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On relative amenability for von Neumann algebras

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Introduction

The concept of correspondence between two von Neumann algebras has been introduced by A. Connes ([8], [9]) as a very useful tool for the study of type II₁ factors. Recently, S. Popa has systematically developed this point of view to get some new insight in the domain [21]. Among many interesting results and remarks, he discussed Connes' classical work on the injective II₁ factor in the framework of correspondences, and he defined and studied a natural notion of amenability for a finite von Neumann algebra $M$ relative to a von Neumann subalgebra $N$. When the Jones' index $[M : N]$ is finite or when $M$ is injective the inclusion $N \subset M$ is amenable, but this situation occurs in many other examples. For instance, if $M$ is the crossed product of a finite von Neumann algebra $N$ by an action of a discrete group $G$ preserving a faithful finite normal trace of $N$, then $N \subset M$ is amenable if and only if $G$ is an amenable group ([21], Th. 3.2.4).

In [28], Zimmer considered a notion of amenable action in ergodic theory, which was extended in [1] to actions on arbitrary von Neumann algebras. We say that the $G$-action $\alpha$ on $N$ is amenable if there exists an equivariant norm one projection from $L^\infty(G) \otimes N$ onto $N$, the $G$-action on $L^\infty(G) \otimes N$ being the tensor product of the action by left translation on $L^\infty(G)$ and the action $\alpha$ on $N$. When there exists a $G$-invariant state on the centre $Z(N)$ of $N$, the amenability of the action is equivalent to the amenability of the group ([1], Prop. 3.6). Otherwise, it is easy to construct amenable actions of non amenable groups. Since Popa's notion of amenable inclusion makes sense for arbitrary von Neumann algebras, he asked ([21], 3.4.2) whether the amenability of the $G$-action $\alpha$ was equivalent to the amenability of the inclusion $N \subset M = N \rtimes G$ in the case of a discrete group $G$ acting on any von Neumann algebra $N$. In this paper we give a positive answer to this question (Prop. 3.4).

As far as we are concerned with non finite von Neumann algebras $M$ and $N$, it seems more convenient to consider a correspondence between $M$ and $N$ as a self-dual right Hilbert $N$-module on which $M$ acts to the left, since it avoids the choice of auxiliary weights. This point of view has been already systematically used in [4] for the general study of the index of conditional expectations. In the
first section we recall the needed background on correspondences and Hilbert modules. In particular, to any inclusion \( N \subset M \) is associated a correspondence \( Y_N \) (see 1.8) between \( M \) and \( M \) which gives informations about the embedding \( N \subset M \). Popa has defined the inclusion to be amenable if the identity correspondence of \( M \) is weakly contained in \( Y_N \).

In Section 2 we consider an action \( \alpha \) of a discrete group \( G \) on a von Neumann algebra \( N \), and we denote by \( M \) the crossed product \( N \rtimes_\alpha G \). The classical notions of positive type functions and group representations can respectively be extended in this context of dynamical systems to notions of positive type functions on \( G \) with respect to \( (N, G, \alpha) \) and of cocycles (2.4 and 2.1). These two concepts are closely related, as in the usual case. For each cocycle \( T \) relative to \( (Z(N), G, \alpha) \) we associate in a natural way a correspondence \( X \) between \( M \) and \( M \) (2.6). A positive type function corresponding to \( T \) gives rise to a normal completely positive map from \( M \) to \( M \), which is a coefficient of the correspondence \( X \) (2.8). The positive type functions relative to \( (Z(N), G, \alpha) \), having finite supports, yield coefficients of the correspondence \( Y_N \) associated to the inclusion \( N \subset M \), and the constant positive type function equal to the unit of \( Z(N) \) gives the identity automorphism of \( M \), which is, of course, a coefficient of the identity correspondence of \( M \). We proved in [3] that the \( G \)-action \( \alpha \) on \( N \) is amenable if and only if this constant function is the limit, for the topology of the \( \sigma \)-weak pointwise convergence, of a net of positive type functions relative to \( (Z(N), G, \alpha) \) with finite supports. Using this fact, we show in Section 3 the equivalence between the amenability of the action and the amenability of the inclusion \( N \subset M \).

1. Preliminaries on correspondences

We recall here some facts on correspondences and Hilbert modules, mostly coming from [8], [9], [4], [21], [20], [22], [23], [24], [17], where the reader will find more details. For simplicity, in this paper we shall only consider \( \sigma \)-finite von Neumann algebras. Let \( M \) and \( N \) be two von Neumann algebras.

1.1. A correspondence between \( M \) and \( N \) is a Hilbert space \( H \) with a pair of commuting normal representations \( \pi_M \) and \( \pi_N^0 \) of \( M \) and \( N^0 \) (the opposite of \( N \)) respectively [8]. Usually the triple \( (H, \pi_M, \pi_N^0) \) will be denoted by \( H \), and for \( x \in M \), \( y \in N \) and \( h \in H \), we shall write \( xhy \) instead of \( \pi_M(x)\pi_N^0(y^0)h \).

Note that \( H \) gives rise to a representation of the binormal tensor product \( M \otimes_{\text{bin}} N^0 \) (see [11] for the definition of bin). Two correspondences \( H \) and \( H' \) are equivalent if they are (unitarily) equivalent when considered as representations of \( M \otimes_{\text{bin}} N^0 \).

We denote by \( \text{Corr}(M, N) \) the set of equivalence classes of correspondences between \( M \) and \( N \), and we shall use the same notation \( H \) for a correspondence
and its class. We shall write $\text{Corr}(M)$ for $\text{Corr}(M, M)$. The standard form [13] of $M$ yields an element $L^2(M)$ of $\text{Corr}(M)$ called the identity correspondence of $M$. We shall sometimes write $L^2(M, \varphi)$ instead of $L^2(M)$, with a fixed faithful normal positive form $\varphi$ on $M$.

1.2. Let us recall now another useful equivalent way of defining correspondences, which has been developed in [4]. Let $X$ be a self-dual (right) Hilbert $N$-module (see [20]). We denote by $\langle, \rangle$ (or $\langle, \rangle_N$ in case of ambiguity) the $N$-valued inner product, and we suppose that it is conjugate linear in the first variable. The von Neumann algebra of all $N$-linear continuous operators from $X$ to $X$ will be denoted by $\mathcal{L}_N(X)$ (or $\mathcal{L}(X)$ when $N = \mathbb{C}$). Following ([4], Def. 2.1), by a $M-N$ correspondence we mean a pair $(X, \pi)$ where $X$ is as above, and $\pi$ is a unital normal homomorphism from $M$ into $\mathcal{L}_N(X)$. More briefly, such a correspondence will be denoted by $X$, and we shall often write $x \xi$ instead of $\pi(x)\xi$ for $x \in M$ and $\xi \in X$. Let us remark that $M-N$ correspondences are what Rieffel has called normal $N$-rigged $M$-modules in ([23], Def. 5.1). Two $M-N$ correspondences $X$ and $X'$ are said to be equivalent if there exists a $M-N$ linear isomorphism from $X$ onto $X'$ preserving the scalar products.

1.3. Let $X$ be a self-dual Hilbert $N$-module. We call $s$-topology the topology defined on $X$ by the family of semi-norms $q_\varphi$, where $\varphi$ is any normal positive form on $N$ and

$$q_\varphi(\eta) = \varphi(\langle \eta, \eta \rangle)^{1/2}, \quad \text{for } \eta \in X.$$ 

We say that a vector $\xi$ in a $M-N$ correspondence $X$ is cyclic if the set $M\xi N = \{x\xi y, x \in M, y \in N\}$ is s-total in $X$.

The set of equivalence classes of $M-N$ correspondences will be denoted by $C(M, N)$, and we shall not make any distinction between a correspondence and its class. We shall write $C(M)$ instead of $C(M, M)$. There is a natural bijection $\wedge$ between $C(M, N)$ and $\text{Corr}(M, N)$, that will be described now.

1.4. Let $X \in C(M, N)$ and let $H_X = X \otimes_N L^2(N)$ be the Hilbert space obtained by inducing the standard representation of $N$ up to $M$ via $X$ ([22], Th. 5.1). The induced representation of $M$ in $H_X$ and the right action of $N$ on $H_X$ defined by

$$(\xi \otimes h)y = \xi \otimes (hy), \quad \text{for } \xi \in X, \quad h \in L^2(N), \quad y \in N,$$

give rise to an element $\wedge(X) = H_X$ of $\text{Corr}(M, N)$.

