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Bases for the homology groups of the Hilbert scheme of points in the plane

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To Edie and Robin Hartshorne

0. Introduction

0.1. In this paper we find a basis for the free Chow group $A_*(\text{Hilb}^d \mathbb{P}^2)$, consisting of classes $\text{cl}(\bar{U})$ for suitable locally closed subschemes U of $\text{Hilb}^d \mathbb{P}^2_{\mathbb{C}}$, whose points parametrize reduced subschemes of $\mathbb{P}^2$. Using this basis, one can compute explicitly and give enumerative applications.

We need some definitions and notations in order to state the theorem precisely. A partition $\mathbf{b} = (b_0, \dots, b_{r-1})$ of a non-negative integer d is a sequence of positive integers $b_0 \geq b_1 \geq \dots \geq b_{r-1}$ with $b_0 + b_1 + \dots + b_{r-1} = d$. We say that r is the length of the partition, and we shall write $b_i = 0$ for $i \geq r$. The transposed partition $\mathbf{a} = (a_0, \dots, a_{s-1})$ is the sequence $a_j = \#\{b_i/b_i \geq j+1\}$. It is easily visualised by means of a “Young tableau”, such as

	a_0	a_1	a_2	a_3	
b_0					$\mathbf{b} = (4, 2, 2) \quad r = 3$
b_1					$\mathbf{a} = (3, 3, 1, 1) \quad s = 4$
b_2					(always $b_0 = s, \quad a_0 = r$)

We denote the set of partitions $\mathbf{b}$ of d by B_d . Note that B_0 consists only of the empty partition, which we denote by $\mathbf{0}$.

A mixed partition $\mathbf{b}^* = (\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2)$ of d is a triple of partitions $\mathbf{b}^k$ of integers d^k ($k = 0, 1, 2$) such that $d^0 + d^1 + d^2 = d$. They form a set which we denote M_d .

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The subset

$$M_{d,n} = \{(\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2) \in M_d / d + s^2 - s^0 = n\} \subseteq M_d$$

will be the indexing set for the different bases of $A_n(\text{Hilb}^d \mathbb{P}^2)$ handled in this paper. (We have adopted, once and for all, the convention that given $\mathbf{b}^*$, all b_i^k, r_k, a_j^k, s^k are defined).

Let $\mathbf{b}^* \in M_{d,n}$ and let $P_0^0 \in L_0^0, \dots, P_{s^0-1}^0 \in L_{s^0-1}^0$ be s^0 points on s^0 lines, let $L_0^1, \dots, L_{s^1-1}^1$ be s^1 lines, and let P^2 be a point, all of them “fixed”. Assume all points and lines are distinct, and no point lies on two lines. We associate to them a locally closed subscheme $U \subseteq \text{Hilb}^d \mathbb{P}^2$ called of type $(0, 1, 2)$ and mixed partition $\mathbf{b}^*$. The points of U correspond to subschemes $Z = Z^0 \cup Z^1 \cup Z^2$ of $\mathbb{P}^2$ defined in the following way:

Z^0 is a set containing $\{P_0^0, \dots, P_{s^0-1}^0\}$ and consisting of d^0 distinct points, with a_j^0 of them lying on the line L_j^0 -and none of them the intersection point of two distinct lines.

Z^1 is a set consisting of d^1 distinct points, with a_j^1 of them lying on the line L_j^1 – and none of them the intersection point of two distinct lines.

Z^2 is a set consisting of d^2 distinct points lying in some s^2 distinct lines $L_0^2, \dots, L_{s^2-1}^2$, meeting at the point P^2 , with a_j^2 points lying on the line L_j^2 – and none coincident with P^2 .

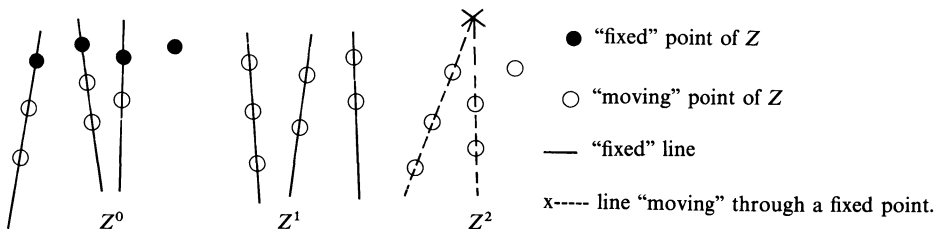
Clearly, Z^0, Z^1 and Z^2 have $d^0 - b_0^0, d^1$ and $d^2 + b_0^2$ degrees of freedom; consequently, the scheme U has dimension $n = d - b_0^0 + b_0^2$.

We say that the closure $\bar{U}$ in $\text{Hilb}^d \mathbb{P}^2$ is a scheme of type $(0, 2)$ and mixed partition $\mathbf{b}^* \in M_d$ (and say $\bar{U}$ is, for instance, of type 1 and partition $\mathbf{b} \in B_d$ if it is of type $(0, 1, 2)$ and mixed partition $\mathbf{b}^* = (0, \mathbf{b}, 0)$). The class $\text{cl}(\bar{U})$ is independent of the choices of points and lines and is denoted by $\sigma_{012}(\mathbf{b}^*)$ (if, for instance, $\mathbf{b}^* = (0, \mathbf{b}, 0)$ we denote $\sigma_{012}(\mathbf{b}^*)$ by $\sigma_1(\mathbf{b}^*)$).

We use the injective map $\sigma_{012}: M_d \rightarrow A.(\text{Hilb}^d \mathbb{P}^2)$ so defined, to state our main

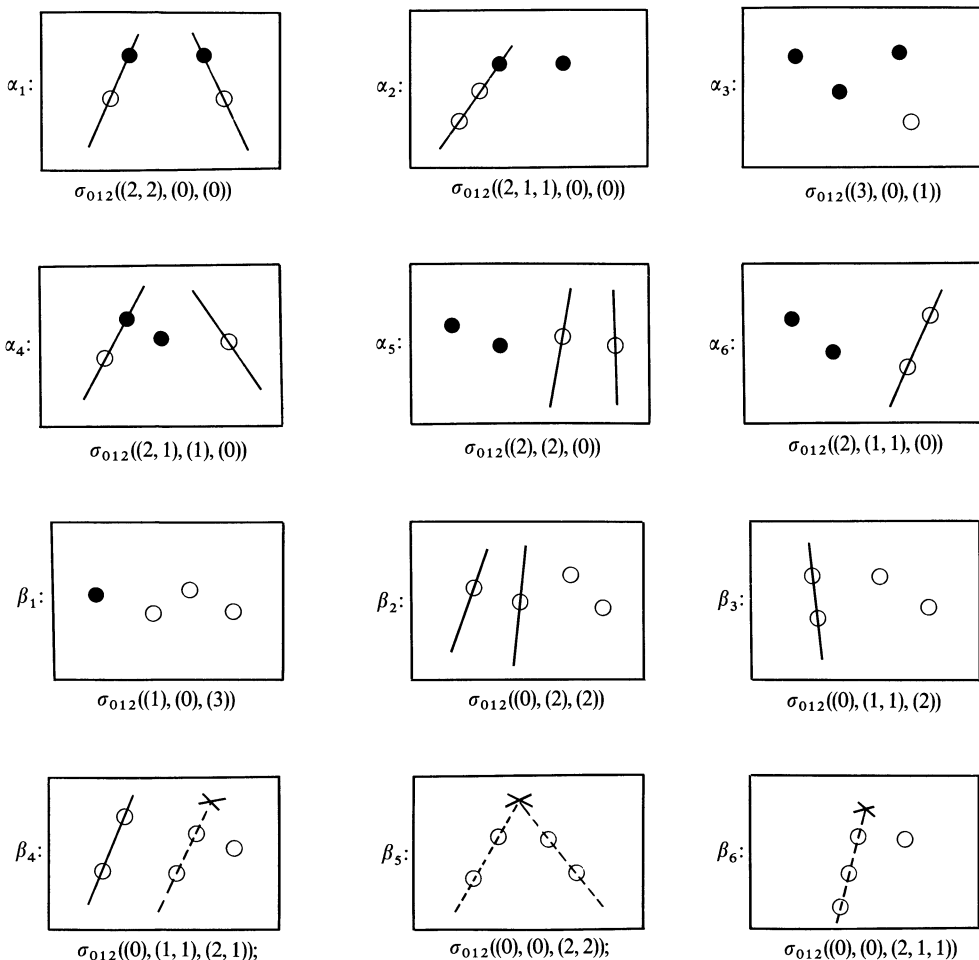
THEOREM 012. *The set $\sigma_{012}(M_{d,n})$ is a basis for $A_n(\text{Hilb}^d \mathbb{P}^2)$.*

For a heuristic example, an element of the basis of $A_{5+7+9}(\text{Hilb}^{9+7+6} \mathbb{P}^2)$ is $\text{cl}(\bar{U})$, where U is the locally closed subscheme of $\text{Hilb}^{22} \mathbb{P}^2$ whose points parametrize length 22 subschemes $Z = Z^0 \cup Z^1 \cup Z^2$ of $\mathbb{P}^2$, as pictured, using conventions of [5] [6]:



Here, $\mathbf{b}^0 = (4, 3, 2)$, $\mathbf{b}^1 = (3, 3, 1)$, $\mathbf{b}^2 = (3, 2, 1)$. (Observe that picturing lines L_j^0 with $a_j^0 = 1$ or lines L_j^2 with $a_j^2 = 1$ would be superfluous).

For another example, our bases of $A_2 = \{\alpha_i\}_{i=1}^6$ and $A_5 = \{\beta_i\}_{i=1}^6$ of $\text{Hilb}^4 \mathbb{P}^2$ are provided by the following configurations:



0.2. Ellingsrud and Strømme observed in [7] that the group $A_n(\text{Hilb}^d \mathbb{P}^2)$ is free, so that rational, homological and numerical equivalence are all the same in $\text{Hilb}^d \mathbb{P}^2$, and they found its rank. In a later paper [8] they gave a cell decomposition of $\text{Hilb}^d \mathbb{P}^2$, which provides a basis of $A_n(\text{Hilb}^d \mathbb{P}^2)$ consisting of the classes of the closures of the cells. However, the main reason for finding a basis of this group, namely its potential application to Enumerative Geometry, is not achieved by their basis, for it consists of classes of the closures of cells whose

generic element corresponds to schemes of length d in $\mathbb{P}^2$ which are, unfortunately, nonreduced. Indeed, in enumerative applications we often need the expression, in a basis of $A_n(\text{Hilb}^d \mathbb{P}^2)$, of the class of a subvariety X of dimension n of $\text{Hilb}^d \mathbb{P}^2$ (in most occasions corresponding to nonreduced subschemes of $\mathbb{P}^2$). This is achieved by computing the intersection numbers $X \cdot \gamma_i$ of X with all elements γ_i of the dual basis of $A_{2d-n}(\text{Hilb}^d \mathbb{P}^2)$, then solving the linear system so obtained. This is, for example, the case for the cycle τ in Section 5.

The task of computing the numbers $X \cdot \gamma_i$ seems hopeless, without a geometric description of the subschemes of $\mathbb{P}^2$ corresponding to γ_i , such as the one we give in Section 5. But even with this geometric description, if the base γ_i also corresponds to nonreduced subschemes of $\mathbb{P}^2$, then the actual computation of the intersection numbers $X \cdot \gamma_i$ leads to difficult problems of multiplicity counting and, even more seriously, it leads too frequently to excess intersections.

The basis we propose avoids this difficulty, as it consists of cycles whose generic points correspond to reduced length d subschemes of $\mathbb{P}^2$.

We obtain a geometric description of the Ellingsrud-Strømme basis as the one given by the partition map which tells how many points of the scheme lie in the same horizontal line of the affine plane (then the cell-decompositions of Ellingsrud-Strømme appear as extensions, outside the origin, of the Briançon classification, by the “escalier vertical” of schemes supported at the origin). Then we are able to change to our basis, which is represented by subschemes of $\mathbb{P}^2$ whose generic points correspond to reduced schemes of $\mathbb{P}^2$. For $d \leq 3$, our basis is just-up to elementary basis changes – the one found by Elencwajg and Le Barz in [4] [6], and for $d = 4$, it is the one of Mallavibarrena in [10]. Enumerative applications may be found in the papers of Elencwajg-Le Barz as well as in second author’s thesis [5] [11] [12], which contains a rigorous proof of the Schubert conjectures [15, Chap. IX] about double contacts between two families of *smooth* plane curves. We state these conjectures as theorems in Section 5, providing only a sketch of the proof of the less elementary one, because the others are analogous.

Our geometric description of the Ellingsrud-Strømme basis simplifies to a simple exercise when $d = 3$. Then, a basis for the group $A_i(W^*)$ for the variety W^* of Schubert triangles can be immediately deduced as an application, obtaining 1, 7, 17, 22, 17, 7, 1 as cardinalities (number of Schubert “Bedingungen” in [15]). We have been recently learned that Roberts and Speiser [14] and also Fulton and Collino (in progress) have independently obtained bases for $A_i(W^*)$. However we have kept the description of our proof in Section 4 because of its simple nature.

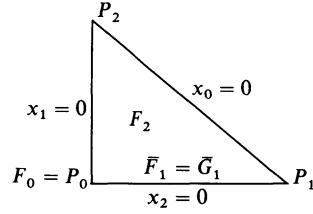
0.3. In Section 1 we give the geometric description of the basis of Ellingsrud and Strømme. In order to state this as a theorem we need some notations:

Fix once and for all homogeneous coordinates x_0, x_1, x_2 in $\mathbb{P}^2$, and corresponding vertices P_0, P_1, P_2 and affine pieces of $\mathbb{P}^2$:

$$U_0 = \text{Spec } \mathbb{C}[x_1/x_0, x_2/x_0] = \text{Spec } \mathbb{C}[x, y]$$

$$U_1 = \text{Spec } \mathbb{C}[x_0/x_1, x_2/x_1] = \text{Spec } \mathbb{C}[x, y]$$

$$U_2 = \text{Spec } \mathbb{C}[x_0/x_2, x_1/x_2] = \text{Spec } \mathbb{C}[x, y]$$



We decompose $\mathbb{P}^2$ in affine spaces in the standard way: $F_2 = \mathbb{P}^2 \setminus \langle P_0 P_1 \rangle = U_2$, $F_1 = \langle P_0 P_1 \rangle \setminus P_0$ is the affine line of U_1 of equation $y = 0$, and $F_0 = \{P_0\}$. We will also denote $G_1 = \langle P_0 P_1 \rangle \setminus P_1$ the affine line of U_0 defined by the equation $y = 0$, so that $\bar{F}_1 = \bar{G}_1$ is the line of $\mathbb{P}^2$ given by the equation $x_2 = 0$.

