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JUAN ELIAS

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Characterization of the Hilbert-Samuel polynomials of curve singularities

JUAN ELIAS
Dept. Algebra i Geometria, Universitat de Barcelona Gran Via 585, 08007 Barcelona, Spain

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0. Introduction

The aim of this paper is to characterize the set of triplets (b, e, p) for which there exists a one-dimensional Cohen-Macaulay local ring with embedding dimension b, multiplicity e, and reduction number ρ.

In general b, e, ρ are related by the following conditions:
(1) $1 \leqslant b \leqslant e$ ([M], Theorem 12.10),
(2) $e-1 \leqslant \rho \leqslant e(e-1)/2$ ([K], Theorem 2).
Few more links are known between (e, b) and ρ, see for instance [M] Chapter 12 and [K].

For every triplet (b, e, ρ) we define the integer: $\rho_{(0,b,e)} = (r+1)e - \binom{r+b}{r}$ where r is the integer such that: $\binom{b+r-1}{r} \leqslant e < \binom{b+r}{r+1}$, and we put $\rho_{1,b,e} = e(e-1)/2 - (b-1)(b-2)/2$.

The main result of this paper is the following:

THEOREM. *There exists a one-dimensional Cohen-Macaulay local ring A, with embedding dimension b, multiplicity e, and reduction number ρ if and only if either* $b = 1$, $e = 1$, $\rho = 0$ *or*:
(1) $2 \leqslant b \leqslant e$, *and*
(2) $\rho_{0,b,e} \leqslant \rho \leqslant \rho_{1,b,e}$.
Moreover: for each triplet (b, e, ρ) satisfying (1) *and* (2) *we can take* $A = \mathbf{k}[[X_1, \ldots, X_N]]/I$ *reduced, with* $\mathbf{k}$ *an algebraically closed field of characteristic zero.*

From this result we deduce that $p(T)$ is the Hilbert-Samuel polynomial of a one-dimensional Cohen-Macaulay local ring if and only if $p(T)$ is the Hilbert-Samuel polynomial of a reduced one dimensional Cohen-Macaulay local ring $A = \mathbf{k}[[X_1, \ldots, X_N]]/I$. Since for a reduced curve singularity X the ring $\mathcal{O}_X$ is Cohen-Macaulay, the main result of this paper gives us a characterization of the Hilbert-Samuel polynomials of reduced curve singularities.

Notice that the ring A is obtained as the completion of $\mathcal{O}_X$ on its maximal ideal, where X is a union of monomial curves (see proof of Proposition 3.1).

Notice that the bound $\rho \leqslant \rho_{1,e,b}$ gives a restriction on the set of numerical functions $F: \mathbf{N} \to \mathbf{N}$ such that $F = FHS_A$ where A is a one-dimensional Cohen-Macaulay local ring. For such a function it holds:

(1) There exist non-negative integers e, ρ such that $F(n) = en - \rho$ for all $n \geqslant 0$,

(2) $\{\Delta F(n)\}_{n \geqslant 1}$ verifies the conditions of Macaulay ([St], Theorem 2.2),

(3) $\Delta F(n) \leqslant e$ for all $n \geqslant 1$ ([M] Proposition 12.15),

(4) $\rho \leqslant \rho_{1,b,e}$.

We will show that these conditions are not sufficient to assure the existence of a one-dimensional Cohen-Macaulay local ring A such that $F = FHS_A$ (Section 4 Remark 2). Hence the problem of characterize the Hilbert-Samuel functions of the one-dimensional Cohen-Macaulay local rings remains open.

Recall that in [E-4] we establish the existence of a $\mathbf{k}$-scheme $H_{N,p(T)}$ parametrizing the curve singularities $X \subset (\mathbf{k}^N, 0)$ with Hilbert-Samuel polynomial $p(T)$. The main result of this paper (Theorem 3.1) enables us to know for which $p(T)$ the scheme $H_{N,p(T)}$ actually occurs.

In the Section 1 we give lower and upper bounds for the reduction number of a Cohen-Macaulay local ring in terms of its embedding dimension and multiplicity. The key tool used in this section is the structure of the finitely generated modules over rings of principal ideals (see for instance [A-B], Chap. 10 Theorem 3.1). In particular we prove that for each degree one superficial element x of A, x^{e-b+1} belongs to the conductor of the extension $A \subset Bl(A)$, where $Bl(A)$ is the Blowing-up of A (Theorem 1.4).

Section 2 is devoted to construct reduced curves with a suitable reduction number; for this we compute, in several cases, the reduction number of $X \cup Y$ in terms of the reduction number of X and Y (Propositions 2.1, 2.4 and 2.6).

The Characterization Theorem is proved in Section 3 (Theorem 3.1). We will do that by induction on (e, b) and we will use the results about construction of curves of the Section 2.

In the last section (Section 4) we compute the Hilbert-Samuel function of the rings A of multiplicity $e \leqslant 5$ (Proposition 4.4). In this section we also study the rigidity of PHS_A and the Cohen-Macaulayness of $Gr(A)$. In particular we show that if $\rho = \rho_{0,b,e}$ (resp. $\rho = \rho_{1,b,e}$ then $Gr(A)$ is (resp. is not) Cohen-Macaulay, and for $b = e - 1$, $Gr(A)$ is Cohen-Macaulay if and only if $\rho = \rho_{0,e-1,e}$ (Proposition 4.6).

Let A be a one-dimensional Cohen-Macaulay local ring with maximal ideal m. We will denote by FHS_A (resp. $PHS_A(T) = eT - \rho$) the Hilbert-Samuel function (resp. polynomial) of A, i.e. $FHS_A(n) = \text{lenght}_A(A/m^n)$ for all $n \geqslant 0$ and $PHS_A(n) = FHS_A(n)$ for $n \gg 0$. We say that e is the multiplicity of A and ρ is the reduction number of A. The embedding dimension of A is $b = \dim_{A/m}(m/m^2)$.

From now we put $A/m = \mathbf{k}$, and we assume that $\mathbf{k}$ is infinite. Let $\hat{A}$ be the

m-adic completion of A; since $b(A) = b(\hat{A})$, and $FHS_A = FHS_{\hat{A}}$, we may assume that A is complete. Recall that the integer $i(A) = \text{Max}\{n \geqslant 0 | PHS_A(n) \neq FHS_A(n)\} + 1$ is the regularity index of A ([Sch]). Throughout this paper R will be the power series ring $\mathbf{k}[[X_1, \ldots, X_N]]$. We denote by M the maximal ideal of R. As usual $Gr_m(A)$ is the graded ring associated to A; if $A = R/I$ then we denote by I^* the homogeneous ideal of $Gr_M(R) = \mathbf{k}[X_1, \ldots, X_N]$ such that $Gr_m(A) = Gr_M(R)/I^*$. We denote by $s(I)$ the integer $s(I) = \text{Max}\{n \geqslant 0 | I \subset M^n\}$ and by $v(I)$ the number of elements of a minimal basis of I.

Given a numerical function $F: \mathbf{N} \to \mathbf{N}$, $\Delta F: \mathbf{N} \to \mathbf{N}$ will be the numerical function defined by $\Delta F(0) = 0$, and for all $n \geqslant 1 \Delta F(n) = F(n) - F(n-1)$. We set $\Delta^h F = \Delta(\Delta^{h-1} F)$ for all $h \geqslant 2$.

1. The bounds

We say that $x \in m \backslash m^2$ is superficial of degree one if and only if x verifies one of the following equivalent conditions ([E-1] Proposition 1).

(1) there exists an integer n_0 such that $(m^{n+1}: x) = m^n$ for all $n \geqslant n_0$,
(2) $(m^{n+1}: x) = m^n$, for all $n \geqslant i(A)$,
(3) $(m^{i(A)+2}: x) = m^{i(A)+1}$,
(4) $m^{n+1} = xm^n$ for all $n \geqslant i(A)$.

Since $\mathbf{k}$ is infinite we may assume that there exists a degree-one superficial element x of A ([S-4], Chap. 1, Proposition 3.2). Notice that x is a non zero divisor of A.

We denote by $Bl(A)$ the ring of the blow-up of A ([M], Chap. 12).

