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CM-fields with all roots of unity

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Let $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{P}$ be the rational number field, the rational integer ring, the set of positive integers, and that of prime numbers, respectively. For each $p \in \mathbb{P}$, let $\mathbb{Q}_p$ denote the p-adic number field and $\mathbb{Z}_p$ the p-adic integer ring. We denote by $\hat{\mathbb{Z}}$ the direct product of all $\mathbb{Z}_p$, $p \in \mathbb{P}$:

$$\hat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p.$$

Let $\mathbb{N}'$ denote the set of at most countable cardinal numbers. Writing ∞ for the countable cardinal number, we then understand that $\mathbb{N}' = \mathbb{N} \cup \{0, \infty\}$. The additive group of each topological ring R will be denoted by the same letter R; for any $\nu \in \mathbb{N}'$, we let $\Pi^\nu R$ and $\oplus^\nu R$ denote respectively the direct product and the direct sum of ν copies of R. Now, let $\mathbb{C}$ be the complex number field, j the complex conjugation of $\mathbb{C}$, and J the Galois group of $\mathbb{C}$ over the real number field; $J = \{1, j\}$. For any (multiplicative) abelian group $\mathfrak{M}$ acted on by J, we put

$$\mathfrak{M}^- = \{\tau \in \mathfrak{M} \mid \tau^j = \tau^{-1}\}.$$

Then, viewing $\mathfrak{M}$ as a module over the group ring $\mathbb{Z}[J]$, we have $(\mathfrak{M}^-)^2 \subseteq \mathfrak{M}^{1-j} \subseteq \mathfrak{M}^-$. We shall suppose, throughout the following, all algebraic number fields to be contained in $\mathbb{C}$. For each algebraic number field F, let C_F denote the ideal class group of F, $\tilde{F}$ the maximal unramified abelian extension over F, and F^+ the maximal real subfield of F. In general, C_F is isomorphic to a subgroup of $\oplus^\infty(\mathbb{Q}/\mathbb{Z})$ while the Galois group $G(\tilde{F}/F)$ of $\tilde{F}/F$ is isomorphic to a topological quotient group of (the additive group of) $\Pi^\infty \hat{\mathbb{Z}}$; hereafter $G(\ )$ will denote the Galois group of the Galois extension in the parenthesis. When F is a CM-field, J acts on C_F and on $G(\tilde{F}/F)$ in the usual manner. We denote by $\mathbb{K}$ the maximal CM-field, so that $\mathbb{K}^+$ is nothing but the maximal totally real algebraic number field. We put

$$\zeta_n = e^{2\pi i/n} \quad \text{for each } n \in \mathbb{N}.$$

As is well known, the maximal abelian extension over $\mathbb{Q}$, which we denote by $\mathbb{Q}_{ab}$, is generated by all ζ_n, $n \in \mathbb{N}$, over $\mathbb{Q}$:

$$\mathbb{Q}_{ab} = \mathbb{Q}(\zeta_n \mid n \in \mathbb{N}).$$

In this paper, introducing first the notion of "wild extension", we shall generalize some results of Uchida [9] on unramified solvable extensions of algebraic number fields. We shall next show that for any CM-field K containing $\mathbb{Q}_{ab}$,

$$G(\tilde{K}/K) \cong \prod^{\infty} \hat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \left(\prod^{\infty} \mathbb{Z}_p \right) \quad \text{and} \quad G(\tilde{K}/K)^- \cong \prod^{\infty} \hat{\mathbb{Z}}.$$

On the other hand, we shall deduce from the above generalization that, given any map $f : \mathbb{P} \to \mathbb{N}'$, there exist infinitely many CM-fields $K \supseteq \mathbb{Q}_{ab}$ such that

$$C_K = C_K^- \cong \bigoplus_{p \in \mathbb{P}} \left(\bigoplus^{f(p)} (\mathbb{Q}_p/\mathbb{Z}_p) \right).$$

Moreover some related results, such as the following, will be added: $C_{\mathbb{K}} = C_{\mathbb{K}}^- = \{1\}$ (cf. [6]) while

$$C_K \cong \bigoplus^{\infty} (\mathbb{Q}/\mathbb{Z}), \quad (C_K^-)^2 = C_K^{1-j} \cong \bigoplus^{\infty} (\mathbb{Q}/\mathbb{Z})$$

for every CM-field $K \supseteq \mathbb{Q}_{ab}$ which is contained in a nilpotent extension over some finite algebraic number field in K^+ (cf. [1]). In the last part of the paper, we shall unite our results on wild extensions with classical results of Iwasawa [3] on solvable extensions.

We conclude this introduction by giving additional notations and remarks. Let F be any algebraic number field and let I_F denote the ideal group of F. An ideal of F, i.e., an element of I_F is considered to be an ideal of any algebraic number field F' containing F via the natural imbedding of I_F into the ideal group of F'. For each algebraic number $\alpha \neq 0$ (in $\mathbb{C}$), the principal ideal of $\mathbb{Q}(\alpha)$ generated by α is a principal ideal of any algebraic number field containing α, in the above sense, and will be denoted by (α). We shall write $F^\times$ for the multiplicative group of F. Throughout the paper, we shall often use basic facts in [8] on Galois cohomology, without mentioning this bibliography.