Conversely, given $H \in \text{Corr}(M, N)$, let $X_H$ be the space $\text{Hom}_{\varphi_0}(L^2(N), H)$ of continuous $N^0$-linear operators from $L^2(N)$ into $H$. Let $N$ acts on the right of $X_H$ by composition of operators and define on $X_H$ a $N$-valued inner product by $\langle r, s \rangle = r^*s$ for $r, s \in X_H$. Then $X_H$ is a self-dual Hilbert $N$-module ([23], Th. 6.5).
Moreover, $M$ acts on the left of $X_H$ by composition of operators and we obtain in this way a $M-N$ correspondence.

The maps $X \mapsto H_X$ and $H \mapsto X_H$ are inverse from each other ([4], Th. 2.2 and [23], Prop. 6.10). In fact, there is a natural isomorphism between the $M-N$ correspondences $X$ and $\text{Hom}_{N^0}(L^2(N), X \otimes_N L^2(N))$, given by assigning to any $\xi \in X$ the element $\Theta_\xi: h \mapsto \xi \otimes h$ of $\text{Hom}_{N^0}(L^2(N), X \otimes_N L^2(N))$.

1.5. Let $M, N, P$ be von Neumann algebras, $X \in C(M, N)$ and $Y \in C(N, P)$. We denote by $X \otimes_N Y$ the self-dual completion (see [20], Th. 3.2) of the algebraic tensor product $X \otimes Y$ endowed with the obvious right action of $P$ and the $P$-valued inner product

$$\langle \xi \otimes \eta, \xi_1 \otimes \eta_1 \rangle = \langle \eta, \langle \xi, \xi_1 \rangle_N \eta_1 \rangle_p, \quad \text{for } \xi, \xi_1 \in X, \quad \eta, \eta_1 \in Y.$$ 

**LEMMA.** (i) For $x \in \mathcal{L}_N(X)$, there is an element $\rho(x)$ in $\mathcal{L}_P(X \otimes_N Y)$ well defined by

$$\rho(x)(\xi \otimes \eta) = (x \xi) \otimes \eta, \quad \text{for } \xi \in X, \eta \in Y.$$ 

We get in this way a normal homomorphism from $\mathcal{L}_N(X)$ into $\mathcal{L}_P(X \otimes_N Y)$.

(ii) If the representation of $N$ into $\mathcal{L}_P(Y)$ is faithful, then $\rho$ is faithful.

(iii) If we take $Y = L^2(N)$, viewed as an element of $C(N, \mathbb{C})$, then $\rho$ is an isomorphism of the von Neumann algebra $\mathcal{L}_N(X)$ onto the commutant $\text{Hom}_{N^0}(H_X, H_X)$ of the right action of $N$ on $H_X = X \otimes_N L^2(N)$.

**Proof.** For the proof of (i) see ([22], Th. 5.9 and [4], Prop. 2.9). Let us show that $\rho$ is isometric under the assumption of (ii). If $\xi \in X$ we define a continuous $P$-linear operator $\Theta_\xi$ from $Y$ into $X \otimes_N Y$ by $\Theta_\xi(\eta) = \xi \otimes \eta$ for $\eta \in Y$. It is easily checked that $$(\Theta_\xi)^*(\xi' \otimes \eta) = \langle \xi, \xi' \rangle_N \eta$$ for $\xi' \in X$ and $\eta \in Y$, so that $\|\Theta_\xi\|^2 = \|\Theta_\xi^* \Theta_\xi\| = \|\langle \xi, \xi \rangle_N\| = \|\xi\|^2$.

Let $x \in \mathcal{L}_N(X)$ and $\varepsilon > 0$, and take $\xi \in X$ with $\|\xi\| = 1$ and $\|x \xi\| \geq \|x\| - \varepsilon$. Now choose $\eta \in Y$ with $\|\eta\| = 1$ and $\|\Theta_{x \xi}(\eta)\| \geq \|x \xi\| - \varepsilon$. Then we have

$$\|\rho(x)(\xi \otimes \eta)\| = \|x \xi \otimes \eta\| \geq \|x \xi\| - \varepsilon \geq \|x\| - 2\varepsilon$$

and

$$\|\xi \otimes \eta\| = \|\Theta_\xi(\eta)\| \leq \|\xi\| \|\eta\| = 1,$$

from which it follows that $\|\rho(x)\| = \|x\|$.

Let us prove (iii) now. Obviously the range of $\rho$ is contained in $\text{Hom}_{N^0}(H_X, H_X)$. Conversely, let $r \in \text{Hom}_{N^0}(H_X, H_X)$ and consider the element $\tilde{r}$ of $\mathcal{L}_N(X)$ such that
\[ \Theta_{r\xi} = r \circ \Theta_{\xi} \] for \( \xi \in X \). Then for \( \zeta \in X \) and \( h \in L^2(N) \) we have
\[ \rho(\tilde{r})(\zeta \otimes h) = \tilde{r}(\zeta) \otimes h = \Theta_{r\xi}(h) = r \circ \Theta_{\xi}(h) = r(\zeta \otimes h), \]
and thus \( \rho(\tilde{r}) = r \).

1.6. Keeping the notations of 1.5, we say that the self-dual Hilbert \( P \)-module \( X \otimes_N Y \) provided with the homomorphism of \( M \) into \( \mathcal{L}_p(X \otimes_N Y) \) given by restricting \( \rho \) is the\textit{ composition correspondence of \( X \) by \( Y \)}. It is the version in the setting of Hilbert modules of the composition of correspondences defined in ([8], §II).

There are other classical operations on correspondences. We shall need the following ones. Let \( H \in \text{Corr}(M, N) \) be a correspondence between \( M \) and \( N \). Let \( \overline{H} \) be the conjugate Hilbert space. If \( h \in H \), we denote by \( \overline{h} \) the vector \( h \) when viewed as an element of \( \overline{H} \). Then \( \overline{H} \) has a natural structure of correspondence from \( N \) to \( M \) by
\[ y \overline{h} x = x^* \overline{h} y^*, \quad \text{for } x \in M, \quad y \in N, \quad h \in H. \]
(see [21], 1.3.7). We call it the\textit{ adjoint correspondence of \( H \)}.

Thanks to the bijection \( \wedge \) between \( C(M, N) \) and \( \text{Corr}(M, N) \), we see that to each \( X \in C(M, N) \) we can associate an element \( \overline{X} \in C(N, M) \), also called the\textit{ adjoint correspondence of \( X \)}. In general we haven’t an explicit description of \( \overline{X} \) (see however 1.8 below).

A\textit{ subcorrespondence} of \( X \in C(M, N) \) is a submodule \( Y \) of \( X \) closed for the\( s \)-topology and stable by the left action of \( M \). There is a natural bijection between the set of subcorrespondences of \( X \) and the set of projections in \( \mathcal{L}_N(X) \) which commute with the range of \( M \) in \( \mathcal{L}_N(X) \) by the left action. If \( X \) and \( Y \) are two \( M \)–\( N \) correspondences, we say that \( Y \) is\textit{ contained in \( X \)} and we write \( Y \subseteq X \) if \( Y \) is equivalent to a subcorrespondence of \( X \).

1.7. We shall have to consider the following special case of composition of correspondences. Let \( H \) be a Hilbert space and \( N \) a von Neumann algebra. Then, in an obvious way, \( H \) is an element of \( C(\mathcal{L}(H), \mathbb{C}) \) and \( N \) is an element of \( C(\mathbb{C}, N) \). Thus we may define the composition correspondence \( H \otimes_c N \), written \( H \otimes N \) afterwards. The\( N \)-valued scalar product in \( H \otimes N \) is given by
\[ \langle h \otimes y, h_1 \otimes y_1 \rangle = \langle h, h_1 \rangle y^* y_1 \quad \text{for } h, h_1 \in H \quad \text{and} \quad y, y_1 \in N. \]
Take an orthonormal basis \( (e_i)_{i \in I} \) in \( H \). Denote by \( l^*_w(I, N) \) the right\( N \)-module of
nets \((y_i)_{i \in I}\) of elements of \(N\) such that \(\sum_{i \in I} y_i^* y_i\) is \(\sigma\)-weakly convergent. Provided with the \(N\)-module inner product \(\langle (x_i)_{i \in I}, (y_i)_{i \in I} \rangle = \sum_{i \in I} x_i^* y_i\), it is a self-dual Hilbert \(N\)-module, and the map which sends \((y_i)_{i \in I}\) on \(\sum_{i \in I} y_i \otimes y_i\) is an isomorphism of Hilbert \(N\)-modules from \(l^2(I, N)\) onto \(H \otimes N\). (See \([20]\), p. 457–459). We shall identify \(l^2(I, N)\) and \(l^2(I) \otimes N\). Remark that \(L^2_N(H \otimes N)\) may be identified to the von Neumann tensor product \(L^2(H) \otimes N\) in a natural way.

