beta_1(P) \subset L$ , and
- (3)  $\beta_t(P_\partial) \subset U_\partial^r$  for all  $t \in I$ .

We can therefore engulf  $P$  (and thus also  $V$ ) with a regular neighborhood of  $L \cup U_0^{r-r} = L \cup U_0^0$  (see the proof of Proposition 1). Let  $N^{(0)}$  be a small collar of  $U_0^0$  together with the inverse image of a regular neighborhood of  $L$  under the engulfing isotopy. Then  $X \subset V \subset N^{(0)} \subset U^0$  and  $N^{(0)} \searrow N_\partial^{(0)} \subset U_\partial^0$ .

To continue, define  $U^{(1)}$  to be the neighborhood  $V \cap U^r$  of  $X$ . From Lemma 3 there is a polyhedron  $K \subset U^{(1)}$  such that  $K \searrow K_1$ . As above, we can produce a PL isotopy  $\text{rel}(M \times I - U^{(1)}) \cup (M \times \partial I)$  which takes  $K$  to  $K^*$  such that  $K^* \cup U_1^{(1)}$  has a regular neighborhood  $N^{(1)}$  with  $X \subset N^{(1)} \subset U^{(1)}$  and  $N^{(1)} \searrow N_\partial^{(1)} \subset U_\partial^{(1)}$ . Continuing this procedure will complete the proof of the Lemma.



*Proof of Proposition 2.* Let  $N^{(0)}$  be a neighborhood of  $X$  (from Lemma 5) such that  $N^{(0)} \searrow N_0^{(0)}$  and let  $K^{(0)}$  be a spine of  $N_1^{(0)}$ . Regular neighborhood theory shows that  $N^{(0)} \cong N_0^{(0)} \times I$ . Thus the projection map  $K^{(0)} \times \{1\} \rightarrow K^{(0)} \subset N_1^{(0)}$  extends to a PL embedding  $e: (K^{(0)} \times I, K^{(0)} \times \{0\}) \rightarrow (N^{(0)}, N_0^{(0)})$ . By [3] there is an isotopy  $h^1$  of  $M \times I$  which pushes  $e(K^{(0)} \times I)$  to the straight vertical copy  $K^{(0)} \times I$  and has compact support. Now use Lemma 5 again to find a smaller neighborhood  $N^{(1)}$  of  $h_1^1(X)$  such that  $N^{(1)} \searrow N_1^{(1)}$  and let  $K^{(1)}$  be a spine of  $N_0^{(1)}$ . Again,  $N^{(1)} \cong N_1^{(1)} \times I$ , so there exists a PL embedding of  $K^{(1)} \times I$  into  $N^{(1)}$ . Apply [3] again to get an isotopy which straightens out this embedding of  $K^{(1)} \times I$ . The composition of the two isotopies will push  $X$  into  $N(K^{(0)}) \times I$ , where  $N(K^{(0)})$  is a small regular neighborhood of  $K^{(0)}$  in  $M_0$ .

Now replace  $M$  with  $N(K^{(0)})$ . Continuing in the fashion described in the preceding paragraph gives an infinite sequence of isotopies whose composition converges, on  $(M \times I) - X$ , to a homeomorphism between  $(M \times I) - X$  and  $(M - X_0) \times I$ .

REMARK 5. Proposition 2 can also be proved by applying the proper  $h$ -cobordism theorem of Siebenmann [13] to the cobordism  $(M \times I - X; M_0 - X_0, M_1 - X_1)$ . First use the fact that  $X$  is a proper ILC concordance to verify conditions  $(\pi_1)_\infty$  and  $(H_*)_\infty$  in [13, Proposition IV] and thus conclude that the cobordism is a proper  $h$ -cobordism. Then use a collapsible defining sequence for  $X$  to prove that both obstructions  $\sigma_\infty$  and  $\tau'$  vanish.

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### References

1. T. A. Chapman, On some applications of infinite-dimensional manifolds to the theory of shape, *Fund. Math.* 76 (1972), 181–193.
2. S. C. Ferry, A stable converse to the Vietoris-Smale theorem with applications to shape theory, *Trans. Amer. Math. Soc.* 261 (1980), 369–386.
3. J. F. P. Hudson, Concordance, isotopy and diffeotopy, *Ann. of Math.* 91 (1970), 425–488.
4. J. F. P. Hudson, *Piecewise Linear Topology*, W. A. Benjamin, Inc., New York and Amsterdam, 1969.
5. L. S. Husch and I. Ivanšić, On shape concordances, in ‘Shape Theory and Geometric Topology,’ *Lecture Notes in Mathematics*, vol. 870, Springer-Verlag, New York, 1981, pp. 135–149.
6. I. Ivanšić and R. B. Sher, A complement theorem for continua in a manifold, *Topology Proceedings* 4 (1979), 437–452.

7. I. Ivanšić, R. B. Sher, and G. A. Venema, Complement theorems beyond the trivial range, *Ill. J. Math.* 25 (1981), 209–220.
8. V. T. Liem and G. A. Venema, Stable shape concordance implies homeomorphic complements, *Fund. Math.* 126 (1986), 123–131.
9. V. T. Liem and G. A. Venema, Neighborhoods of  $S^1$ -like continua in 4-manifolds, *Mich. Math. J.* (to appear).
10. S. Mardešić and J. Segal, *Shape Theory*, North-Holland, Amsterdam, 1982.
11. R. B. Sher, A complement theorem for shape concordant compacta, *Proc. Amer. Math. Soc.* 91 (1984), 123–132.
12. R. B. Sher, Complement theorems in shape theory, II, in ‘Geometric Topology and Shape Theory,’ *Lecture Notes in Mathematics*, vol. 1283, Springer-Verlag, New York, 1987, pp. 212–220.
13. L. C. Siebenmann, Infinite simple homotopy types, *Indag. Math.* 73 (1970), 479–495.
14. E. H. Spanier, *Algebraic Topology*, McGraw-Hill Book Co., New York, 1966.
15. G. A. Venema, Embeddings of compacta with shape dimension in the trivial range, *Proc. Amer. Math. Soc.* 55 (1976), 443–448.
16. G. A. Venema, Neighborhoods of compacta in Euclidean space, *Fund. Math.* 109 (1980), 71–78.
17. G. A. Venema, Neighborhoods of compacta in 4-manifolds, *Topology and its Appl.* 31 (1989), 83–97.