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1. Let k be any algebraic number field. An algebraic extension K over k is called wild when K/k is a Galois extension, every infinite prime of k is unramified in K, and for each finite prime $\mathfrak{B}$ of K, the inertia group of $\mathfrak{B}$ for K/k coincides with the ramification group of $\mathfrak{B}$ for K/k. As easily seen from this definition, the following lemma holds.

LEMMA 1. *With k as above, let $\mathfrak{s}$ be a set of finite primes of k and $\mathfrak{F}$ a family of algebraic extensions over k. If all fields in $\mathfrak{F}$ are wild extensions over k unramified outside $\mathfrak{s}$, then the composite of fields in $\mathfrak{F}$ is also a wild extension over k unramified outside $\mathfrak{s}$.*

Thus, given a set $\mathfrak{s}$ of finite primes of an algebraic number field k, there exists the maximal wild extension over k unramified outside $\mathfrak{s}$. We then denote by $k^{\mathfrak{s}}_{\mathrm{ws}}$ the intersection of this field and the maximal solvable extension over k: $k^{\mathfrak{s}}_{\mathrm{ws}}$ is nothing but the maximal wild solvable extension over k unramified outside $\mathfrak{s}$.

Next, for any positive integer m, we take the abelian extension

$$\mathfrak{C} = \mathbb{Q}(\zeta_q \mid q \in \mathbb{P},\ \equiv 1 \pmod m)$$

over $\mathbb{Q}$, and denote by $\mathbb{Q}^{(m)}$ the minimal intermediate field of $\mathfrak{G}/\mathbb{Q}$ such that $G(\mathfrak{G}/\mathbb{Q}^{(m)})^m = \{1\}$:

$$\mathbb{Q}^{(m)} = \{\alpha \in \mathfrak{G} \mid \alpha^\sigma = \alpha \text{ for all } \sigma \in G(\mathfrak{G}/\mathbb{Q}) \text{ with } \sigma^m = 1\}.$$

Let us now prove

THEOREM 1. *Let F be an algebraic number field containing $\mathbb{Q}^{(m)}$ for some $m \in \mathbb{N}$ and let $\mathfrak{S}$ be a set of finite primes of F. Then the cohomological dimension of the Galois group of $F^{\mathfrak{S}}_{\mathrm{ws}}$ over F is at most equal to* 1:

$$\mathrm{cd}\, G(F^{\mathfrak{S}}_{\mathrm{ws}}/F) \leqslant 1.$$

Proof. Let p be any prime number, S the set of prime numbers obtained by restricting the primes in $\mathfrak{S}$ on $\mathbb{Q}$, and K an intermediate field of $F^{\mathfrak{S}}_{\mathrm{ws}}/F$ such that $G(F^{\mathfrak{S}}_{\mathrm{ws}}/K)$ is a Sylow p-subgroup of $G(F^{\mathfrak{S}}_{\mathrm{ws}}/F)$. It suffices to show that

$$\mathrm{cd}\, G(F^{\mathfrak{S}}_{\mathrm{ws}}/K) \leqslant 1. \tag{1}$$

However, in the case $p \notin S$, this follows immediately from Theorem 1 of [9].

Indeed $F_{ws}^{\mathfrak{S}}$ is then the maximal unramified p-extension over K and K contains $\mathbb{Q}^{(m)}$ by the assumption.

Assume now that $p \in S$. In this case, we can prove (1) by modifying the proof of Theorem 1 of [9], as follows. Let L be any finite Galois extension over K in $F_{ws}^{\mathfrak{S}}$. For simplicity, we put

$$\mathfrak{G} = G(L/K).$$

Let W_p denote the group of pth roots of unity in $\mathbb{C}$: $W_p = \langle \zeta_p \rangle \cong \mathbb{Z}/p\mathbb{Z}$. Let us identify $G(L(\zeta_p)/K(\zeta_p))$ with $\mathfrak{G}$ so that $\mathfrak{G}$ acts on $L(\zeta_p)^\times$ and, trivially, on W_p. Assuming that

$$H^2(\mathfrak{G}, W_p) \neq \{1\}, \quad \text{i.e.,} \quad \mathfrak{G} \neq \{1\},$$

we take any 2-cocycle $\delta: \mathfrak{G} \times \mathfrak{G} \to W_p$ whose cohomology class in $H^2(\mathfrak{G}, W_p)$ is not trivial. Let

$$\{1\} \to W_p \to \mathfrak{F} \xrightarrow{\psi} \mathfrak{G} \to \{1\}$$

be the group extension of $\mathfrak{G}$ by W_p corresponding to δ, with the natural projection $\psi: \mathfrak{F} \to \mathfrak{G}$. For the proof of (1), it is now sufficient to find a Galois extension L' over K containing L such that there exists a $\mathfrak{G}$-isomorphism $\iota: G(L'/K) \xrightarrow{\sim} \mathfrak{F}$ for which $\iota(G(L'/L)) = W_p$ and the composite $\psi \circ \iota$ coincides with the restriction map $G(L'/K) \to \mathfrak{G}$.