For schemes $X \supseteq Y$, we denote by $\text{Hilb}^d(X, Y)$ the subscheme of $\text{Hilb}^d X$ which parametrizes schemes with support contained in Y .

DEFINITION. The scheme of type $(0', 1', 2)$ and mixed partition $\mathbf{b}' \in M_d$ is the closure of the subscheme of $\text{Hilb}^d \mathbb{P}^2$ parametrizing schemes $Z = Z^0 \cup Z^1 \cup Z^2$ with

$$\text{Supp } Z^0 = \{P_0\} \text{ and length } (Z^0 \cap iG_1) = b_0^0 + \cdots + b_{i-1}^0$$

$$\text{Supp } Z^1 \subseteq F_1 \setminus \{P_0\} \text{ and length } (Z^1 \cap jF_1) = a_0^1 + \cdots + a_{j-1}^1$$

$$\text{Supp } Z^2 \subseteq F_2 \text{ and length } (Z^2 \cap L_j^2) = a_j^2 \text{ for lines } L_j^2 \text{ of } F_2 \text{ with } P_0 \in \bar{L}_j^2.$$

Here, for instance, $jF_1 = F_1 + \cdots + F_1$ denotes the j th infinitesimal neighbourhood of F_1 in the affine plane U_1 , with ideal (y^j) (by the assertion in *Remark 1.4*, any scheme supported at P_0 will turn out to be of type $0'$ and any scheme supported at F_1 will be of type $1'$).

The class of this scheme is denoted $\sigma_{0'1'2}(\mathbf{b}')$. Just as before, we speak also, for instance, of a scheme of type $1'$ and partition $\mathbf{b} \in B_d$ and its class $\sigma_{1'}(\mathbf{b})$.

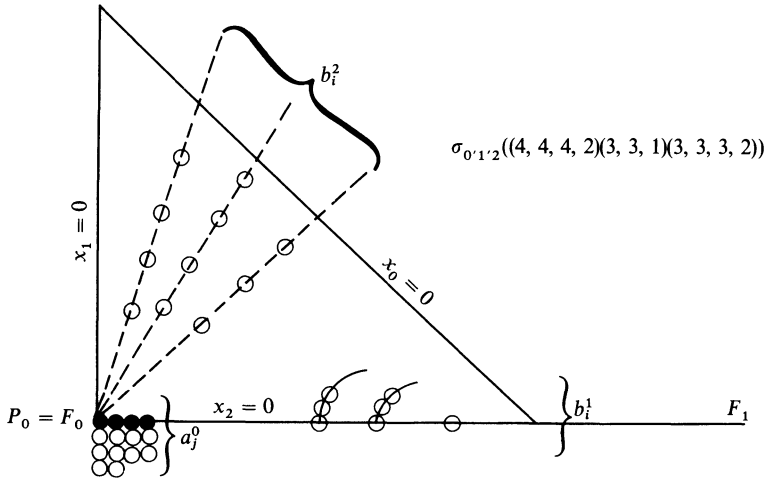
This is the geometric interpretation of the basis of Ellingsrud-Strømme we will get in Section 2:

THEOREM 0'1'2. *The set $\sigma_{0'1'2}(M_{d,n})$ is a basis for $A_n(\text{Hilb}^d \mathbb{P}^2)$.*

For example, we picture below the configuration corresponding to an element of the Ellingsrud-Strømme basis for $A_{31}(\text{Hilb}^{32} \mathbb{P}^2)$.

Although the part of type $0'$ is a scheme supported at one point P_0 , we have done our best to represent it as a limit of nonreduced schemes with the same partition.

The definitions of the classes $\sigma_{0'12}(\mathbf{b}')$ are obvious, so we leave them to the reader. We devote Section 2 to prove



THEOREM 0'12. *The set $\sigma_{0'1'2}(M_{d,n})$ is a basis for $A_n(\text{Hilb}^d \mathbb{P}^2)$.*

In Section 3 we derive Theorem 012 from theorem 0'12.

1. Geometric description of the basis of Ellingsrud-Strømme (type (0', 1', 2))

1.1. We first recall the basis of $A_i(\text{Hilb}^d \mathbb{P}^2)$ given by the classes of the closures of the cells in the cell decomposition of $\text{Hilb}^d \mathbb{P}^2$ in [8].

In order to avoid the inconvenience of referring repeatedly to that paper, we review here exactly those notations necessary for the precise statement of their theorem.

Given a partition $\mathbf{b} \in B_d$, the graph of $\mathbf{b}$ is the set $S(\mathbf{b}) = \{(0, b_0)\} \cup \{(i, j)/1 \leq i \leq r \text{ and } b_{i-1} \geq j \geq b_i\}$. The cardinality of $S(\mathbf{b})$ is $b_0 + r + 1$. Let $k = b_0 + r$ and number the elements of $S(\mathbf{b})$ "from upper left to lower right" $\mathbf{c}_i = (e_i, f_i)$:

$$\mathbf{c}_0 = (0, b_0), \mathbf{c}_1 = (1, b_0), \dots, \mathbf{c}_{k-1} = (r, 1), \mathbf{c}_k = (r, 0).$$

More precisely, we note that for each $i = 1, \dots, k$ the difference $\mathbf{c}_i - \mathbf{c}_{i-1}$ is either $(1, 0)$ (and then we say i is a horizontal index) or $(0, -1)$ (and then we say i is vertical). Let H (resp. V) be the subset of $1, \dots, k$ corresponding to horizontal (resp. vertical) indexes. Observe that the subsequence $(f_v)_{v \in V}$ of $(f_0, \dots, f_k)$ is precisely $(b_0 - 1, b_0 - 2, \dots, 1, 0)$. Whenever a partition $\mathbf{b}$ is given, we view all these as given integers.

Next define another sequence of pairs $\mathbf{n}_i$ ($1 \leq i \leq k$) by the rule $\mathbf{n}_i = \max(\mathbf{c}_i, \mathbf{c}_{i-1})$ where the maximum means the one with larger euclidean norm. Then

$\mathbf{n}_i = \mathbf{c}_i$ if $i \in H$, and $\mathbf{n}_i = \mathbf{c}_{i-1}$ if $v \in V$. Now, let us define the following sets

$$\Delta^1 = \{(i, j) \mid i \in V, j \in H \text{ and } i > j\}$$

$$\Delta^2 = \{(i, j) \mid i + 1 \in H, j \in V \text{ and } i + 1 < j\}$$

and $\Delta = \Delta^1 \cup \Delta^2$. A small calculation shows that $\text{Card } \Delta^1 = \text{Card } \Delta^2 = d$. Let $\mathfrak{D} \subseteq \Delta^1$ be the set $\{(i, j) \in \Delta^1 \mid j + 1, \dots, i \in V\}$, of cardinality b_0 .

We attach in turn to $\mathbf{b}$ a $k \times (k + 1)$ matrix $\alpha(\mathbf{b}) = \alpha = (\alpha_{ij})$ ($0 \leq i \leq k = b_0 + r$, $1 \leq j \leq k$) with entries in $S[x, y] = \mathbb{C}[s_{ij}, x, y]$, where the $2d = (d - b_0) + b_0 + d$ variables s_{ij} are indexed by $\Delta = (\Delta^1 \setminus \mathfrak{D}) \cup \mathfrak{D} \cup \Delta^2$ in the following way: $\alpha_{ij} = m(\mathbf{n}_j - \mathbf{c}_i)$ if $j - 1 \leq i \leq j$ (this denotes the corresponding monomial in x, y), s_{ij} if $(i, j) \in \Delta$, and 0 otherwise. Clearly $\alpha_{vv} = y$, $\alpha_{(v-1)v} = 1$ for $v \in V$, and $\alpha_{hh} = 1$, $\alpha_{(h-1)h} = x$ for $h \in H$.

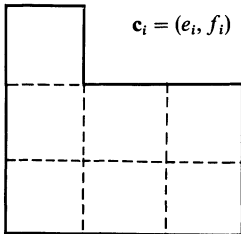
If, for example, $\mathbf{b} = (3, 2, 2)$, then α is the 6×7 matrix

$$\begin{vmatrix} x & \delta_{02}^2 & & & \delta_{05}^2 & \delta_{06}^2 & \\ & 1 & 1 & & & & \\ & d_{21} & y & x & & \delta_{25}^2 & \delta_{26}^2 \\ & & & 1 & x & \delta_{35}^2 & \delta_{36}^2 \\ & & & 1 & 1 & & \\ \delta_{51}^1 & & \delta_{53}^1 & d_{54} & & y & 1 \\ \delta_{61}^1 & & \delta_{63}^1 & d_{64} & & y & \end{vmatrix}$$

where we have represented s_{ij} by either δ_{ij}^1 , d_{ij} or δ_{ij}^2 according whether (i, j) belongs to $\Delta^1 \setminus \mathfrak{D}$, $\mathfrak{D}$ or Δ^2 .

In this case the above integers attached to $\mathbf{b}$ are

$$\mathbf{b} = (3, 2, 2); \quad \mathbf{a} = (3, 3, 1)$$



$$\begin{aligned} \mathbf{c}_0 &= (0, 3) \mathbf{c}_1 = (1, 3) \mathbf{c}_2 = (1, 2) \mathbf{c}_3 = (2, 2) \\ \mathbf{c}_4 &= (3, 2) \mathbf{c}_5 = (3, 1) \mathbf{c}_6 = (3, 0) \\ \mathbf{n}_1 &= (1, 3) \mathbf{n}_2 = (1, 3) \mathbf{n}_3 = (2, 2) \mathbf{n}_4 = (3, 2) \\ \mathbf{n}_5 &= (3, 2) \mathbf{n}_6 = (3, 1) \end{aligned}$$

$$H = \{1, 3, 4\}; V = \{2, 5, 6\}$$

Write $S_2 = S/(s_{ij} \mid (i, j) \notin \Delta^2 \cup \mathfrak{D})$, and $S_1 = S/(s_{ij} \mid (i, j) \notin \Delta^1)$, and $S_0 = S/(s_{ij} \mid (i, j) \notin \Delta^1 \setminus \mathfrak{D})$. Write $S(\mathbf{b})$, $\Delta^i(\mathbf{b})$, etc. . . if the partition $\mathbf{b}$ is to be made explicit. Let $\mathbb{A}^2$ be $\text{Spec } \mathbb{C}[x, y]$, and $\mathbb{A}^1$ the line of equation $y = 0$, and 0 the origin of $\mathbb{A}^2$. It is proved in [8] that we have an open neighbourhood U of the origin of $\text{Spec } S$ such that the closed subscheme of $\mathbb{A}^2 \times U$ defined by the maximal minors, i.e. $k \times k$ minors, of the $k \times (k + 1)$ matrix α has fibers over U of constant length d . Therefore it defines a morphism $\gamma: U \rightarrow \text{Hilb}^d \mathbb{A}^2$ which is shown in [8] to be étale.

THEOREM 1.1. (Ellingsrud-Strømme) (0) *As $\mathbf{b}$ runs through the set B_d , the locally closed sets $\gamma(\text{Spec } S_0(\mathbf{b}))$ of dimension $d - b_0$ form a cell decomposition of $\text{Hilb}^d(\mathbb{A}^2, 0)$.*

(1) *As $\mathbf{b}$ runs through B_d , the locally closed sets $\gamma(\text{Spec } S_1(\mathbf{b}))$ of dimension d form a cell decomposition of $\text{Hilb}^d(\mathbb{A}^2, \mathbb{A}^1)$.*

(2) *As $\mathbf{b}$ runs through B_d , the locally closed sets $\gamma(\text{Spec } S_2(\mathbf{b}))$ of dimension $d + b_0$ form a cell decomposition of $\text{Hilb}^d \mathbb{A}^2$.*

From the proof of 1.1 in [8] we only need to recall the structure of the cell $\gamma(\text{Spec } S_0(\mathbf{b}))$ (resp. $\text{Spec } S_1(\mathbf{b})$, resp. $\text{Spec } S_2(\mathbf{b})$). It consists of those $Z \in \text{Hilb}^2 \mathbb{A}^2$ of ideal I_Z , such that

$$\lim_{t \rightarrow 0} t \cdot I_Z = (y^{b_0}, xy^{b_1}, \dots, x^{r-1}y^{b_{r-1}}, x^r)$$

under an action $t \cdot x^e y^f = t^{\alpha e + \beta f} x^e y^f$ of G_m on $\mathbb{C}[x, y]$ with $\alpha \gg \beta > 0$ (resp. $\alpha < 0$, $\beta > 0$, resp. $\alpha \ll \beta < 0$).