1.8. Next, we shall give fundamental examples of correspondences, related to completely positive maps. Let \(X \in C(M, N)\) and \(\xi \in X\). Then \(\Phi: x \mapsto \langle \xi, x\xi \rangle\) is a completely positive normal map from \(M\) into \(N\). We shall say that \(\Phi\) is a coefficient of \(X\), or is associated to \(X\).

Conversely, given a completely positive normal map \(\Phi\) from \(M\) into \(N\), by the Stinespring construction we get a \(M-N\) correspondence \(X_\Phi\). The self-dual Hilbert \(N\)-module is obtained by separation and self-dual completion of the right \(N\)-module \(M \otimes N\) (algebraic tensor product) gifted with the \(N\)-module inner product

\[
\langle m \otimes n, m_1 \otimes n_1 \rangle = n^* \Phi(m^* m_1) n_1, \quad \text{for } m, m_1 \in M, \quad n, n_1 \in N.
\]

The normal representation \(\pi_\Phi\) of \(M\) into \(L^2_N(X_\Phi)\) is given by

\[
\pi_\Phi(x)(m \otimes n) = x m \otimes n \quad \text{for } x, m \in M, \quad n \in N.
\]

If \(\xi_\Phi\) denotes the class of \(1 \otimes 1\) in \(X_\Phi\), we have \(\Phi(x) = \langle \xi_\Phi, x\xi_\Phi \rangle\) for each \(x \in M\), and \(\xi_\Phi\) is a cyclic vector for the correspondence \(X_\Phi\). We shall say that \(X_\Phi\) is the correspondence associated to \(\Phi\).

If \(X\) is a \(M-N\) correspondence and \(\xi\) is a cyclic vector in \(X\), then it is easily seen that \(X\) is equivalent to the correspondence \(X_\Phi\), where \(\Phi\) is the coefficient of \(X\) given by \(\xi\). Furthermore, every \(M-N\) correspondence is a direct sum of cyclic correspondences, so that, as pointed out by A. Connes in \([8]\), the notions of completely positive maps and correspondences are closely related.

When \(\Phi\) is a normal conditional expectation from \(M\) onto a von Neumann subalgebra \(N\), it is easily checked that \(X_\Phi\) is equivalent to the separated, self-dual completion of the right \(N\)-module \(M\) with \(N\)-valued inner product \((m, m_1) \mapsto \Phi(m^* m_1)\), endowed with the obvious left action of \(M\). More generally, to every semi-finite normal operator valued weight \(\Phi\) from \(M\) to \(N\) (see \([14]\), Def. 2.1), one can associate a \(M-N\) correspondence \(X_\Phi\) which extends the classical Gelfand-Segal construction for usual normal semi-finite weights (see \([4]\), Prop. 2.8).

The right \(M\)-module \(M\) endowed with its inner product \(\langle m, m_1 \rangle = m^* m_1\) is self-dual. Gifted with its natural left \(M\)-module structure, it is the \(M-M\) correspondence associated to the identity homomorphism of \(M\). It will be called
the identity $M-M$ correspondence, and denoted by $X_M$ or $M$; of course
$\Lambda(X_M) = L^2(M)$.

Let now $\rho$ be a normal homomorphism from $M$ into a von Neumann algebra $N$. It is straightforward to show that $X_\rho$ is equivalent to the Hilbert $N$-subspace $\rho(1)N$ of the right Hilbert $N$-module $N$, with left action of $M$ given by

$$x.n = \rho(x)n, \quad \text{for} \; x \in M, \; n \in \rho(1)N.$$ 

Suppose next that $N$ is a von Neumann subalgebra of $M$. The $N-M$ correspondence associated to the inclusion $i: N \to M$ will be denoted by $X_N$. Note that $X_N$ is obtained from $X_M = M$ by restricting to $N$ the left action of $M$. Remark also that $\Lambda(X_N)$ is $L^2(M)$ where we restrict to $N$ the standard representation of $M$ and keep the right action of $M$. Let $E$ be a faithful normal conditional expectation from $M$ onto $N$. It has been noticed in [4] that the (equivalence class of the) $M-N$ correspondence $X_E$ is the adjoint correspondence $\overline{X}_N$ of $X_N$. Indeed, it is shown in ([4], Corol. 2.14) that $\Lambda(X_E)$ is equivalent to $L^2(M)$ considered as a $M-N$ bimodule by restricting to $N$ the right action of $M$, and this correspondence is easily seen to be equivalent to the adjoint of $\Lambda(X_N)$, thanks to the antilinear involutive isometry $J$ of $L^2(M)$. (In fact, this remark remains true when $E$ is any faithful normal semi-finite operator valued weight from $M$ to $N$).

Even if there doesn't exist any conditional expectation from $M$ onto $N$, we may consider $\overline{X}_N$. Note that by Lemma 1.5(iii), $\mathcal{L}_N(\overline{X}_N)$ is isomorphic to the commutant of the right action of $N$ on $L^2(M)$, since $\Lambda(\overline{X}_N) = L^2(M)$ viewed as $M-N$ bimodule. It follows that the normal homomorphism from $M$ into $\mathcal{L}_N(\overline{X}_N)$ which appears in the definition of the $M-N$ correspondence $\overline{X}_N$ is injective, because it comes from the standard representation of $M$.

The $M-M$ correspondence $\overline{X}_N \otimes_M X_N$ will be denoted by $Y_N$. It has been introduced by Popa ([21], 1.2.4) in the finite case, as a very useful tool for the study of the inclusion $N \subset M$. When there exists a normal faithful conditional expectation $E$ from $M$ onto $N$, then $Y_N = X_E \otimes_N X_N$ and $Y_N$ is also the $M-M$ correspondence associated to $E$ viewed as a completely positive map from $M$ to $M$ (see [4], Th. 2.12).

Let us remark that $Y_M = X_M = M$. For $N = \mathbb{C}$, the $\mathbb{C}-M$ correspondence $X_\mathbb{C}$ is the Hilbert $M$-module $M$ with obvious action of $\mathbb{C}$, and $\overline{X}_\mathbb{C}$ is the Hilbert space $L^2(M)$ with the standard representation of $M$. Thus $Y_\mathbb{C} = \overline{X}_\mathbb{C} \otimes_\mathbb{C} X_\mathbb{C} = L^2(M) \otimes M$ is the coarse $M-M$ correspondence (see [8], Def. 3).

1.9. For later use, we prove the following result (see [21], Prop. 1.2.5.(ii)).

**LEMMA.** Let $M$ be a von Neumann algebra and $N$ a finite dimensional von Neumann subalgebra of $M$. Then we have $Y_N \subset Y_\mathbb{C}$.
Proof. Let $z_1, \ldots, z_k$ be the minimal projections of the centre $Z(N)$, and $(e^i_{pq})_{1 \leq p,q \leq n_j}$ a matrix units system for $Nz_j$ where $j = 1, \ldots, k$. Let $u^i_p = e^i_{pj}$ for $p = 1, \ldots, n_j$ and $j = 1, \ldots, k$. We choose a normal faithful state $\varphi$ on $M$ and we put $\alpha_j = \varphi(e^i_{11})$ for $j = 1, \ldots, k$. Then one easily checks that the map $E$ on $M$ defined by

$$E(x) = \sum_{1 \leq p,q \leq n_j} \sum_{j=1}^k \alpha_j \frac{1}{\alpha_j} u^i_p \varphi(u^i_p \star xu_q^i)u_q^i$$

is a normal faithful conditional expectation from $M$ onto $N$.

