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Convergence theorem for Riemannian manifolds with boundary

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1. Introduction

Given a Riemannian manifold M , we denote by K_M , d_M and V_M , its sectional curvature, diameter and volume, respectively. Let $\{M_i; i = 1, 2, 3, \dots\}$ be a sequence of connected closed n -dimensional Riemannian manifolds satisfying the following conditions:

$$|K_{M_i}| \leq \Lambda, \quad d_{M_i} \leq d \quad \text{and} \quad V_{M_i} \geq V,$$

for each i .

Then, M. Gromov's convergence theorem ([7], [10], [6], [14]) asserts that there is a subsequence $\{j\} \subset \{i\}$ such that M_j converges to an n -dimensional differentiable manifold M_∞ with a metric g_∞ of $C^{1+\alpha}$ Hölder class in Lipschitz distance.

The purpose of this paper is to generalize this result to several classes of Riemannian manifolds M with non-empty boundary ∂M . We establish various convergence theorems of the Gromov's type under some additional assumption on the boundary ∂M .

Let $\lambda_i(p)$, $i = 1, 2, \dots, n - 1$, be the principal curvatures at $p \in \partial M$ with respect to the inwardly pointed unit normal vector N_p at p . The following simple examples (1), (2) illustrate the necessary condition on the principal curvatures to establish a reasonable convergence theorem.

Let $D_r(a, b)$ be a disk of radius r centered at (a, b) in the Euclidean two-plane R^2 . Here $r, a, b \in R$.

EXAMPLE. (1) Let A_ε be a convex hull of $D_1(0, 0) \cup D_\varepsilon(2, 0)$. Let ε tend to 0. Then a corner appears at $(2, 0)$. It can be ruled out by an upper bound on λ_i .

(2) Put $B_\varepsilon = D_1(0, 0)/D_\varepsilon(0, 0)$. Let ε tend to 0. Then the boundary of $D_\varepsilon(0, 0)$ tends to a point. It can be ruled out by a lower bound on λ_i .

(3) Put $C_\varepsilon = D_1(0, 0)/D_{1/2}(\frac{1}{2} - \varepsilon, 0)$. Let ε tend to 0. Then two boundaries contact at $(1, 0)$.

The phenomenon occurring in example (3) can not be ruled out by bounds on the principal curvatures.

In this paper, we will show that if we give some additional conditions on the boundary then a convergence theorem holds for Riemannian manifolds with boundary whose principal curvatures are bounded. (Theorems B and B'. See Section 2).

Throughout this paper, all manifolds and their boundaries are assumed to be compact and smoothly differentiable.

1. Notations and definitions

Let M be a compact Riemannian manifold with nonempty boundary ∂M .

(1) We denote by d_M the distance function of M , by $d_{\partial M}$ the distance from the boundary ∂M and by $B_M(p, r)$ the metric ball of radius r around $p \in M$. We put, for $0 \leq t < t' \leq +\infty$,

$$M(t) = d_{\partial M}^{-1}(t), \quad M[t, t'] = d_{\partial M}^{-1}[t, t'].$$

(2) We denote by N_p (N , resp.) the inwardly pointed unit normal vector at $p \in \partial M$ (vector field on ∂M , resp.), by $T_p \partial M$ ($T\partial M$, resp.) the tangent space of ∂M at p (the tangent bundle of ∂M , resp.) and by v_p^+ (v^+ , resp.) the half space of the normal space v_p at $p \in \partial M$ (the normal bundle v of ∂M , resp.) which consists of inwardly pointed vectors. The second fundamental form S at $p \in \partial M$ is defined by $S(X) = -\nabla_X N$ for $X \in T_p \partial M$, where ∇ is the Riemannian connection of M . The principal curvatures $\lambda_i(p), i = 1, 2, \dots, n-1$, at p are eigenvalues of S . For example, the principal curvature of unit n -sphere as a boundary of unit $(n+1)$ -disk is equal to 1.

(3) We define positive $i_{\text{int}}, i_{\partial}$ and i_M by $i_{\text{int}} = \sup\{r > 0 \mid \text{For any } p \in M, \text{ any normal geodesic } \gamma(t), 0 \leq t \leq t_\gamma, \text{ from } p \text{ is distance minimizing up to the distance } \min(t_\gamma, r)\}$,

$$i_{\partial} = \sup\{r > 0 \mid \exp: v^+(r) \rightarrow M[0, r] \text{ is diffeomorphic.}\}$$

and

$$i_M = \min\{i_{\partial}, i_{\text{int}}\},$$

respectively, where $v^+(r) = \{v \in v^+ \mid \|v\| < r\}$.

2. Statement of results

First we consider the following class of Riemannian manifolds.

DEFINITION 2.1. Let $\mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$ be the class of n -dimensional compact Riemannian manifolds M with nonempty boundary ∂M with the sectional curvature $|K_M| \leq \Lambda$, the second fundamental form $\lambda^- \leq S \leq \lambda^+$ and $i_M \geq i_0$. Our condition on the second fundamental form is equivalent to the fact that any principal curvature $\lambda_i(p)$ at $p \in \partial M$ satisfies $\lambda^- \leq \lambda_i(p) \leq \lambda^+$.

Because of the bounds λ^+, λ^- on the principal curvature, we can avoid the phenomena occurring in example (1) and (2), and by the lower bound i_0 on i_M , we can avoid the phenomenon occurring in example (3).

In fact, we have the following main theorem.

THEOREM A. In $\mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$, the Hausdorff and Lipschitz topology coincide. More precisely, given $\rho > 0$, there exists a $\zeta(n, \Lambda, \lambda^+, \lambda^-, i_0) > 0$ such that $d_H(M, \bar{M}) < \zeta$ implies $d_L(M, \bar{M}) < \rho$ for all $M, \bar{M} \in \mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$.

The proof of Theorem A will be given in Section 3. Here, we recall the definitions of the Hausdorff and Lipschitz distances.

DEFINITION 2.2. (1) Let X, Y be metric spaces and $f: X \rightarrow Y$ a Lipschitz map. The $\text{dil } f := \sup_{x \neq x'} d(f(x), f(x'))/d(x, x')$ is called dilatation of f . $d_L(X, Y) := \inf\{|\ln \text{dil } f| + |\ln \text{dil } f^{-1}| \mid f \text{ is an arbitrary bi-Lipschitz homeomorphism}\}$ is the Lipschitz distance between X and Y , if bi-Lipschitz homeomorphisms exist. Otherwise $d_L(X, Y) = \infty$.

(2) Let Z be a metric space and A, B be subspaces of Z . Let $U_\varepsilon(A) := \{z \in Z \mid d(z, A) < \varepsilon\}$ and let $d_H^Z(A, B) := \inf\{\varepsilon > 0 \mid U_\varepsilon(A) \subset B \text{ and } U_\varepsilon(B) \subset A\}$. Now, for arbitrary two metric spaces X and Y , we define the Hausdorff distance $d_H(X, Y)$ by $\inf\{d_H^Z(f(X), g(Y))\}$, where the inf is taken over all metric spaces Z and all isometries $f: X \rightarrow Z, g: Y \rightarrow Z$.

But our condition on i_M is too strong and not natural. Next, we consider the following classes of Riemannian manifolds.

DEFINITION 2.3. (1) Let $\mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, d, V)$ be the class of n -dimensional compact Riemannian manifolds M with nonempty boundary ∂M with the sectional curvature $|K_M| \leq \Lambda$, the second fundamental form $\lambda^- \leq S \leq \lambda^+$, the diameter $d_M \leq d$ and the volume $V_M \geq V$.

(2) We put $\mathbf{M}_c(n, \Lambda, \lambda^+, d, V) = \mathbf{M}(n, \Lambda, 0, \lambda^+, d, V)$. This is the class of Riemannian manifolds with convex boundary.