PROPOSITION 1.1. *A and $Bl(A)$ are free $\mathbf{k}[[x]]$-modules of* rank e.

Proof. Since A is complete and x is a non-zero divisor, we have $\mathbf{k}[[x]] \subset A$ and we can consider A and $Bl(A)$ as $\mathbf{k}[[x]]$-modules without torsion. Recall that for all $n \geqslant i(A)$ we have $m^{n+1} = xm^n$, so A is a finitely generated $\mathbf{k}[[x]]$-module. Since $Bl(A)$ is a finitely generated A-module ([M], Proposition 12.1) we have that A and $Bl(A)$ are free $\mathbf{k}[[x]]$-modules ([A-B], Chap. 10 Proposition 3.1).

On the other hand $\text{rank}(A) = \dim_{\mathbf{k}}(A/xA) = e$ ([M], Proposition 12.5), and from [M], Proposition 12.5, we also get that $\text{rank}(Bl(A)) = e$.

PROPOSITION 1.2. *There exists an isomorphism of $\mathbf{k}[[x]]$-modules*

$$A/(x^{b-2}) \cong \bigoplus_{i=1}^{e} \mathbf{k}[[x]]/(x^{b-2}).$$

Proof. From [M], Proposition 12.5, we have that $T = A/(x^{b-2})$ is an A-module of lenght $(b-2)e$. By [A-B], Chap. 10 Proposition 5.6, we know that $T \cong \oplus_{i=1}^{r} \mathbf{k}[[x]]/(x^{n_i})$. Since $x^{b-2} T = 0$ we get that $n_i \leqslant b - 2$ for all

$i = 1, \ldots, r$. Recall that $r = \text{lenght}_A(T/xT) = e$ ([M], Proposition 12.5), from this it is easy to find the claim.

PROPOSITION 1.3. $m^{e-1} \subset (x^{b-2})$.

Proof. Let π_l be the l-th projection $T = \oplus_{i=1}^{e} \mathbf{k}[[x]]/(x^{b-2}) \to \mathbf{k}[[x]]/(x^{b-2})$, $1 \leqslant l \leqslant e$. Assume that $m^{e-1}T \neq 0$, so there exists l_0 such that $\pi_{l_0}(m^{e-1}T) \neq 0$. Hence for all $n = 1, \ldots, e-2$ we have that $\pi_{l_0}(m^{n+1}T) \not\subset \pi_{l_0}(m^n T)$. From this it is easy to see that $\dim_{\mathbf{k}}(\pi_{l_0}(m^{e-1}T)) = 0$, so we get a contradiction.

THEOREM 1.4. $x^{e-b+1}Bl(A) \subset A$.

Proof. Notice that we can write $Bl(A) = m^{e-1}/x^{e-1}$ (see proof of [M], Theorem 12.1(1)), so by Proposition 1.3 we obtain $x^{e-b+1}Bl(A) = m^{e-1}/x^{b-2} \subset A$.

REMARK 1. Notice that Theorem 1.4 could be written as follows: x^{e-b+1} belongs to the conductor of the extension $A \subset Bl(A)$.

The following result is inspired in the proof of Theorem 2.1 of [H-W].

PROPOSITION 1.5. *For all* $t \geqslant 1$ *and* $r \geqslant 0$ *there exist* $F_1, \ldots, F_s \in m^t$, $s = \text{Min}\{t+1, e\}$, *such that the cosets of* $x^r F_1, \ldots, x^r F_s$ *in* m^{t+r}/m^{t+r+1} *form a* $\mathbf{k}$*-linear independent set.*

Proof. Let ω be a superficial element of degree one of A. Assume that $1 \leqslant t \leqslant i(A) - 1$. J. Herzog and R. Waldi in [H-W], Theorem 2.1, proved that if we have elements $x_1, \ldots, x_{i(A)} \in m$ such that $x_1 \cdots x_{i(A)} \notin \omega m^{i(A)-1} + m^{i(A)+1}$ then the elements $F_i = \omega^i x_{i+1} \ldots x_{i(A)}$, $i = 0, \ldots, t$ verify the following conditions: the cosets of F_i, $i = 0, \ldots, t$, in m^t/m^{t+1} form a $\mathbf{k}$-linear independent set and the cosets of $x_{t+1} \ldots x_{i(A)} F_i$ in $m^{i(A)}/m^{i(A)+1}$ form also a $\mathbf{k}$-linear independent set.

First step is to show that we can take $x_1 = \cdots = x_{i(A)} \in m$. Assume that for all $L \in m \backslash m^2$ it holds $L^{i(A)} \in \omega m^{i(A)-1} + m^{i(A)+1}$. Then we have that $m^{i(A)} \subset \omega m^{i(A)-1} + m^{i(A)+1}$, from this it is easy to deduce that $\omega m^{i(A)-1} = m^{i(A)}$. Hence we get that $\dim_{\mathbf{k}}(m^{i(A)-1}/m^{i(A)}) = e$ ([M], Proposition 12.10). But this gives us a contradiction with the definition of $i(A)$, so we can assume that there exist $L \in m \backslash m^2$ such that $L^{i(A)} \notin \omega m^{i(A)-1} + m^{i(A)+1}$. Notice that after a linear change we can suppose $L = x$.

From the definition of superficial element we obtain that the map

$$m^n/m^{n+1} \xrightarrow[.x]{} m^{n+1}/m^{n+2},$$

is an isomorphism for all $n \geqslant i(A)$. From this we get the claim.

DEFINITION. We denote by $\rho_{0,b,e}$ the integer $\rho_{0,b,e} = (r+1)e - \binom{r+b}{r}$, where r is the integer such that $\binom{b+r-1}{r} \leqslant e < \binom{b+r}{r+1}$. We put $\rho_{1,b,e} = e(e-1)/2 - (b-1)(b-2)/2$.

THEOREM 1.6. *Let A be a one-dimensional Cohen-Macaulay local ring with embedding dimension b, multiplicity e, and reduction number ρ. Then*

$$\rho_{0,b,e} \leqslant \rho \leqslant \rho_{1,b,e}.$$

Proof. First of all we will prove $\rho \leqslant \rho_{1,b,e}$.

Since $Bl(A)/A$ is a torsion $\mathbf{k}[[x]]$-module (Theorem 1.4,), there exist integers $a_1 \leqslant \cdots \leqslant a_e$ such that

$$Bl(A)/A \cong \bigoplus_{i=1}^{e} \mathbf{k}[[x]]/(x^{a_i}),$$

([A-B], Chap. 10 Theorem 5.6). Hence

$$\sum_{i=1}^{e} a_i = \rho.$$

LEMMA 1.7. *For all $t, e-1 \geqslant t \geqslant 0$,*

$$\dim_{\mathbf{k}}(A + x^t Bl(A)/A + x^{t+1} Bl(A)) \leqslant \operatorname{Max}\{0, e-t-1\}.$$

Proof of lemma 1.7. Let $F_1, \ldots, F_{t+1}$ be the elements of Proposition 1.5. Since $F_i = x^t(F_i/x^t)$ and $F_i/x^t \in Bl(A)$ we can consider the coset of $F_1, \ldots, F_{t+1}$ in $A + x^t Bl(A)/A + x^{t+1} Bl(A)$. Assume that this cosets are $\mathbf{k}$-linear dependent, this means that there exist $\lambda_1, \ldots, \lambda_{t+1} \in \mathbf{k}$, $r \geqslant 0$, and $z \in m^r$ such that

$$\sum_{i=1}^{t+1} \lambda_i F_i = x^{t+1}(z/x^r),$$

from this we deduce that in A it holds:

$$\sum_{i=1}^{t+1} \lambda_i x^r F_i = x^{t+1} z \in m^{r+t+1}.$$

From Proposition 1.5 we get $\lambda_1 = \cdots = \lambda_{t+1} = 0$, so the cosets of $F_1, \ldots, F_{t+1}$ in $A + x^t Bl(A)/A + x^{t+1} Bl(A)$ form a $\mathbf{k}$-linear independent set. From this and [M], Theorem 12.5, we deduce Lemma 7.