For any triple (d^0, d^1, d^2) of nonnegative integers with $d = d^0 + d^1 + d^2$, let $W(d^0, d^1, d^2)$ be the locally closed subset of $\text{Hilb}^d \mathbb{P}^2$ corresponding to subschemes $Z = Z^0 \cup Z^1 \cup Z^2$ with $\text{Supp } Z^i \subseteq F_i$ and $\text{length } Z^i = d^i$ for $i = 0, 1, 2$. Clearly

$$\text{Hilb}^d \mathbb{P}^2 = \cup \{W(d^0, d^1, d^2) \mid d^0 + d^1 + d^2 = d\}$$

$$\text{and } W(d^0, d^1, d^2) \simeq W(d^0, 0, 0) \times W(0, d^1, 0) \times W(0, 0, d^2).$$

The part 0 (resp. 1, resp. 2) of Theorem 1.1 applied to $\mathbb{A}^2 = \text{Spec } \mathbb{C}[x, y] = \text{Spec } \mathbb{C}[x_2/x_0, x_1/x_0]$ (resp. $\text{Spec } \mathbb{C}[x_0/x_1, x_2/x_1]$, resp. $\text{Spec } \mathbb{C}[x_0/x_2, x_1/x_2]$) provides a cell decomposition of $W(d^0, 0, 0)$ (resp. $W(0, d^1, 0)$, resp. $W(0, 0, d^2)$). This yields a cell decomposition of each $W(d^0, d^1, d^2)$, thus of the whole $\text{Hilb}^d \mathbb{P}^2$. Therefore, *the classes of the closures of the cells form a basis for the free group $A.(\text{Hilb}^d \mathbb{P}^2)$, which we call the Ellingsrud-Strømme basis.*

1.2. Besides the notations borrowed from [8] in 1.1, we need some others for the sequel.

Let $\mathbf{b}$ be a partition of d , let $L_0, \dots, L_{m-1}$ be parallel lines of $\mathbb{A}^2$ -not necessarily distinct – and let $(L_0 + \dots + L_{m-1})$ be the scheme associated to its sum, as divisors. We denote by $\text{Hilb}^{\mathbf{b}}(L_0, \dots, L_{m-1})$ the locally closed subset (with reduced scheme structure) of $\text{Hilb}^d(L_0 + \dots + L_{m-1})$ parametrizing schemes Z such that

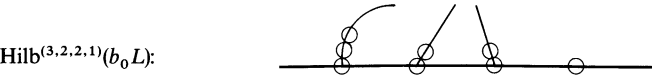
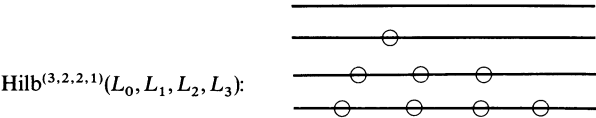
$$\text{length } Z \cap (L_0 + \dots + L_j) - \text{length } Z \cap (L_0 + \dots + L_{j-1}) = a_j$$

or equivalently

$$\text{length } Z \cap (L_0 + \dots + L_j) = a_0 + \dots + a_j,$$

for $j = 0, \dots, m - 1$. Observe that $\text{Hilb}^{\mathbf{b}}(L_0, \dots, L_{m-1})$ depends on the order of the lines and it is nonempty if and only if $m \geq b_0$. If all $L_j = L$ we just write $\text{Hilb}^{\mathbf{b}}(b_0 L)$.

We may picture this as follows:



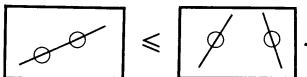
(n adjacent points denote an element of the n th infinitesimal neighbourhood of a point P on a smooth curve C , given in the local ring of the plane at P by the sum of the ideal of C and the n th power of the ideal of a line L transverse to C at P). It is clear that the subschemes just defined are not necessarily closed since, for instance, a doubled point in a line L belongs to $\overline{\text{Hilb}^{(2)}(2L)} \setminus \text{Hilb}^{(2)}(2L)$.

In these notations, the scheme of type $0'1'2$ is the closure of the subscheme of $\text{Hilb}^d \mathbb{P}^2$ parametrizing those $Z = Z^0 \cup Z^1 \cup Z^2 \subseteq \mathbb{P}^2$ with

$$\begin{aligned} Z^0 &\in \text{Hilb}^{a^0}(a_0 G_1, P_0) \subseteq \text{Hilb}^d U_0 \\ Z^1 &\in \text{Hilb}^{b^1}(b_0 F_1) \subseteq \text{Hilb}^d U_1 \\ Z^2 &\in \text{Hilb}^{b^2}(L_0^2, \dots, L_{s-1}^2) \subseteq \text{Hilb}^d U_2 \text{ for distinct lines } L_i \subseteq U_2 \text{ with } P_0 \in \bar{L}_i. \end{aligned}$$

In order to be able, later in the paper, to express basis changes by matrices, we need to order the bases.

We partially order the set B_d of d -partitions by taking $\mathbf{b}' \leq \mathbf{b}$ if and only if $a'_0 + \cdots + a'_j \geq a_0 + \cdots + a_j$.

Thus, for instance, $(1, 1) \leq (2)$ in B_2 , i.e. .

Clearly $\mathbf{b}' \leq \mathbf{b}$ implies that $\mathbf{b}'$ is lexicographically prior to $\mathbf{b}$ since $a_0 + \cdots + a_j = d - \sum_i \max\{0, b_i - j - 1\}$.

Let $(\mathbb{A}^{2^\vee})$ be the punctured projective plane consisting of all lines in affine space $\mathbb{A}^2$. For another integer $m \geq 1$, let $F_{d,m}$ be the locally closed subscheme of $\text{Hilb}^d \mathbb{A}^2 \times ((\mathbb{A}^{2^\vee})^m)$ consisting of those $(Z, (L_0, \dots, L_{m-1}))$ such that $L_0, \dots, L_{m-1}$ are parallel and $Z \in \text{Hilb}^{\mathbf{b}}(L_0, \dots, L_{m-1})$ for some $\mathbf{b} \in B_d$. Since clearly $\text{Hilb}^{\mathbf{b}}(L_0, \dots, L_{m-1}) \cap \text{Hilb}^{\mathbf{b}'}(L_0, \dots, L_{m-1}) = \emptyset$ for $\mathbf{b} \neq \mathbf{b}'$, there is an obvious “partition” map $\pi: F_{d,m} \rightarrow B_d$.

PROPOSITION 1.2. The “partition” map $\pi: F_{d,m} \rightarrow B_d$ is lower semicontinuous, i.e. $\pi^{-1}\{\mathbf{b}' \in B_d / \mathbf{b}' \leq \mathbf{b}\}$ is closed in $F_{d,m}$ for all $\mathbf{b} \in B_d$.

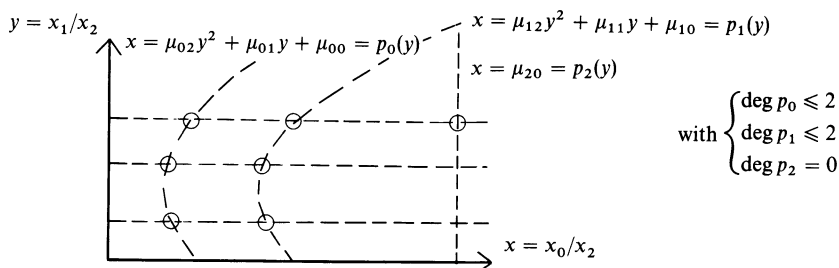
Proof. The inverse image by the partition map of $\{\mathbf{b}' \in B_d / \mathbf{b}' \leq \mathbf{b}\}$ consists of those $(Z, (L_0, \dots, L_{m-1})) \in \text{Hilb}^d \mathbb{A}^2 \times (\mathbb{A}^{2^\vee})^m$ satisfying the closed condition $a'_0 + \cdots + a'_j := \text{length } Z \cap (L_0 + \cdots + L_j) \geq a_0 + \cdots + a_j$ (Note we do not need the lines L_j to be distinct). Q.E.D.

We totally order $B = \bigcup_{d \geq 0} B_d$ according to d with lexicographic priority for equal d . Then we totally order the set M_d of mixed d -partitions $(\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2)$ of a given d according to the lexicography of the triple. This induces a total order in the subset $M_{d,n}$ indexing our bases of $A_n(\text{Hilb}^d \mathbb{P}^2)$.

To make this clearer, observe that the elements of the bases of A_2 and A_6 of $\text{Hilb}^4 \mathbb{P}^2$ given in 0.1, are strictly decreasing, $\alpha_1 > \alpha_2 > \cdots$, in this order.

1.3. In the second part of this section we prove Theorem 0'1'2, via the Proposition 1.3 below.

We indicate the flavour of the proof of this proposition, observe that the schemes of type 2 and partition $\mathbf{b} = (3, 3, 1)$, for instance, are the closures of the schemes parametrizing subschemes of $\mathbb{P}^2$ as pictured:



The ideals of these subschemes are

$$\begin{aligned} & ((y - v_0)(y - v_1)(y - v_2), x - p_0(y)) \cap \\ & \cap ((y - v_0)(y - v_1)(y - v_2), x - p_1(y)) \cap (y - v_0, x - p_2(y)) \\ & = ((y - v_0)(y - v_1)(y - v_2), (y - v_0)(x - p_0(y))(x - p_1(y)), \\ & (x - p_0(y))(x - p_1(y))(x - p_2(y))). \end{aligned}$$

Under the action $(x, y) \rightarrow (t^\alpha x, t^\beta y)$ $\alpha \ll \beta < 0$, of G_m on $\mathbb{C}[x, y]$, all of them approach the same fixed ideal (y^3, x^2y, x^3) , so they lie in the corresponding cell of the Ellingsrud-Strømme cell decomposition of $\text{Hilb}^7 \mathbb{P}^2$.

Having the same dimension 10, our scheme of type 2 and partition $(3, 3, 1)$ is just the closure of this cell. The dimension 10 corresponds to parameters for the choice of three lines $y = v_i$ and 3, 3, 1 parameters for the choices of coefficients of the polynomials p_0, p_1, p_2 , respectively.

From now on $L_v \subseteq \mathbb{A}^2 = \text{Spec } \mathbb{C}[x, y]$ denotes the affine line of equation $y = v$, so that the line $\mathbb{A}^1$ of equation $y = 0$ is L_0 . Denote by L' the line of equation $x = 0$.

PROPOSITION 1.3. *Keeping the notations of Theorem 1.1 we have, in $\text{Hilb}^d \mathbb{A}^2$*

- (0) $\gamma(\text{Spec } S_0(\mathbf{b})) = \text{Hilb}^a(a_0 L', 0)$
- (1) $\gamma(\text{Spec } S_1(\mathbf{b})) = \text{Hilb}^b(b_0 \mathbb{A}^1)$
- (2) $\gamma(\text{Spec } S_2(\mathbf{b})) = \bigcup \{ \text{Hilb}^b(L_{v_0}, \dots, L_{v_{b_0-1}}) \mid v_i \in \mathbb{C} \}.$

Therefore, all three are irreducible, of dimensions $d - b_0$, d , $d + b_0$, and providing cell decompositions of $\text{Hilb}^d(\mathbb{A}^2, 0)$, $\text{Hilb}^d(\mathbb{A}^2, \mathbb{A}^1)$ and $\text{Hilb}^d \mathbb{A}^2$ respectively.

Proof of (2). We call for short $\bigcup_{\mathbf{b}}$ to the union in the statement. For generic choices of $v_0, \dots, v_{b_0-1} \in \mathbb{C}^{b_0}$ and polynomials $p_0, \dots, p_{r-1} \in \mathbb{C}[y]$ with $\deg p_i < b_i$, the scheme in U with ideal

$$I = \bigcap_{i=0}^{r-1} \left(\prod_{k=0}^{b_i-1} (y - v_k), x - p_i \right) = \sum_{i=0}^{r-1} \left(\prod_{k=0}^{b_i-1} (y - v_k) \cdot \prod_{h=0}^{i-1} (x - p_h) \right)$$

approaches under an action $t \cdot x^e y^f = t^{\alpha e + \beta f} x^e y^f$, with $\alpha \ll \beta < 0$ of G_m on $\mathbb{C}[x, y]$, the scheme with ideal

$$\lim_{t \rightarrow 0} t \cdot I = (y^{b_0}, xy^{b_1}, \dots, x^{r-1} y^{b_{r-1}}, x^r) = (x^{a_0}, yx^{a_1}, y^2 x^{a_2}, \dots, y^s).$$

These ideals I thus describe a subset of the cell $\gamma(\text{Spec } S_2(\mathbf{b}))$ (by Theorem 1.1) and they depend on $b_0 + d$ parameters, which is the dimension of the cell, so it is a dense subset, i.e. $\gamma(\text{Spec } S_2(\mathbf{b}))$ has a dense intersection with $\bigcup_{\mathbf{b}}$. Indeed, the coefficients of polynomials $p_0, \dots, p_{r-1}$ provide $b_0 + \dots + b_{r-1} = d$ parameters,

and the choice of the lines $y - v_0, \dots, y - v_{b_0-1}$, i.e. the choice of coefficients of the polynomial $(y - v_0) \cdots (y - v_{b_0-1})$ provide b_0 parameters.

Let Z be any scheme in $\gamma(\text{Spec } S_2(\mathbf{b}))$. Let $Z_\lambda \in \text{Hilb}^{\mathbf{b}}(L_{v_0(\lambda)}, \dots, L_{v_{b_0-1}(\lambda)})$, $\lambda \in \mathbb{C}^*$, be a one-parameter flat family of schemes in $\bigcup_{\mathbf{b}} \gamma(\text{Spec } S_2(\mathbf{b}))$ of which Z is the limit as λ goes to zero. Let $L_{v_0}, \dots, L_{v_{b_0-1}}$ be the limit of the lines, so that clearly $Z \subseteq L_{v_0} + \dots + L_{v_{b_0-1}}$ since the incidence relation is closed. Furthermore $Z \in \text{Hilb}^{\mathbf{b}'}(L_{v_0}, \dots, L_{v_{b_0-1}})$ for some $\mathbf{b}' \leq \mathbf{b}$ because of the semicontinuity of the partition map.