(3) Let $\mathbf{M}_\partial(n, \Lambda, \lambda^-, \lambda^+, d, V; d_\partial)$ be a class of elements $M \in \mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, d, V)$ satisfying that the sum $d_{\partial M}$ of the intrinsic diameters of all components of ∂M is not greater than d_∂ .

The conditions on the sectional curvature, diameter and the volume are natural. But the phenomenon occurring in example (3) can not be ruled out by only a bound on the principal curvatures. Therefore the class $\mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, d, V)$ is not adequate to establish a reasonable convergence theorem. But in smaller classes, we have the following.

PROPOSITION 2.4. *There exist $\lambda_0^- < 0$ and $i_0 > 0$ depending on $n, \Lambda, \lambda^+, d, V$ and d_δ such that the classes $\mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$ and $\mathbf{M}_\delta(n, \Lambda, \lambda_0^-, \lambda^+, d, V; d_\delta)$ are contained in $\mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$.*

PROPOSITION 2.5. *The classes $\mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$ and $\mathbf{M}_\delta(n, \Lambda, \lambda_0^-, \lambda^+, d, V; d_\delta)$ are precompact with respect to the Hausdorff topology.*

The proof of Proposition 2.4 will be given in Section 6 in more explicit form. The proof of Proposition 2.5 will be given in Section 7.

Now, Theorem A, Proposition 2.4 and Proposition 2.5 imply the following convergence theorems.

THEOREM B. *Any sequence in $\mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$ contains a subsequence converging with respect to the Lipschitz topology to an n -dimensional differential manifold M with metric g of C^0 class.*

THEOREM B'. *There exists $\lambda_0^- = \lambda_0^-(n, \Lambda, \lambda^+, d, V; d_\delta) < 0$ such that if $\lambda_0^- < \lambda^- \leq 0$ then any sequence in $\mathbf{M}_\delta(n, \Lambda, \lambda^-, \lambda^+, d, V; d_\delta)$ contains a subsequence converging with respect to the Lipschitz topology to an n -dimensional differential manifold M with metric g of C^0 class.*

As consequences of Theorem B and B', we have the following finiteness theorems.

THEOREM C. *$\mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$ contains only a finite number of diffeomorphism classes. In particular, the number of the connected components of the boundary ∂M of any $M \in \mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$ is uniformly bounded from above.*

In fact, these numbers can be estimated explicitly in terms of n, Λ, λ^+, d and V .

THEOREM C'. *$\mathbf{M}_\delta(n, \Lambda, \lambda_0^-, \lambda^+, d, V; d_\delta)$ contains only a finite number of diffeomorphism classes. In particular, the number of the connected components of the boundary ∂M of any $M \in \mathbf{M}_\delta(n, \Lambda, \lambda_0^-, \lambda^+, d, V; d_\delta)$ is uniformly bounded from above.*

In fact, these numbers can be estimated explicitly in terms of $n, \Lambda, \lambda^+, d, V$ and d_δ .

REMARK 2.6. As a consequence of Theorem B, we can derive Gromov's convergence theorem as follows. Let $\mathbf{m}(n, \Lambda, d, V)$ be a class of closed Riemannian

manifolds of dimension n with the sectional curvature $|K_M| \leq \Lambda$, diameter $d_M \leq d$ and the volume $V_M \geq V$. For $M \in \mathbf{m}(n, \Lambda, d, V)$, $M \times [0, 1]$ is an element of $\mathbf{M}_c(n + 1, \Lambda, \lambda^+, \sqrt{d^2 + 1}, V)$. Let $\mathbf{m}(n, \Lambda, d, V) \times [0, 1]$ be a class of such $M \times [0, 1]$'s. Then $\mathbf{m}(n, \Lambda, d, V) \times [0, 1] \subset M_c(n + 1, \Lambda, \lambda^+, \sqrt{d^2 + 1}, V)$ and the convergence theorem for the class $\mathbf{m}(n, \Lambda, d, V)$ follows from that of $\mathbf{M}_c(n + 1, \Lambda, \lambda^+, \sqrt{d^2 + 1}, V)$. We can also derive Cheeger's finiteness theorem ([1]) from Theorem C.

3. Comparison lemmas

In the first part of this section, we give comparison lemmas (Lemmas 3.1 and 3.2) and, in the second part, two simple consequences (Lemmas 3.3 and 3.4) which will be used in later sections.

Let M be a compact n -dimensional Riemannian manifolds with boundary ∂M satisfying $|K_M| \leq \Lambda$, $\lambda^+ \leq S = -\nabla N \leq \lambda^-$ and $-\lambda^+ \leq \lambda^- \leq 0$. Let $c(t)$, $0 \leq t$, be a geodesic of M with the initial point $c(0) \in \partial M$ and the initial vector $\dot{c}(0) = N_{c(0)}$. Let $\mu(t)$ be the fiber of the normal bundle of the geodesic c at $c(t)$. The Jacobi tensor $J(t): T_{c(0)}\partial M \rightarrow \mu(t)$ satisfies the Jacobi equation

$$J''(t) + R(t) \circ J(t) = 0, \quad J(0) = I, \quad J'(t) = -S,$$

$$\Lambda \geq R(t) \geq -\Lambda,$$

where $R(t) \circ J(t)(X) = R(J(t)X, \dot{c}(t))\dot{c}(t)$ and $I(X) = X$, for $X \in T_{c(0)}\partial M$. Here we identify $T_{c(0)}\partial M$ with $\mu(t)$ by the parallel transformation. Here and hereafter, for two operators P and P' , the inequality $P > P'$ ($P \geq P'$, resp.) means that $P - P'$ is positive definite (nonnegative semidefinite, resp.).

Put

$$a(t) = -\frac{\lambda^+}{\sqrt{\Lambda}} \sin \sqrt{\Lambda}t + \cos \sqrt{\Lambda}t,$$

$$b(t) = -\frac{\lambda^-}{\sqrt{\Lambda}} \sinh \sqrt{\Lambda}t + \cosh \sqrt{\Lambda}t.$$

Then they are the solutions of

$$a''(t) + \Lambda a(t) = 0, \quad a(0) = 1, \quad a'(0) = -\lambda^+,$$

$$b''(t) - \Lambda b(t) = 0, \quad b(0) = 1, \quad b'(0) = -\lambda^-,$$

respectively. The first zero point of $a(t)$ appears at $t_0 = 1/\sqrt{\Lambda} \arctan \sqrt{\Lambda}/\lambda^+$.

Let $U(t) = J'(t)/J(t)$. Then $U(t)$ satisfies the Riccati equation

$$U'(t) + U^2(t) + R(t) = 0, \quad U(0) = -S.$$

$U(t)$ is symmetric with respect to the Riemannian inner product and coincides with the second fundamental form of $M(t)$ (cf. [5]). $A(t) = a'(t)/a(t)$ and $B(t) = b'(t)/b(t)$ are the solutions of

$$\begin{aligned} A'(t) + A^2(t) + \Lambda &= 0, & A(0) &= -\lambda^+, \\ B'(t) + B^2(t) - \Lambda &= 0, & B(0) &= -\lambda^-, \end{aligned}$$

respectively. $A(t)$ is strictly decreasing and has a pole at t_0 . $B(t)$ is monotonous and tends to $\sqrt{\Lambda}$ at infinity. Our condition that $-\lambda^+ \leq \lambda^- \leq 0$ implies that $B(t) \leq -A(t)$ for $0 \leq t < t_0$.

In this paper, we use the following version of comparison lemma. Our proof is based on that of J.H. Eschenburg [4, Lemma 3.4].

LEMMA 3.1. For $0 < t < t_0$,

$$b(t) \geq J(t) \geq a(t) \tag{3.1}$$

$$B(t) \geq U(t) \geq A(t). \tag{3.2}$$

The first inequalities are valid for $t_0 \leq t$. Hence the focal distance along c is not less than t_0 .