We take for $L^2(M)$ the standard form $L^2(M, \varphi)$ of the identity correspondence given by $\varphi$, and we identify $M$ to a subspace of $L^2(M, \varphi)$. Let

$$\xi = \sum_{1 \leq p \leq n_j} \sum_{j=1}^k \alpha_j \frac{1}{\alpha_j} u^i_p \otimes u^i_p \in Y_C = L^2(M, \varphi) \otimes M.$$ 

We have, for $x \in M,$

$$\langle \xi, x\xi \rangle = \sum_{p,q} \sum_{i,j} (1/\alpha_j^{1/2} \alpha_i^{1/2}) \langle u_p^i \otimes u_p^i, xu_p^i \otimes u_p^i \rangle$$

$$= \sum_{p,q} \sum_{i,j} (1/\alpha_j^{1/2} \alpha_i^{1/2}) u_p^i \varphi(u_p^i \star xu_q^i)u_q^i$$

$$= E(x) = \langle \xi, x\xi \rangle,$$

where $\Phi$ is $E$ considered as a completely positive map from $M$ to $M$. Thus, $x\xi y \mapsto x\Phi(y)$, with $x, y \in M$, induces an equivalence between the subcorrespondence of $L^2(M, \varphi) \otimes M$ having $\xi$ as cyclic vector and $Y_N$ which is the $M$-$M$ correspondence associated to $\Phi$.

Notice that $\Phi$ appears as a completely positive map which is a finite sum of completely positive maps factored by $\varphi$ in the sense of ([19], Def. 1).

1.10. LEMMA. A correspondence $X$ contains the identity correspondence $M$ if and only if there exists a non zero central and separating vector $\xi$ in $X$ (i.e. $\xi x = x\xi$ for all $x \in M$ and if $\xi x = 0$ then $x = 0$).

Proof. The necessity of the existence of $\xi$ is obvious. Conversely suppose that there is a non zero separating central vector $\xi$ in $X$. Then $\langle \xi, \xi \rangle$ belongs to $Z(M)$ and its support is 1. Consider the polar decomposition $\xi = \eta \langle \xi, \xi \rangle^{1/2}$ of $\xi$ (see [20], Prop. 3.11). Then $\eta$ is central and since $\langle \eta, \eta \rangle$ is the support of $\langle \xi, \xi \rangle$, we have $\langle \eta, \eta \rangle = 1$. Now it is easy to prove that $\eta M$ defines a subcorrespondence of $X$ equivalent to $M$. \qed
1.11. REMARK. In ([21], Prop. 1.2.5) Popa has shown that for type II₁ factors $N \subset M$ the properties $[M:N] < \infty$ and $M \subset Y_N$ are closely related, where $[M:N]$ denotes as usually the Jones' index. More generally, let $E$ be a faithful normal conditional expectation from a von Neumann algebra $M$ onto a von Neumann subalgebra $N$. In [4], the index of $E$ has been defined to be finite if there exists $k > 0$ such that the map $t \mapsto E - t|d_M$ from $M$ to $M$ is completely positive ($t$ being the injection of $N$ into $M$). This definition is equivalent to the one given by Kosaki [18] when $M$ and $N$ are factors, and extends Jones' definition. It follows easily from ([4] Th. 3.5) and Lemma 1.10 that $M \subset Y_N$ when the index of $E$ is finite, and that, conversely, if $M \subset Y_N$ with $N' \cap M = \mathbb{C}$ then the index of $E$ is finite. Thus, Popa's result remains true in general.

1.12. Recall that in [9] a topology has been defined on $\text{Corr}(M, N)$, described by its neighbourhoods in the following way.

DEFINITION. Let $H_0 \in \text{Corr}(M, N)$, $\epsilon > 0$, $E \subset M$ and $F \subset N$ two finite sets, and $S = \{h_1, \ldots, h_n\}$ a finite subset of $H_0$. We denote by $U(H_0; \epsilon, E, F, S)$ the set of $H \in \text{Corr}(M, N)$ such that there exist $k_1, \ldots, k_n \in H$ with $|\langle h_i, xk_j y \rangle - \langle h_i, xh_j y \rangle| < \epsilon$ for all $x \in E$, $y \in F$ and $i, j = 1, \ldots, n$. The we consider the well defined topology on $C(M, N)$ for which these sets $U$ are basis of neighbourhoods.

Note that if we consider correspondences as representations of $M \otimes_{\text{bin}} N^0$ (the binormal ones), then it is easily verified that the above topology on $\text{Corr}(M, N)$ is induced by the quotient topology introduced in [11] on the set of (unitary equivalence classes of) representations of $M \otimes_{\text{bin}} N^0$.

We shall now give an equivalent way of defining this topology on $C(M, N)$.

DEFINITION. Let $X_0 \in C(M, N)$, $\mathcal{V}$ a $\sigma$-weak neighbourhood of 0 in $N, E$ a finite subset of $M$ and $S = \{\xi_1, \ldots, \xi_n\}$ a finite subset of $X_0$. We denote by $V(X_0; \mathcal{V}, E, S)$ the set of $X \in C(M, N)$ such that there exist $\eta_1, \ldots, \eta_n \in X$ with $\langle \eta_i, x\xi_j \rangle - \langle \xi_i, x\eta_j \rangle \in \mathcal{V}$ for all $x \in E$ and $i, j = 1, \ldots, n$. We provide $C(M, N)$ with the topology having such sets as basis of neighbourhoods.

PROPOSITION. The bijection $\land : C(M, N) \to \text{Corr}(M, N)$ is an homeomorphism.

Proof. Let $X_0 \in C(M, N)$ and $H_0 = X_0 \otimes_N L^2(N, \varphi)$, where $\varphi$ is a fixed faithful normal state on $N$. Denote by $h_\varphi$ the canonical cyclic vector in $L^2(N, \varphi)$. Consider a neighbourhood $U = U(H_0; \epsilon, E, F, S)$ of $H_0$. Then we may suppose that $S = \{\xi_1 \otimes h_\varphi, \ldots, \xi_n \otimes h_\varphi\}$ with $\xi_1, \ldots, \xi_n$ in $X_0$, since the subspace $\{\xi \otimes h_\varphi, \xi \in X_0\}$ is dense in $H_0$. Let:

$$S' = \{\xi_1, \ldots, \xi_n\} \quad \text{and} \quad \mathcal{V} = \{x \in N, |\langle h_\varphi, xh_\varphi y \rangle| < \epsilon \quad \text{for} \quad y \in F\}.$$  

Then we shall prove that the image of $V = V(X_0; \mathcal{V}, E, S')$ by $\land$ is contained in $U$. 

Take $\lambda \in V$ and let $\lambda = \lambda \otimes_{\mathcal{N}} L^2(N, \varphi)$. There exist $\eta_1, \ldots, \eta_n \in \lambda$ with

$$|\langle \eta_i, (\eta_j, \lambda \eta_j) - \langle \zeta_i, \lambda \zeta_j \rangle \rangle \lambda y \rangle | < \varepsilon$$
for $x \in E$, $y \in F$, $1 \leq i, j \leq n$,

so that

$$|\langle \eta_i \otimes \lambda \varphi, \eta_j \otimes \lambda \varphi \rangle \rangle y \rangle - \langle \zeta_i \otimes \lambda \varphi, \zeta_j \otimes \lambda \varphi \rangle | < \varepsilon$$

for $x \in E$, $y \in F$, $1 \leq i, j \leq n$; hence $\lambda \in U$.

Conversely, consider a neighbourhood $V = V(X_0, \mathcal{V}, E, S)$ of $X_0$, where $S = \{ \xi_1, \ldots, \xi_n \} \subset X_0$ and $\mathcal{V} = \{ x \in N, |\varphi_i(x)| < 1, 1 \leq i \leq p \}$, with $\varphi_1, \ldots, \varphi_p$ given normal positive forms on $N$. Let $\psi$ be a faithful normal positive form on $N$ with $\varphi_i \leq \psi$ for $1 \leq i \leq p$. By ([10], Prop. 2.5.1) there exist $y_i \in N$ such that

$$\varphi_i(x) = \langle \psi, x \varphi_i y_i \rangle, \quad \text{for } x \in N.$$

We may suppose that $H_0 = X_0 \otimes N L^2(N, \psi)$ and $S' = \{ \xi_1 \otimes \psi, \ldots, \xi_n \otimes \psi \}$, and let us show that the image of $U = U(H_0 ; 1, E, F, S')$ by $\mathcal{V}^{-1}$ is contained in $V$. Consider $H \in \text{Corr}(M, N)$ such that there exist $h_1, \ldots, h_n \in H$ with

$$|\langle h_i, x h_j y \rangle - \langle \zeta_i \otimes \lambda \varphi, \zeta_j \otimes \lambda \varphi \rangle | < 1$$

for $x \in E$, $y \in F$, $1 \leq i, j \leq n$.