The transform Z' of Z by the above action of $t \in G_m$ on $\mathbb{A}^2$ belongs to $\text{Hilb}^{\mathbf{b}'}$ $(L_{v_0(t)}, \dots, L_{v_{b_0-1}(t)})$ since t acts on $\mathbb{A}^2$ transforming x, y into $t^\alpha x, t^\beta y$, thus as an affinity transforming horizontal lines L_{v_i} into horizontal lines $L_{v_i(t)}$, and the partition $\mathbf{b}'$ of Z with respect to the lines is clearly preserved by any affinity. We obtain by semicontinuity that $Z^0 \in \text{Hilb}^{\mathbf{b}''}(b_0 L_0)$, for some $\mathbf{b}'' \leq \mathbf{b}'$ for the limit Z^0 of the Z' as t goes to zero. But $Z \in \gamma(\text{Spec } S_2(\mathbf{b}))$ so this limit has ideal $I_{Z^0} = (y^{b_0}, xy^{b_1}, \dots, x^{r-1}y^{b_{r-1}}, x^r)$ which clearly is in $\text{Hilb}^{\mathbf{b}}(b_0 L_0)$, so we get $\mathbf{b} = \mathbf{b}'' \leq \mathbf{b}' \leq \mathbf{b}$. This proves that $\gamma(\text{Spec } S_2(\mathbf{b})) \subseteq \bigcup_{\mathbf{b}} \text{Hilb}^{\mathbf{b}}(\mathbb{A}^2)$ as $\mathbf{b}$ runs through B_d , while the second member yields disjoint subschemes of $\text{Hilb}^d \mathbb{A}^2$.

It would have been more natural to prove directly – by manipulating minors – that the schemes defined by a matrix in $\text{Spec } S_2(\mathbf{b})$ have partition $\mathbf{b}$, without using the limit tricks above. Such a proof exists, but the notation needed is cumbersome. (However, let us just say that the scheme defined by the general matrix in $\text{Spec } S_2(\mathbf{b})$ written in 1.4 below is contained in the union of horizontal lines whose equation is the product of the dotted minors of size $b_i - b_{i-1} + 1$).

Proof of (1). The inclusion is clear by just applying (2) after permutation of the x, y coordinates – so that the transform of $\text{Spec } S_2(\mathbf{b})$ contains $\text{Spec } S_1(\mathbf{b})$ – just recall that $\text{Spec } S_1(\mathbf{b}) \subseteq \text{Hilb}^d(\mathbb{A}^2, \mathbb{A}^1)$. The equality follows immediately from the inclusion as above.

Proof of (0). The inclusion follows from (1) applied to the transposed partition $\mathbf{a}$ and interchanging x, y (so that Δ^1 and Δ^2 are also interchanged). This yields the equality, as above.

REMARK 1.4. This implies that any length d subscheme Z of $\mathbb{A}^2$ supported at $\mathbb{A}^1$ (of equation $y = 0$) belongs to $\text{Hilb}^{\mathbf{b}}(b_0 \mathbb{A}^1)$ for some $\mathbf{b} \in B_d$. Since of course, $\mathbb{A}^1$ may be taken to be any line, we find that for a length d subscheme Z of the affine plane $\mathbb{A}^2$ supported on a line $\mathbb{A}^1$, the numbers a_j defined by length $(Z \cap j\mathbb{A}^1) = a_0 + \dots + a_{j-1}$, are always decreasing: $a_0 \geq \dots \geq a_{s-1}$. So, for instance, for the scheme Z of ideal (y^2, xy, x^2) and the line $\mathbb{A}^1$ of equation $y = 0$, it is

$$\text{length}(Z \cap \mathbb{A}^1) = \text{length}(y^2, xy, x^2, y) = \text{length}(x^2, y) = 2$$

$$\text{length}(Z \cap 2\mathbb{A}^1) = \text{length}(y^2, xy, x^2, y^2) = \text{length}(y^2, xy, x^2) = 3$$

thus $\mathbf{a} = (a_0, a_1) = (2, 1)$, so that $Z \in \text{Hilb}^{\mathbf{b}}(2\mathbb{A}^1)$ for $\mathbf{b} = (b_0, b_1) = (2, 1)$.

Another fact worth noting is that our stratification by all $\mathbf{b} \in B_d$ of $\text{Hilb}^d(\mathbb{A}^2, 0)$ is just the one given by all possible “escaliers verticaux” in Briançon’s work [2], Section 1.3, with analytic methods. Indeed, the schemes Z supported at the origin corresponding to an “escalier” have the partition map $\mathbf{b} \in B_d$ given by the heights of the ‘escalier’, by just computing the lengths $a_0 + \dots + a_j$ of (I_Z, x^j) with the help of Corollary I.1.12 [2]. Therefore the given cell decompositions of $\text{Hilb}^d(\mathbb{A}^2, \mathbb{A}^1)$ and $\text{Hilb}^d \mathbb{A}^2$ may be seen as extending the one of $\text{Hilb}^d(\mathbb{A}^2, 0)$ in Briançon’s paper, so this is an algebraization of Briançon’s techniques, extending them beyond the origin. In fact, the Ellingsrud-Strømme matrices are an extended version of the Briançon matrices.

Now Theorem 0’1’2 is proved by observing that $\sigma_{0'1'2}(M_{d,n})$ is just the basis of Ellingsrud-Strømme – quoted at the end of 1.1 – because of Proposition 1.3. (0) (resp. (1) (2)) applied to $\mathbb{A}^2 = \text{Spec } \mathbb{C}[x, y] = \text{Spec } \mathbb{C}[x_2/x_0, x_1/x_0]$ (resp. $\text{Spec } \mathbb{C}[x_0/x_1, x_2/x_1]$, resp. $\text{Spec } \mathbb{C}[x_0/x_2, x_1/x_2]$).

1.4. We observe, to compute intersections, that the classes $\sigma_{0'1'2}(\mathbf{b}')$ do not depend on the particular choice of the line $\bar{F}_1$ and point F_0 . For an example, which we will need later, the class $\sigma_0(\mathbf{b}) \in A_{d-b_0}(\text{Hilb}^d \mathbb{P}^2)$ has intersection number zero with any element $\sigma_{0'1'2}(\mathbf{b}')$ of the basis for $A_{d+b_0}(\text{Hilb}^d \mathbb{P}^2)$, unless $(\mathbf{b}') = (0, 0, \mathbf{b})$. We show now the intersection number is 1 in this case. We need this in order to show later (3.2) that our basis candidate is expressed in Ellingsrud-Strømme’s basis as a triangular matrix with all diagonal entries equal to 1. Move the center of the cycle of type 2 and partition $\mathbf{b}$ to the point at infinity $P_2 = (0, 0, 1)$ of the x -axis of affine plane $U_0 = \text{Spec } \mathbb{C}[x_2/x_0, x_1/x_0] = \text{Spec } \mathbb{C}[x, y]$. The cycle of type 0’ and partition $\mathbf{b}$ is $\gamma(\text{Spec } S_0(\mathbf{b}))$ and the cycle of type 2 is $\gamma(\text{Spec } S_2(\mathbf{b}))$. The intersection is the scheme defined by maximal minors of the $(k+1) \times k$ matrix α with $\alpha_{ij} = m(\mathbf{n}_j - \mathbf{c}_i)$ if $j-1 \leq i \leq j$ and $\alpha_{ij} = 0$ otherwise. To check that the multiplicity is 1, we simply observe that locally this is the intersection, at

$$\left| \begin{array}{cccc} x & & & \\ 1 & 1 & & \\ & y & x & \\ & & 1 & x \\ & & & 1 & 1 \\ \delta_{51}^1 & \delta_{53}^1 & & y & 1 \\ \delta_{61}^1 & \delta_{63}^1 & & & y \end{array} \right| \quad \left| \begin{array}{cccccc} x & \delta_{02}^2 & & \delta_{05}^2 & \delta_{06}^2 & \\ \boxed{\begin{array}{cc} 1 & 1 \\ d_{21} & y \end{array}} & x & & \delta_{25}^2 & \delta_{26}^2 & \\ & 1 & x & \delta_{35}^2 & \delta_{36}^2 & \\ & & \boxed{\begin{array}{ccc} 1 & 1 & \\ d_{54} & y & 1 \\ d_{64} & & y \end{array}} & & & \end{array} \right|$$

the origin of $\mathbb{C}^{2d}$, of a linear subspace of dimension $d - b_0$ and a complementary linear subspace of dimension $d + b_0$. If, for instance, $\mathbf{b} = (3, 2, 2)$ this is the intersection at the origin, in the space $\mathbb{C}^{14}$ of 6×7 matrices as pictured in 1.1, of the complementary subspaces $\mathbb{C}^4$ and $\mathbb{C}^{10}$ consisting of the above matrices

We end this section with a lemma describing the generic configuration of pure type 1'.

LEMMA. *Let $\mathbf{b} \in B_d$ and $F_1 \subseteq \text{Spec } \mathbb{C}[x_0/x_1, x_2/x_1]$ as above. The generic point of $\text{Hilb}^{\mathbf{b}}(b_0 F_1)$ corresponds to a subscheme Z with ideal*

$$I_Z = \bigcap_{i=0}^{r-1} (y^{b_i}, x - p_i(y))$$

for polynomials $p_i(y) \in \mathbb{C}[y]$ with $\deg p_i(y) < b_i$ and all $p_i(0)$ distinct.

Proof. First observe that the above intersection of ideals, being coprime, is

$$I_Z = \bigcap_{i=0}^{r-1} (y^{b_i}, x - p_i(y)) = (y^{b_0}, y^{b_1}(x - p_0(y)), \\ y^{b_2}(x - p_0(y))(x - p_1(y)), \dots, (x - p_0(y))(x - p_1(y)) \dots (x - p_{r-1}(y)))$$

since clearly both ideals define the disjoint union of the b_i th infinitesimal neighbourhoods of each curve $x = p_i(y)$ at the point $(p_i(0), 0)$.

For an action $t \cdot x^e y^f = t^{\alpha e + \beta f} x^e y^f$ of G_m on $\mathbb{C}[x, y]$ with $\alpha \ll 0$, $\beta > 0$ it is

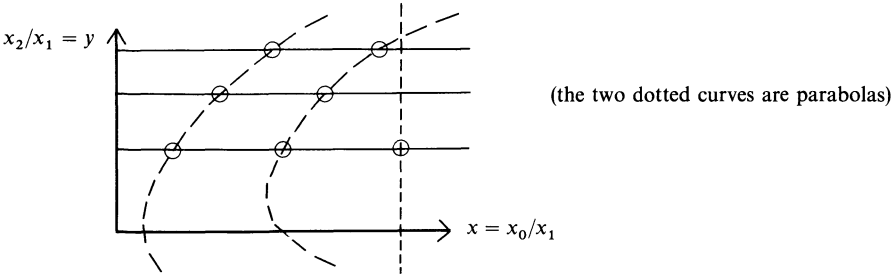
$$\lim_{t \rightarrow 0} t \cdot I_Z = (y^{b_0}, xy^{b_1}, \dots, x^r).$$

Therefore Z lies in $\gamma_1(\text{Spec } S_1(\mathbf{b})) = \text{Hilb}^{\mathbf{b}}(b_0 F_1)$ (cfr. Remark 1.4). Since the choice of the scheme Z depends on d parameters – the coefficients of the polynomials $p_i(y)$ – we can assume, by comparing dimensions, that this represents the generic point of $\text{Hilb}^{\mathbf{b}}(b_0 F_1)$.

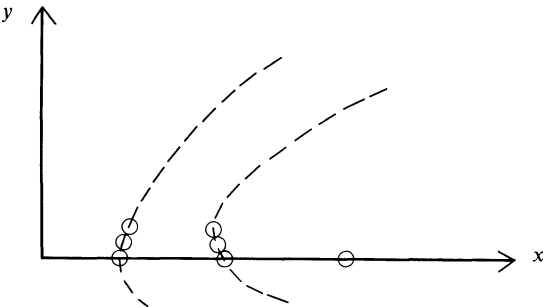
2. From type 1' to type 1

In this section we deduce Theorem 0'12 from Theorem 0'1'2.

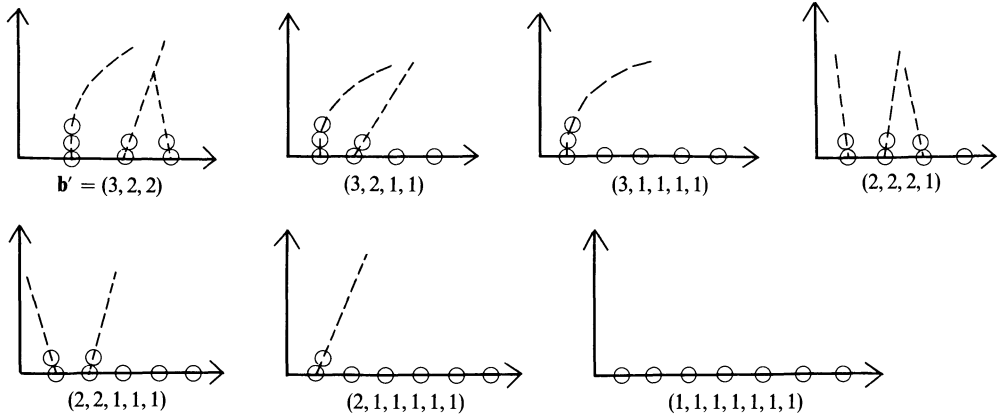
2.1. The heuristic idea of the proof is as follows: A scheme of type 1 and partition $\mathbf{b} = (3, 3, 1)$



specializes in a flat $\mathbb{P}^1$ -family to a rationally equivalent scheme of type 1' and same partition $\mathbf{b} = (3, 3, 1)$, with multiplicity 1,



plus other schemes – with irrelevant multiplicities $n(\mathbf{b}, \mathbf{b}')$ – of type 1' and partitions $\mathbf{b}' \leq \mathbf{b}$ (by semicontinuity).



This implies that $\sigma_1(\mathbf{b}) = \sigma_{1'}(\mathbf{b}) + \sum_{\mathbf{b}' < \mathbf{b}} n(\mathbf{b}, \mathbf{b}') \sigma_{1'}(\mathbf{b}')$.

By the same argument, but holding fixed the $(0, 2)$ part along the specialization, we obtain

$$\sigma_{0'12}(\mathbf{b}^0, \mathbf{b}, \mathbf{b}^2) = \sigma_{0'1'2}(\mathbf{b}^0, \mathbf{b}, \mathbf{b}^2) + \sum_{\mathbf{b}' < \mathbf{b}} n(\mathbf{b}, \mathbf{b}') \sigma_{0'1'2}(\mathbf{b}^0, \mathbf{b}', \mathbf{b}^2).$$

Therefore the set $\sigma_{0'12}(M_{d,n})$ is expressed in the basis $\sigma_{0'1'2}(M_{d,n})$ by a triangular matrix with diagonal entries ± 1 , so it is also a basis.