Proof. Without loss of generality, we can assume $\Lambda > R(t) > -\Lambda$ and $\lambda^- < S < \lambda^+$. First, we prove that $B(t) > U(t)$ for $0 \leq t$. Since

$$(B(t) - U(t))' + B^2(t) - U^2(t) = R(t) + \Lambda > 0,$$

and so,

$$(B(t) - U(t))' > -(B(t) - U(t))(B(t) + U(t)),$$

$$B(0) - U(0) = -\lambda^- + S > 0.$$

Suppose $t_1 > 0$ is the first zero point of $\langle (B - U)x, x \rangle$ where $x \in T_{c(0)}\partial M$. Since $B - U$ is symmetric, we can assume that x is the eigenvector of $(B - U)(t_1)$. Then we have

$$\begin{aligned} \langle (B - U)x, x \rangle'(t_1) &= \langle (B - U)'x, x \rangle(t_1) \\ &> -\langle (B + U)(B - U)x, x \rangle(t_1) = 0. \end{aligned}$$

This implies that t_1 cannot be the first zero point of $\langle (B - U)x, x \rangle$. Therefore $B - U$ is positive definite for $t \geq 0$.

Next we prove that $U(t) > A(t)$ for $0 \leq t < t_0$. Since

$$(U - A)' + U^2 - A^2 = -R + \Lambda > 0,$$

and so,

$$(U - A)' > -(U - A)(U + A), U(0) - A(0) = -S + \lambda^+ > 0.$$

Since $U + A < U - B < 0$ and $U \pm A$ are symmetric, $(U - A)'$ is positive definite as long as $U - A$ is positive definite. Therefore $U - A$ is positive definite for $0 \leq t < t_0$.

Finally we prove (3.1). Put $X(t) = J(t)/a(t) - 1$. Then we have

$$X(0) = 0, \quad X'(t) = (J'a - a'J)/a^2 = (U - A)J/a > 0,$$

for $0 \leq t < t_0$. Thus $X(t) > 0$ for $0 \leq t < t_0$. And then $J(t) > a(t)$ for $0 \leq t < t_0$. Similarly we can verify $b(t) > J(t)$ for $0 \leq t$. q.e.d.

We define the norm of $J(t)$ as $|J(t)| = \sup\{|J(t)x| \mid x \in T_{c(0)}\partial M, |x| = 1\}$. By Lemma 3.1, we have the following comparison lemmas for $|J(t)|$ and $|J(t)/J(t')|$. The proofs are easy and hence omitted. Here, we note that our comparison lemmas are included in more general comparison theorem due to T. Yamaguchi [19, Theorem 1]. The author is grateful to Professor T. Yamaguchi and N. Innami who kindly pointed out this fact to the author.

LEMMA 3.2. For $0 \leq t < t' < t_0$,

$$b(t) \geq |J(t)| \geq a(t), \tag{3.3}$$

$$\frac{b(t')}{b(t)} \geq \left| \frac{J(t')}{J(t)} \right| \geq \frac{a(t')}{a(t)}, \quad \frac{a(t)}{a(t')} \geq \left| \frac{J(t)}{J(t')} \right| \geq \frac{b(t)}{b(t')}. \tag{3.4}$$

LEMMA 3.3. For fixed $t_1 (< t_0)$ and any $t, t' \in [0, t_1]$, we have

$$1 - |t' - t|\Delta(t_1) < \left| \frac{J(t')}{J(t)} \right| < 1 + |t' - t|\Delta(t_1). \tag{3.6}$$

Here $\Delta(t_1) = \max\{a'(t_1)/a^2(t_1), \max(-\lambda^-, \sqrt{\Lambda})b(t_1)\}$.

Let $c(s)$, $0 \leq s \leq s_1$, be a normal geodesic in $M[0, t_2]$, for fixed $t_2 \leq \min(t_0, i_M)$. Put $\ell(s) = d(\partial M, c(s))$.

LEMMA 3.4. For $0 \leq s \leq s_1$, we have

$$B(\ell(s)) \geq \ell''(s) \geq A(\ell(s)). \quad (3.7)$$

Assume $\ell(0) = r$ and $\ell'(0) = 0$. Then

$$s \max(\sqrt{\Lambda}, -\lambda^-) \geq \ell'(s) \geq sA(t_2), \quad (3.8)$$

$$r + \frac{s^2}{2} \max(\sqrt{\Lambda}, -\lambda^-) \geq \ell(s) \geq r + \frac{s^2}{2} A(t_2). \quad (3.9)$$

Furthermore assume $c(s_1) \in \partial M$. Then

$$\sqrt{2r/A(t_2)} \leq s_1. \quad (3.10)$$

Proof. Let \tilde{c} be a curve on ∂M such that $d(\tilde{c}(s), c(s)) = d(\partial M, c(s)) = \ell(s)$. We define a variation $\alpha: [0, s_1] \times [0, 1] \rightarrow M$ by $\alpha(s, t) = \exp_{\tilde{c}(s)} t \ell(s) N$ and variation fields $T = d\alpha(\partial/\partial t)$, $V = d\alpha(\partial/\partial s)$. By the first and the second variation formula, we have

$$\begin{aligned} \ell'(s) &= \frac{1}{\ell(s)} \langle V, T \rangle|_0^1, \\ \ell''(s) &= \frac{1}{\ell(s)} \left\{ \langle \nabla_V V, T \rangle|_0^1 + \int_0^1 \|\nabla_T N\|^2 + \langle R(V, T)V, T \rangle dt \right\} \\ &= \frac{1}{\ell(s)} \{ \langle \nabla_V V, T \rangle|_0^1 + \langle V^\perp, \nabla_T V^\perp \rangle|_0^1 \} \\ &= \frac{1}{\ell(s)} \{ \langle \nabla_{\tilde{c}(s)} \dot{\tilde{c}}(s), N \rangle + \langle V^\perp, \nabla_T V^\perp \rangle|_{t=1} - \langle \dot{\tilde{c}}(s), \nabla_{\tilde{c}(s)} N \rangle \} \\ &= \frac{1}{\ell(s)} \langle V^\perp, \nabla_T V^\perp \rangle|_{t=1}, \end{aligned}$$

where $V^\perp = V - \langle V, T \rangle T$ and $N = \text{grad } d_{\partial M}$. Let $J(t)$ be the Jacobi tensor along $\exp_{\tilde{c}(s)} tN$ with $J(0) = I$ and $J'(0) = -S$.

Then $V^\perp(s, t) = J(\ell(s)t)\dot{c}(s)$, and so, we find

$$\begin{aligned} \langle V^\perp, \nabla_T V^\perp \rangle|_{t=1} &= \ell(s)\langle V^\perp, U(\ell(s))V^\perp \rangle \\ &= \ell(s)\langle \dot{c}(s), U(\ell(s))\dot{c}(s) \rangle. \end{aligned}$$

Then, by (3.2), we have (3.7) which implies (3.8) and (3.9). q.e.d.

4. Proof of Theorem A

In this section, we prove the following theorem which implies Theorem A. (cf. [7], Proposition 3.5) From now on, without loss of generality, we always assume $-\lambda^+ \leq \lambda^- \leq 0$.

THEOREM A'. *Given $\delta > 0$. There exists $\varepsilon_0(\delta) = \varepsilon_0(n, \Lambda, \lambda^+, \lambda^-, i_0; \delta) > 0$ such that if $\varepsilon < \varepsilon_0(\delta)$ and $M, M' \in \mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$ have ε -dense and $\varepsilon/10$ -discrete nets $\{z_i\}_{i=1}^{n_\varepsilon} \subset M, \{z'_i\}_{i=1}^{n_\varepsilon} \subset M'$ which satisfy*

$$1 - \varepsilon < \frac{d(z_i, z_j)}{d(z'_i, z'_j)} < 1 + \varepsilon \tag{*}$$

for $0 < d(z_i, z_j) < \frac{1}{2} \min(i_0, \pi/\sqrt{\Lambda}, t_0)$, then there exists a diffeomorphism $\tilde{F}: M \rightarrow M'$ such that

$$1 - \delta < |dF| < 1 + \delta.$$

Here we recall the definition of net. A subset $\{z_i\}_{i \in I}$ in a metric space Z is called an ε -dense net if $\bigcup_{i \in I} U_\varepsilon(z_i) = Z$ and called ε -discrete if $d(z_i, z_j) > \varepsilon$ for any $i \neq j$. Throughout this section, $C_i(\varepsilon_i(\delta), r_i(\delta)$, resp.), $i = 1, 2, 3, \dots$, is a constant depending only on $n, \Lambda, \lambda^+, \lambda^-$ and i_0 ($n, \Lambda, \lambda^+, \lambda^-, i_0$ and δ , resp.).