We may suppose that $H = X \otimes N L^2(N, \psi)$ with $X = \mathcal{V}^{-1}(H)$, and since the set $\{ \eta \otimes \lambda \varphi, \eta \in \lambda \}$ is a dense subspace of $H$, we may take $h_i = \eta_i \otimes \lambda \varphi$ with $\eta_i \in \lambda$, for $i = 1, \ldots, n$. Then we have

$$|\varphi_k(\langle \eta_i, \lambda \eta_j \rangle - \langle \zeta_i, \lambda \zeta_j \rangle | = |\langle h_\psi, (\lambda \eta_j, \psi) - \langle \zeta_i, \lambda \zeta_j \rangle \rangle \lambda y_k \rangle |$$

$$= |\langle \eta_i \otimes \lambda \varphi, \lambda \eta_j \otimes \lambda \varphi \rangle \rangle y_k \rangle - \langle \zeta_i \otimes \lambda \varphi, \lambda \zeta_j \otimes \lambda \varphi \rangle |< 1$$

for $x \in E$, $1 \leq i, j \leq n, k = 1, \ldots, p$, so that $X \in V$.

1.13. REMARKS. (a) Let $X_0 \in C(M, N)$ with a cyclic vector $\xi_0$. Then it is easy to see that $X_0$ has a basis of neighbourhoods of the form $V(X_0, \mathcal{V}, E, \{ \xi_0 \})$. In particular, the identity correspondence $Y_M = M$ has $V(M, \mathcal{V}, E)$ as basis of neighbourhoods, where $\mathcal{V}$ is a $\sigma$-weak neighbourhood of $O$ in $M, E$ is a finite subset of $M, and V(M, \mathcal{V}, E)$ is the set of $X \in C(M, M)$ such that there exists $\eta \in X$ with $\langle \eta, x \eta \rangle - x \in \mathcal{V}$ for $x \in E$.

(b) $\lambda = \lambda \otimes_{\mathcal{N}} L^2(N, \psi)$.
Φ: M → N be also a normal completely positive map. If Φₙ(x) converges σ-weakly to Φ(x) for all x ∈ M, then obviously Xₙ tends to Xₜ in C(M, N).

1.14. As it has already been pointed out in [19] and [21], the notions of irreducibility, weak containment, type, still apply to M–N correspondences when the latter are regarded as representations of the C*-algebra $M \otimes_{\text{bin}} N^0$.

DEFINITION. We say that a correspondence $X ∈ C(M, N)$ is irreducible if the commutant of $M$ in $L_N(X)$ is reduced to the scalar operators.

Thanks to the Lemma 1.5 (iii), this means that the associated representation of $M \otimes_{\text{bin}} N^0$ is irreducible.

DEFINITION. We say that a correspondence $X ∈ C(M, N)$ is weakly contained in $Y ∈ C(M, N)$ if the associated representation $π_X$ of $M \otimes_{\text{bin}} N^0$ is weakly contained in the representation $π_Y$, that is $\text{Ker } π_X ⊃ \text{Ker } π_Y$.

This means that $π_X$ (resp. $X$) belongs to the closure of the set of finite direct sums of copies of $π_Y$ (resp. $Y$) in the set of representations of $M \otimes_{\text{bin}} N^0$ gifted with the quotient topology of Fell (resp. in $C(M, N)$) ([12], Th. 1.1). When $X$ is irreducible, this is equivalent to the fact that $π_X$ belongs to the closure of $\{π_Y\}$, or to the fact that $X$ is in the closure of $\{Y\}$ in $C(M, N)$ (see [12], or ([10], §3.4)).

2. Cocycles, positive type functions and correspondences

In this section we consider a $(W^*−)$ dynamical system $(N, G, x)$ where $G$ is a discrete group and $x$ is an homomorphism from $G$ into the group of automorphisms of $N$.

2.1. DEFINITION. Let $K$ be a Hilbert space. A map $g → T_g$ from $G$ into the unitary group of $L(K) \otimes N = L_N(K \otimes N)$ such that

$$T_{st} = T_s(I_K \otimes x_s)(T_t), \quad \text{for } s, t ∈ G$$

will be called a unitary cocycle for $(N, G, x)$.

We denote by $Z(N, G, x)$ the set of such cocycles, where of course the Hilbert space $K$ may vary.

To every unitary representation $π$ of $G$ in $H_π$, we can associate the cocycle $T: s → π(s) \otimes 1$, with values in the unitary group of $L(H_π) \otimes N$. When $π$ is the trivial representation of $G$ we obtain the identity cocycle $I: s → 1 ∈ N$. The left regular representation of $G$ is denoted by $λ$ as well as the associated cocycle $s → λ(s) \otimes 1$, with values in the unitary group of $L(l^2(G)) \otimes N$. It is called the (left) regular cocycle for $(N, G, x)$.

Consider now the special case where $N$ is an abelian von Neumann algebra.
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Then there exist (in an essentially unique way) a probability space \((X, \mu)\) and a Borel \(G\)-action \((x, s) \mapsto xs\) leaving \(\mu\) quasi-invariant such that

\[(\alpha_s f)(x) = f(xs) \mu\text{-a.e., } \text{ for } f \in L^\infty(X, \mu).\]

Let \(T\) be a cocycle for \((N, G, \alpha)\) with values in the unitary group of \(\mathcal{L}(K) \otimes N = L^\infty(K, \mathcal{L}(K))\). Let \(T_s(x) = \beta(x, s)\mu\text{-a.e. for all } s \in G\). Then the cocycle equality becomes

\[\beta(x, st) = \beta(x, s)\beta(xs, t)\mu\text{-a.e. for all } s, t \in G.\]

Thus the elements of \(Z(N, G, \alpha)\) are the unitary cocycles considered by Zimmer in [27].

2.2. DEFINITION (see [6]). Let \(X\) be a Hilbert \(N\)-module. An homomorphism \(v: s \mapsto v_s\) from \(G\) into the group of \(\mathbb{C}\)-linear, bijective, bicontinuous maps of \(X\) onto itself will be called an action of \(G\) on \(X\). We say that the action is \(\alpha\)-equivariant if

\[\alpha_t(\eta, \xi) = \langle v_t \eta, v_t \xi \rangle, \quad \forall t \in G, \quad \eta, \xi \in X,\]

\[v_t(\xi x) = v_t(\xi)\alpha_t(x), \quad \forall t \in G, \quad \xi \in X, \quad x \in N.\]

2.3. Let \(K\) be a Hilbert space. It is easily checked that we can define an \(\alpha\)-equivariant action \(\hat{\alpha}\) (or more simply \(\hat{\alpha}\)) of \(G\) on \(K \otimes N\) by

\[\hat{\alpha}_s(k \otimes x) = k \otimes \alpha_s(x), \quad \text{for } k \in K, \quad x \in N.\]

Consider now a cocycle \(T\) for \((N, G, \alpha)\), with values in the unitaries of \(\mathcal{L}(K) \otimes N\). Then \(s \mapsto T_s \circ \hat{\alpha}_s\) is an \(\alpha\)-equivariant action of \(G\) on \(K \otimes N\), since we have

\[(I_K \otimes \alpha_s)(S) = \hat{\alpha}_s \circ S \circ \hat{\alpha}_s^{-1}\]

for \(S \in \mathcal{L}(K) \otimes N\) and \(s \in G\). Conversely, if \(v\) is an \(\alpha\)-equivariant action of \(G\) on \(K \otimes N\), then \(s \mapsto T_s = v_s \circ \hat{\alpha}_s^{-1} \in \mathcal{L}'_N(K \otimes N) = \mathcal{L}(K) \otimes N\) is a unitary cocycle. In this way, we obtain a natural bijection between \(Z(N, G, \alpha)\) and the set of \(\alpha\)-equivariant \(G\)-actions on Hilbert \(N\)-modules of the form \(K \otimes N\).

2.4. Recall from [3] that a map \(s \mapsto f(s)\) from \(G\) into \(N\) is said to be of positive type (with respect to \(\alpha\)) if for every \(s_1, \ldots, s_n \in G\), the matrix \((\alpha_{s_i}(f(s_i^{-1}s_j)))_{i,j} \in M_n(N)\) is positive.