2.2. Then, we prove Theorem 0'12 as a consequence of the

PROPOSITION 2.1. *For a partition $\mathbf{b} \in B_d$, it is $\sigma_1(\mathbf{b}) = \sigma_{1'}(\mathbf{b}) + \sum_{\mathbf{b}' < \mathbf{b}} n(\mathbf{b}, \mathbf{b}') \sigma_{1'}(\mathbf{b}')$, for some $n(\mathbf{b}, \mathbf{b}') \in \mathbb{Z}$.*

Proof. We will prove this by showing that for the parallel lines $L_0, \dots, L_{b_0-1}$ of $U_1 = \text{Spec } \mathbb{C}[x_0/x_1, x_2/x_1] = \text{Spec } \mathbb{C}[x, y]$ we have the rational equivalence of cycles in $\text{Hilb}^d \mathbb{P}^2$

$$\overline{[\text{Hilb}^{\mathbf{b}}(L_0, \dots, L_{b_0-1})]} \sim \overline{[\text{Hilb}^{\mathbf{b}} F_1]} + \sum_{\mathbf{b}' < \mathbf{b}} n(\mathbf{b}, \mathbf{b}') \overline{[\text{Hilb}^{\mathbf{b}'}(b'_0 F_1)]}.$$

Let $\mathbf{b} = (b_0, \dots, b_{r-1})$ be a d -partition and let $D \subseteq (\text{Hilb}^d U_1) \times \mathbb{C}^*$ be the subscheme flat over $\mathbb{C}^*$ whose fibre over $t \neq 0$ is $D_t = \text{Hilb}^{\mathbf{b}}(L_0, L_t, L_{2t}, \dots, L_{(b_0-1)t})$ where L_{jt} is the affine line of equation $y = jt$ in U_1 . Clearly $D \cong D_1 \times \mathbb{C}^*$ and $D_1 = \text{Hilb}^{\mathbf{b}}(L_0, L_1, \dots, L_{b_0-1}) = \Pi \text{Sym}^{a_j} L_j$ is irreducible. Let $\hat{D}$ and $\tilde{D} = \hat{D} \cap (\text{Hilb}^d \mathbb{P}^2 \times \mathbb{C}^*)$ be the closures of D in $\text{Hilb}^d \mathbb{P}^2 \times \mathbb{C}$ and $\text{Hilb}^d \mathbb{P}^2 \times \mathbb{C}^*$ (the upper bar keeps denoting closures in $\text{Hilb}^d \mathbb{P}^2$). The closure $\hat{D}$ is flat over $\mathbb{C}$ by, for instance, III.9.8 [9], applied to $\tilde{D}$ (using $\text{Hilb}^d \mathbb{P}^2$ is a projective variety). Therefore the fibre $(\hat{D})_0$ defines a cycle rationally equivalent to the one defined by any fibre $(\hat{D})_t$, with $t \neq 0$. Let us see that $(\hat{D})_t = \tilde{D}_t$ for the generic value of $t \neq 0$ (which, of course, we can assume to be $t = 1$). Call $\delta = \dim D = \dim \tilde{D} = \dim \hat{D}$. All components of the fibres of the flat projection $\pi: \tilde{D} \rightarrow \mathbb{C}^*$ have same dimension $\delta - 1$. Now $\dim(\tilde{D} \setminus D) < \delta$ and we learn from the restriction map $(\tilde{D} \setminus D) \rightarrow \mathbb{C}^*$ that $\dim(\tilde{D} \setminus D)_t = \dim(\tilde{D})_t \setminus D_t < \delta - 1$, so that $(\tilde{D})_t \setminus D_t$ cannot, by dimensionality, be an irreducible component of $(\tilde{D})_t$. In other words, D_t is dense in $(\tilde{D})_t = (\hat{D})_t$, as wanted.

We prove first that the irreducible components of $((\hat{D})_0)_{\text{red}}$ are the d -dimensional varieties $\overline{\text{Hilb}^{\mathbf{b}'}(b'_0 F_1)}$ of type 1' and all partitions $\mathbf{b}' \leq \mathbf{b}$. Observe that it is enough to prove that all these schemes are contained in $(\hat{D})_0$ i.e. in the closure of D in $(\text{Hilb}^d \mathbb{P}^2) \times \mathbb{C}$. Indeed, any other point of $(\hat{D})_0$ is a limit of points of D_t , for $t \neq 0$, so it is in fact a point in $\overline{\text{Hilb}^{\mathbf{b}'}(b'_0 F_1)}$ for some $\mathbf{b}' \leq \mathbf{b}$ by semicontinuity of the partition map.

We shall need an auxiliary definition and lemma.

DEFINITION. An incidence diagram ε of “input” d -partition $\mathbf{b}$ is a double sequence $(\varepsilon_{ij})_{i,j \geq 0}$ with entries $\varepsilon_{ij} \in \{0, 1\}$ summing $\sum_{i \geq 0} \varepsilon_{ij} = a_j$ for all $j \geq 0$, and such that the sequence $b'_i = \sum_{j \geq 0} \varepsilon_{ij}$ is non-increasing. We call $\mathbf{b}' = (b'_0, b'_1, \dots, b'_{r'-1})$ its “output” d -partition.

For example, for both incidence diagrams

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \quad \begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

the input partition is $\mathbf{b} = (3, 3, 1)$, thus having $\mathbf{a} = (3, 2, 2)$ (sum of the columns), and the output partition is $\mathbf{b}' = (2, 2, 2, 1)$ (sums of the rows).

LEMMA. Given d -partitions $\mathbf{b}' \leq \mathbf{b}$, there is at least one incidence diagram having input $\mathbf{b}$ and output $\mathbf{b}'$.

Proof. To set up an induction, we call two d -partitions $\mathbf{b}^1 < \mathbf{b}^2$ consecutive if there are indexes $m < n$ such that

$$b_i^1 = b_i^2 \quad \text{for all } i, \text{ excepted } b_m^1 = b_m^2 - 1 \quad \text{and} \quad b_n^1 = b_n^2 + 1.$$

It is clear that for given $\mathbf{b}' \leq \mathbf{b}$, there is a finite sequence of consecutive d -partitions starting by $\mathbf{b}$ (which owns an evident incidence diagram of input $\mathbf{b}$ and output $\mathbf{b}$) and ending by consecutive $\tilde{\mathbf{b}}' > \mathbf{b}'$. We can thus assume by our induction hypothesis that $\tilde{\mathbf{b}}'$ possesses an incidence diagram $(\tilde{\varepsilon}_{ij})$ of input $\mathbf{b}$ and output $\tilde{\mathbf{b}}'$.

Let $m < n$ be the indexes such that $b'_m = \tilde{b}'_m - 1$ and $b'_n = \tilde{b}'_n + 1$ so that $\tilde{b}'_n \leq \tilde{b}'_m - 2$. This implies the existence of an index l such that $\tilde{\varepsilon}_{ml} = 1$ and $\tilde{\varepsilon}_{nl} = 0$. We define a new incidence diagram (ε_{ij}) of input partition $\mathbf{b}$ by $\varepsilon_{ij} = \tilde{\varepsilon}_{ij}$ if $(i, j) \neq (m, l), (n, l)$ and $\varepsilon_{ml} = 0, \varepsilon_{nl} = 1$. The output of this diagram is clearly $\mathbf{b}'$, which proves our lemma.

Turning the main proof, it suffices to show, for $\mathbf{b}' \leq \mathbf{b}$, that the generic point Z_0 of $\text{Hilb}^{\mathbf{b}'}(b'_0 F_1)$ is a limit of points Z_t of D_t with $t \neq 0$. Recall from the Lemma in 1.4 that we can assume Z_0 has ideal

$$I_{Z_0} = \bigcap_{i=0}^{r'-1} (y^{b'_i}, x - p_i(y))$$

with all $p_i(0)$ distinct and $\deg p_i(y) < b'_i$.

Let Z_t , for all $t \in \mathbb{C}$, be the schemes of length d and ideal

$$\bigcap_{i=0}^{r'-1} \left(\prod_{\varepsilon_{ij}=1} (y - jt), x - p_i \right)$$

where (ε_{ij}) is an incidence diagram of input $\mathbf{b}$ and output $\mathbf{b}'$. They form a flat family whose special fibre is Z_0 and whose generic fibre Z_t for $t \neq 0$ lies in $\text{Hilb}^{\mathbf{b}}(L_0, L_t, L_{2t}, \dots, L_{b_0-1}t) = D_t$ since clearly $\text{length}(Z \cap L_{jt}) = \sum \varepsilon_{ij} = a_j$ for all $j \geq 0$. This verifies our description of the irreducible components of $((\hat{D})_0)_{\text{red}}$.

We show next that $\text{Hilb}^{\mathbf{b}}(b_0 F_1)$ is generically smooth in $(\hat{D})_0$. We have remarked in the lemma of Section 1.4 that a dense open set W of $\text{Hilb}^{\mathbf{b}}(b_0 F_1) \subseteq \text{Hilb}^d U_1$ corresponds to ideals

$$\bigcap_{i=0}^{r-1} (y^{b_i}, x - p_i(y))$$

with $\deg p_i < b_i$ and all $p_i(0)$ distinct, i.e. the intersection is coprime. (More precisely, $W \subseteq \text{Hilb}^{\mathbf{b}} F_1$ consists of the schemes that are curvilinear, i.e. contained in a smooth curve, and transversal to F_1).

We now pick a generic point in W not lying in other irreducible components. We obtain an analytical chart of $\hat{D}$ at this point by choosing an analytical neighbourhood V small enough to consist of points of $\text{Hilb}^d U_1$ representing schemes whose ideal is coprime intersection

$$\bigcap_{i=0}^{r-1} \left(y^{b_i} - \sum_{j=1}^{b_i} \lambda_{ij} y^{b_i-j}, x - p_i(y) - \sum_{j=1}^{b_i} \mu_{ij} y^{b_i-j} \right)$$

for some parameters λ_{ij} and μ_{ij} describing an analytical open set V' of affine space A^{2d} isomorphic to V . Clearly $V \cap (\text{Hilb}^{\mathbf{b}}(b_0 F_1)) \subseteq W$.

The ideal of a subscheme of U_1 corresponding to a point of $\hat{D} \cap (V \times \mathbb{C})$ lying on $t \in \mathbb{C}$ must contain the ideal $\bigcap_{j=0}^{r-1} (y - jt)$, so $\hat{D}$ admits local equations in $V \times \mathbb{C}$

$$\lambda_{ij} = (-1)^{j+1} \sum_{0 \leq k_1 \leq \dots \leq k_j \leq b_i-1} (k_1 t) \cdots (k_j t) = (-1)^{j+1} t^j s_{ij},$$

where $s_{ij} = \sum_{0 \leq k_1 \leq \dots \leq k_j \leq b_i-1} k_1 \cdots k_j$. (Don't forget that D is equipped with a reduced structure). This defines a $\mathbb{C}$ -family of linear spaces in $V' \subseteq \mathbb{A}^{2d}$

describing a curve in the corresponding Grassmann variety which is smooth when $t = 0$ since $s_{i1} = 0 + 1 + 2 + \dots + (b_i - 1) \neq 0$ for some i . (Except in the trivial case where $b_0 = 1$). Therefore the flat limit of the linear spaces of equations $\lambda_{ij} = (-1)^{j+1} s_{ij} t^j$ for $t \neq 0$ is the linear space of equations $\lambda_{ij} = 0$ just taken with multiplicity one. In other words, (via the analytical isomorphism of V and V'), the flat limit $(\hat{D})_0 \cap V$ of the family $(\hat{D})_t \cap V$ for $t \neq 0$ is just $W \cap V = \text{Hilb}^{\mathbf{b}}(b_0 F_1) \cap V$ taken with multiplicity one. Therefore an irreducible component of $(\hat{D})_0$ is $\text{Hilb}^{\mathbf{b}}(b_0 F_1)$, with eventual embedded components which are obliterated by the associated cycle.

This proves that the cycle $[\text{Hilb}^{\mathbf{b}}(L_0, L_1, \dots, L_{b_0-1})]$ is rationally equivalent to the cycle $[\text{Hilb}^{\mathbf{b}}(b_0 F_1)]$ plus an integer combination* of cycles $[\text{Hilb}^{\mathbf{b}'}(b'_0 F_1)]$, with $\mathbf{b}' < \mathbf{b}$. Indeed, we have shown that the two closed schemes defining both cycles appear (neglecting embedded components) as two scheme fibers of the same flat $\mathbb{C}$ -family, thus a same flat $\mathbb{P}^1$ -family, of closed subschemes of $\text{Hilb}^d \mathbb{P}^2$. Q.E.D.

2.3. We remark that the inequalities $\mathbf{b}' < \mathbf{b}$ imply that $\mathbf{b}'$ comes before $\mathbf{b}$ in the lexicographic total order of B_d . The argument of the proof of Proposition 2.1 could be performed so as to conclude that a cycle of type $(0', 1, 2)$ of partition $(\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2)$ is rationally equivalent to a cycle of type $(0', 1', 2)$ of same partition plus a sum of cycles of type $(0', 1', 2)$ and prior partition $(\mathbf{b}^0, \mathbf{b}^{1'}, \mathbf{b}^2)$, for all $\mathbf{b}^{1'} \leq \mathbf{b}^1$, with irrelevant coefficients: just specialize the $1'$ -part as in that proof and leave invariant, along the specialization, the 2 and $0'$ parts of the mixed configuration.

Therefore, we have expressed the set $\sigma_{0'12}(M_{d,n})$, in terms of the basis $\sigma_{0'1'2}(M_{d,n})$, by a triangular matrix of integers with all diagonal entries equal to 1, so $\sigma_{0'12}(M_{d,n})$ is also a basis for $A_n(\text{Hilb}^d \mathbb{P}^2)$. This proves Theorem 0'12.