When M and M' are closed manifolds, such a diffeomorphism has been constructed by M. Gromov [7], A. Katsuda [10], S. Peters [14] and R. Green, H. Wu [6]. Their constructions are applicable to our case. In fact, according to [10, section 8], we have the following.

FACT 4.1. *Given $1 > \delta > 0$. There exist positive $\varepsilon_1(\delta), r_1(\delta)$ such that if $\varepsilon < \varepsilon_1(\delta)$ and $M, M' \in \mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$ have ε -dense and $\varepsilon/10$ -discrete nets $\{z_i\}_{i=1}^{n_\varepsilon} \subset M, \{z'_i\}_{i=1}^{n_\varepsilon} \subset M'$ which satisfy (*) then there exists an embedding $F = F(\delta): M[\delta, +\infty) \rightarrow M'$ with the following properties:*

$$(1) \quad 1 - \delta < |dF| < 1 + \delta,$$

- (2) $|d_M(z_i, z) - d_{M'}(z'_i, F(z))| < C_1\varepsilon, \quad i = 1, 2, \dots, n$, where $z \in M[\delta, +\infty)$.
- (3) Choose $z_{i_j}, j = 1, 2, \dots, n$, such that

$$r_1(\delta) \geq d_M(z_{i_j}, z) \geq r_1(\delta)/10, \quad j = 1, 2, \dots, n,$$

$$|u_{i_j}| \geq 1 - 1/100n^2,$$

$$|\langle u_{i_j}, u_{i_k} \rangle| \leq 1/100n^2, \quad j \neq k,$$

where $u_{i_j} = \exp_z^{-1} z_{i_j} / |\exp_z^{-1} z_{i_j}|$. (Such a choice of $z_{i_j}, j = 1, 2, \dots, n$, is possible when $\varepsilon_1(\delta)$ tends to zero.) Let $\{e_j\}_{j=1}^n$ be an orthonormal base of $T_z M$ obtained from $\{u_{i_j}\}_{j=1}^n$ by Schmidt orthogonalization. Similarly we obtain an orthonormal base $\{e'_j\}_{j=1}^n$ of $T_{F(z)} M'$ from $u'_{i_j} = \exp_{F(z)}^{-1} z_{i_j} / |\exp_{F(z)}^{-1} z_{i_j}|, j = 1, 2, \dots, n$.

Then the isometry $I: T_z M \rightarrow T_{F(z)} M'$ given by $I(e_j) = e'_j, j = 1, 2, \dots, n$, satisfies

$$|dF - I| < \delta.$$

Furthermore, by choosing sufficiently small $\varepsilon_1(\delta)$, we can assume

$$\frac{\sqrt{\varepsilon_1(\delta)}}{r_1(\delta)} \leq C_2\delta^2, \quad r_1(\delta) \leq C_3\delta. \tag{4.1}$$

We extend the vector field N (N' , resp.) on ∂M ($\partial M'$, resp.) to a unit normal vector field on $M[0, i_0]$ ($M'[0, i_0]$, resp.) by

$$N = \text{grad } d_{\partial M} \quad (N' = \text{grad } d_{\partial M'}, \text{ resp.}),$$

where $d_{\partial M} = d(\partial M, *)$ ($d_{\partial M'} = d(\partial M', *)$, resp.).

From Fact 4.1, we can derive the following.

FACT 4.1'. There exists $\varepsilon_2(\delta) > 0$ such that if $\varepsilon < \varepsilon_2(\delta)$ then the embedding $F: M[r_1(\delta), +\infty) \rightarrow M'$ given in Fact 4.1 satisfies the following:

$$F(M(t)) \subset M'[t - C_4\sqrt{\varepsilon}, t + C_4\sqrt{\varepsilon}], \tag{2'}$$

for $0 \leq t < R := \frac{1}{3} \min(i_0, \pi/\sqrt{\Lambda}, t_0)$

$$|N'_{F(z)} - dF_z(N)| < C_5\delta, \quad \text{for } z \in M[r_1(\delta), R). \tag{3'}$$

Fact 4.1'(2') is a direct consequence of Fact 4.1(2) and the following lemma which will be proved in the next section.

LEMMA 4.2. *There exists $\varepsilon_3 > 0$ such that if $\varepsilon < \varepsilon_3$ then, for any $t \in [0, R]$ and $z_i \in M[t - \varepsilon, t + \varepsilon]$, $z'_i \in M'[t - C_6\sqrt{\varepsilon}, t + C_6\sqrt{\varepsilon}]$.*

Let $\Phi: M[0, i_0] \rightarrow \partial M \times [0, i_0]$ be a diffeomorphism defined by $\Phi(y) = (x, t)$ for $y = \exp_x tN$, $x \in \partial M$. Then we define two diffeomorphisms:

$$\begin{aligned} \Phi'_{t'}: M(t) &\rightarrow M(t'): (x, t) \rightarrow (x, t'), \\ \Phi_{t,t'}: M[t, t'] &\rightarrow M(t) \times [t, t']: (x, s) \rightarrow ((x, t), s), \end{aligned}$$

for $t \leq s \leq t'$. Since $d\Phi'_{t'} = J(t')/J(t)$, Lemma 3.3 implies the following.

LEMMA 4.3. *For $0 \leq t < t' < R$,*

$$\begin{aligned} 1 - C_7|t' - t| &< |d\Phi'_{t'}| < 1 + C_7|t' - t|, \\ 1 - C_7|t' - t| &< |d\Phi_{t,t'}| < 1 + C_7|t' - t|, \end{aligned}$$

Here we assume that $M(t) \times [t, t']$ is equipped with the product metric.

Now, we verify Fact 4.1'(3').

Proof of Fact 4.1'(3'). For $z \in M(t)$, $r_1(\delta) \leq t \leq R$, choose z_i in the ε -neighborhood of $y = \exp_z(-r_1/2)N$, and put

$$\tilde{N} = -\exp_z^{-1}z_i/|-\exp_z^{-1}z_i|, \quad \tilde{N}' = -\exp_{F(z)}^{-1}z'_i/|-\exp_{F(z)}^{-1}z'_i|.$$

Then by Fact 4.1(1), (3), (4.1) and Rauch comparison theorem, we have

$$\begin{aligned} |dF(N) - \tilde{N}'| &\leq |dF(N) - dF(\tilde{N})| + |dF(\tilde{N}) - \tilde{N}'| \\ &\leq |dF| |N - \tilde{N}| + |(dF - I)(\tilde{N})| \\ &\leq (1 + \delta)\text{dil}_{r, \varepsilon}(r_1/2 - \varepsilon)^{-1} + \delta \leq C_8\delta. \end{aligned} \tag{4.2}$$

Here $\text{dil}_r \leq \sinh \sqrt{\Lambda r}/\sqrt{\Lambda r}$ is the dilatation of $\exp_z: T_z M \rightarrow B(z, r)$.