From the last Lemma it is straightforward that

$$a_i \leqslant \operatorname{Min}\{i-1, e-b+1\},$$

for $i = 1, \ldots, e-1$, so we have

$$\rho = \sum_{i=1}^{e-1} a_i \leqslant \rho_{1,b,e}.$$

From [K], Theorem 2, we get $PHS_A(e-1) = FHS_A(e-1)$, by [M], Theorem 12.10, we obtain $\rho_{0,b,e} \leqslant \rho$.

REMARK 2. Notice that from Theorem 1.6 we recover the following well known results: for $b = 2$ we have $\rho = \rho_{0,2,e} = \rho_{1,2,e} = e(e-1)/2$. For $b = 3$ we get $\rho \leqslant e(e-1)/2 - 1$ ([K], Corollary 2), and for $b = e$ we obtain $\rho = \rho_{0,e,e} = \rho_{1,e,e} = e - 1$ ([M], Theorem 12.15).

2. Construction of curves

In this section we will assume that $\mathbf{k}$ is an algebraically closed field.

A curve of $(\mathbf{k}^N, 0) = \mathrm{Spec}(R)$ is a one-dimensional, Cohen-Macaulay closed subscheme X of $(\mathbf{k}^N, 0)$, i.e. $X = \mathrm{Spec}(R/I)$ where $I = I(X)$ is a perfect height $N-1$ ideal of R; we put $\mathcal{O}_X = R/I$. A branch is an integral curve. From now we will denote by $(n_1, \ldots, n_s)$ the monomial curve $(\mathbf{k}^s, 0)$ defined by $(t^{n_1}, \ldots, t^{n_s})$.

If X is a reduced curve of $(\mathbf{k}^N, 0)$ we denote by $\delta(X)$ the dimension over $\mathbf{k}$ of the quotient $\tilde{\mathcal{O}}_X/\mathcal{O}_X$ where $\tilde{\mathcal{O}}_X$ is the integral closure of $\mathcal{O}_X$. If r is the number of branches of X then we define the Milnor number of X by $\mu(X) = 2\delta - r + 1$.

Let X be a reduced curve and let Q be an infinitely near point of X, see [E-Ch], [V der W]. It is known that there exists a unique sequence $Q_{i=0,\ldots,s}$ of infinitely near points of X such that $Q_0 = 0, \ldots, Q_s = Q$, and that Q_{i+1} belongs to the first neighbourhood of Q_i for $i = 0, \ldots, s-1$. We denote by (X, Q) the union of the irreducible components throught Q of the proper transform of X by the composition of the blowing-up centered at Q_i for $i = 0, \ldots, s-1$. We denote by $e(X, Q)T - \rho(X, Q)$ the Hilbert polynomial of the local ring $\mathcal{O}_{(X,Q)}$. We put $i(X) = i(\mathcal{O}_X)$, and $s(X) = s(I(X))$.

Let $\mathcal{T}(X)$ be the set of infinitely near points Q of X such that its multiplicity $e(X, Q)$ is greater than one. From [C] we obtain that

$$\delta(X) = \sum_{Q \in \mathcal{T}(X)} \rho(X, Q). \tag{F1}$$

Hence if X is a curve such that the only singular infinitely near point is 0, then we have $\delta(X) = \rho(X)$. In particular the monomial curve X defined by $(t^e, t^{e+1}, t^{s(3)}, \ldots, t^{s(N)})$ with $e + 1 \leqslant s(3) \leqslant \cdots \leqslant s(N)$ verifies

$$\delta(X) = \rho(X) = \mathrm{Card}(\mathbf{N} \backslash \langle e, e+1, s(3), \ldots, s(N) \rangle).$$

REMARK 1. Recall ([K], Theorem 2) that $\rho(A)$ take values in $[e-1, e(e-1)/2]$, so for all $e \geqslant 1$ and $e-1 \leqslant \rho \leqslant e(e-1)/2$, there exists a monomial ring A with $PHS_A = eT - \rho$. To prove this consider the semigroup Γ generated by e and $e+1$, and the increasing sequence of integers $(n_i)_{i=1,\ldots,(e-1)(e-2)/2}$ defined by $\mathbf{N}\backslash(\Gamma \cup \{1, 2, \ldots, e-1\})$.

The monomial ring defined by $e, e+1, n_i i = \rho - e + 2, \ldots, (e-1)(e-2)/2$ has reduction number ρ.

Let X, Y be the curves of $\mathbf{k}$, we denote by $(X.Y)$ the number $\dim_{\mathbf{k}}(R/I(X) + I(Y))$([H]).

It is well known (see for example [H]) that if $Y_1, \ldots, Y_r$ are the branches of X then it holds:

$$\delta(X) = \sum_{i=1}^{r} \delta(Y_i) + \sum_{i=1}^{r-1} \left(Y_i \cdot \left(\bigcup_{j=i+1}^{r} Y_j \right) \right). \tag{F2}$$

Notice that if X and Y only share the origin as infinitely near point we have (F1 and F2):

$$\rho(X \cup Y) = \rho(X) + \rho(Y) + (X \cdot Y). \tag{F3}$$

From this we will construct curves with suitable reduction number (Propositions 2.1, 2.4 and 2.6).

PROPOSITION 2.1. *Let X be a reduced curve. Given a general hyperplane H and a reduced curve $Y \subset H$ such that $i(X) \leqslant s(Y) - 1$, it holds*

$$\rho(X \cup Y) = \rho(X) + \rho(Y) + e(X).$$

Proof. Let $h \in R$ be an equation of H, and assume that the coset of h in $\mathcal{O}_X$ is a degree one superficial element. First of all we will prove:

CLAIM. $I(Y) \subset I(X) + (h)$.

Proof of the claim. From [S-4], Chapter 2 Theorem 3.1, we get

$$FHS_{R/I(X)+(h)}(n) = \Delta FHS_{R/I(X)}(n) + \dim_{\mathbf{k}}((m^n : h)/(m^{n-1})).$$

By [E-1], Proposition 1, we obtain

$$FHS_{R/I(X)+(h)}(n) = \Delta FHS_{R/I(X)}(n).$$

for all $n \geqslant i(X) + 1$. Hence

$$FHS_{R/I(X)+(h)}(n) = e(X),$$

for all $n \geqslant i(X)+1$. Since $\dim_{\mathbf{k}}(R/I(X) + (h)) = e(X)$, ([M], Theorem 12.5) from this we deduce the claim.

From the claim we get $(X.Y) = e(X)$; since H is general we can assume that X and Y only share, as infinitely near point, the origin so we have (F3):

$$\rho(X \cup Y) = \rho(X) + \rho(Y) + e(X).$$

DEFINITION. We denote by $T(X)$ the triplet $(b(X), e(X), \rho(X))$.

COROLLARY 2.2. *Let X be a reduced curve. There exists a reduced curve Y such that*

$$T(Y) = T(X) + (1, 1, 1).$$

Proof. We put $b = b(X)$. Assume that X is contained in the hyperplane $Z = 0$ of $(\mathbf{k}^{b+1}, 0) = \mathbf{k}[[X_1, \ldots, X_b, Z]]$. Consider the line L defined by $X_1 = \cdots = X_b = 0$. From the Proposition 2.1 we deduce the claim for $Y = X \cup L$.

To end this section we will give the second set of curves with a suitable reduction number. For this we need some preliminar results:

PROPOSITION 2.3. *If we denote by $\Gamma = \langle e, e+1 \rangle$ the numerical semigroup generated by e and $e+1$, then $e(e-1)$ is the conductor of Γ and it holds*:

$$\Gamma \backslash e(e-1) + \mathbf{N} = \bigcup_{j=0}^{e-2} \{ej, \ldots, (e+1)j\}.$$

Proof. Straightforward.