Let \(v\) be an \(\alpha\)-equivariant action of \(G\) on a Hilbert \(N\)-module \(X\), and take \(\xi \in X\). Then \(s \mapsto \langle \xi, v_s \xi \rangle\) is a positive type function with values in \(N\). Conversely, every
positive type function comes in such a way from an $\alpha$-equivariant action ([3], Prop. 2.3). We may consider only self-dual modules, and even of the type $K \otimes N$:

**Lemma.** Let $f$ be a positive type map from $G$ into $N$. There exist an Hilbert space $K$, an $\alpha$-equivariant action $v$ of $G$ on $K \otimes N$ and a vector $\xi \in K \otimes N$ such that $f(s) = \langle \xi, v_s \xi \rangle$ for $s \in G$.

**Proof.** By ([3], Prop. 2.3), there exist a Hilbert $N$-module $E$, an $\alpha$-equivariant action $w$ on $E$, and a vector $\eta \in E$ such that $f(s) = \langle \eta, w_s \eta \rangle$ for $s \in G$. We denote by $X$ the self-dual completion of $E$, which can be viewed as the set of $N$-module bounded maps of $E$ into $N$ (see [20], §3). Then it is easily shown that $w$ may be extended to an $\alpha$-equivariant action $\tilde{w}$ on $X$ by

$$\tilde{w}_s(\tau)(\xi) = \alpha_s(\tau(w_{s^{-1}} \xi)) \quad \text{for } s \in G, \xi \in E, \tau \in X.$$

By ([20], Th. 3.12) $X$ is isomorphic to a self-dual Hilbert $N$-submodule of $l^2(I) \otimes N$, where $I$ is a well chosen infinite set of indices, and thus the Hilbert $N$-modules $X \oplus (l^2(I) \otimes N)$ and $l^2(I) \otimes N$ are isomorphic. Let $v$ be the $\alpha$-equivariant action on $l^2(I) \otimes N$ transferred by such an isomorphism from the action on $X \oplus (l^2(I) \otimes N)$ which is equal to $w$ on $X$ and to $\alpha$ on $l^2(I) \otimes N$. If $\xi$ is the vector in $l^2(I) \otimes N$ which corresponds to $\eta \in X \oplus (l^2(I) \otimes N)$, then we have $f(s) = \langle \xi, v_s \xi \rangle$ for $s \in G$. \qed

2.5. In the rest of Section 2, we denote by $M$ the crossed product $N \times_\alpha G$. Recall that $M$ is generated by $N$ and by the range of an homomorphism $s \mapsto u_s$ from $G$ into the unitary group of $M$ such that $u_s x u_{s^{-1}} = x_s$ for $x \in N$ and $s \in G$. More precisely, every element of $M$ may be written in a unique way as a $\sigma$-weakly convergent sum $\sum_{s \in G} u_s x_s$, where $x_s \in N$ for $s \in G$. We denote by $E$ the faithful normal conditional expectation of $M$ onto $N$ such that $E(\sum_{s \in G} u_s x_s) = x_e$, where $e$ is the neutral element of $G$. Let $(\xi_s)_{s \in G}$ be the canonical orthonormal basis of $l^2(G)$. It is straightforward to check that the Hilbert $N$-modules $X_E$ and $l^2(G) \otimes N$ are isomorphic by the map sending $\sum_{s \in G} u_s x_s \in M \subset X_E$ onto $\sum_{s \in G} \xi_s \otimes x_s$. Hence, we may identify $L_H(X_E)$ with $L(\ell^2(G)) \otimes N$, and it is easy to see that when we make this identification, an element $x = \sum_{s \in G} u_s x_s \in M \subset L_H(X_E)$ becomes the matrix $(x_{s,t})$ where $x_{s,t} = x_{s^{-1}}(x_t x_{s^{-1}})$ for $s, t \in G$. In other words, the embedding $M \subset L_H(X_E)$ is the well known embedding of $N \times_\alpha G$ in $L(\ell^2(G)) \otimes N$ (see [25] for instance).

Take a normal faithful state $\varphi$ on $N$ and let $\psi = \varphi \circ E$. The Hilbert space $L^2(M, \psi)$ is isomorphic to $l^2(G) \otimes L^2(N, \varphi)$ by the map which sends $\sum_{s \in G} u_s x_s \in M \subset L^2(M, \psi)$ onto $\sum_{s \in G} \xi_s \otimes x_s$ (where $N$ is here viewed as a subspace of $L^2(N, \varphi)$). With this identification, $x \in N \subset M \subset L(\ell^2(M, \psi))$ becomes the operator sending $\xi \in \ell^2(G) \otimes L^2(N, \varphi)$ onto $\sum_{s \in G} (x(\xi)) \otimes x_s$ and $u_s \in M$ becomes the operator $\lambda_s \otimes 1$. 
2.6. To each cocycle for \((Z(N), G, \alpha)\) we can associate in a natural way a \(M-M\) correspondence. This has already been noticed in ([2], Prop. 4.3), and extends a construction of ([9], proof of Th. 2) where \(N = \mathbb{C}\).

**PROPOSITION.** Let \(T\) be a cocycle for \((Z(N), G, \alpha)\) with values in the unitary group of \(\mathcal{L}(K) \otimes Z(N)\) and let \(X = K \otimes M\). There exists a normal homomorphism \(\pi\) of \(M\) into \(\mathcal{L}_M(K \otimes M) = \mathcal{L}(K) \otimes M\) such that

\[
\pi(x) = 1_K \otimes x, \forall x \in N,
\]

\[
\pi(u_s) = T_s \cdot (1_K \otimes u_s), \forall s \in G.
\]

Thus \((X, \pi)\) is a \(M-M\) correspondence, which will be said to be associated to \(T\).

**Proof.** We identify \((K \otimes M) \otimes_M L^2(M)\) to the Hilbert space tensor product \(K \otimes L^2(M)\) in the obvious way, and we denote by \(\rho\) the canonical injective normal homomorphism from \(\mathcal{L}_M(X)\) into \(\mathcal{L}(X \otimes_M L^2(M)) = \mathcal{L}(K \otimes L^2(M))\) (see 1.5). We shall prove that \(\pi\) comes from a normal homomorphism from \(M\) into \(\mathcal{L}(K \otimes L^2(M))\) via \(\rho\). For each \(S \in \mathcal{L}_M(X) = \mathcal{L}(K) \otimes M\) we have \(\rho(S) = S\) considered as acting on \(K \otimes L^2(M)\) in the natural way, since this is clearly true for decomposable elements of \(\mathcal{L}(K) \otimes M\).

We take \(L^2(M) = l^2(G) \otimes L^2(N)\) (see 2.5) and we write the elements \(\xi\) of \(K \otimes l^2(G) \otimes L^2(N)\) as maps from \(G\) into \(K \otimes L^2(N)\). Then we have

\[
(\rho(\pi(x))\xi)(s) = (1_K \otimes \alpha_{x^{-1}}(x))\xi(s) \quad \text{for } x \in N,
\]

\[
(\rho(\pi(u_t))\xi)(s) = (I_K \otimes \alpha_{x^{-1}}(T_t))\xi(t^{-1}s) \quad \text{for } t \in G,
\]

where \(1_K\) is the unit of \(\mathcal{L}(K)\) and \(I_K\) the identity automorphism of \(\mathcal{L}(K)\). Denote by \(w\) the unitary operator on \(K \otimes l^2(G) \otimes L^2(N)\) such that

\[
(w(\pi(x)))\xi(s) = (I_K \otimes \alpha_{x^{-1}}(x))\xi(s) \quad \text{for } s \in G.
\]

Since \((I_K \otimes \alpha_{x^{-1}}(x)) = (I_K \otimes \alpha_{x^{-1}})(1_K \otimes x)\) and \((I_K \otimes \alpha_{x^{-1}})(T_s)\) commute for \(x \in N\) and \(s \in G\), we see that \(w(\pi(x))w = \rho(\pi(x))\). On the other hand, for \(\xi \in K \otimes l^2(G) \otimes L^2(N)\) and \(x \in G\) we have

\[
(w(\pi(x)))\xi(s) = [(I_K \otimes \alpha_{x^{-1}})(T_s^*T_t)](I_K \otimes \alpha_{x^{-1}})(T_{s^{-1}}s)\xi(t^{-1}s) = \xi(t^{-1}s)
\]

by the cocycle property on \(T\). Hence \(\pi\) is the normal homomorphism from \(M\) into \(\mathcal{L}_M(X)\) such that \(\rho(\pi(x)) = w(1_K \otimes x)w^* \in \mathcal{L}(K \otimes L^2(M))\) for each \(x \in M \subset \mathcal{L}(L^2(M))\).

\[\square\]

2.7. **PROPOSITION.** (i) If \(T\) is the identity cocycle for \((Z(N), G, \alpha)\), the associated \(M-M\) correspondence is the identity correspondence.
(ii) \( Y_N \) is the \( M-M \) correspondence associated to the regular cocycle for \((Z(N), G, \alpha)\).