Next, we estimate $|\tilde{N}' - N'|$. Put $t' = d(\partial M', F(z))$ and $t'' = d(\partial M', z'_i)$. Then, by Fact 4.1'(2') and Lemma 4.2, we have

$$|t' - t| \leq C_4\sqrt{\varepsilon}, |t'' - (t - r_1/2)| \leq C_6\sqrt{\varepsilon} + \varepsilon,$$

and so,

$$|(t'' - t') - r_1/2| \leq (C_4 + C_6)\sqrt{\varepsilon} + \varepsilon.$$

On the other hand, by Fact 4.1(2), we find

$$|d(z'_i, F(z)) - r_1/2| \leq (C_1 + 1)\varepsilon.$$

Put $y' = \exp_{F(z)} - (t' - t'')N'$. By Rauch comparison theorem, we obtain

$$|N' - \tilde{N}'| \leq (r_1/2 - C_8\sqrt{\varepsilon})^{-1} \cdot \text{dil}_{r_1} \cdot d(z'_i, y'). \tag{4.3}$$

Consider $\Phi'_{t'', t'} : M'[t'', t'] \rightarrow M'(t'') \times [t'', t']$ and apply Lemma 4.3. Then we find

$$\begin{aligned} & d(\Phi'(z'_i), \Phi'(y'))^2 \\ &= d(\Phi'(z'_i), \Phi'(F(z)))^2 - d(\Phi'(F(z)), \Phi'(y'))^2 \\ &\leq \{(t' - t'') + (C_4 + C_6)\sqrt{\varepsilon} + (C_1 + 2)\varepsilon\}^2 \cdot (1 + |t' - t''|C_7)^2 - (t' - t'')^2 \\ &\leq r_1/2 \cdot C_9\sqrt{\varepsilon} + C_{10}r_1^2\sqrt{\varepsilon}, \end{aligned}$$

and so,

$$d(z'_i, y') \leq C_{11}(\sqrt{r_1}\varepsilon^{1/4} + r_1\varepsilon^{1/4}) \leq C_{12}\sqrt{r_1}\varepsilon^{1/4}. \tag{4.4}$$

Combining (4.3) with (4.1) and (4.4), we get

$$|N' - \tilde{N}'| \leq C_{13}\varepsilon^{1/4}r_1^{-1/2} \leq C_{14}\delta. \tag{4.5}$$

By (4.2) and (4.5), we get Fact 4.1'(3'). q.e.d.

Proof of Theorem A'. For $z = (x, t) \in \partial M \times [r_1, R/2]$, we put $z' = F(z) = (x', t') \in \partial M' \times [r_1 - C_4\sqrt{\varepsilon}, R/2 + C_4\sqrt{\varepsilon}]$. Here we identify $M[r, r']$ ($M'[r, r']$, resp.) with $\partial M \times [r, r']$ ($\partial M' \times [r, r']$, resp.), for $0 \leq r < r'$, through Φ .

Step 1. We define a map $G: F(M[r_1, R/2]) \rightarrow M'[r_1, R/2 + C_4\sqrt{\varepsilon}]$ by

$$G(z') = z'' = (x', t''), \quad t'' = t + \psi(t)(t - t'). \tag{4.6}$$

Here $\psi: [r_1, R/2] \rightarrow [0, 1]$ is a strictly increasing smooth function such that $\psi(r_1) = 0$, $\psi(R/2) = 1$ and $\psi'(t) < 4/R$, for $r_1 \leq t \leq R/2$. Then we have:

ASSERTION 4.4. $G \circ F: M[r_1, R/2] \rightarrow M'[r_1, R/2 + C_4\sqrt{\varepsilon}]$ satisfies

$$1 - C_{15}\delta < |d(G \circ F)| < 1 + C_{15}\delta. \tag{4.7}$$

Proof. Let $X \in T_z M$ be a unit vector orthogonal to N . We have an orthogonal decomposition of $dF(N), dF(X)$ at $T_{F(z)}M'$:

$$\begin{aligned} dF(N) &= N_1 \oplus N_2, \quad \text{where } N_1 = \langle dF(N), N' \rangle N', \\ dF(X) &= X_1 \oplus X_2, \quad \text{where } X_1 = \langle dF(X), N' \rangle N', \end{aligned}$$

respectively. By Fact 4.1', a direct computation yields

$$\begin{aligned} -C_{16}\delta &< \langle dF(N), dF(X) \rangle < C_{16}\delta, \\ 1 - C_{16}\delta &< |N_1| < 1 + C_{16}\delta, \\ |N_2| &< C_{16}\delta, \end{aligned} \tag{4.8}$$

$$\begin{aligned} |X_1| &< C_{16}\delta, \\ 1 - C_{16}\delta &< |X_2| < 1 + C_{16}\delta. \end{aligned} \tag{4.8}$$

In order to compute $d(G \circ F)$, we introduce the following two maps:

$$\begin{aligned} F' : \partial M \times [r_1, R/2] &\rightarrow \partial M' \times [r_1 - C_4\sqrt{\varepsilon}, R/2 + C_4\sqrt{\varepsilon}] \times \\ &\times [-C_4\sqrt{\varepsilon}, C_4\sqrt{\varepsilon}] : (x, t) \rightarrow (x', t, t' - t), \\ G' : \partial M' \times [r_1 - C_4\sqrt{\varepsilon}, R/2 + C_4\sqrt{\varepsilon}] &\times [-C_4\sqrt{\varepsilon}, C_4\sqrt{\varepsilon}] \rightarrow \partial M' \times \\ &\times [r_1, R/2 + C_4\sqrt{\varepsilon}] : (x', t, s) \rightarrow (x', t + \psi(t)s) := (x', t''). \end{aligned}$$

Then we have $G \circ F = G' \circ F'$. Since

$$\begin{aligned} dF'(N) &= (N_2, N', N_1 - N'), \\ dF(X) &= (X_2, 0, X_1), \end{aligned}$$

and $dt' = (1 + \psi'(t)s) dt + \psi(t) ds$, it follows that

$$\begin{aligned} d(G' \circ F')(N) &= (N_2, \{1 + \psi'(t)(t' - t)\}N' + \psi(t)(N_1 - N')), \\ d(G' \circ F')(X) &= (X_2, \psi(t)X_1). \end{aligned} \tag{4.9}$$

Combining (4.9) with (4.8) and the fact $|t' - t| < C_4\sqrt{\varepsilon}$, we obtain

$$\begin{aligned} 1 - C_{17}\delta &< |d(G \circ F)(N)| < 1 + C_{17}\delta, \\ 1 - C_{17}\delta &< |d(G \circ F)(X)| < 1 + C_{17}\delta, \\ |\langle d(G \circ F)(N), d(G \circ F)(X) \rangle| &< C_{17}\delta. \end{aligned} \tag{4.10}$$

From (4.10), Assertion 4.4 follows immediately. q.e.d.

ASSERTION 4.5. $G \circ F$ is one to one.

Proof. Let $z_1 = (x_1, t_1), z_2 = (x_2, t_2)$ be two distinct points in $M[r_1, R/2]$. If $t_1 = t_2$, then $z_i \in F(M(t_i)), i = 1, 2$, and so, $x_1 \neq x_2, z_1'' \neq z_2''$. Next we assume that $x_1 = x_2$ and $t_1 < t_2$. Then, since $F(M[t_1, +\infty)) \supset F(M[t_2, +\infty))$, we find $t_1' < t_2'$. Put $c(t') = (x_1', t')$, for $t_1' \leq t' \leq t_2'$ and put $c_0 = F^{-1}(c), c_1 = G(c)$. By fact 4.1(1), we see $1 - 2\delta < |\dot{c}_0| < 1 + 2\delta$. Hence, by Fact 4.1', $1 - C_{18}\delta < |\dot{c}_1| =$

$|G \circ F(c_0)| < 1 + C_{18}\delta$. Therefore $t'' = d(\partial M', c_1(t'))$ is monotonous and so $t''_1 < t''_2, z''_1 \neq z''_2$. q.e.d.