Since $K(\zeta_p) \supseteq F \supseteq \mathbb{Q}^{(m)}$, Lemma 1 of [9] implies that the local degree of $K(\zeta_p)/\mathbb{Q}$ at each finite prime of $K(\zeta_p)$ is divisible by p^∞. Furthermore all infinite primes of $K(\zeta_p)$ are unramified in $L(\zeta_p)$. Hence, as in the proof of Lemma 5 of [11], we obtain

$$H^2(\mathfrak{G}, L(\zeta_p)^\times) = \{1\}.$$

In particular, δ is considered to be a 2-coboundary $\mathfrak{G} \times \mathfrak{G} \to L(\zeta_p)^\times$, namely, there exists a homomorphism $\beta: \mathfrak{G} \to L(\zeta_p)^\times$ such that

$$\delta(\sigma, \tau) = \beta(\tau)^\sigma \beta(\sigma\tau)^{-1} \beta(\sigma), \quad \sigma, \tau \in \mathfrak{G}.$$

Here, since each $\delta(\sigma, \tau)$ is in W_p and, as is well known, $H^1(\mathfrak{G}, L(\zeta_p)^\times) = \{1\}$, there also exists an element η of $L(\zeta_p)^\times$ such that

$$\beta(\sigma)^p = \eta^{\sigma - 1} \quad \text{for all } \sigma \in \mathfrak{G}.$$

Let $n = [L(\zeta_p): L]$, let ρ be a generator of the cyclic group $G(L(\zeta_p)/L)$, and choose an integer r satisfying

$$\zeta_p^{\rho} = \zeta_p^{r}, \quad r^n \equiv 1 \pmod{p}, \quad r^n \not\equiv 1 \pmod{p}.$$

The group ring $\mathbb{Z}[G(L(\zeta_p)/L)]$ acts on $L(\zeta_p)^{\times}$ in the obvious manner. By Lemma 2 of [9], we may assume that

$$\eta = \omega^{\theta}\xi^{p} \quad \text{for suitable } \omega, \xi \in L(\zeta_p)^{\times},$$

where θ is the element of $\mathbb{Z}[G(L(\zeta_p)/L)]$ defined by

$$\theta = \sum_{\nu=0}^{n-1} r^{n-\nu}\rho^{\nu}.$$

Let m_0 denote the product of distinct prime divisors of m different from p. As K contains $\mathbb{Q}^{(m)}$, there exists a Galois extension L_0/K_0 of finite algebraic number fields with the following properties:

(i) $L_0 \cap K = K_0$, $L_0 K = L$, $[L_0(\zeta_p): L_0] = n$,

(ii) L_0 is unramified over K_0 outside p; further, all prime ideals of K_0 dividing m_0 are completely decomposed in L_0,

(iii) η, ω, ξ, and all $\beta(\sigma)$, $\sigma \in \mathfrak{G}$, lie in $L_0(\zeta_p)$.

By (ii) above, the approximation theorem guarantees the existence of an element a of $K_0(\zeta_p)^{\times}$ such that, for each prime ideal $\mathfrak{v}$ of $K_0(\zeta_p)$ dividing m_0, ω/a is a pth power in the $\mathfrak{v}$-adic completion of $K_0(\xi_p)$ and $\mathfrak{w}(\omega/a) > 0$ for every real archimedian valuation $\mathfrak{w}$ of $L_0(\zeta_p)$. Then the same discussion as in page 314 of [9] shows that the principal ideal $(\eta a^{-\theta})$ is expressed in the form

$$(\eta a^{-\theta}) = \mathfrak{n}^{\theta}\mathfrak{a}^{p}\mathfrak{b}.$$

Here $\mathfrak{n}$ is an ideal of $K_0(\zeta_p)$ prime to mp, $\mathfrak{a}$ an ideal of $L_0(\zeta_p)$ prime to p, and $\mathfrak{b}$ that of $L_0(\zeta_p)$ whose numerator and denominator are products of prime ideals of $L_0(\zeta_p)$ dividing p. With t the order of the Frobenius automorphism

$$\left(\frac{K_0(\zeta_{mp})/K_0(\zeta_p)}{\mathfrak{n}}\right),$$

let K_1 be an extension of degree t over K_0 contained in K. By the Tschebotareff density theorem, there exists a prime ideal $\mathfrak{q}$ of $K_1(\zeta_p)$ unramified for $K_1(\zeta_p)/\mathbb{Q}$, of degree 1 over $\mathbb{Q}$, and belonging to the class of $\mathfrak{n}$ in the ray class group of $K_1(\zeta_p)$ modulo $(mp)\mathfrak{r}_{\infty}$ where $\mathfrak{r}_{\infty}$ is the product of all real infinite primes of $K_1(\zeta_p)$. It

follows that $\mathfrak{q}\mathfrak{n}^{-1} = (b)$ for some $b \in K_1(\zeta_p)$ with $b \equiv 1 \pmod{(mp)\mathfrak{r}_\infty}$. The field $L(\zeta_p, \sqrt[p]{\eta a^{-\theta} b^{\theta}}) = L(\zeta_p, \sqrt[p]{(\omega a^{-1} b)^{\theta}})$ is then an abelian extension of degree np over L. Furthermore the cyclic extension of degree p over L in that field becoms a Galois extension over K, which can be taken as the before-mentioned field L'. To prove this final assertion, one may only check the last part of the proof of Theorem 1 in [9]; so we omit the detail.