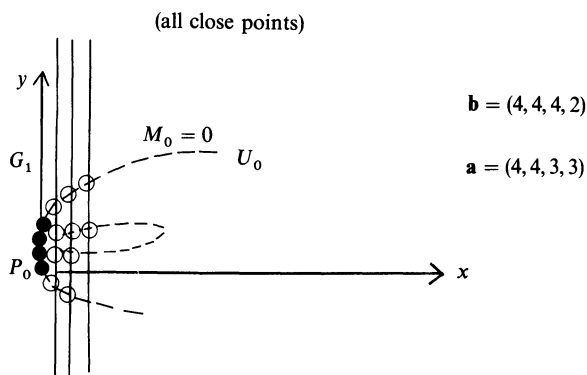
3. From type $0'$ to type 0

In this section we deduce Theorem 012 from Theorem 0'12 i.e. replace the basic classes $\sigma_{0'12}$ by the classes σ_{012} .

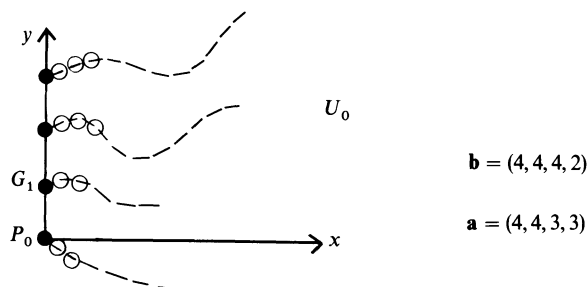
3.1. First we explain heuristically the plan of the proof. Just as before, it is enough to show we can replace the class $\sigma_0, (\mathbf{b})$ of the scheme of type $0'$ and partition $\mathbf{b}$. Recall that this last is the closure $\bar{U}'$ in $\text{Hilb}^d \mathbb{P}^2$ of the scheme $U' = \text{Hilb}^a(a_0 G_1, P_0) \subseteq \text{Hilb}^d U_0$ (for $U_0 = \text{Spec } \mathbb{C}[x_1/x_0, x_2/x_0] =$

* In fact, a more complicated version of the given argument, working with a different analytic chart for each $\mathbf{b}' < \mathbf{b}$, shows that the coefficient $n(\mathbf{b}, \mathbf{b}')$ of $[\text{Hilb}^{\mathbf{b}'}(b'_0 F_1)]$ appearing in this combination is the number of incidence diagrams of input $\mathbf{b}$ and output $\mathbf{b}'$. This is irrelevant to our purpose.

$\text{Spec } \mathbb{C}[x, y]$ parametrizing subschemes of $\mathbb{P}^2$ supported at the origin as

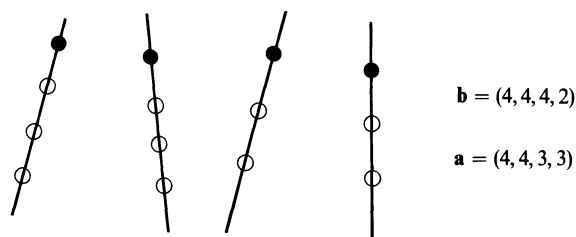


(See the picture as the limit when $\lambda \rightarrow 0$ of the corresponding one with $M_0^\lambda = 0$ and lines $x = 0, x = \lambda, x = 2\lambda, x = 3\lambda$). We show that the scheme $\bar{U}'$ is numerically equivalent to the scheme $\bar{U}''$ of type $0''$ and same partition $\mathbf{b}$ for U'' parametrizing subschemes of $\mathbb{P}^2$ as

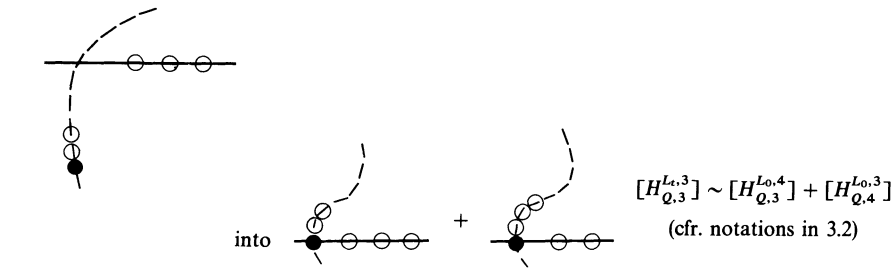


The reason is that both have the same intersection numbers with elements $\sigma_{0'12}(\mathbf{b}')$ of the basis for $A_{2d-n}(\text{Hilb}^d \mathbb{P}^2)$: More precisely, both have intersection 1 with $\sigma_{0'12}(0, 0, \mathbf{b})$ and intersection 0 with all the others.

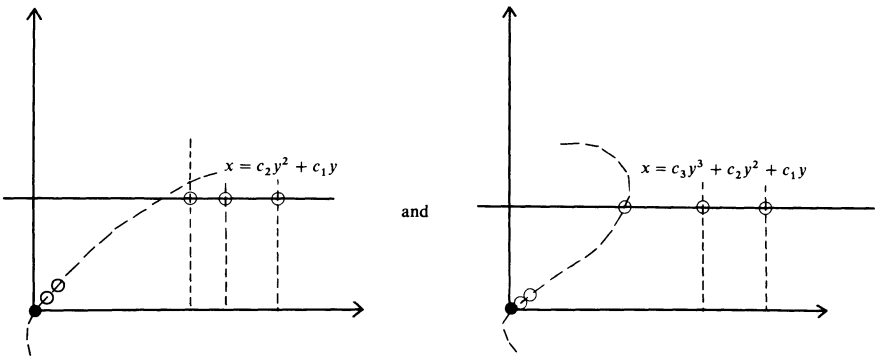
This cycle of type $0''$ and partition $\mathbf{b}$ is shown to be rationally equivalent – up to coefficient ± 1 – to the cycle of type 0 of same partition $\mathbf{b}$ (see the picture below) plus cycles of prior type, thus leading to a triangular matrix with diagonal entries ± 1 .



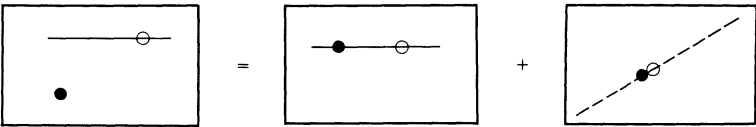
This is proved by iterating the specialization in a flat $\mathbb{P}^1$ -family (thus in the same class of rational equivalence) of



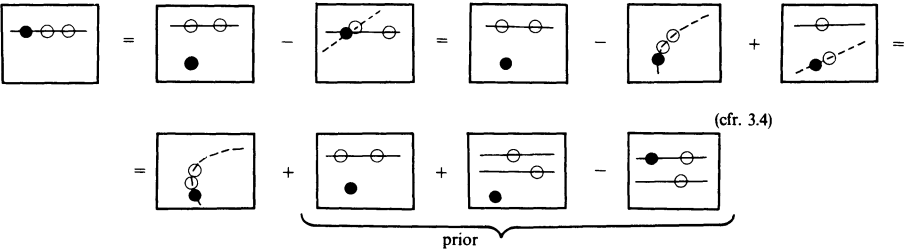
corresponding to its expression in two analytical charts (cfr. 3.2)



Such iteration is, for example



As another less trivial example



3.2. Let L, Q be a line and a point of $\mathbb{P}^2$ and ℓ, q nonnegative integers. We define the closed subsets of $\text{Hilb}^d \mathbb{P}^2$

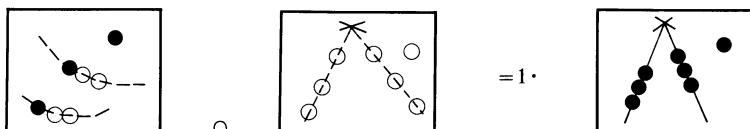
$$H^{L,\ell} = \{Z \in \text{Hilb}^d \mathbb{P}^2 / \text{length}(Z \cap L) \geq \ell\}$$

$$H_{Q,q} = \{Z \in \text{Hilb}^d \mathbb{P}^2 / \text{length}(Z)_Q \geq q\}$$

$$H_{Q,q}^{L,\ell} = H^{L,\ell} \cap H_{Q,q}.$$

Observe that in case $Q \notin L$ and $\ell + q = d$, the inequalities defining $H_{Q,q}^{L,\ell}$ are in fact equalities.

As a technical step, we define the scheme of type $0''$ and d -partition $\mathbf{b} = (b_0, \dots, b_{r-1})$ associated to s distinct points $Q_0, \dots, Q_{s-1}$ of $\mathbb{P}^2$, as $H_{Q_0, a_0} \cap \dots \cap H_{Q_{s-1}, a_{s-1}}$. Observe that here again the inequalities defining the scheme are all forced to be equalities, since $\sum a_j = d$. We see it is numerically equivalent to the scheme $\overline{\text{Hilb}^s(a_0 G_1, F_0)}$ of type $0'$ and d -partition $\mathbf{b}$. Indeed, both schemes have the same intersection number with $\sigma_{0'1'2}(\mathbf{b}')$ for all $\mathbf{b}' \in M_{d,d+b_0}$. This number is zero unless $\mathbf{b}' = (\mathbf{0}, \mathbf{0}, \mathbf{b})$, being 1 in this case (cfr. 1.4, remarking that a scheme $H_{Q_0, a_0} \cap \dots \cap H_{Q_{s-1}, a_{s-1}}$ is a disjoint union of schemes of lengths $a_0, \dots, a_{s-1}$, all of type $0'$). This is, for instance, if $\mathbf{b} = (3, 2, 2)$, thus $\mathbf{a} = (3, 3, 1)$, as in the example of 1.4,



By a trivial extension of this argument we can conclude that a scheme of type $(0'', 1, 2)$ of mixed d -partition $(\mathbf{b}^0, \mathbf{b}^1, \mathbf{b}^2)$ is numerically equivalent to a scheme of type $(0', 1, 2)$ of same mixed d -partition. We refer to this fact writing $\sigma_{0'12}(\mathbf{b}) = \sigma_{0''12}(\mathbf{b})$. From this we will show that $\sigma_{012}(M_{d,n})$ is also a basis.

3.3. The proof of Theorem 012 will rest on the following

PROPOSITION 3.1. *Let Q be a point of $\mathbb{P}^2$, L a line not containing Q and M a line containing Q . Let ℓ and q be nonnegative integers. There is an equivalence of cycles in $\text{Hilb}^{\ell+q} \mathbb{P}^2$*

$$[H_{Q,q}^{L,\ell}] \sim [H_{Q,q+1}^{M,\ell}] + [H_{Q,q}^{M,\ell+1}].$$

Proof. We take homogeneous coordinates x_0, x_1, x_2 in $\mathbb{P}^2$ so that Q is the point $(1, 0, 0)$ and L, M have equations $x_2 = x_0, x_2 = 0$. Let L_t be the line of equation $x_2 = tx_0$ for each $t \in \mathbb{C}$ so that $M = L_0, L = L_1$ and let U_0 be the

affine piece $\text{Spec } \mathbb{C}[x_1/x_0, x_2/x_0] = \text{Spec } \mathbb{C}[x, y]$. The closure of $\hat{H}_{Q,q}^{L_0, \ell} = H_{Q,q}^{L_0, \ell} \cap \text{Hilb}^{\ell+q} U_0$ is clearly $H_{Q,q}^{L_0, \ell}$. Let D be the subscheme of $(\text{Hilb}^{\ell+q} \mathbb{P}^2 \times \mathbb{C}^*)$, flat over $\mathbb{C}^*$, whose fibre on $t \neq 0$ is $H_{Q,q}^{L_t, \ell}$. The closure $\bar{D}$ in $(\text{Hilb}^{\ell+q} \mathbb{P}^2) \times \mathbb{C}$ is flat over $\mathbb{C}$ with generic fibre $(\bar{D})_t = \bar{D}_t = H_{Q,q}^{L_t, \ell}$ for $t \neq 0$.

We prove first that the support of the special fibre $(\bar{D})_0$ is contained in $H_{Q,q+1}^{L_0, \ell} \cup H_{Q,q}^{L_0, \ell+1}$. A closed point Z_0 of this fibre $(\bar{D})_0$ is in the closure of D , so it is special point of a curve $\mathfrak{z} \subseteq D$ whose generic point Z_t lies in $\bar{D}_t$ for $t \neq 0$. We still denote by Z_0, Z_t the corresponding subschemes of $\mathbb{P}^2$. From $\text{Supp } Z_t \subseteq \{Q\} \cup L_t$ and length $(Z_t)_Q = q$ and length $(Z_t \cap L_t) = \ell$ for $t \neq 0$, we obtain that $\text{supp } Z_0 \subseteq L_0$ and length $(Z_0)_Q \geq q$ and length $(Z_0 \cap L_0) \geq \ell$ (since the condition length $(Z \cap L) \geq \ell$ defines a closed subset of $(\text{Hilb}^{\ell+q} \mathbb{P}^2) \times (\mathbb{P}^2)^\vee$).

Assume length $(Z_0)_Q = q$. Then length $(Z_t)_Q \leq q$ for generic $t \in \mathbb{C}$ since this is an open condition, so length $(Z_t)_Q = q$. In other words, $\mathfrak{z}$ is a curve in the locally closed subscheme $H_{Q,(\ell+q)}$ of $\text{Hilb}^{\ell+q} \mathbb{P}^2$ parametrizing schemes of length $\ell + q$ having length q at the point Q . There is an obvious morphism $H_{Q,(\ell+q)} \rightarrow \text{Hilb}^{\ell}(\mathbb{P}^2 \setminus \{Q\})$ and the image $\mathfrak{z}'$ of the curve $\mathfrak{z}$ by this morphism is a curve of $\text{Hilb}^{\ell}(\mathbb{P}^2 \setminus \{Q\})$ having $Z_0 \cap (\mathbb{P}^2 \setminus \{Q\})$ as a special point and $Z_t \cap (\mathbb{P}^2 \setminus \{Q\})$, with $t \neq 0$, as generic point. For generic $t \neq 0$, the scheme $Z_t \cap (\mathbb{P}^2 \setminus \{Q\}) \cap (L_t \setminus \{Q\})$ is just $Z_t \cap (\mathbb{P}^2 \setminus \{Q\})$, thus of length ℓ . Therefore, for the special value $t = 0$, we obtain length $(Z_0 \cap (\mathbb{P}^2 \setminus \{Q\}) \cap (L_0 \setminus \{Q\})) \geq \ell$ i.e. length $Z_0 \cap L_0 \geq \ell + 1$, which means $Z_0 \in H_{Q,q}^{L_0, \ell+1}$.