**Proof.** (i) is obvious. Let us prove (ii). The Hilbert \( M \)-module \( Y_N = X_E \otimes_N X_N \) is isomorphic to \( (l^2(G) \otimes N) \otimes_N X_N \) (see 2.5) and thus to \( l^2(G) \otimes M \) by the map which sends \( (\sum_{s \in G} u_s x_s) \otimes y \in X_E \otimes_N X_N \) onto \( \sum_{s \in G} \xi_s \otimes x_s y \). If we identify \( Y_N \) and \( l^2(G) \otimes M \) thanks to this isomorphism we see that the left action \( \pi' \) of \( M \) on \( Y_N \) becomes the action on \( l^2(G) \otimes M \) given by

\[
\begin{align*}
\pi'(x)(\xi)(s) &= \alpha_{s^{-1}}(x)\xi(s) \\
\pi'(u_s)(\xi)(s) &= \xi(t^{-1}s)
\end{align*}
\]

for \( \xi \in l^2(G) \otimes M, x \in N \) and \( s, t \in G \).

Let \( w \) be the automorphism of \( l^2(G) \otimes M \) such that \( (w\xi)(s) = u_s \xi(s) \). Then we have

\[
\begin{align*}
w\pi'(x)w^* &= 1_{l^2(G)} \otimes x, \quad \forall x \in N, \\
w\pi'(u_s)w^* &= \lambda_s \otimes u_s, \quad \forall s \in G.
\end{align*}
\]

Therefore, \( Y_N \) is equivalent to the \( M-M \) correspondence associated to the regular cocycle for \((Z(N), G, \alpha)\).

\[\square\]

2.8. The following proposition extends the construction of completely positive maps carried out by Haagerup in ([15], Lemma 1.1).

**PROPOSITION.** Let \( f \) be a positive type map from \( G \) into \( Z(N) \) with respect to \( \alpha \). Then there exists a unique normal completely positive map \( \Phi_f \) from \( M \) into \( M \) such that

\[
\Phi_f(u_s x) = f(s)u_s x \quad \text{for } s \in G \quad \text{and } x \in N,
\]

and \( \Phi_f \) is \( N \)-bilinear.

More precisely, suppose that \( f \) is given by \( f(s) = \langle \xi, v_s \xi \rangle \) as in lemma 2.4 but with \( N \) replaced by \( Z(N) \). Then, denoting by \( T \) the cocycle corresponding to \( v \), \( \Phi_f \) is the coefficient of the \( M-M \) correspondence associated to \( T \), which is defined by \( \xi \in K \otimes Z(N) = K \otimes M \).

**Proof.** The unicity of \( \Phi_f \) is obvious. Let \( (K \otimes M, \pi) \) be the \( M-M \) correspondence associated to \( T \). For \( x \in N \) and \( t \in G \), we have

\[
\begin{align*}
\langle \xi, \pi(u_t)x\xi \rangle_M &= \langle \xi, \pi(u_t)\xi x \rangle_M \text{ since } \xi \in K \otimes Z(N) \\
&= \langle \xi, T_t \circ \hat{\alpha}_t(\xi) \rangle u_t x = f(t)u_t x.
\end{align*}
\]
Thus $y \mapsto \langle \xi, \pi(y)\xi \rangle_M$ is a $N$-bilinear normal completely positive map with the required property.

2.9. REMARK. Suppose that $G$ is freely acting on $N$ in the sense of [16] and let $\Phi$ be a $N$-bilinear normal completely positive map from $M$ to $M$. For $s \in G$, put $f(s) = \Phi(u_s u_s^*)$. We easily check that $f(s) \in N' \cap M$, which is equal to $Z(N)$ since the action $\alpha$ is free. Now $f$ is a positive type map because we have, for $a_1, \ldots, a_n$ in $Z(N)$ and $s_1, \ldots, s_n$ in $G$,

$$\sum_{i,j=1}^n a_i^* \alpha_{s_i}(f(s_i^{-1} s_j)) a_j = \sum_{i,j=1}^n a_i^* u_{s_i} \Phi(u_{s_i} u_{s_j}) u_{s_j}^* a_j \geq 0$$

by the complete positivity of $\Phi$.

Thus, when the $G$-action $\alpha$ is free, every $N$-bilinear normal completely positive map $\Phi$ from $M$ to $M$ comes from a positive type function as indicated in 2.8.

2.10. Of course, if $f$ is the constant map with value equal to the unit of $Z(N)$, the associated completely positive map is the identity automorphism of $M$.

PROPOSITION. Let $f$ be a positive type map from $G$ to $Z(N)$ with finite support. Then the associated completely positive map $\Phi_f$ is a coefficient of the $M-M$ correspondence $Y_N$.

Proof. Let $\tilde{\alpha}$ be the $\alpha$-equivariant action of $G$ on $l^2(G) \otimes Z(N)$, associated to the regular cocycle $\lambda$, which means that $(\tilde{\alpha}_h)(s) = \alpha_t(h(t^{-1}s))$ for $h \in l^2(G) \otimes Z(N)$ and $s, t \in G$. Since $f$ has a finite support, by ([3], Prop. 2.5) there exists $h \in l^2(G) \otimes Z(N)$ such that $f(s) = \langle h, \tilde{\alpha}_s h \rangle$. Then the result follows from Propositions 2.8 and 2.7(ii).

2.11. We denote by $PT_1(Z(N), G, \alpha)$ the set of positive type maps from $G$ to $Z(N)$ with respect to $\alpha$, such that $\sup_{s \in G} \|f(s)\| \leq 1$ (or, equivalently $f(e) \leq 1$ ([3], Prop. 2.4)), and we endow this set with the topology of pointwise $\sigma$-weak convergence. The space of normal completely positive maps from $M$ to $M$ will be denoted by $CP(M)$ and equipped similarly with the topology of pointwise $\sigma$-weak convergence.

PROPOSITION. The map $f \mapsto \Phi_f$ from $PT_1(Z(N), G, \alpha)$ into $CP(M)$ is continuous.

Proof. We show the continuity at $f_0 \in PT_1(Z(N), G, \alpha)$. Let $\mathcal{V}$ be a $\sigma$-weak neighbourhood of $0$ in $M$ and $\{x^1, \ldots, x^n\}$ a finite subset of $M$. We write $x^i = \sum_{s \in G} u_s x_s$ for $1 \leq i \leq n$.

We choose a faithful normal state $\varphi$ on $N$, and for $a \in N$ and $s \in G$, we denote by $\varphi_{a,s}$ the form $x \mapsto \varphi(E(a u_s x))$ on $M$. When $(a, s)$ describes $N \times G$, we get a total family of elements in the predual $M_*$, with respect to the norm. Hence, we may
find $a_1, \ldots, a_p$ in $N$ and $s_1, \ldots, s_p$ in $G$ such that for every $y \in M$ satisfying
\[ \|y\| \leq 2 \sup_{1 \leq j \leq n} \|x^j\| \quad \text{and} \quad |\varphi_{a_i,s_i}(y)| < 1, \quad \text{for } i = 1, \ldots, p \]

we have $y \in \mathcal{V}$. 