For any algebraic number field k, let k_{nil} denote the maximal nilpotent extension over k. The proof of Theorem 2 in [9], together with the above theorem, yields the following result.

THEOREM 2. *Let F be an algebraic number field such that*

$$\mathbb{Q}^{(m)} \subseteq F \subseteq k_{\mathrm{nil}}$$

for some positive integer m and some finite algebraic number field k in F. Let $\mathfrak{S}$ be a set of finite primes of F. Then $G(F_{\mathrm{ws}}^{\mathfrak{S}}/F)$ is isomorphic to the solvable completion of a free group with countable free generators.

Finally we add a result which follows immediately from the definition of a wild extension.

LEMMA 2. *Let k be an algebraic number field and $\mathfrak{s}$ a set of finite primes of k. Then:*
(i) *for any intermediate field F of $k_{\mathrm{ws}}^{\mathfrak{s}}/k$,*

$$F_{\mathrm{ws}}^{\mathfrak{S}} = k_{\mathrm{ws}}^{\mathfrak{s}}$$

where $\mathfrak{S}$ is the set of all primes of F lying above primes in $\mathfrak{s}$,
(ii) *if k is totally real, then so is $k_{\mathrm{ws}}^{\mathfrak{s}}$.*

2. For any multiplicative abelian group M on which J acts, we let

$$\mathfrak{M}^{+} = \{\tau \in \mathfrak{M} \mid \tau^{j} = \tau\},$$

so that $(\mathfrak{M}^{+})^2 \subseteq \mathfrak{M}^{1+j} \subseteq \mathfrak{M}^{+}$, $\mathfrak{M}^{1+j} \cong \mathfrak{M}/\mathfrak{M}^{-}$ and $\mathfrak{M}^{1-j} \cong \mathfrak{M}/\mathfrak{M}^{+}$. The purpose of this section is to prove the following.

THEOREM 3. *Let K be any* CM*-field containing $\mathbb{Q}_{\mathrm{ab}}$. Then, as profinite groups,*

$$G(\tilde{K}/K)^{-} \cong \prod^{\infty} \hat{\mathbb{Z}}, \quad G(\tilde{K}/K) \cong \prod^{\infty} \hat{\mathbb{Z}}.$$

Furthermore

$$G(\tilde{K}/K)^{+} \cong \prod^{\infty} \hat{\mathbb{Z}}$$

if K is contained in k_{nil} for some finite algebraic number field k in K^+.

For the proof of the above, we need

LEMMA 3. *Let L be a* CM*-field. Then*

(i) $G(\tilde{L}/L)^- \supseteq G(\tilde{L}/\tilde{L}\cap\mathbb{K}) \supseteq G(\tilde{L}/L)^{1-j}$,

(ii) *for any* CM*-field $L' \supseteq L$, $G(\tilde{L}/L)^{1-j}$ is contained in the image of $G(\widetilde{L'}/L')^-$ under the restriction map $G(\widetilde{L'}/L') \to G(\tilde{L}/L)$.*

Proof. Let F be any CM-field in L of finite degree. Since C_F^- contains the kernel of the norm map $C_F \to C_{F^+}$, it follows from class field theory that $G(\tilde{F}/F)^-$ contains $G(\tilde{F}/F\widetilde{F^+})$, the kernel of the restriction map $G(\tilde{F}/F) \to G(\widetilde{F^+}/F^+)$. Thus we have $G(\tilde{L}/L)^- \supseteq G(\tilde{L}/L\widetilde{L^+})$, which implies $G(\tilde{L}/L)^- \supseteq G(\tilde{L}/\tilde{L}\cap\mathbb{K})$ by $L\widetilde{L^+} \subseteq \tilde{L}\cap\mathbb{K}$. Furthermore, since $\tilde{L}\cap\mathbb{K}$ is a CM-field and an abelian extension over L, it is also an abelian extension over L^+ so that $G(\tilde{L}/\tilde{L}\cap\mathbb{K}) \supseteq G(\tilde{L}/L)^{1-j}$. This completes the proof of (i). We obtain (ii) from (i), noting that the restriction max in (ii) induces a surjective homomorphism $G(\widetilde{L'}/\widetilde{L'}\cap\mathbb{K}) \to G(\tilde{L}/\tilde{L}\cap\mathbb{K})$.

Proof of Theorem 3. Let A be any non-trivial finite abelian group. We can then take a cyclotomic field F such that $G(\tilde{F}/F)^{1-j}$ has a subgroup isomorphic to A (see, e.g., [2]). Hence it follows from Lemma 3 that there exists a group homomorphism of $G(\tilde{K}/K)^-$ onto A. On the other hand, $G(\tilde{K}/K)^-$ is torsion-free since so is $G(\tilde{K}/K)$ by Theorem 1 of [9]. Consequently

$$G(\tilde{K}/K)^- \cong \prod^{\infty}\hat{\mathbb{Z}}, \quad G(\tilde{K}/K) \cong \prod^{\infty}\hat{\mathbb{Z}}.$$

As K^+ includes $\mathbb{Q}^{(2)}$ and $G(\widetilde{K^+}/K^+)^2$ is the image of $G(\tilde{K}/K)^{1+j}$ under the restriction map $G(\tilde{K}/K) \to G(\widetilde{K^+}/K^+)$, the last assertion of Theorem 3 is now an immediate consequence of Theorem 2 in [9].