If we assume, on the contrary, that length $(Z_0)_Q \geq q + 1$, then clearly $Z_0 \in H_{Q,q+1}^{L_0, \ell}$ since we saw that length $(Z_0 \cap L_0) \geq \ell$.

We prove next that the support of the fibre $(\bar{D})_0$ is $H_{Q,q+1}^{L_0, \ell} \cup H_{Q,q}^{L_0, \ell+1}$. Let $Z_0 \in \hat{H}_{Q,q+1}^{L_0, \ell}$ (resp. $Z_0 \in \hat{H}_{Q,q}^{L_0, \ell+1}$) and recall this scheme has closure $H_{Q,q+1}^{L_0, \ell}$ (resp. $H_{Q,q}^{L_0, \ell+1}$). Applying the result of Briançon [2, Theorem V.3.2] about the structure of the generic subschemes of $\mathbb{P}^2$ of length q supported in a point Q , we see immediately that the generic element of $H_{Q,q+1}^{L_t, \ell}$, for $t \neq 0$, can be described by an ideal

$$(y^q(y-t), x - c_q y^q - \cdots - c_1 y) \cap (y-t, x - \lambda_1) \cap \cdots \cap (y-t, x - \lambda_{l-1}) \\ (\text{resp. } (y^q, x - c_{q-1} y^{q-1} - \cdots - c_1 y) \cap (y-t, x - \lambda_1) \cap \cdots \cap (y-t, x - \lambda_\ell) (*))$$

for arbitrary $c_i, \lambda_j \in \mathbb{C}$.

Each element Z_0 of a dense open set W of $\hat{H}_{Q,q+1}^{L_0, \ell}$ (resp. $\hat{H}_{Q,q}^{L_0, \ell+1}$) has ideal defined by this same expression with $t = 0$, so it belongs to the closure $\bar{D}$ of D in $(\text{Hilb}^{\ell+q} \mathbb{P}^2) \times \mathbb{C}$. This proves that $H_{Q,q+1}^{L_0, \ell} \cup H_{Q,q}^{L_0, \ell+1}$ is contained in the support of $(\bar{D})_0$ so it is equal to this support.

In order to investigate the scheme fibre $(\bar{D})_0$, we now take Z_0 general enough, so as to assure that all $\lambda_i \neq 0$, and that they are all distinct, i.e. that the above intersection is coprime. Let V be an analytic neighbourhood of Z_0 in $\text{Hilb}^d \mathbb{P}^2$,

small enough so that all its points correspond to coprime intersection ideals

$$\begin{aligned} & (y^{q+1} - \gamma_q y^q - \gamma_{q-1} y^{q-1} - \dots - \gamma_0), x - c_q y^q - \dots - c_0) \cap \\ & \cap (y - \mu_1, x - \lambda_1) \cap \dots \cap (y - \mu_{\ell-1}, x - \lambda_{\ell-1}) \\ & (\text{resp. } (y^q - \gamma_{q-1} y^{q-1} - \dots - \gamma_0, x - c_{q-1} y^{q-1} - \dots - c_0) \cap \\ & \cap (y - \mu_1, x - \lambda_1) \cap \dots \cap (y - \mu_{\ell}, x - \lambda_{\ell})) \end{aligned}$$

for parameters $\gamma_0, \dots, \gamma_q, c_0, \dots, c_q, \lambda_1, \dots, \lambda_{\ell-1}, \mu_1, \dots, \mu_{\ell-1}$ (resp. $\gamma_{q_0}, \dots, \gamma_{q-1}, c_0, \dots, c_{q-1}, \lambda_1, \dots, \lambda_{\ell}, \mu_1, \dots, \mu_{\ell}$). These parameters describe an analytic open set V' of the affine space $\mathbb{A}^{2(\ell+q)}$ so that V is a chart of $\text{Hilb}^d \mathbb{P}^2$ near Z_0 . Clearly $V \cap \hat{H}_{Q,q+1}^{L_0, \ell}$ (resp. $V \cap \hat{H}_{Q,q}^{L_0, \ell+1}$) lies in W . All the points of $\hat{D} \cap (V \times \mathbb{C})$ correspond to ideals expressed as coprime intersections as above (*), so $\hat{D}$ admits in $V \times \mathbb{C}$ the local equations

$$\begin{aligned} & \gamma_0 = 0, \dots, \gamma_{q-1} = 0, \gamma_q = t, c_0 = 0, \mu_1 = t, \dots, \mu_{\ell-1} = t \\ & (\text{resp. } \gamma_0 = 0, \dots, \gamma_{q-1} = 0, c_0 = 0, \mu_1 = t, \dots, \mu_{\ell} = t). \end{aligned}$$

This defines a linear pencil (parametrized by $t \in \mathbb{C}$) of linear spaces of dimension $\ell + q - 1$ in $V' \subseteq \mathbb{A}^{2(\ell+q)}$. The special one corresponding to $t = 0$ is the flat limit of the generic one corresponding to $t \neq 0$. Therefore the fibre $(\hat{D})_0$ is reduced at the generic point of each one of its two components. Since the definition of the cycle associated to a closed scheme overlooks all possible embedded components we obtain that

$$[H_{Q,q}^{L_1, \ell}] = [\bar{D}_1] = [(\hat{D})_1] \sim [(\hat{D})_0] = [H_{Q,q+1}^{L_0, \ell}] + [H_{Q,q}^{L_0, \ell+1}]$$

as $(\hat{D})_1$ and $(\hat{D})_0$ are both fibres of a same flat $\mathbb{C}$ -family (thus $\mathbb{P}^1$ -family) of subschemes of $\text{Hilb}^{\ell+q} \mathbb{P}^2$. Q.E.D.

3.4. Iterating the proposition and keeping its notation we obtain as corollary the rational equivalence of cycles

$$\begin{aligned} & [H_{Q,q}] \sim [H_{Q,q}^{M,1}] \sim [H_{Q,q-1}^{L,1}] - [H_{Q,q-1}^{M,2}] \sim [H_{Q,q-1}^{L,1}] - [H_{Q,q-2}^{L,2}] + \\ & + [H_{Q,q-2}^{M,3}] \sim \dots \sim \left(\sum_{i=1}^q (-1)^{i+1} [H_{Q,q-i}^{L,i}] \right) + (-1)^{q-1} [H_{Q,1}^{M,q}]. \end{aligned}$$

Therefore $[H_{Q,1}^{M,q}]$ is, up to a sign, $[H_{Q,q}]$ plus a sum of cycles $[H_{Q,q-i}^{L,i}]$ of type $(0'', 1)$ with $d^{0''} = q - i, d^1 = i$ (for $i = 1, \dots, q$) thus strictly prior in the total order of the basis $\sigma_{0''12}(M_{d,n})$.

A cycle of type 0 and partition $\mathbf{b}$ as defined in 0.1 is of the form

$$[H_{P_{0,1}}^{L_{0,a_0}} \cap \cdots \cap H_{P_{s-1,1}}^{L_{s-1,a_{s-1}}}] .$$

By a trivial extension of the proposition and its corollary, this cycle is rationally equivalent to the cycle $[H_{P_{0,a_0}} \cap \cdots \cap H_{P_{s-1,a_{s-1}}}]$ of type $0''$ and same partition $\mathbf{b}$, plus a sum of cycles of type $(0'', 1)$ which are prior in the total order of the basis $\sigma_{0''12}(M_{d,n})$. Extending the argument further, we observe that a cycle of type $(0, 1, 2)$ and mixed partition $\mathbf{b}' \in M_{d,n}$ is rationally equivalent to the sum of a cycle – with coefficient ± 1 – of type $(0'', 1, 2)$ and same mixed partition $\mathbf{b}'$ plus other cycles of type $(0'', 1, 2)$ and mixed d -partitions $\mathbf{b}'$ prior in the total order of $M_{d,n}$. Therefore the set $\sigma_{012}(M_{d,n})$ is expressed, relative to the basis $\sigma_{0''12}(M_{d,n})$, by a triangular matrix with diagonal entries ± 1 , so it is also a basis. This proves the main Theorem 012.

4. Bases for the Chow groups of the variety of Schubert triangles

This smooth projective variety, denoted W^* has been formalized in [13] as the closure in $(\mathbb{P}^2)^3 \times (\mathbb{P}^{2\vee})^3 \times \text{Gr}(2, 5)$ of those $(a, b, c, \alpha, \beta, \gamma, n)$ such that the points a, b, c are distinct, the lines α, β, γ are distinct, $b, c \in \alpha; a, c \in \beta; b, c \in \gamma$, and n is the net of conics passing by a, b, c . This clearly translates as the subvariety of $(\mathbb{P}^2)^3 \times (\mathbb{P}^{2\vee})^3 \times \text{Hilb}^3 \mathbb{P}^2$ consisting of those $(a, b, c, \alpha, \beta, \gamma, t)$ such that $b, c \in \alpha; a, c \in \beta; b, c \in \gamma$ and the triple t is of one of the types (and their permutations)

- (1) If a, b, c are distinct, then $t = \{a, b, c\}$
- (2) If $a \neq b = c$ then t consists of a and point b doubled on line α
- (3) If $a = b = c$ then t is m_a^2 unless $\alpha = \beta = \gamma$, in which case t can also be a curvilinear triple at a of tangent α .

In case 1, all α, β, γ are distinct (general case φ) or equal (degeneration ε). In case 2, either $\alpha \neq \beta = \gamma$ (degeneration v_a) or $\alpha = \beta = \gamma$ (degeneration ω_a). In case 3, all α, β, γ may be distinct (degeneration τ) or $\alpha \neq \beta = \gamma$ (degeneration ω_a) or $\alpha = \beta = \gamma$ with $t = m_a^2$ (degeneration ξ) or t being curvilinear (degeneration ψ) and more particularly inflexional, i.e. with $t \subseteq \alpha$ (degeneration η). Each one of these degenerations defines – by taking the closure – a closed subscheme, denoted similarly as are the corresponding cycle and rational equivalence class.

Still following Schubert's notation in [15], we will consider also “positional” classes of cycles: a (resp. α) is the class of the cycle associated to the scheme parametrizing Schubert triangles with the vertex a in a given line (resp. the side α passing through a given point). Thus, a^2 (resp. α^2) corresponds to fixing the first vertex (resp. the first side).

That the classes of cycles of type $(0', 1', 2)$ provide a basis for A . ($\text{Hilb}^3 \mathbb{P}^2$) can be proved by an elementary variant of our general method. Consider just one

G_m -action on $\mathbb{P}^2$, namely $\lambda(x_0, x_1, x_2) = (\lambda^3 x_0, \lambda x_1, x_2)$ with fixed points P_0, P_1, P_2 and corresponding cell decomposition $\mathbb{P}^2 \setminus F_1, F_1 \setminus F_0, F_0$. The induced action on $\mathbb{P}^{2^\vee}$ is $\lambda(u_0, u_1, u_2) = (\lambda^{-3} u_0, \lambda^{-1} u_1, u_2)$ with fixed lines $L_0 = \langle P_1 P_2 \rangle, L_1 = \langle P_0 P_2 \rangle, L_2 = \langle P_0 P_1 \rangle$ and corresponding cell decomposition $\mathbb{P}^{2^\vee} \setminus F_0^\vee, F_0^\vee \setminus F_1^\vee, F_1^\vee$. There is an induced action of G_m on the set $\text{Hilb}^3 \mathbb{P}^2$ of triples t , and the limiting behaviour, as λ approaches 0, is easily found in each case, so we arrive immediately at the $(0', 1', 2)$ cell decomposition described in Section 1. Knowing the limiting behaviour of the action of G_m on points, lines and triples we also know the limiting behaviour on the smooth projective variety of triangles so we get by Bialynicki-Birula theorem an explicitly described cell decomposition: Writing down (at left) all fixed triangles we obtain (at right) a basis for all groups $A_n(W^*)$ – of same cardinals 1, 7, 17, 22, 17, 7, 1 as Schubert “bedingungen” – by just writing the rational equivalence class of the closure of the cell of triangles approaching each fixed triangle:

A_0 :	$(P_0, P_0, P_0, L_2, L_2, L_2, ((3), (0), (0)))$	$\eta a^2 \alpha$
A_1 :	$(P_0, P_0, P_0, L_1, L_2, L_2, ((2, 1), (0), (0))) (\times 3)$	$\omega_a a^2 \beta (\times 3)$ i.e. three index permutations
	$(P_0, P_0, P_0, L_2, L_2, L_2, ((2, 1), (0), (0)))$	$\psi a^2 \alpha$
	$(P_1, P_0, P_0, L_2, L_2, L_2, ((2), (1), (0))) (\times 3)$	$\omega_a b^2 \beta (\times 3)$
A_2 :	$(P_1, P_1, P_1, L_2, L_2, L_2, ((0), (2, 1), (0))) (\times 3)$	$\psi \alpha^2$
	$(P_0, P_0, P_0, L_2, L_1, L_1, ((2, 1), (0), (0))) (\times 3)$	$\tau a^2 \alpha (\times 3)$
	$(P_0, P_0, P_0, L_1, L_1, L_1, ((1, 1, 1), (0), (0)))$	ψa^2
	$(P_0, P_1, P_1, L_0, L_2, L_2, ((1), (1, 1), (0))) (\times 3)$	$\nu_a a^2 \beta (\times 3)$
	$(P_0, P_1, P_1, L_2, L_2, L_2, ((1), (2), (0))) (\times 3)$	$\varepsilon a^2 \alpha (\times 3)$
	$(P_1, P_0, P_0, L_1, L_2, L_2, ((1, 1), (1), (0))) (\times 3)$	$\nu_a a^2 \beta (\times 3)$
	$(P_2, P_0, P_0, L_2, L_1, L_1, ((1, 1), (0), (1))) (\times 3)$	$\nu_a b^2 \alpha (\times 3)$
A_3 :	$(P_2, P_2, P_2, L_1, L_1, L_1, ((0), (0), (2, 1))) (\times 3)$	$\psi \alpha$
	$(P_1, P_1, P_1, L_0, L_2, L_2, ((0), (2, 1), (0))) (\times 3)$	$\nu_a \alpha (\times 3)$
	$(P_1, P_1, P_1, L_2, L_0, L_0, ((0), (2, 1), (0))) (\times 3)$	$\tau \alpha^2 (\times 3)$
	$(P_1, P_1, P_1, L_0, L_0, L_0, ((0), (1, 1, 1), (0)))$	ψa
	$(P_1, P_1, P_1, L_2, L_2, L_2, ((0), (3), (0)))$	$\varepsilon \alpha^2$
	$(P_0, P_0, P_0, L_1, L_1, L_1, ((2, 1), (0), (0)))$	τa^2
	$(P_0, P_2, P_2, L_1, L_1, L_1, ((1), (0), (2))) (\times 3)$	$\varepsilon a^2 (\times 3)$
	$(P_2, P_0, P_0, L_1, L_1, L_1, ((1, 1), (0), (1))) (\times 3)$	$\nu_a b^2 (\times 3)$
	$(P_0, P_1, P_2, L_0, L_1, L_2, ((1), (1), (1))) (\times 6)$	$a^2 \beta (\times 6)$