Let $\mathcal{W}$ be the $\sigma$-weak neighbourhood of $O$ in $Z(N)$ given by
\[ \mathcal{W} = \{x \in Z(N), |\varphi(a_i x_{s_i}(x)x_{s_i}^{-1})| < 1 \text{ for } 1 \leq i \leq p \text{ and } 1 \leq j \leq n\}. \]

We shall show that if $f \in PT_1(Z(N), G, \alpha)$ satisfies
\[ f(s_i^{-1}) - f_0(s_i^{-1}) \in \mathcal{W} \quad \text{for } i = 1, \ldots, p, \]
then $\Phi_f(x^j) - \Phi_0(x^j) \in \mathcal{V}$ for $j = 1, \ldots, n$ (where $\Phi_0 = \Phi_{f_0}$), and this will end the proof. We have
\[ |\varphi_{a_i,s_i}(\Phi_f(x^j)) - \Phi_0(x^j)| = \left| \sum_{t \in G} \varphi(a_i E(u_s(f(t) - f_0(t))u_s,x^j_s)) \right| \]
\[ = |\varphi(a_i x_{s_i}(f(s_i^{-1}) - f_0(s_i^{-1}))x_{s_i}^{-1} x_{s_i}^j)| < 1 \]
for $i = 1, \ldots, p$ and $j = 1, \ldots, n$. As $\Phi_f$ and $\Phi_0$ are contractions, we get
\[ \|\Phi_f(x^j) - \Phi_0(x^j)\| \leq 2\|x^j\|, \quad \text{and therefore we have } \Phi_f(x^j) - \Phi_0(x^j) \in \mathcal{V}. \quad \square \]

## 3. Amenability

### 3.1. DEFINITION (see [21] Def. 3.1). Let $N \subset M$ be von Neumann algebras. We say that $M$ is amenable relative to $N$ (or that the inclusion is amenable) if the identity correspondence $Y_M = M$ is weakly contained in $Y_N$.

Note that when there exists a faithful normal conditional expectation from $M$ onto $N$ with finite index, the inclusion is amenable since $Y_M$ is then contained in $Y_N$ (see 1.11).

Consider now the case $N = \mathbb{C}$. The representation of $M \otimes_{\text{bin}} M^0$ defined by the identity correspondence is $x \otimes y^0 \mapsto x J y^* J$ acting on $L^2(M)$, where, as usual, $J$ is the antilinear involution on $L^2(M)$ given by the Tomita–Takesaki theory. The representation of $M \otimes_{\text{bin}} M^0$ associated to the coarse correspondence is $x \otimes y^0 \mapsto x \otimes J y^* J$ acting on $L^2(M) \otimes L^2(M)$. Thus the inclusion $\mathbb{C} \subset M$ is amenable if and only if the map $x \otimes y \mapsto xy$ from the algebraic tensor product $M \otimes M'$ into the C*-subalgebra of $\mathcal{L}(L^2(M))$ generated by $M$ and $M'$ is continuous when $M \otimes M'$ is equipped with the minimal C*-norm. It is proved in
([11]), Prop. 4.5) that this property is equivalent to semi-discreteness, and by [11], [7], [5] and [26] it is equivalent to injectivity.

The following result, which extends a part of Popa’s Theorem 3.2.3 in [21], shows that relative amenability implies a relative injectivity property.

3.2. PROPOSITION. Let \( N \subseteq M \) be an amenable inclusion. Then there exists a norm one projection from \( \mathcal{L}_N(\overline{X}_N) \) onto \( M \) (naturally identified to a von Neumann subalgebra of \( \mathcal{L}_N(\overline{X}_N) \)).

Proof. By hypothesis, \( Y_M \) belongs to the closure in \( C(M) \) of the set of finite direct sums of copies of \( Y_N \). Hence there exists a net \( (\eta_i)_{i \in I} \), where each \( \eta_i \) is a finite sequence \( \eta^i_1, \ldots, \eta^i_{p_i} \) of elements of \( Y_N \), such that for each \( x \in M \)

\[
\sum_{1 \leq j \leq p_i} \left\langle \eta^i_j, x\eta^i_j \right\rangle \text{ converges } \sigma\text{-weakly to } x.
\]

Choose an ultrafilter \( \mathcal{U} \) finer than the filter obtained from the directed set \( I \). Let \( \varphi \) be a normal positive form on \( M \) and take \( x \in \mathcal{L}_N(\overline{X}_N) \) (identified to the von Neumann subalgebra \( \rho(\mathcal{L}_N(\overline{X}_N)) \) of \( \mathcal{L}_M(Y_N) \) by Lemma 1.5(ii)). We have

\[
\left| \varphi\left( \sum_{1 \leq j \leq p_i} \left\langle \eta^i_j, x\eta^i_j \right\rangle \right) \right| \leq \|x\| \varphi\left( \sum_{1 \leq j \leq p_i} \left\langle \eta^i_j, \eta^i_j \right\rangle \right), \quad \text{for } i \in I.
\]

This allows us to define

\[
S(\varphi, x) = \lim_{\mathcal{U}} \varphi\left( \sum_{1 \leq j \leq p_i} \left\langle \eta^i_j, x\eta^i_j \right\rangle \right)
\]

and we get

\[
|S(\varphi, x)| \leq \|x\| \lim_{\mathcal{U}} \varphi\left( \sum_{1 \leq j \leq p_i} \left\langle \eta^i_j, \eta^i_j \right\rangle \right) = \|x\| \varphi(1) = \|x\| \|\varphi\|.
\]

It follows that \( (\varphi, x) \mapsto S(\varphi, x) \) is a bilinear continuous form on \( M_\ast \times \mathcal{L}_N(\overline{X}_N) \). Thus, for each \( x \in \mathcal{L}_N(\overline{X}_N) \) there is an element \( \Phi(x) \) in \( M_\ast \) well defined by

\[
\varphi(\Phi(x)) = S(\varphi, x), \quad \text{for } \varphi \in M_\ast.
\]

Obviously \( \Phi \) is positive with \( \Phi(x) = x \) for all \( x \in M \), and therefore it is a norm one projection from \( \mathcal{L}_N(\overline{X}_N) \) onto \( M \) (see [24] Th. 3.1).

3.3. REMARKS. (1) It follows from Proposition 3.2 that if \( N \subseteq M \) is an amenable inclusion, and if \( N \) is an injective von Neumann algebra, then \( M \) is also injective, since it is the case for \( \mathcal{L}_N(\overline{X}_N) \).
(2) The converse of the above proposition has been proved by Popa in ([21] Th. 3.2.3) when $M$ is a finite factor. When $N = \mathbb{C}$, one has $\mathcal{L}_N(\mathcal{X}_N) = \mathcal{L}(l^2(M))$, and the converse of Proposition 3.2 is the fact that injectivity implies semi-discreteness. The following proposition gives another case where this converse is true.

3.4. Let $(N, G, \alpha)$ be a dynamical system as in Section 2. In [1] we have defined a notion of amenability for the action $\alpha$, generalizing the corresponding notion introduced by Zimmer [28] in ergodic theory. For $G$ discrete we have shown that the action $\alpha$ is amenable if and only if there exists a norm one projection from $\mathcal{L}(l^2(G)) \otimes N$ onto $N \times G$ (canonically embedded into $\mathcal{L}(l^2(G)) \otimes N$) (see [1], Prop. 3.11).

PROPOSITION. Let $(N, G, \alpha)$ be a dynamical system with $G$ discrete. The following conditions are equivalent:

(i) the inclusion $N \subseteq M = N \times G$ is amenable;

(ii) the action of $G$ on $N$ is amenable;

(iii) there is a norm one projection from $\mathcal{L}_N(\mathcal{X}_N) = \mathcal{L}(l^2(G)) \otimes N$ onto $M$.

Proof. Remark that the embedding of $M$ into $\mathcal{L}_N(\mathcal{X}_N)$ identified to $\mathcal{L}(l^2(G)) \otimes N$ is the usual embedding in the theory of crossed products (see 2.5). Then the equivalence between (ii) and (iii) follows from ([1] Prop. 3.11). The implication (i) $\Rightarrow$ (iii) has been proved in Proposition 3.2. So it remains to see that (ii) $\Rightarrow$ (i). By ([3], Th. 3.3) there exist a net $(f_i)_{i \in I}$ of elements of $PT_1(Z(N), G, \alpha)$ with finite support such that $f_i(s)$ converges to 1 $\sigma$-weakly for every $s \in G$. For $i \in I$, denote by $\Phi_i$ the completely positive map associated to $f_i$, and let $X_i = X_{\Phi_i}$. We have $X_i \subseteq Y_N$ since $\Phi_i$ is a coefficient of $Y_N$ by Proposition 2.10. Furthermore, it follows from Proposition 2.11 that $\Phi_i(x)$ tends to $x$ $\sigma$-weakly for all $x \in M$, and thus $\lim_i X_i = Y_M$ in $C(M)$. This proves that $Y_M$ belongs to the closure of $Y_N$ in $C(M)$.

References