3. The main result of the present section is as follows.

THEOREM 4. *For any given map $f\colon \mathbb{P} \to \mathbb{N}'$, there exist infinitely many* CM*-fields K containing $\mathbb{Q}_{\mathrm{ab}}$ such that*

$$C_K = C_K^- \cong \bigoplus_{p\in\mathbb{P}} \left(\bigoplus^{f(p)} (\mathbb{Q}_p/\mathbb{Z}_p) \right).$$

To prove this, we prepare some notations and show two lemmas.

Let F be any algebraic number field. We then denote by F_{ws} the maximal wild solvable extension over F, namely, put

$$F_{\mathrm{ws}} = F_{\mathrm{ws}}^{\mathfrak{U}}$$

where $\mathfrak{U}$ is the set of all finite primes of F. We denote by M_F the maximal abelian

extension over F in F_{ws}. For each $p \in \mathbb{P}$, let $C_F(p)$ and $M_{F,p}$ denote respectively the p-primary component of C_F and the maximal p-extension over F in M_F, i.e., the maximal abelian p-extension over F unramified outside p; so that if F is a CM-field, $C_F(p)$ and $G(M_{F,p}/F)$, as well as $G(M_F/F)$, naturally become J-modules. Here, by a J-module, we mean of course an abelian group on which J acts. For any profinite group H, we let H^{ab} denote the maximal abelian quotient of H, i.e., the quotient group of H modulo the topological commutator subgroup of H. When H itself is a profinite abelian group, we let H^* denote the Pontryagin dual of H.

LEMMA 4. *Let p be any prime number. Let K be a* CM*-field containing $\mathbb{Q}^{(m)}$ for* some $m \in \mathbb{N}$ *and $\mathbb{Q}(\zeta_{p^n})$ for all $n \in \mathbb{N}$. Then $C_K(p)$ is a divisible group and, as discrete groups,*

$$(C_K(p)^-)^2 = C_K(p)^{1-j} \cong G(M_{K^+,p}/K^+)^*.$$

Proof. It is obvious that $G(M_{K,p}/K)$ is isomorphic to the Sylow p-subgroup of $G(K_{ws}/K)^{ab}$. However, since $K \supseteq \mathbb{Q}^{(m)}$ with $m \in \mathbb{N}$, Theorem 1 implies that $\operatorname{cd} G(K_{ws}/K) \leqslant 1$. Therefore $G(M_{K,p}/K)$ becomes a torsion-free $\mathbb{Z}_p$-module. Similarly, noticing $K^+ \supseteq \mathbb{Q}^{(2m)}$, we can see again from Theorem 1 that $G(M_{K^+,p}/K^+)$ is a torsion-free $\mathbb{Z}_p$-module.

The rest of the proof is devoted to essentially known discussions on the Kummer extension $M_{K,p}$ over K (cf. [5]). We let $\mathfrak{R}$ denote the quotient of the subgroup

$$\{\alpha \in M_{K,p} \mid \alpha^{p^n} \in K^\times \text{ for some integer } n \geqslant 0\}$$

of $M_{K,p}^\times$ modulo $K^\times$, which becomes a J-module in the obvious manner. Let L be the maximal abelian extension over K^+ in $M_{K,p}$, namely, the intermediate field of $M_{K,p}/K$ such that $G(M_{K,p}/L) = G(M_{K,p}/K)^{1-j}$. Then the natural isomorphism $\mathfrak{R} \xrightarrow{\sim} G(M_{K,p}/K)^*$ in Kummer theory induces

$$\mathfrak{R}^- \cong (G(M_{K,p}/K)/G(M_{K,p}/K)^{1-j})^* \cong G(L^+/K^+)^*.$$

Here $\mathfrak{R}$ is a divisible group; indeed we have shown that $G(M_{K,p}/K)$ is a torsion-free $\mathbb{Z}_p$-module. Hence

$$\mathfrak{R}^{1-i} = (\mathfrak{R}^-)^2 \cong (G(L^+/K^+)^*)^2. \tag{2}$$

Now let z be any class in $\mathfrak{R}$. We take an element α of z, so that $\alpha^{p^r} \in K^\times$ for some integer $r \geqslant 0$. Since all $\mathbb{Q}(\alpha^{p^r}, \zeta_{p^n})$, $n \in \mathbb{N}$, are subfields of K, there exists an

intermediate field k of $K/\mathbb{Q}(\alpha^{p^r}, \zeta_{p^r})$ with finite degree such that $k(\alpha)$ is unramified over k outside p and that each prime ideal of $\mathbb{Q}(\alpha^{p^r})$ dividing p is a p^rth power in the ideal group I_k of k. Therefore

$$(\alpha^{p^r}) = \mathfrak{a}^{p^r} \quad \text{for some } \mathfrak{a} \in I_K.$$

We then denote by c_z the ideal class in $C_K(p)$ containing $\mathfrak{a}$, which actually does not depend on the choice of α, $\mathfrak{a}$.