$A_4:$	$(P_2, P_2, P_2, L_0, L_1, L_1, ((0), (0), (2, 1))) (\times 3)$	$\omega_\alpha (\times 3)$
	$(P_2, P_2, P_2, L_1, L_1, L_1, ((0), (0), (3)))$	$\varepsilon\alpha$
	$(P_1, P_1, P_1, L_0, L_0, L_0, ((0), (2, 1), (0)))$	$\tau\alpha^2$
	$(P_2, P_1, P_1, L_2, L_0, L_0, ((0), (2), (1))) (\times 3)$	$\alpha^2 (\times 3)$
	$(P_2, P_1, P_1, L_0, L_0, L_0, ((0), (1, 1), (1))) (\times 3)$	$\nu_a b (\times 3)$
	$(P_1, P_2, P_2, L_1, L_0, L_0, ((0), (1), (2))) (\times 3)$	$a\alpha (\times 3)$
	$(P_0, P_2, P_2, L_0, L_1, L_1, ((1), (0), (1, 1))) (\times 3)$	$a^2 (\times 3)$
$A_5:$	$(P_2, P_2, P_2, L_1, L_0, L_0, ((0), (0), (2, 1))) (\times 3)$	$\alpha (\times 3)$
	$(P_2, P_2, P_2, L_0, L_0, L_0, ((0), (0), (2, 1)))$	τ
	$(P_1, P_2, P_2, L_0, L_0, L_0, ((0), (1), (1, 1))) (\times 3)$	$a (\times 3)$
$A_6:$	$(P_2, P_2, P_2, L_0, L_0, L_0, ((0), (0), (1, 1, 1)))$	W^*

5. Enumerative applications

We end with some enumerative applications of our result, concerning the case $d = 4$. This is the proof, in the case of smooth curves, of some formulae stated by Schubert as “very probable” in his classical article [15]. Schubert says that two plane curves have a double contact (zweimal zweifach Kontakt) when they share the tangent at two common points – not necessarily distinct. Given two families Σ_i ($i = 1, 2$) of plane curves of degrees n_1, n_2 (flat over a smooth and complete variety), he defines a double contact between Σ_1, Σ_2 to be a double contact between a curve of Σ_1 and a curve of Σ_2 . Let μ_i be the rational equivalence class of the cycle of Σ_i consisting of curves passing through a general point. If Σ_i is one-dimensional, then $\mu_i \in \mathbb{Z}$. If Σ_i is two-dimensional, then $\mu_i^2 \in \mathbb{Z}$. Assuming the general curves of the families Σ_i are smooth, the Schubert formulae we want to prove – the first two already proved by Zeuthen [16] – are these:

F1. Let Σ_1 be a smooth plane curve, and Σ_2 a two-dimensional family. The number of double contacts between Σ_1 and Σ_2 is

$$\mu_2^2 n_1 [n_1^3 - 6n_1^2 - n_1 + 30 + 4n_1^2 n_2 + 4n_1 n_2^2 - 12n_1 n_2 - 12n_2].$$

F2. If Σ_1, Σ_2 are both one-dimensional families, the number of double contacts between Σ_1, Σ_2 is

$$2\mu_1 \mu_2 [5n_1^2 n_2^2 + 2n_1 n_2^3 + 2n_1^3 n_2 - 9n_1 n_2^2 - 9n_1^2 n_2 - 11n_1 n_2 - 6n_1^2 - 6n_2^2 + 30n_1 + 30n_2 - 24].$$

F3. If Σ_1 is one dimensional and Σ_2 is two-dimensional, the degree of the plane curve of points of double contact between them is

$$\mu_1 \mu_2^2 [2n_1^3 - 9n_1^2 - 11n_1 + 30 + 5n_1 n_2^2 - 15n_1 n_2 - 12n_2 + 8n_1^2 n_2].$$

F4. If Σ_1, Σ_2 are both two-dimensional, the number of pairs of points of double contact, one of the two points being fixed, is

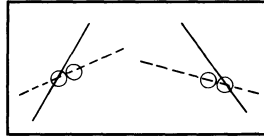
$$\mu_1^2 \mu_2^2 [n_1^2 + n_2^2 + 4n_1 n_2 - 3n_1 - 3n_2 - 10].$$

F5. Let Σ_1, Σ_2 as in F4. The degree of the curve described by one of the two points of a double contact while the other one moves in a fixed line of $\mathbb{P}^2$ is

$$\mu_1^2 \mu_2^2 [-1 + 4n_1^2 + 4n_2^2 - 12n_1 - 12n_2 + 10n_1 n_2].$$

We sketch here the proof of the more difficult assertion F5. (For details, and for the analogous proofs of the other formulae, see [12]).

Proof of F5. We note by $\tau \in A_4(\text{Hilb}^4 \mathbb{P}^2)$ the rational equivalence class of the closure of the scheme parametrizing length four subschemes of $\mathbb{P}^2$ expressed in our conventions by the picture

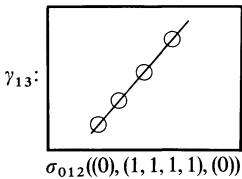
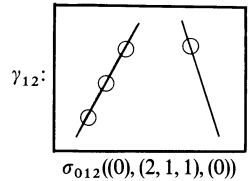
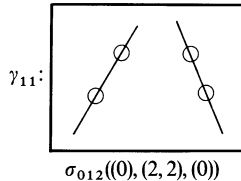
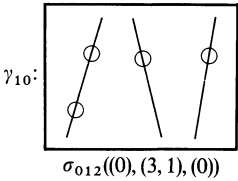
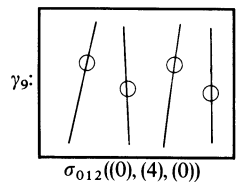
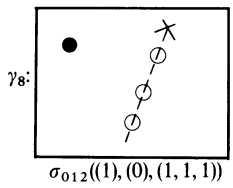
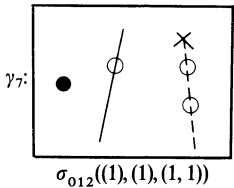
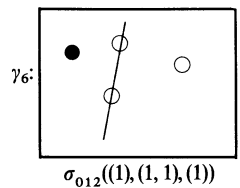
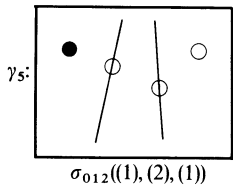
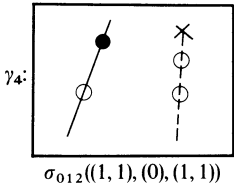
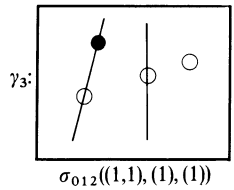
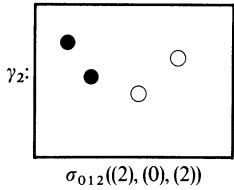
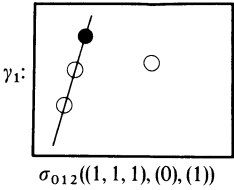


(i.e. disjoint unions of length two subschemes supported in two points moving in two fixed lines). The number η we want to obtain is the intersection number of three classes

$$\eta := \tau \left[\text{Hilb}^4 \sum_1 \right] \cdot \left[\text{Hilb}^4 \sum_2 \right]$$

Here $[\text{Hilb}^d \Sigma_i]$ denotes the rational class of the scheme consisting of length d subschemes of $\mathbb{P}^2$ contained in a curve of Σ_i . Observe both of them are in $A_6(\text{Hilb}^4 \mathbb{P}^2)$.

Our first goal is to express each of these three classes in terms of the bases we have obtained for the Chow groups A_4 and A_6 of $\text{Hilb}^4 \mathbb{P}^2$. These are the unknowns of the linear system of equations with coefficient matrix $A_4 \cdot A_4$ (resp. $A_2 \cdot A_6$) and independent terms $\tau \cdot \alpha_i$ (resp. $[\text{Hilb}^4 \Sigma_i] \cdot \beta_j$) where α_i, β_j are the elements of the bases of A_2, A_6 described in the introduction and that of A_4 :



	β_1	β_2	β_3	β_4	β_5	β_6
α_1	0	2	1	2	1	0
α_2	0	1	0	1	0	0
α_3	1	0	0	0	0	0
α_4	0	2	1	3	1	1
α_5	0	2	1	4	2	2
α_6	0	1	0	2	1	0

[illegible]

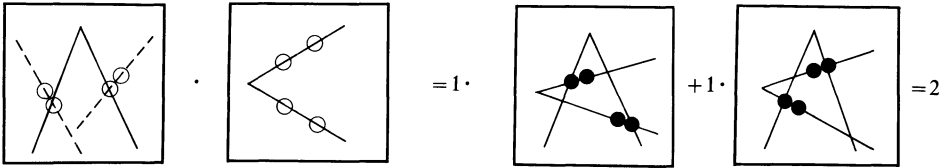
To illustrate how easily all the entries have been obtained, we give the computation $\tau \cdot \gamma_{11} = 2$.

Let $L_i, M_i, i = 1, 2$ be the lines of $\mathbb{P}^2$ defining the closed subschemes of $\text{Hilb}^4 \mathbb{P}^2$ corresponding to the classes τ and γ_{11} . Clearly, the support of the scheme intersection corresponding to $\tau \cdot \gamma_{11}$ consists of two points, x_1, x_2 of $\text{Hilb}^4 \mathbb{P}^2$ corresponding to subschemes X_1, X_2 of $\mathbb{P}^2$ with

$$\text{Supp } X_1 = (L_1 \cap M_1) \cup (L_2 \cap M_2)$$

$$\text{Supp } X_2 = (L_1 \cap M_2) \cup (L_2 \cap M_1).$$

We want to show that the multiplicity of $\tau \cdot \gamma_{11}$ at each x_i is actually 1 (so $\tau \cdot \gamma_{11}$ has length 2, as needed), i.e. that



It is enough to prove this in the case $i = 1$.

Let x, y be coordinates of an affine piece $\mathbb{A}^2$ of $\mathbb{P}^2$ containing X_1 and X_2 such that the lines M_1, M_2, L_1, L_2 have equations $x = 0, y = 0, y = 1, x = 1$. The ideal of X_1 in $\mathbb{A}^2$ is $((y - 1)^2, x) \cap ((x - 1)^2, y)$. An open neighbourhood of the corresponding $x_1 \in \text{Hilb}^4 \mathbb{A}^2$ consists of points determined by length 4 subschemes of $\mathbb{P}^2$ of ideal

$$((y - 1)^2 - ay - b, x - cy - d) \cap ((x - 1)^2 - ex - f, y - gx - h)$$

for arbitrary $a, b, c, d, e, f, g, h \in \mathbb{C}$. This is a chart of $\text{Hilb}^4 \mathbb{A}^2$ near x_1 , in which γ_{11} and τ have local equations $c = 0, d = 0, g = 0, h = 0$ and $a = 0, b = 0, e = 0, f = 0$, so that $\gamma_{11} \cdot \tau$ consists of the origin x_1 with multiplicity 1.

Observe that not only $\tau \cdot \gamma_{11}$ but also the intersections in $A_2 \cdot A_6$ or $A_4 \cdot A_4$ (or any $A_i \cdot A_{2d-i}$) are given by points with multiplicity one, which confirm the geometric naturality of our bases.

To continue our proof we solve the linear systems directly to obtain the following expressions for τ and $[\text{Hilb}^4 \Sigma_i]$:

$$\tau = \gamma_9 - 4\gamma_{10} + 4\gamma_{11}$$

$$\begin{aligned} \left[\text{Hilb}^4 \sum_i \right] &= (\mu_i^2 (n_i - 2)(n_i - 3)/2) \beta_2 + \mu_i^2 (n_i - 2) \beta_3 + \\ &+ \mu_i^2 (n_i - 2) \beta_4 + \mu_i^2 \beta_5 + \mu_i^2 \beta_6 \end{aligned}$$

Now, we only need to compute the products $\gamma_i \beta_j \beta_l$ for $i = 9, 10, 11$ and $j, l = 2, \dots, 6$. These are easily found to be

24	12	48	24	24
12	6	24	12	12
48	24	60	36	36
24	12	36	15	12
24	12	36	12	12

$\gamma_9 \beta_j \beta_l$

12	5	20	8	6
5	2	8	3	2
20	8	29	10	7
8	3	10	5	0
6	2	7	0	3

$\gamma_{10} \beta_j \beta_l$

6	2	8	4	0
2	1	2	1	0
8	2	8	4	0
4	1	4	1	0
0	0	0	0	0

$\gamma_{11} \beta_j \beta_l$

We are indebted to prof. P. Le Barz for proposing to the first author the proof of Schubert conjectures as an application of our result.

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