Step 2. We define a diffeomorphism $H: M[0, +\infty) \rightarrow M[r_1, +\infty)$ by $H(z) = (x, \phi(t))$ for $z = (x, t)$. Here $\phi: [0, +\infty) \rightarrow [r_1, +\infty)$ is a strictly increasing smooth function such that

$$\phi(t) = \begin{cases} r_1 & \text{if } t = 0 \\ t & \text{if } t \geq R \end{cases}, \quad \frac{R - r_1}{R} \leq \phi'(t) \leq 1. \tag{4.11}$$

Similarly we define $H': M'[0, +\infty) \rightarrow M'[r_1, +\infty)$. Then we have:

ASSERTION 4.6. $||dH| - 1| \leq C_{19}\delta$.

Proof. By (4.11), we find

$$\frac{R - r_1}{R} \cdot t \leq \phi(t) \leq t,$$

and so, $|\phi(t) - t| \leq r_1/R$. Since $dH = \Phi'_{\phi(t)} \oplus \phi'(t)$, by Lemma 4.3 and (4.11), we get

$$\min\left(1 - \frac{r_1}{R}C_7, \frac{R - r_1}{R}\right) \leq |dH| \leq 1 + \frac{r_1}{R}C_7.$$

Then put $C_{19} = \frac{C_3}{R} \min(C_7, 1)$. q.e.d.

Step 3. Define $\overline{G \circ F}: M[r_1, +\infty) \rightarrow M'[r_1, +\infty)$ by

$$\overline{G \circ F}(z) = \begin{cases} G \circ F(z) & \text{if } z \in M[r_1, R/2] \\ F(z) & \text{if } z \in M(r/2, +\infty) \end{cases}$$

and put

$$\tilde{F}(\delta) = H'^{-1} \circ \overline{G \circ F}(\delta) \circ H: M \rightarrow M'.$$

Then, Fact 4.1', Assertions 4.4, 4.5 and 4.6 imply that $\tilde{F}(\delta): M \rightarrow M'$ is a Lipschitz diffeomorphism satisfying

$$1 - C_{20}\delta < |d\tilde{F}(\delta)| < 1 + C_{20}\delta.$$

Put $\tilde{F} = \tilde{F}(\delta/C_{20}): M \rightarrow M'$. Then \tilde{F} satisfies the condition of Theorem A'.

q.e.d.

5. Proof of Lemma 4.2

Let M, M' be an element of $\mathbf{M}_0(n, \Lambda, \lambda^+, \lambda^-, i_0)$ with ε -dense net $\{z_i\}_{i=1}^{n_\varepsilon}, \{z'_i\}_{i=1}^{n_\varepsilon}$, respectively, which satisfies

$$1 - \varepsilon < \frac{d(z_i, z_j)}{d(z'_i, z'_j)} < 1 + \varepsilon, \tag{*}$$

for $0 < d(z_i, z_j) < t_1 < \frac{1}{2} \min(i_0, t_0)$, where $t_0 = 1/\sqrt{\Lambda} \arctan \sqrt{\Lambda}/\lambda^+$.

Then we have the following lemma which implies Lemma 4.2.

LEMMA 5.1. *There exists $\varepsilon' = \varepsilon'(n, \Lambda, \lambda^+, \lambda^-, i_0) > 0$ such that if $\varepsilon < \varepsilon'$ then, for any $t \in [0, \frac{3}{4}t_1]$ and $z_i \in M[t - \varepsilon, t + \varepsilon]$, we have $z'_i \in M'[t - C\sqrt{\varepsilon}, t + C\sqrt{\varepsilon}]$. Here $C = C(n, \Lambda, \lambda^+, \lambda^-, i_0)$.*

Proof. Let $z_i \in M[0, t_1]$ and put $r = d(\partial M, z_i)$. First, we prove lemma for small r .

Let $\gamma(t)$ be a geodesic such that $\gamma(0) \in \partial M, \dot{\gamma}(0) = N$ and $\gamma(r) = z_i$. Put

$$B^+(z_i, R) = \exp_{z_i} \{v \in T_{z_i}M \mid |v| \leq R, \langle v, N \rangle \geq 0\},$$

$$B^0(z_i, R) = \exp_{z_i} \{v \in T_{z_i}M \mid |v| \leq R, \langle v, N \rangle = 0\}.$$

For $x \in B^0(z_i, R)$, define $\bar{x} \in \partial M$ by $d(\bar{x}, x) = d(\partial M, x)$. Put

$$D = \{\exp_{\bar{x}} tN \mid x \in B^0(z_i, R), 0 \leq t \leq d(\bar{x}, x)\}.$$

If r is sufficiently small then Lemma 3.4 implies that

- (a) $B^0(z_i, R), B^+(z_i, R)$ and D are well defined for $R = \sqrt{2/A(t_1)}\sqrt{r}$
- (b) $d(\bar{x}, x) \leq \frac{1}{2}\{A(t_1) + \max(\sqrt{\Lambda}, -\lambda^-)\}R^2 := \ell(R)$ for any $x \in B^0(z_i, R)$,
- (c) $B(z_i, R') \subset D \cup B^+(z_i, R')$, where $R' = R - \ell(R)$.

Choose z_{i_1} in the ε -neighborhood of $\exp_{z_i}(R' - 3\varepsilon)\dot{\gamma}(r)$. Then, by Rauch comparison theorem, any $y \in B(z_i, R')$ satisfies

$$d(y, z_{i_1}) \leq \ell(R) + \text{dil}_R \sqrt{2R}, \tag{5.1}$$

where $\text{dil}_R \leq \sinh \sqrt{\Lambda}R / \sqrt{\Lambda}R$ is the dilatation of $\exp_{z_i}: T_{z_i}M \rightarrow B(z_i, R)$.

Assume $z'_i \in M'[R, \infty)$ and r is sufficiently small. Since $(R' - 4\varepsilon)(1 - \varepsilon) <$

$d(z'_i, z'_{i_1}) < (R' - 2\varepsilon)(1 + \varepsilon)$, we can take $z'_{i_2} \in B(z'_i, R')$ in the ε -neighborhood of $\exp_{z'_i}(-\exp_{z'_i} z'_{i_1})$. Then we have

$$d(z'_{i_1}, z'_{i_2}) \geq 2(R' - 4\varepsilon)(1 - \varepsilon). \tag{5.2}$$

Since $d(z'_i, z'_{i_2}) < (R' - 2\varepsilon)(1 + \varepsilon) + \varepsilon$, we find $z_{i_2} \in B(z_i, R')$ and, by (5.1) and (5.2),

$$\{2(R' - 4\varepsilon)(1 - \varepsilon) - \varepsilon\}(1 - \varepsilon) < d(z_{i_1}, z_{i_2}) < \ell(R) + \text{dil}_R \sqrt{2R}. \tag{5.3}$$

Now, let r tend to zero. Then R tends to zero and, in (5.3), we find

$$\begin{aligned} \text{the left side} &\rightarrow 2R + O(R^2), \\ \text{the right side} &\rightarrow \sqrt{2R} + O(R^2). \end{aligned}$$

Hence there exists $0 < \varepsilon_1 \ll 1$ such that if $r < \varepsilon_1$ then (5.3) does not hold. Therefore for $r \leq \varepsilon < \varepsilon_1$ we have $z'_i \in M'[0, R] \subset M'[0, \sqrt{2/A(t_1)}\sqrt{\varepsilon}]$.

Nextly, we prove lemma for $r \in [t - \varepsilon, t + \varepsilon]$, $0 < t < \frac{3}{4}t_1$. Choose $z_j \in M[0, \varepsilon]$, $z'_k \in M'[0, \varepsilon]$ such that

$$|d(z_j, z_i) - d(\partial M, z_i)| < \varepsilon, \quad |d(z'_k, z'_i) - d(\partial M', z'_i)| < \varepsilon.$$

Then we have

$$\begin{aligned} d(\partial M', z'_i) &\geq -d(z'_k, \partial M') + d(z'_k, z'_i) \geq -\varepsilon + (1 - \varepsilon)d(z_k, z_i) \\ &\geq -\varepsilon + (1 - \varepsilon)(t - \varepsilon - \sqrt{2/A(t_1)}\sqrt{\varepsilon}) \geq t - C\sqrt{\varepsilon} \end{aligned}$$

and

$$\begin{aligned} d(\partial M', z'_i) &\leq d(\partial M', z'_j) + d(z'_j, z'_i) \\ &\leq \sqrt{2/A(t_1)}\sqrt{\varepsilon} + (1 + \varepsilon)(t + 2\varepsilon) \leq t + C\sqrt{\varepsilon}, \end{aligned}$$

where $C = \sqrt{2/A(t_1)} + 6 + i_0$. q.e.d.