Thus, letting each class z' in $\mathfrak{R}$ correspond to $c_{z'}$, we obtain a J-module homomorphism $\mathfrak{R} \to C_K(p)$. Let E denote the unit group of K and define a J-module $\mathfrak{E}$ by

$$\mathfrak{E} = \{\alpha \in M_{K,p} \mid \alpha^{p^n} \in E \text{ for some } n \in \mathbb{Z}, \geqslant 0\}/E.$$

As easily seen, the above homomorphism induces the following exact sequence of J-modules:

$$\{1\} \to \mathfrak{E} \to \mathfrak{R} \to C_K(p) \to \{1\}. \tag{3}$$

In particular, it follows that $C_K(p)$ is a divisible group, whence

$$(C_K(p)^-)^2 = C_K(p)^{1-j}. \tag{4}$$

We also have

$$(\mathfrak{E}^-)^2 = \mathfrak{E}^{1-j} = \{1\}, \tag{5}$$

because the group of roots of unity in K is p-divisible. Therefore, in the case $p > 2$, the last assertion $C_K(p)^{1-j} \cong G(M_{K^+,p}/K^+)^*$ follows from (2), (3), (5), and the fact $L^+ = M_{K^+,p}$.

In the case $p = 2$, L is the maximal abelian 2-extension over K^+ unramified outside the primes of K^+ which are infinite or lie above 2. Hence L^+ is an abelian extension over $M_{K^+,2}$ such that $G(L^+/M_{K^+,2})^2 = \{1\}$. We can therefore view $G(M_{K^+,2}/K^+)^*$ as a subgroup of $G(L^+/K^+)^*$ containing $(G(L^+/K^+)^*)^2$. However $G(M_{K^+,2}/K^+)^*$ is a divisible group and, by (2), so is $(G(L^+/K^+)^*)^2$. Consequently we have $G(M_{K^+,2}/K^+)^* = (G(L^+/K^+)^*)^2$. This together with (2), (3), (4), and (5) completes the proof of Lemma 4 for the case $p = 2$.

The following lemma is an immediate consequence of Lemma 4.

LEMMA 5. *For any* CM-*field* $K \supseteq \mathbb{Q}_{ab}$, C_K *is divisible and*

$$(C_K^-)^2 = C_K^{1-j} \cong G(M_{K^+}/K^+)^*.$$

Proof of Theorem 4. Let F be any totally real finite Galois extension over $(\mathbb{Q}_{ab})^+$ such that $G(F/(\mathbb{Q}_{ab})^+)$ is isomorphic to a non-abelian simple group; for example, we may take as F a composite field of $(\mathbb{Q}_{ab})^+$ and a finite real Galois extension over $\mathbb{Q}$ with Galois group a symmetric group of degree $\geqslant 5$. Since $\mathbb{Q}^{(2)} \subseteq F \subseteq \mathbb{Q}(\alpha)_{nil}$ for any primitive element α of $F/(\mathbb{Q}_{ab})^+$, Theorem 2 implies that $G(F_{ws}/F)$ is isomorphic to a free pro-solvable group with countable free generators.

Next, let p be any prime number and T an inertia group for F_{ws}/F of a prime of F_{ws} lying above p. As every Sylow p-subgroup of $G(F_{ws}/F)$ is free, T is a free pro-p-group. With n being any positive integer, let q be a prime number $\equiv 1 \pmod{p^n}$ such that p is not a pth power (mod q); the existence of q is guaranteed by Tschebotareff's density theorem. Let N be the cyclic extension of degree p over $\mathbb{Q}$ with conductor q. We note that p remains prime in N. Let N_∞ denote the basic $\mathbb{Z}_p$-extension over N. It then follows from [4] that the unique prime of N_∞ above p is fully ramified in $M_{N_\infty,p}$. Furthermore, by [10], $G(M_{N_\infty,p}/N)$ is isomorphic to $\Pi^r \mathbb{Z}_p$ where

$$r = (p-1)\left(\operatorname{ord}_p \frac{q^{2(p-1)}-1}{4} - 2\right) \geqslant (p-1)(n-3),$$

ord_p denoting the p-adic exponential valuation. Hence T has at least r free generators, while n is an arbitrary positive integer. Thus T must be a free pro-p-group with countable free generators.

Now, let f be any map $\mathbb{P} \to \mathbb{N}'$. By the above discussion, we can take for each $p \in \mathbb{P}$, an intermediate field F_p of F_{ws}/F such that $G(F_{ws}/F_p)$ is contained in an inertia group, for F_{ws}/F, of a prime of F_{ws} above p and has exactly $f(p)$ free generators as a free pro-p-group. Let K be the composite of $\mathbb{Q}_{ab}$ and the intersection of all F_p, $p \in \mathbb{P}$. It is clear that

$$K \subseteq \mathbb{K}, \quad K^+ = \bigcap_{p \in \mathbb{P}} F_p.$$

However, as $\widetilde{K^+} \subseteq F_p$ for all $p \in \mathbb{P}$, we have $\widetilde{K^+} = K^+$. Therefore we see easily from the principal ideal theorem that

$$C_{K^+} = \{1\} \quad \text{whence} \quad C_K = C_K^-. \tag{6}$$

On the other hand, it follows from the choices of F_p, $p \in \mathbb{P}$, that

$$G(F_{ws}/K^+)^{ab} \cong \prod_{p \in \mathbb{P}} \left(\prod^{f(p)} \mathbb{Z}_p \right).$$