PROPOSITION 2.4. *For all integers $e, b, 2 \leqslant b \leqslant e$, there exists a monomial curve $C_{e,b} \subset (\mathbf{k}^b, 0) = \operatorname{Spec}(\mathbf{k}[[X, Y, X_1, \ldots, X_{b-2}]])$ such that $T(X) = (b, e, \rho_{1,b,e})$.*

Proof. Let us consider the monomial curve $\alpha_j = (e, e+1, (e+1)j, \ldots, e(j+1)-1)$, for $j = 1, \ldots, e-2$. We denote by $\Gamma(\alpha_j)$ the semigroup generated by $e, e+1, (e+1)j, \ldots, e(j+1)-1$. Notice that $e(\alpha_j) = e$, $b(\alpha_j) = e-j+1$ and $ej + \mathbf{N} \subset \Gamma(\alpha_j)$ (Proposition 2.1). Since $\rho(\alpha_j) = \delta(\alpha_j) = \operatorname{Card}\{\mathbf{N} \backslash \Gamma(\alpha_j)\}$, we get that $\rho(\alpha_j) = e(e-1)/2 - (e-j)(e-j-1)/2$. From this we deduce that it suffices to take $C_{e,b} = \alpha_{e+1-b}$ in order to obtain $\rho(C_{e,b}) = \rho_{1,b,e}$.

PROPOSITION 2.5. *For all $e, b, 2 \leqslant b \leqslant e$ it holds*:

$$t^* = \operatorname{Min}\{n \in \mathbf{N} \mid G \in (X, X_1, \ldots, X_{b-2}), Y^n - G \in I(C_{e,b})\} \geqslant e + 1 - b.$$

Proof. Assume that there exists $G \in (X, X_1, \ldots, X_{b-2})$ such that $Y^n - G \in I(C_{e,b})$ with $n < e+1-b$. Notice that $order_t(X_i) \geqslant (e+1)(e+1-b)$, see proof of Proposition 2.4, so there exists $H(X) \in k[[X]]$ such that $Y^n - H(X) \in I(C_{e,b})$.

Since $b \geqslant 2$ we get $n < e - 1$, by the other hand the ideal of the monomial curve $(e, e+1)$ is generated by $Y^e - X^{e+1}$. From this we obtain a contradiction.

PROPOSITION 2.6. *Given integers* $e \geqslant 2, 2 \leqslant b < e, 1 \leqslant n \leqslant e - b$, *there exist a non-singular curve* $\gamma \subset (\mathbf{k}^{b+1}, 0)$ *such that*:

$$T(C_{e,b} \cup \gamma) = (b + 1, e + 1, \rho_{1,b,e} + n).$$

Proof. We put $(\mathbf{k}^{b+1}, 0) = \mathrm{Spec}(R)$ where $R = \mathbf{k}[[X, Y, X_1, \ldots, X_{b-2}, T]]$. Let γ be the non-singular curve defined by the ideal $(X, X_1, \ldots, X_{b-2}, T - Y^n)$. Let us consider $C_{e,b}$ as curve of $(\mathbf{k}^{b+1}, 0)$ via the immersion defined by the projection $R \to R/(T)$.

We only need to prove that $b(\beta) = b + 1$ and $\rho(\beta) = \rho_{1,b,e} + n$ where $\beta = C_{e,b} \cup \gamma$. From the definition of $C_{e,b}$ it is easy to see that the reduced tangent cone of this curve is the line $Y = X_1 = \cdots = X_{b-2} = Y = 0$, by the other hand the tangent cone of γ is $X = X_1 = \ldots, X_{b-2} = T = 0$. Hence the only infinitely near point shared by $C_{e,b}$ and γ is the origin, so

$$\rho(\beta) = \rho(C_{e,b}) + (C_{e,b} \cdot \gamma).$$

LEMMA 2.7. $(C_{e,b} \cdot \gamma) = n$.

Proof of lemma 2.7. A straightforward computation gives us

$$I(C_{e,b}) + I(\gamma) = (X, X_1, \ldots, X_{b-2}, T) + (Y^s),$$

where $s = \mathrm{Min}\{t^*, n\}$. From Proposition 2.5 we deduce the claim.

To end the proof of Proposition 2.6 it suffices to prove that $b(\beta) = b + 1$. Assume that $b(\beta) \leqslant b$, so there exists a non-singular hypersurface H of $(\mathbf{k}^{b+1}, 0)$ containing β. Let G be an equation of H, since H contains γ it holds

$$G = G' + a(T - Y^n),$$

where $a \in R$ and $G' \in (X, X_1, \ldots, X_{b-2})$.

Assume $a(0) \neq 0$. Since $G \in I(C_{e,b})$ we get

$$T - Y^n + a^{-1}G' \in I(C_{e,b}),$$

and then

$$Y^n - a^{-1}G' \in I(C_{e,b}).$$

From the Proposition 2.5 we deduce $n \geqslant e + 1 - b$, so we obtain a contradiction with the assumption $n \leqslant e - b$.

Suppose $a(0) = 0$, since H is non-singular we have that order$(G') = 1$. From this we deduce that $G(X, Y, X_1, \ldots, X_{b-2}, 0)$ belongs to the ideal of $C_{e,b} \subset (\mathbf{k}^b, 0)$, so the embedding dimension of $C_{e,b}$ is not greater than $b - 1$. Hence we obtain a contradiction with Proposition 2.4.

REMARK 2. We proved (Proposition 2.4) that given b, e there exists a reduced curve, $C_{e,b}$, with maximal number reduction $\rho_{1,b,e}$. On the other hand A.V. Geramita and F. Orecchia, [G-O], Theorem 4, prove that, given b, e, there exists a reduced curve with maximal Hilbert-Samuel function ([O-1], [O-2]), it is easy to see that such a curve has minimal number reduction $\rho_{0,b,e}$

3. Characterization of the triplets (b, e, ρ)

The aim of this section is to prove the main result of this paper:

THEOREM 3.1. *There exists a one-dimensional Cohen-Macaulay local ring A, with multiplicity e, embedding dimension b and reduction number ρ if and only if either $b = 1$, $e = 1$, $\rho = 0$ or:*
(1) $2 \leqslant b \leqslant e$, *and*
(2) $\rho_{0,b,e} \leqslant \rho \leqslant \rho_{1,b,e}$.
Moreover: for each triplet (b, e, ρ) satisfying (1) *and* (2) *we can take $A = \mathbf{k}[[X_1, \ldots, X_b]]/I$ reduced, with $\mathbf{k}$ an algebraically closed filed of characteristic zero.*

Proof. The "only if part" follows from Theorem 1.6 and [M], Theorem 12.10 and Proposition 12.16.

We will prove the existencial part by induction on the pair (b, e).

For $e \leqslant 4$ it suffices to take the following monomial curves:

e	b	ρ	*curve*
1	1	0	(1)
2	2	1	(2, 3)
3	2	3	(3, 4)
3	3	2	(3, 4, 5)
4	2	6	(4, 5)
4	3	5	(4, 5, 11)
4	3	4	(4, 5, 6)
4	4	3	(4, 5, 6, 7)

The case $e = 5$ is studied in Proposition 4.4, so we can assume $e \geqslant 6$.

If $b = 1$ then $e = 1$ and $\rho = 0$, this case is included in the table.

If $b = 2$ then we can take $X \subset (\mathbf{k}^2, 0)$ a reduced plane curve of multiplicity e.

Notice that in this case we have $\rho_{0,b,e} = \rho_{1,b,e} = e(e-1)/2$ (see Section 1 Remark 2).

First step is to prove the result for $b = 3$, after this we will prove the result for a general $b \leqslant 4$.

LEMMA 3.2. *Let ρ be an integer such that $\rho_{0,3,e} \leqslant \rho \leqslant \rho_{1,3,e}$. There exists a reduced curve X such that $T(X) = (3, e, \rho)$.*

Proof. We set $t_0 = [e/2]$. We will prove the Lemma in three steps according with $\rho \in [p_{0,3,e}, \rho^*], [\rho^*, p_{0,2,e-1} + 1], [\rho_{0,2,e-1} + 1, \rho_{1,3,e}]$.

STEP 1. We will construct reduced curves Z such that $T(Z) = (3, e, \rho)$ with $\rho \in [p_{0,3,e}, p^*]$. We will use Proposition 2.1.

Let r be the integer such that: $\binom{r+2}{2} \leqslant e < \binom{r+3}{2}$. Notice that $r \leqslant t_0 - 1$, so $e - r - 1 \geqslant t_0$.

For each $t = t_0, \ldots, e - r - 1$ we will denote by $i(t)$ the regularity index of a curve singularity X such that $T(X) = (3, t, \rho_{0,3,t})$ (Section 2, Remark 2). Notice that $i(t) \leqslant r + 1$.