6. Estimates on i_M

In this section, we prove the following two propositions which imply Proposition 2.4.

PROPOSITION 6.1. For any $M \in \mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$,

$$i_\partial \geq \min\left(\frac{1}{\sqrt{\Lambda}} \arctan \frac{\sqrt{\Lambda}}{\lambda^+}, \frac{\pi V}{\omega_n} \left\{ \frac{1}{\sqrt{\Lambda}} \sinh\left(\sqrt{\Lambda} \cdot \min\left(d, \frac{\pi}{2\sqrt{\Lambda}}\right)\right) \right\}^{1-n}\right),$$

$$i_{\text{int}} \geq \min\left(\frac{\pi}{\sqrt{\Lambda}}, \frac{\pi V}{\omega_n} \left\{ \frac{1}{\sqrt{\Lambda}} \sinh\left(\sqrt{\Lambda} \cdot \min\left(d, \frac{\pi}{2\sqrt{\Lambda}}\right)\right) \right\}^{1-n}\right).$$

Here ω_n is the volume of the unit n sphere S^n .

PROPOSITION 6.2. There exist $\lambda_0^- < 0, i_0 > 0$ depending on $n, \Lambda, \lambda^+, d, V$ and d_∂ such that for any $\lambda_0^- < \lambda^- \leq 0$ and $M \in \mathbf{M}_\partial(n, \Lambda, \lambda_0^-, \lambda^+, d, V; d_\partial)$, it holds that

$$i_M \geq i_0.$$

Following two lemmas are slight extension of [2, Lemma 5.6, Lemma 5.7].

LEMMA 6.3. If i_∂ is not the forcal radius of ∂M , then i_∂ is half the length of a geodesic orthogonally intersects ∂M at the end points.

Proof. Let p be a cut point of a normal geodesic γ_1 from $p_1 \in \partial M$ such that $\gamma_1(i_\partial) = p$. If p is not a forcal point of ∂M , then there exists another geodesic γ_2 from $p_2 \in \partial M$ to p . Let $h_i, i = 1, 2$, be the distance function from ∂M restricted near $\gamma_i, i = 1, 2$, respectively. Locally $h_i, i = 1, 2$, can be naturally extended beyond $p = \gamma_i(i_\partial), i = 1, 2$, respectively. Then $\dot{\gamma}_i(i_\partial) = \text{grad } h_i(p), i = 1, 2$, and $H = (h_1 - h_2)^{-1}(0)$ is a smooth hypersurface near p . If we assume $\text{grad}(h_1 + h_2)(p) \neq 0$ then $\text{grad}(h_1 + h_2)(p)$ be a tangent vector of S at p . Since

$$\langle \text{grad } h_1, \text{grad}(h_1 + h_2) \rangle_p > 0,$$

in the direction of $-\text{grad}(h_1 + h_2)$, there is a point $p' \in S$ such that $h_1(p') = h_2(p') < i_\partial$. Then p' is a cut point of ∂M and $d(\partial M, p') < i_\partial$. This is a contradiction and hence

$$\dot{\gamma}_1(i_\partial) = -\dot{\gamma}_2(i_\partial),$$

and so $\gamma_1 \cup (-\gamma_2)$ is a smooth geodesic and lemma holds. q.e.d.

LEMMA 6.4. If i_{int} is not the conjugate radius then it is the half of the length of a simple closed geodesic of M .

Proof. Let q be a cut point of $p \in M$ along two normal geodesics $\gamma_i(t), 0 \leq t \leq i_{\text{int}}, i = 1, 2$, satisfying $\gamma_i(0) = p$ and $\gamma_i(i_{\text{int}}) = q$. If $q \notin \partial M$ then, by the

same argument in the proof of Lemma 6.4, we have $\dot{\gamma}_1(i_{\text{int}}) = -\dot{\gamma}_2(i_{\text{int}})$. Assume $q \in \partial M$ and $\dot{\gamma}_1(i_{\text{int}}) \neq -\dot{\gamma}_2(i_{\text{int}})$. Then, since

$$\langle -(\dot{\gamma}_1(i_{\text{int}}) + \dot{\gamma}_2(i_{\text{int}})), N \rangle \geq 0,$$

in the direction of $-(\dot{\gamma}_1(i_{\text{int}}) + \dot{\gamma}_2(i_{\text{int}}))$ there exists a cut point q' of p so that $d(p, q') < i_{\text{int}}$ and this is a contradiction. Thus $\dot{\gamma}_1(i_{\text{int}}) = -\dot{\gamma}_2(i_{\text{int}})$. Similarly $\dot{\gamma}_1(0) = -\dot{\gamma}_2(0)$ and so $\gamma_1 \cup (-\gamma_2)$ is a simple closed geodesic. q.e.d.

We have already known that, for any $M \in \mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, d, V)$, the conjugate radius at any $p \in M$, forcal radius of ∂M is not less than $\pi/\sqrt{\Lambda}$, $1/\sqrt{\Lambda} \arctan \sqrt{\Lambda}/\lambda^+$, respectively. Therefore we can assume that $i_{\text{int}}, i_{\partial}$ is attained by the half of a geodesic given in Lemma 6.3, Lemma 6.4, respectively.

Proof of Proposition 6.1. Let γ be the geodesic of length ℓ given in Lemma 6.3. By the convexity of ∂M , $\text{exp}_\gamma: \mu \rightarrow M$ is surjective, where μ is the normal bundle of γ . By Heintz-Karcher's comparison theorem [8], we get

$$V \leq \text{vol}(M) \leq \omega_{n-2} \int_0^\ell dt \int_0^{\min(d, \pi/2\sqrt{\Lambda})} \left(\frac{\sinh \sqrt{\Lambda} r}{\sqrt{\Lambda}} \right)^{n-2} \cosh \sqrt{\Lambda} r \, dr,$$

and so,

$$i_{\partial} \geq \frac{1}{2}\ell \geq \frac{\pi V}{\omega_n} \cdot \left(\frac{\sinh(\sqrt{\Lambda} \cdot \min(d, \pi/2\sqrt{\Lambda}))}{\sqrt{\Lambda}} \right)^{1-n}.$$

Similarly we have

$$i_{\text{int}} \geq \frac{\pi V}{\omega_n} \cdot \left(\frac{\sinh(\sqrt{\Lambda} \cdot \min(d, \pi/2\sqrt{\Lambda}))}{\sqrt{\Lambda}} \right)^{1-n}.$$

q.e.d.