Since $(K^+)_{ws} = F_{ws}$ by Lemma 2, we also have $G(M_{K^+}/K^+) \cong G(F_{ws}/K^+)^{ab}$. Hence, by Lemma 5 and (6),

$$C_K = (C_K^-)^2 \cong (G(F_{ws}/K^+)^{ab})^* \cong \bigoplus_{p \in \mathbb{P}} \left(\bigoplus^{f(p)} (\mathbb{Q}_p/\mathbb{Z}_p) \right).$$

Furthermore, for any finite Galois extension F' over $(\mathbb{Q}_{ab})^+$ in $\mathbb{K}^+$ with $G(F'/(\mathbb{Q}_{ab})^+)$ a non-abelian simple group, the composite $F'_{ws}\mathbb{Q}_{ab}$ contains K if and only if $F' = F$. Theorem 4 is therefore proved.

Of course, for CM-fields containing $\mathbb{Q}_{ab}$ but not "so large", we can get a result analogous to that of Brumer [1].

PROPOSITION 1. *Let K be a* CM*-field containing $\mathbb{Q}_{ab}$ such that $K \subseteq k_{nil}$ for some finite algebraic number field k in K^+. Then $(C_K^-)^2 = C_K^{1-j}$ is isomorphic to the direct sum of countably infinite copies of $\mathbb{Q}/\mathbb{Z}$.*

Proof. This follows immediately from Theorem 2 and Lemma 5.

REMARK. Under the hypothesis of Proposition 1, we also have

$$C_K \cong \bigoplus^{\infty} (\mathbb{Q}/\mathbb{Z}).$$

Moreover it might be remarkable that $C_F^+ = C_{F^+} = \{1\}$ holds for every CM-field $F \supseteq \mathbb{Q}_{ab}$ if the so-called Greenberg conjecture in Iwasawa theory is generally true.

We next consider when the ideal class group of a CM-field $\supseteq \mathbb{Q}_{ab}$ vanishes.

LEMMA 6. *Let p and K be the same as in Lemma* 4. *Then the three conditions $C_K(p) = \{1\}$, $C_K(p)^- = \{1\}$, and $M_{K^+,p} = K^+$ are equivalent.*

Proof. By Lemma 4, the condition $M_{K^+,p} = K^+$ is a necessary one for $C_K(p)^- = \{1\}$. So it suffices to prove that $M_{K^+,p} = K^+$ implies $C_K(p) = \{1\}$. The principal ideal theorem shows, however, that $C_{K^+}(p) = \{1\}$ holds if K^+ coincides with the maximal unramified abelian p-extention over K^+. Hence, in the case $M_{K^+,p} = K^+$, we certainly have $C_{K^+}(p) = \{1\}$ so that $C_K(p) = C_K(p)^-$. We have further, by Lemma 4, $C_K(p)^2 = C_K(p)$ and $(C_K(p)^-)^2 = \{1\}$. Then $C_K(p)$ vanishes as desired.

We thus obtain

PROPOSITION 2. *For any* CM*-field $K \supseteq \mathbb{Q}_{ab}$, the following conditions are equivalent.*

(i) $C_K = \{1\}$,
(ii) $C_K^- = \{1\}$,
(iii) $M_{K^+} = K^+$,

(iv) $(K^+)_{ws} = K^+$,
(v) $K^+ = k_{ws}$ for some subfield k of $\mathbb{K}^+$.
In particular, $C_{\mathbb{K}} = \{1\}$ (cf. [6]).

4. In this final section, we generalize some results of the preceding sections.

Let F be any algebraic number field, $\mathfrak{T}$ a set of finite primes of F, and $\mathfrak{S}$ a subset of $\mathfrak{T}$. We take the family $\mathscr{G}$ of all Galois extensions F' over F unramified outside $\mathfrak{T}$ such that for each prime $\mathfrak{B}$ of F' whose restriction on F lies in $\mathfrak{S}$, the first ramification field of $\mathfrak{B}$ for F'/F coincides with the inertia field of $\mathfrak{B}$ for F'/F. Let $\Omega_F^{\mathfrak{S},\mathfrak{T}}$ denote the composite of all fields in $\mathscr{G}$. Then, as easily seen, $\Omega_F^{\mathfrak{S},\mathfrak{T}}$ also belongs to $\mathscr{G}$, i.e., $\Omega_F^{\mathfrak{S},\mathfrak{T}}$ is the maximal field in $\mathscr{G}$. We denote by $F_{sol}^{\mathfrak{S},\mathfrak{T}}$ the intersection of $\Omega_F^{\mathfrak{S},\mathfrak{T}}$ and the maximal solvable extension over F. Note that $F_{sol}^{\mathfrak{S},\mathfrak{S}} = F_{ws}^{\mathfrak{S}}$. The discussions of [9] and section 1 now lead us to the following result, which implies Theorems 6, 7 of [3] as well as our Theorems 1, 2.