CLAIM 1. The following statements hold:

(i) $\rho_{0,3,t} + e - t - 1 - i(t) \leqslant \rho_{1,3,t}$,

(ii) $e - t - 1 - i(t) \geqslant 0$.

Proof of the Claim 1: (i) Since $\rho_{1,3,t} - \rho_{0,3,t} \geqslant t - 2$, we need to prove $t - 2 \geqslant e - t - 1 - i(t)$.

Since $e \geqslant 6$ we have $i(t) \geqslant 2$, so it suffices to prove $2t \geqslant e - 1$. From $t \geqslant t_0$ we get (i).

The inequality (ii) follows from the assumption $t \leqslant e - r - 1$ and the fact $i(t) \leqslant r + 1$.

The Claim 1 enable us to consider reduced curves X_ρ with $T(X_\rho) = (3, e, \rho)$, $\rho \in I_t = [\rho_{0,3,t}, \rho_{0,3,t} + e - t - 1 - i(t)]$. Notice that such a curve verifies $i(X_\rho) \leqslant e - t - 1$. Let Y_t be a reduced plane curve of multiplicity $e - t$, so $s(Y_t) = e - t$. From Proposition 2.1 we get that for all $t = t_0, \ldots, e - r - 1$ and every $\rho \in I_t$ there exists a reduced curve $Z = X_\rho \cup Y_t$ such that $T(Z) = (3, e, \rho + t + \rho_{0,2,e-t})$.

To obtain Lemma 3.2 we only need to prove that

$$\bigcup_{t_0}^{e-r+1} I_t + t + \rho_{0,2,e-t} = [\rho_{0,3,e}, \rho^*].$$

CLAIM 2. The following statements hold:

(i) $\rho_{0,3,e-r-1} + \rho_{0,2,r+1} + e - r - 1 = \rho_{0,3,e}$,

(ii) $\mathrm{Max}(I_{t+1}) + 1 = \mathrm{Min}(I_t)$, for $t \pm t_0, \ldots, e - r - 1$,

(iii) $\rho_{1,3,t_0} + \rho_{0,2,e-t_0} + t_0 = \rho^*$.

Proof of Claim 2: (i) Notice that $e - r - 1 \leqslant \binom{r+2}{2}$.

If $e-r-1=\binom{r+2}{2}$ then $\rho_{0,3,e-r-1}=(r+1)(e-r-1)-\binom{r+3}{3}$, so (i) follows by a straightforward computation.

If $e-r-1<\binom{r+2}{2}$ then $\rho_{0,3,e-r-1}=r(e-r-1)-\binom{r+2}{3}$, from this it is easy to find (i).

(ii) It suffices to prove

$$\rho_{0,3,t+1}+e-i(t+1)+\rho_{0,2,e-t-1}=\rho_{0,3,t}+\rho_{0,2,e-t}+t. \tag{1}$$

Let j be the integer such that $\binom{j+2}{2}\leqslant t+1<\binom{j+3}{2}$, so $\rho_{0,3,t+1}=(j+1)(t+1)-\binom{j+3}{3}$. Moreover $i(t+1)\leqslant j+1$.

Let h be the integer such that $\binom{h+2}{2}\leqslant t<\binom{h+3}{2}$, so $\rho_{0,3,t}=(h+1)t-\binom{h+3}{3}$. Notice that $j-1\leqslant h\leqslant j$, and that $h=j-1$ if and only if $t=\binom{j+2}{2}-1$.

Assume $h=j$. Since $t+1>\binom{j+2}{2}$ we have $i(t+1)=j+1$.

In this case (1) takes the form

$$(j+1)(t+1)-\binom{j+3}{3}+e-j-1+(e-t-1)(e-t-2)/2$$
$$=(j+1)t-\binom{j+3}{3}+(e-t)(e-t-1)/2+t.$$

From this we get (ii).

Suppose $h=j-1$. In this case we have $t=\binom{j+2}{2}-1$ and $i(t+1)=j$. A similar argument are done in the previous case show us (ii).

The equality (iii) follows from the definition of ρ^*.

STEP 2. Let ρ^* be the integer $\rho^*=(e-t_0)(e-t_0-1)/2+t_0(t_0-1)/2+t_0$. We will prove that for each $\rho\in[\rho^*, p_{0,2,e-1}+1]$ there exists a reduced curve Z with $T(Z)=(3,e,\rho)$.

Let us consider the reduced plane curves of $(\mathbf{k}^3,0)$ defined by:

$$Z_1=\operatorname{Spec}(R/(X_3,X_2^{e-t}-X_1^{e-t+1}))\quad\text{and}\quad Z_2=\operatorname{Spec}(R/(X_1,X_3^t-X_2^n))$$

with $t\leqslant n$, and $1\leqslant t\leqslant t_0$.

Notice that $(Z_1.Z_2)=\dim(R/J)$ where $J=(X_1,X_3,X_2^{e-t},X_2^n)$, so $(Z_1,Z_2)=\operatorname{Min}\{e-t,n\}$. If we take $t\leqslant n\leqslant e-t$ then we get $(Z_1.Z_2)=n$.

On the other hand Z_1 and Z_2 only share the origin as infinitely near point, so

$$\rho(Z_1\cup Z_2)=(e-t)(e-t-1)/2+t(t-1)/2+n$$

with $1\leqslant t\leqslant t_0, t\leqslant n\leqslant e-t$. Hence for every $t, 2\leqslant t\leqslant t_0$, and for all $\rho\in[(e-t)(e-t-1)/2+t(t-1)/2+t,(e-t)(e-t-1)/2+t(t-1)/2+e-t]$ there exists a reduced curve $Z=Z_1\cup Z_2$ such that $T(Z)=(e,3,\rho)$.

A straightforward computation show us

$$(e-t)(e-t-1)/2 + t(t-1)/2 + e - t$$
$$= (e-t+1)(e-t)/2 + (t-1)(t-2)/2 + t - 1,$$

so we have that for all $\rho \in [p^*, p_{0,2,e-1} + 1]$ there exists a reduced curve Z with $T(Z) = (3, e, \rho)$.

STEP 3. From Propositions 2.4 and 2.6 we deduce that for all $\rho \in [\rho_{0,2,e-1} + 1, \rho_{1,3,e}]$ there exists a reduced curve Z with $T(Z) = (3, e, \rho)$.

From now we will assume $e \geqslant 6$ and $b \geqslant 4$. Notice that by Proposition 2.4 for all $b \geqslant 3$ there exists a reduced curve $C_{b,b}$ such that $T(C_{b,b}) = (b, b, \rho_{1,b,b})$. Recall that in this case we have $\rho_{1,b,b} = \rho_{0,b,b} = b - 1$ ([M], Theorem 12.15), so we can assume $b < e$.

We will consider two cases:

Case 1. $b > [e/2]$. We consider $X_i = C_{i,i}$ for $i = 1, 2, \ldots, e - b + 1$, by from [M], Theorem 12.15 we obtain that $i(X_i) = 1$. Notice that $e - b + 1 \leqslant b$.

By induction hypothesis we know that for all integers $i = 1, 2, \ldots, e - b + 1$ and $\rho \in [\rho_{0,b-1,e-i}, \rho_{1,b-1,e-i}]$ there exists a reduced curve Y_ρ with $T(Y_\rho) = (b - 1, e - i, \rho)$, since $b \geqslant 3$ we have $s(Y_\rho) \geqslant 2$.
From Proposition 2.1 we deduce that for all integer ρ with

$$\rho \in [\rho_{0,b-1,e-i}, \rho_{1,b-1,e-i}] + 2i - 1$$

there exists a reduced curve $Z = X_i \cup Y_\rho$ of $(\mathbf{k}^b, 0)$ with $T(Z) = (b, e, \rho)$.

CLAIM 3. (i) $\rho_{0,b-1,b-1} + 2(e - b + 1) - 1 = \rho_{0,b,e}$,
(ii) For all $i = 1, 2, \ldots, e - b$ it holds $\rho_{0,b-1,e-i} + 2i - 1 \leqslant \rho_{1,b-1,e-i-1} + 2(i + 1) - 1$.