Proof of Proposition 6.2. Let γ be the geodesic of length ℓ given in Lemma 6.3. Put $M_\gamma = \text{exp}_\gamma(\mu)$, where μ is the normal bundle of γ . Then, by Heintze-Karcher's comparison theorem [8], we have

$$\text{vol}(M_\gamma) \leq \ell \cdot \frac{\omega_n}{2\pi} \left(\frac{\sinh(\sqrt{\Lambda} \cdot \min(d, \pi/2\sqrt{\Lambda}))}{\sqrt{\Lambda}} \right)^{n-1}. \tag{6.1}$$

Let q be a point in M/M_γ and $\tau = \tau_1 \cup \tau_2$ be the distance minimizing path from γ to q , where $\tau_2(s), 0 \leq s \leq s_1$, be a normal geodesic of M such that $\tau_2(0) =$

$q' \in \partial M$ and $\tau_2(s_1) = q$. Put $\ell(s) = d(\partial M, \tau_2(s))$. Then, by Lemma 3.4, we have

$$\ell(0) = \ell'(0) = 0, \quad \ell''(s) \leq B(\ell(s)), \quad \text{for } 0 \leq s \leq s_1 \leq d,$$

whenever $\ell(s) \leq t_0$. On the other hand, we have

$$\begin{aligned} \{B(\ell(s))\}' &= B'(\ell(s))\ell'(s) = \{\Lambda - B^2(\ell(s))\}\ell'(s) \\ &\leq \Lambda\ell'(s) \end{aligned}$$

and $\ell''(0) \leq B(0) = -\lambda^-$. Therefore $\ell'(s)$ is less than or equal to the solution

$$u(s) := \frac{-\lambda^- \sinh \sqrt{\Lambda} s}{\sqrt{\Lambda}}$$

of

$$u''(s) = \Lambda u(s), \quad u(0) = 0, \quad u'(0) = -\lambda^-.$$

Hence we obtain

$$\ell(s) \leq -\lambda^- \cdot \frac{\cosh \sqrt{\Lambda} d - 1}{\Lambda}$$

and so,

$$M/M_\gamma \subset M[0, -\lambda^- (\cosh \sqrt{\Lambda} d - 1)/\Lambda]. \tag{6.2}$$

By the equation of Gauss [3, Chapter 2], the intrinsic sectional curvature $K_{\partial M}$ of ∂M satisfies

$$|K_{\partial M}| \leq \Lambda + \max(\lambda^{+2}, \lambda^{-2}) := \tilde{\Lambda},$$

and so,

$$\text{vol}(\partial M) \leq \omega_{n-2} \int_0^{d_\partial} \left(\frac{\sinh \sqrt{\tilde{\Lambda}} r}{\sqrt{\tilde{\Lambda}}} \right)^{n-2} dr. \tag{6.3}$$

By (6.1), (6.2) and (6.3), we obtain

$$\ell \cdot \frac{\omega_n}{2\pi} \left(\frac{\sinh(\sqrt{\Lambda} \cdot \min(d, \pi/2\sqrt{\Lambda}))}{\sqrt{\Lambda}} \right)^{n-1}$$

$$\begin{aligned} &\geq V_M - \text{vol}(M[0, -\lambda^-(\cosh\sqrt{\Lambda}d - 1)/\Lambda]) \\ &\geq V - \omega_{n-2} \int_0^{d_0} \left(\frac{\sinh\sqrt{\Lambda}r}{\sqrt{\Lambda}}\right)^{n-2} dr \int_0^{-\lambda^-(\cosh\sqrt{\Lambda}d - 1)/\Lambda} b^{n-1}(r) dr. \end{aligned} \tag{6.4}$$

If $|\lambda^-|$ is sufficiently small then the right-hand side of (6.4) is positive and hence $i_{\partial} = \ell/2$ has a positive lower bound which depends on $n, \lambda^-, \lambda^+, d, V$ and d_{∂} . Similarly we have a positive lower bound on i_{int} . q.e.d.

7. Precompactness

In this section, we prove Proposition 2.5. Firstly, Proposition 5.2 of [7] implies the following.

LEMMA 7.1. *Let $\mathbf{M}(n, a(\varepsilon), V, d)$ be the class of compact n -dimensional Riemannian manifolds M such that the volume of M is not less than V , the volume of any metric ball $B(p, \varepsilon), p \in M$, of radius ε is not less than $a(\varepsilon)$, and the diameter of M is not larger than d . Then $\mathbf{M}(n, a(\varepsilon), V, d)$ is precompact with respect to the Hausdorff topology.*

Then, by Lemma 7.1 and Proposition 2.4, following two Lemmas 7.2 and 7.3 imply Proposition 2.5.

LEMMA 7.2. *For any $M \in \mathbf{M}_0(n, \Lambda, \lambda^-, \lambda^+, i_0)$ and $p \in M$, the volume $\text{vol}(B(p, \varepsilon))$ of metric ball $B(p, \varepsilon)$ of radius ε satisfies*

$$\text{vol}(B(p, \varepsilon)) \geq \text{vol}(S_{\pi/4}^{n-1}) \int_0^{\varepsilon} \left(\frac{\sinh\sqrt{\Lambda}r}{\sqrt{\Lambda}}\right)^{n-1} dr. \tag{7.1}$$

Here $S_{\pi/4}^{n-1}$ is the spherical cap of S^{n-1} of radius $\pi/4$.

Proof. Let $c(s), 0 \leq s \leq \varepsilon$, be a geodesic emanating from $p = c(0) \in \partial M$. Put $\ell(s) = d(\partial M, c(s))$. By Lemma 3.4, we have

$$\ell(s) \geq s \cos \varphi + (s^2/2)A(t_1),$$

where $\cos \varphi = \ell'(0) = \langle \dot{c}(0), N \rangle$ and $t_1 = \min(t_0, i_0)/2$. Hence if $0 < \varepsilon < \min\{-1/A(t_1), t_1\}$ and $0 \leq \tau\varphi \leq \pi/4$, then

$$\ell(s) > (\cos \varphi - \frac{1}{2})s \geq 0 \quad \text{for } 0 \leq s \leq \varepsilon,$$

and so $c(s), 0 < s \leq \varepsilon$, does not hit ∂M . Then, volume comparison theorem

implies (7.1) for $p \in \partial M$. Similarly we can verify that (7.1) holds for any $p \in M$.
 q.e.d.

LEMMA 7.3. Let $V_M, V_{M'}$ be the volume of element $M \in \mathbf{M}_c(n, \Lambda, \lambda^+, d, V)$, $M' \in \mathbf{M}_\delta(n, \Lambda, \lambda^-, \lambda^+, d, V: d_\delta)$, respectively. Then

$$V_M \leq \omega_{n-1} \int_0^d \left(\frac{\sinh \sqrt{\Lambda} r}{\sqrt{\Lambda}} \right)^{n-1} dr \tag{7.2}$$

$$V_{M'} \leq \omega_{n-2} \int_0^{d_\delta} \left(\frac{\sinh \sqrt{\tilde{\Lambda}} r}{\sqrt{\tilde{\Lambda}}} \right)^{n-2} dr \int_0^d b_{n-1}(r) dr. \tag{7.3}$$

Here $\tilde{\Lambda} := \Lambda + \max(\lambda^{+2}, \lambda^{-2})$.

Proof. Since the boundary of M is convex, $\exp_p: T_p M \rightarrow M$ is surjective and so the volume comparison theorem implies (7.2). Similarly, (6.3) and Heintze-Karcher's comparison theorem [8] implies (7.3).
 q.e.d.

8. Remarks

(1) When the sectional curvature K_M of M is positive and the boundary ∂M is p -convex ($1 \leq p \leq n$), the topology of M is completely classified by the works of J-P. Sha [16, 17], H-B. Lawson [11], H-B. Lawson and M-L. Michelsohn [12], H. Wu [18] and J. D. Moore and T. Schulte [13].

(2) Recently, author is able to refine the argument in section 4 and find the regularity of the metric g_∞ of the limit manifold M_∞ . He is also able to obtain an universal upper bound on the volume V_M for any $M \in \mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, i_0, d)$. Here $\mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, i_0, d) = \{M \in \mathbf{M}_0(n, \Lambda, \lambda^-, \lambda^+, i_0) \mid d_M \leq d\}$. Then, by Lemma 7.1 and 7.2, $\mathbf{M}(n, \Lambda, \lambda^-, \lambda^+, i_0, d)$ is precompact in Hausdorff distance, and, by Theorem A and the regularity of the metric g_∞ of M_∞ , we find that any sequence $\{M_i\}$ contains a subsequence $\{j\} \subset \{i\}$ such that M_j converges in Lipschitz distance to a n -dimensional Riemannian manifold M_∞ with metric g_∞ which is $C^{1+\alpha}$, $0 < \alpha < 1$, Hölder continuous in the interior of M_∞ .

The proof of this fact will appear elsewhere.

(3) It is likely that the restriction d_δ in Theorem B could be removed.

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