THEOREM 5. *If* $F \supseteq \mathbb{Q}^{(m)}$ *for some* $m \in \mathbb{N}$, *then*

$$\operatorname{cd} G(\Omega_F^{\mathfrak{S},\mathfrak{T}}/F) \leqslant 1, \quad \operatorname{cd} G(F_{sol}^{\mathfrak{S},\mathfrak{T}}/F) \leqslant 1.$$

If, furthermore, $F \subseteq k_{nil}$ *for some finite algebraic number field* k in F, then $G(F_{sol}^{\mathfrak{S},\mathfrak{T}}/F)$ *is isomorphic to a free pro-solvable group with countable free generators.*

To weaken lastly the hypothesis of Proposition 1, we start with proving

PROPOSITION 3. *Let* F *be an algebraic number field containing* $\mathbb{Q}^{(m)}$ *for some* $m \in \mathbb{N}$. *Then* C_F *is a divisible group.*

Proof (cf. [1]). Let n be any positive integer and c any ideal class in C_F. It suffices to show that

$$x^n = c \quad \text{for some} \quad x \in C_F. \tag{7}$$

We write u for the order of c. Now there exists an element α of $\mathbb{Q}(\zeta_{mn})$ satisfying $F(\alpha) = F(\zeta_{mn}) \cap \tilde{F}$. There also exists an intermediate field k of $F/F \cap \mathbb{Q}(\zeta_{mn})$ with finite degree such that c contains an ideal $\mathfrak{a}$ of k whose uth power is principal in k and that α lies in $\tilde{k}$ whence $k(\alpha) = \tilde{k} \cap k(\zeta_{mn})$. Let q be a prime number $\equiv 1 \pmod{mu}$ not dividing the discriminant of k. Let k' be the composite of k and the cyclic extension of degree u over $\mathbb{Q}$ with conductor q. Note that F contains k'. Obviously the norm of $\mathfrak{a}$ for k'/k is $\mathfrak{a}^u$, a principal ideal of k. Hence, by class field theory, we have

$$\left(\frac{\tilde{k}'/k'}{\mathfrak{a}}\right) \in G(\tilde{k}'/k'\tilde{k}).$$

Since $\tilde{k}' \cap k'(\zeta_{mn}) = k'(\alpha) = k'(\tilde{k} \cap k(\zeta_{mn})) \subseteq k'\tilde{k}$, Tschebotareff's density theorem

shows that there exists a prime ideal $\mathfrak{J}$ of k' unramified for $k'/\mathbb{Q}$, of degree 1 over $\mathbb{Q}$, belonging to the ideal class of $\mathfrak{a}$ in $C_{k'}$, and completely decomposed in $k'(\zeta_{mn})$. Let l be the prime number divisible by $\mathfrak{J}$, so that $l \equiv 1 \pmod{mn}$. Let k'' be the composite of k' and the cyclic extension of degree n over $\mathbb{Q}$ with conductor l. As k'' is an intermediate field of F/k' of degree n over k' in which $\mathfrak{J}$ is fully ramified, we can then take, as x of (7), the ideal class in C_F that contains the prime ideal of k'' dividing $\mathfrak{J}$.

THEOREM 6. *Let K be a* CM*-field such that*

$$\mathbb{Q}(\zeta_{2p} \mid p \in \mathbb{P}) \subseteq K \subseteq k_{\mathrm{nil}}$$

with a subfield k of K^+ of finite degree. Then

$$(C_K^-)^2 = C_K^{1-j} \cong \bigoplus^{\infty} (\mathbb{Q}/\mathbb{Z}), \quad C_K \cong \bigoplus^{\infty} (\mathbb{Q}/\mathbb{Z}).$$

Proof. Let L be the composite of the maximal unramified Kummer extensions of exponents $2p$ over K for all $p \in \mathbb{P}$. Let E denote the unit group of K and E' the subgroup of $L^\times$ generated by the $2p$th roots in $L^\times$ of elements of E for all $p \in \mathbb{P}$. As J acts on $G(L/K)$ and on the quotient group E'/E in the obvious manner, we obtain from Kummer theory the following exact sequence of J-modules:

$$\{1\} \to E'/E \to G(L/K)^* \to C_K$$

(see the proof of Lemma 3 in [1] or of Lemma 4). This induces an exact sequence

$$\{1\} \to (E'/E)^- \to G(L_0/K^+)^* \to C_K^-,$$

where L_0 denotes the maximal abelian extension over K^+ in L^+. However, $((E'/E)^-)^2 \subseteq (E'/E)^{1-j} \subseteq WE/E$ with W the group of roots of unity in L while L_0 contains all unramified abelian extensions of degrees $2p$, $p \in \mathbb{P}$, over the intermediate field K^+ of $k_{\mathrm{nil}}/\mathbb{Q}^{(2)}$. Hence, by Theorem 2 of [9], C_K^- has a subgroup isomorphic to

$$\bigoplus_{p \in \mathbb{P}} \left(\bigoplus^{\infty} (\mathbb{Z}_p/2p\mathbb{Z}_p) \right).$$

Thus Proposition 3 completes the proof of Theorem 6.

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