Proof of the Claim 3. (i) from [M], Theorem 12.15 we get $\rho_{0,b-1,b-1} = b - 2$, so we need to prove $2e - b - 1 = \rho_{0,b,e}$. This equality follows from the assumption $b > [e/2]$. (ii) follows from a straightforward computation.

From the Claim 1 we deduce that for all $\rho = \rho_{0,b,e}, \ldots, \rho_{1,b-1,e-1} + 1$ there exists a reduced curve Z such that $T(Z) = (b, e, \rho)$. By Propositions 2.4 and 2.6 we obtain the result for $b > [e/2]$.

Case 2. $b \leqslant [e/2]$. Let r be the integer such that: $\binom{b+r-1}{r} \leqslant e < \binom{b+r}{r+1}$, and put $e = \binom{b+r-1}{r} + v$.

Let K be the integer: $K = \binom{b+r-1}{r}$ if $v \geqslant \binom{b+r-2}{r}$, and $K = e - \binom{b+r-2}{r}$ if $v < \binom{b+r-2}{r}$.

First of all notice that $b \leqslant K$; if $v \geqslant \binom{b+r-2}{r}$ then $b \leqslant \binom{b+r-1}{r} = K$. On the other hand if $v < \binom{b+r-2}{r}$ then $K = e - \binom{b+r-2}{r} \geqslant \binom{b+r-1}{r} - \binom{b+r-2}{r} = \binom{b+r-2}{r-1}$. For $r \geqslant 2$ we get $b \leqslant K$.

Assume $r = 1$, in this case $e - \binom{b+r-2}{r} = e - b + 1 \geqslant b$ because $b \leqslant [e/2]$.

Let i be an integer of $[b, K]$, and let $j(i)$ be the integer such that $\binom{b+j(i)-1}{j(i)} \leqslant i < \binom{b+j(i)}{j(i)+1}$. We denote by $s(i)$ the integer such that $\binom{b+s(i)-2}{s(i)} \leqslant e - i < \binom{b+s(i)-1}{s(i)+1}$.

CLAIM 4. The following statements hold:

(i) $e - i \geqslant b - 1$,

(ii) $s(i) \geqslant j(i)$,

(iii) $\rho_{0,b-1,e-i} + s(i) - j(i) \leqslant \rho_{1,b-1,e-i}$.

Proof of Claim 4. (i) $e - i \geqslant e - K \geqslant \binom{b+r-2}{r} \geqslant b - 1$.

(ii) Since $i \leqslant \binom{b+r-1}{r}$ we get $j(i) \leqslant r$. From the fact $e - i \geqslant \binom{b+r-2}{r}$ we deduce $s(i) \geqslant r$, so we have proved $j(i) \leqslant r \leqslant s(i)$. Hence we find (ii).

If $s(i) = 1$ then $j(i) = 1$ and we obtain (iii).

Assume $s(i) = 2$. In this case we have $j(i) = 1, 2$ and $e - i > b - 1$. From this it is easy to get $\rho_{1,b-1,e-i} - \rho_{0,b-1,e-i} \geqslant 1 \geqslant s(i) - j(i)$.

Suppose now $s(i) \geqslant 3$, since $\rho_{1,b-1,e-i} - \rho_{0,b-1,e-i} \geqslant s(i)$ we get (iii).

By induction hypothesis and Claim 4 we can consider curves Y_ρ such that $T(Y_\rho) = (b - 1, e - i, \rho)$ with $\rho = \rho_{0,b-1,e-i}, \ldots, \rho_{0,b-1,e-i} + s(i) - j(i)$

CLAIM 5. If Y is a curve with $T(Y) = (b - 1, e - i, \rho)$, $\rho \leqslant \rho_{0,b-1,e-i} + s(i) - j(i)$ then $s(Y) \geqslant j(i) + 2$.

Proof of the Claim 5. Assume that $s(Y) \leqslant j(i)$, then

$$\rho \geqslant \rho_{0,b-1,e-i} + \binom{b + s(i) - j(i) + 1}{b} \geqslant \rho_{0,b-1,e-i} + s(i) - j(i) + 2.$$

Hence we get a contradiction.

Let X_i be a reduced curve such that $T(X_i) = (b, i, \rho_{0,b,i})$ (Section 2 Remark 2), since $i(X_i) \leqslant j(i) + 1$, by Proposition 2.1 we deduce that for every $i = b, \ldots, K$ and $\rho \in I_i = [\rho_{0,b-1,e-1}, \ldots, \rho_{0,b-1,e-i} + s(i) - j(i)]$ there exists a reduced curve $Z = X_i \cup Y_\rho$ with $T(Z) = (b, e, \rho + i + \rho_{0,b,i})$.

CLAIM 6. (i) $\rho_{0,b-1,e-K} + K + \rho_{0,b,K} = \rho_{0,b,e}$.

(ii) $\mathrm{Min}(I_i) \leqslant \mathrm{Max}(I_{i+1})$ for all $i = b, \ldots, K$.

Proof of the Claim 6. (i) If $v \geqslant \binom{b+r-2}{r}$ then $e - K = v$; from the fact $e < \binom{b+r}{r+1}$ we get that r is the integer such that: $\binom{b+r-2}{r} \leqslant v < \binom{b+r-1}{r+1}$, so $\rho_{0,b-1,v} = (r + 1)v - \binom{b+r-1}{r}$. From this we get (i).

Assume that $v < \binom{b+r-2}{r}$, so $e - K = \binom{b+r-2}{r}$. Hence we have $\rho_{0,b-1,e-K} = (r + 1)(e - K) - \binom{b+r-1}{r}$. Now we need to compute $\rho_{0,b,K}$; let s be the integer such that $\binom{b+s-1}{s} \leqslant K < \binom{b+s}{s+1}$.

From the assumption $v < \binom{b+r-2}{r}$ we deduce $s = r - 1$, so $\rho_{0,b,K} = rK - \binom{b+r-1}{r-1}$. From this we obtain (i).

To prove (ii) it suffices to show that

$$\rho_{0,b-1,e-i} + \rho_{0,b,i} \leqslant \rho_{0,b-1,e-i-1} + \rho_{0,b,i+1} + s(i + 1) - j(i + 1) + 1. \qquad (1)$$

From the definition of $s(i)$ and $j(i)$ we get

$\rho_{0,b,i} = (j(i)+1)i - \binom{j(i)+b}{j(i)}$,
$\rho_{0,b-1,e-i} = (s(i)+1)(e-i) - \binom{b+s(i)-1}{s(i)}$,
$j(i) \leqslant j(i+1) \leqslant j(i)+1$, and $s(i)-1 \leqslant s(i+1) \leqslant s(i)$.

Hence (1) takes the form:

$$(s(i)+1)(e-i) - \binom{b+s(i)-1}{s(i)} + (j(i)+1)i - \binom{j(i)+b}{j(i)} \tag{2}$$
$$\leqslant (s(i+1)+1)(e-i-1) - \binom{b+s(i+1)-1}{s(i+1)} +$$
$$+ (j(i+1)+1)(i+1) - \binom{j(i+1)+b}{j(i+1)} + s(i+1) - j(i+1) + 1.$$

Assume $j(i) = j(i+1)$, so (2) is equivalent to

$$(s(i)+1)(e-i) - \binom{b+s(i)-1}{s(i)} + \binom{b+s(i+1)-1}{s(i+1)} \leqslant \tag{3}$$
$$\leqslant s(i+1) - j(i+1) + 1.$$

If $s(i+1) = s(i)$ then (3) holds.

Suppose $s(i+1) = s(i) - 1$, in this case (3) takes the form

$$e - i \leqslant \binom{b+s(i)-2}{s(i)} + 1. \tag{4}$$

Notice that if $s(i+1) = s(i) - 1$ then $e - i = \binom{b+s(i)-2}{s(i)}$, so in this case (4) also holds. Assume $j(i+1) = j(i) + 1$, so $i = \binom{b+j(i)}{j(i)+1} - 1$. Hence (2) becomes

$$(s(i)+1)(e-i) - \binom{b+s(i)-1}{s(i)} + \binom{b+s(i+1)-1}{s(i+1)} \tag{5}$$
$$\leqslant (s(i+1)+1)(e-i-1) - \binom{j(i)+b}{j(i)+1} + s(i+1) + i + 2.$$

If $s(i) = s(i+1)$ then (5) holds.

Suppose $s(i+1) = s(i) - 1$. Hence (5) takes the form

$$e - 2i - 1 \leqslant \binom{b+s(i)-2}{s(i)} - \binom{j(i)+b}{j(i)+1}. \tag{6}$$

Notice that from $s(i+1) = s(i) - 1$ one can deduce $e - i = \binom{b+s(i)-2}{s(i)}$ so (6) is equivalent to $\binom{j(i)+b}{j(i)+1} \leqslant i+1$. Recall that this is an equality, so we obtain Claim 6.

From the Claim 6 we get that for every integer $\rho = \rho_{0,b,e}, \ldots, \rho_{0,b,b} + b + \rho_{0,b-1,e-b} + s(b) - j(b)$ there exists a reduced curve Z such that $T(Z) = (b, e, \rho)$.

Notice that for $i = b$ we have $j(b) = 1$, so

$$\rho_{0,b,b} + b + \rho_{0,b-1,e-b} + s(b) - j(b) \geqslant \rho_{0,b-1,e-b} + 2b - 2.$$

Hence for all $\rho = \rho_{0,b,e}, \ldots, \rho_{0,b-1,e-b} + 2b - 2$ there exists a reduced curve X with $T(X) = (b, e, \rho)$.

We know for each integer $i = 1, 2, \ldots, b$ and for all $\rho = \rho_{0,b-1,e-i}, \ldots, \rho_{1,b-1,e-i}$ there exists a reduced curve Y_ρ with $T(Y_\rho) = (b-1, e-i, \rho)$. Let X be a reduced curve with $T(X_i) = (i, i, \rho_{0,i,i})$. Notice that $i(X_i) = 1$ and that $s(Y_\rho) \geqslant 2$, so we can apply Proposition 2.1. From the Claim 3 we deduce that for all $\rho = \rho_{0,b-1,e-b} + 2b - 1, \ldots, \rho_{1,b-1,e-1} + 1$ there exists a reduced curve Z such that $T(Z) = (b, e, \rho)$. By Propositions 2.4 and 2.6 we obtain the result in the case 2.

4. Small multiplicities

The aim of this section is twofold: to compute the Hilbert-Samuel functions of the rings of multiplicity less or equal than 5, and to study the Cohen-Macaulayness of $Gr(A)$ where A is a ring with extremal reduction number.

DEFINITION. We say that a polynomial $p(T)$ is rigid if there exists a numerical function $F: \mathbf{N} \to \mathbf{N}$ such that, if A is a Cohen-Macaulay local ring, with Hilbert-Samuel polynomial $PHS_A = p$ then its Hilbert-Samuel function is F.

In the following result we will give rigid polynomials for the one-dimensional Cohen-Macaulay rings. From this we compute the Hilbert-Samuel functions of the rings of multiplicity $e \leqslant 4$. In particular we prove that every polynomial $p = eT - \rho$ with $e \leqslant 4$ is rigid.

PROPOSITION 4.1. (1) *The polynomials* $p = eT - \rho$ for $\rho = e-1, e, e(e-1)/2 - 1, e(e-1)/2$ *are rigid and the associated functions are the following*:

p	$\Delta(F)$
$P_1 = eT - (e-1)$	$1, e, \ldots$
$P_2 = eT - e$	$1, e-1, e, \ldots$
$P_3 = eT - (e(e-1)/2 - 1)$	$1, 3, 4, \ldots, e, \ldots$
$P_4 = eT - (e(e-1)/2)$	$1, 2, 3, \ldots, e, \ldots$

(2) *For all integers* $1 \leqslant e \leqslant 4$ *and* $e - 1 \leqslant \rho \leqslant e(e-1)/2$ *the polynomial* $p =$

$eT - \rho$ is rigid. The associated functions are the following:

e	ρ	$\Delta(F)$
1	0	$1, 1, \ldots$
2	1	$1, 2, \ldots$
3	2	$1, 3, \ldots$
3	3	$1, 2, 3, \ldots$
4	3	$1, 4, \ldots$
4	4	$1, 3, 4, \ldots$
4	5	$1, 3, 3, 4, \ldots$
4	6	$1, 2, 3, 4, \ldots$

Proof. From [M] Proposition 12.15 we deduce that P_1 is rigid and we get its associated function. For P_2 see [K] Corollary 3; for P_3 see [K] Corollary 4; for P_4 see [K] Corollary 2. Hence the first part is proved. The second part follows from the first one.

From Propositions 3.1 and 4.1 it is easy to prove:

PROPOSITION 4.2. *Let $p(T) = eT - \rho$ be a polynomial with non-negative coefficients. Then $p(T)$ is rigid if and only if $p = eT - \rho$ with $\rho = e - 1, e, e(e-1)/2 - 1, e(e-1)/2$.*

PROPOSITION 4.3. *Let A be a one dimensional Cohen-Macaulay ring.*
(1) *If $\Delta FHS_A(n) = n$ then $\Delta FHS_A(t) = t$ for $t = n, \ldots, e$.*
(2) *If $\Delta FHS_A(n) \leqslant n + 1$ then $\Delta FHS_A(t) = t + 1$ for $t = n, \ldots, e$.*

Proof. Follows [St], Theorem 2.2, and Remark c to this result.

PROPOSITION 4.4. *Let A be a one dimensional local ring of multiplicity 5. Then $b(A)$ and $\rho(A)$ determine the Hilbert-Samuel function of A*:

ρ	b	ΔFHS_A	*curve*
4	5	$1, 5, \ldots$	(5, 6, 7, 8, 9)
5	4	$1, 4, 5, \ldots$	(5, 6, 7, 8)
6	3	$1, 3, 5, \ldots$	(5, 6, 8)
6	4	$1, 4, 4, 5, \ldots$	(5, 6, 9, 13)
7	3	$1, 3, 4, 5, \ldots$	(5, 6, 9)
7	4	$1, 4, 4, 4, 5, \ldots$	(5, 6, 13, 14)
8	3	$1, 3, 4, 4, 5, \ldots$	(5, 6, 14)
9	3	$1, 3, 3, 4, 5, \ldots$	(5, 6, 19)
10	2	$1, 2, 3, 4, 5, \ldots$	(5, 6)

Proof. From Proposition 4.3(1) we obtain the cases $\rho = 4, 5, 9, 10$.

Since $2 \leqslant b(A) \leqslant e$, by Proposition 4.4 we deduce that if $\rho = 6, 7, 8$ then $b(A) = 3, 4$.

LEMMA 4.5. *If* $\rho = 8$ *then* $b(A) = 3$.

Proof. In this case FHS_A can take two values (Proposition 4.3):

(1) $FHS_A = F_1$, where $F_1(0) = 0$, $F_1(1) = 1, F_1(2) = 5$, $F_1(3) = 8$ and $F_1(n) = 5n - 8$ for $n \geqslant 4$.

(2) $FHS_A = F_2$, where $F_2(0) = 0, F_2(1) = 1$, $F_2(2) = 4$, $F_2(3) = 7$ and $F_2(n) = 5n - 8$ for $n \geqslant 4$.

We will show that the second case is not possible. Suppose that $FHS_A = F_2$, then we have:

$$\dim_{\mathbf{k}}(m^2/m^3) = \dim_{\mathbf{k}}(\mathbf{k}[[X, Y, Z]]/I^*(2)) = 3,$$

Hence there exists h_1, h_2, h_3, elements of I, such that their initial forms, say H_1, H_2, H_3, form a $\mathbf{k}$-basis of $I^*(2)$. From [E-2], Theorem 3, we get that every minimal basis of I has 3 elements, so $\{h_1, h_2, h_3\}$ is a minimal basis of I.

Let $(a_{ij})_{i=1,2,3;\, j=1,2}$ be a matrix with entries in $\mathbf{k}[[X, Y, Z]]$ such that its maximal minors are h_1, h_2, h_3 ([Bu] Theorem 5). If A_{ij} is the initial form of a_{ij} then it is easy to see that the maximal minors of (A_{ij}) are H_1, H_2, H_3.

Notice that $J = (H_1, H_2, H_3)$, is a height two ideal of $\mathbf{k}[X, Y, Z]$, if $ht(J) = 1$ then we have $\dim_{\mathbf{k}}(m^3/m^4) \leqslant 4$. Hence we can assume that $ht(J) = 2$, so J is a perfect ideal ([E-N], Theorem 1).

From [Rob-V], Corollary 4.4, we get that $\{h_1, h_2, h_3\}$, standard basis of I, so we have a homogeneous minimal resolution of $\mathbf{k}[X, Y, Z]/I^*$([E-N])

$$0 \to S(-3)^2 \xrightarrow[(A_{ij})]{} S(-2)^3 \xrightarrow[(H_i)]{} S \to S/I^* \to 0$$

where $S = \mathbf{k}[X, Y, Z]$. From this resolution we have $\dim_{\mathbf{k}}(m^3/m^4) = 3$ so we obtain a contradiction with $F_2(4) = 5$.

REMARK 1. Now we will prove that in general the triplet (b, e, ρ) does not determine the Hilbert-Samuel function. Let Y the union of 5 straight lines in a plane H and a straight line not contained in H. From [G-M-R], Corollary 2.8, we obtain that the Hilbert-Samuel function of Y is $\{1, 3, 4, 5, 6, \ldots\}$ so its Hilbert-Samuel polynomial is $6T - 11$. On the other hand the monomial curve $(6, 7, 11)$ has Hilbert-Samuel function $\{1, 3, 5, 5, 5, 6, \ldots\}$ and Hilbert-Samuel polynomial $6T - 11$. Hence the triplet $(3, 6, 11)$ does not determine the Hilbert-Samuel function. Recall that the Hilbert-Samuel function of the curve $(6, 7, 11)$ can be computed by hand or taking an explicit basis of $I(6, 7, 11)$ ([H]) and then using Macaulay system ([B-S]). Finally from Proposition 4.3 and [E-2], Theorem 3, it is easy to prove that the triplets $(3, 6, \rho)$, $\rho \neq 11$ determines the Hilbert-Samuel function.

REMARK 2. Let A be one dimensional Cohen-Macaulay local ring of

embedding dimension 3, multiplicity 6 and reduction number 12. From Proposition 4.3 and [M], Theorem 12.10, we get that the Hilbert-Samuel function of A must be either $F_1 = \{1, 3, 4, 5, 5, 6, \ldots\}$ or $F_2 = \{1, 3, 5, 4, 5, 6, \ldots\}$. By [E-2], Theorem 3, we deduce that $FHS_A = F_1$. Notice that F_1 and F_2 verifies the conditions (1), (2), (3) and (4) of the introduction, so these conditions are not sufficient to characterize the Hilbert-Samuel functions of the one dimensional Cohen-Macaulay local ring.

As we said in the introduction we will study Cohen-Macaulayness of $Gr(A)$, where A is a one-dimensional Cohen-Macaulay local ring with extremal reduction number.

Let A be a one-dimensional Cohen-Macaulay local ring. If $e = 1$ then it is well known that $Gr(A)$ is isomorphic to $(A/m)[X]$, so $Gr(A)$ is Cohen-Macaulay. Hence we may assume $e \geqslant 2$. From [S-1], Corollary 3, we have that $Gr(A)$ is Cohen-Macaulay for $e = 2, 3$. By the other hand if $b = 2$ then $Gr(A)$ is Cohen-Macaulay ([S-1]), and for $b = e$ the ring $Gr(A)$ is also Cohen-Macaulay ([S-1], Theorem 2).

PROPOSITION 4.6. *Let $e \geqslant 4$ be an integer. Let A be a one-dimensional Cohen-Macaulay local ring of multiplicity e, embedding dimension b, $3 \leqslant b \leqslant e - 1$, and reduction number ρ, then it holds*:

(1) *If $\rho = \rho_{0,b,e}$ then $Gr(A)$ is Cohen-Macaulay,*

(2) *If $\rho = \rho_{1,b,e}$ then $Gr(A)$ is not Cohen-Macaulay,*

(3) *If $b = e - 1$ then $Gr(A)$ is Cohen-Macaulay if and only if $\rho = \rho_{0,e-1,e}$.*

Proof. (1) If A has minimal reduction number then its Hilbert-Samuel function of A is maximal, by [O-1], Theorem 3.2, we obtain that Gr(A) is Cohen-Macaulay.

Assume that $Gr(A)$ is Cohen-Macaulay, then there exists degree one superficial element $x \in A$ such that:

$$FHS_{A/xA}(n) = \dim_k(m^{n-1}/m^n) + \dim_k((m^n : x)/m^{n-1}).$$

for all $n \geqslant 1$ ([S-4], Chap. 2, Note to Theorem 3.1). In particular we get that FHS_A is not decreasing.

Since FHS_A is not decreasing, it holds $FHS_A(n) \geqslant F(n)$ for all $n \geqslant 0$ where F is the numerical function defined by:

$$F(n) = \begin{cases} 1 & \text{for } n = 1 \\ b & \text{for } 2 \leqslant n \leqslant b - 1 \\ n + 1 & \text{for } b \leqslant n \leqslant e - 1 \\ e & \text{for } n \leqslant e. \end{cases}$$

If $\sigma = \sigma_{1,b,e}$ then $FHS_A = F$, so $FHS_{A/xA}(2) = 0$ and $EHS_{A/xA}(3) = 1$. By Nakayama's Lemma this is not possible. Hence we get (2).

Suppose now that $b = e - 1$ and $\dim_{\mathbf{k}}(m^2/m^3) \neq e$. Since $FHS_A(n) \geqslant F(n)$ for all $n \geqslant 0$ and [M], Theorem 12.10, we deduce $\dim_{\mathbf{k}}(m^2/m^3) = e - 1$ and $FHS_{A/xA}(2) = FHS_{A/xA}(3) = e - 1$. Hence $\text{Length}_A(A/xA) = e - 1$. From [M], Proposition 12.5, we obtain a contradiction, so $\dim_{\mathbf{k}}(m^2/m^3) = e$. By [M]. Proposition 12.5 we deduce $\rho = \rho_{0,e-1,e}$.

If $\rho = \rho_{0,e-1,e}$ then by (1) we deduce $Gr(A)$ Cohen-Macaulay.

REMARK 3. From [S-2], Theorem 1, we get that A is not Gorenstein for $b(A) = e - 1$ and $\rho(A) > \rho_{0,e-1,e}$ (see [S-3] for other results in this subject).

REMARK 4. Assume $\rho(A) = \rho_{0,b,e}$ and that there exists a non-negative integer with $e = \binom{b+i-1}{i}$. In this case we have that $Gr(A)$ is extremely compressed of type $e.Z$ ([F-L], Proposition 8), and that A is compressed of type $e.Z$ ([E-I]).

REMARK 5. Assume that $b(A) = 3$ and $A = R/I$. In [E-2], Theorem 5, we proved that $e \geqslant \binom{v(I)}{2}$. Moreover if $e = \binom{v(I)}{2}$ then $\rho = 2\binom{v}{3} = \rho_{0,3,e}$ and the ring $Gr(A)$ is Cohen-Macaulay.

REMARK 6. Let γ be a branch of $(\mathbf{k}^N, 0)$ and denote by $\hat{\gamma} \subset (\mathbf{k}^2, 0)$ the generic plane projection of γ. In [E-3] we proved that

$$\delta(\gamma) \leqslant \delta(\hat{\gamma}) \leqslant (e-1)\delta(\gamma) - (e-1)(e-2)/2. \tag{1}$$

We also proved that we have equality in (1) if and only if γ is isomorphic to the branch α_1 of Proposition 3.4, i.e. $\gamma \cong \alpha_1 = (e, e+1, \ldots, 2e-1)$. From this it is easy to prove that the following statements are equivalent:

(1) $\gamma \cong \alpha_1$,
(2) $\delta(\gamma) \leqslant \delta(\hat{\gamma}) \leqslant (e-1)\delta(\gamma) - (e-1)(e-2)/2$,
(3) $b = e$, $\rho = e - 1$,
(4) $\Delta FHS_\gamma(1) = 1$ and $\Delta FHS_\gamma(n) = e$ for all $n \geqslant 2$.

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